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ADVANCED FIELD-THEORETICAL METHODS IN STOCHASTIC DYNAMICS AND THEORY OF DEVELOPED TURBULENCE

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Selected recent contributions involving fluctuating velocity fields to the rapidly developing domain of stochastic field theory are reviewed. Functional representations for solutions of stochastic differential equations and master equations are worked out in detail with an emphasis on multiplicative noise and the inherent ambiguity of the functional method. Application to stochastic models of isotropic turbulence of multi-parameter expansions in regulators of dimensional and analytic renormalization is surveyed. Effects of the choice of the renormalization scheme are investigated. Special attention is paid to the rôle and properties of the minimal subtraction scheme. Analysis of the consequences of symmetry breaking of isotropic turbulence with the use of the renormalization-group method is demonstrated by the effects due to helicity, strong and weak anisotropy. A careful description is given of the influence of turbulent advection on paradigmatic reaction-diffusion problems.

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Contents

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1 Solution of stochastic problems in field theory

In 1928 the relativistic equation of motion for an electron was postulated by P. A. M. Dirac [1]. This gave rise to tremendous development in theoretical physics, which ultimately led to a formulation of relativistic quantum mechanics. From the historical point of view the first example of such a theory is quantum electrodynamics (QED), which describes quantum theory of electromagnetic interactions between electrons, positrons and photons. New theoretical methods became available for calculation of differential cross sections for elementary processes in the high energy physics, which offered a controlled perturbation calculation in the form of asymptotic series in a small parameter – the fine-structure constant. However, it turned out that integrals, which terms of these series contain, are divergent both in the infrared (IR) and in the ultraviolet (UV) regions and mathematical expressions loose their meaning. It was realized that this problem is a property of any quantum field theory, not only quantum electrodynamics.

A method for the removal of ultraviolet divergences was put forward in the 50s by Stückelberg and Petermann [2] giving rise to the creation of a solid theoretical approach in the form of the renormalization group (RG). Group properties in this approach were found and mathematically rigorously defined by N. N. Bogolyubov and D. V. Shirkov [3–5].

The methods of perturbative renormalization group, UV renormalization and Feynman diagrammatic technique have become a standard part of theoretical physics. One of the most important observations originated from RG was the discovery of running constants. Roughly speaking, this means that physical quantities such as charge of the electron, its mass etc. are not constant, but their values depend on the scale at which they are measured. For instance, the fine-structure constant, which is approximately equal to $1/137$ at energy scales of order $100\ \mathrm{eV}$, attains value $1/128$ at energies of order $100\ \mathrm{GeV}$. This growth leads to a modification of the Coulomb law. QED is an example of abelian gauge theory. Yang and Mills generalized the principle of gauge invariance to non-abelian theories [6], which are the main ingredients of models of weak interactions and quantum chromodynamics (QCD). Gauge theories have played a crucial role in the explanation of properties of elementary particles and are the basis of the standard model, which is the most successful attempt of mankind for a reductionist description of the universe (without gravity).

The growth of the fine-structure constant in QED affects results of calculations, which can be experimentally observed, e.g., in the Lamb shift of the energy levels $2s_{1/2}$ and $2p_{1.2}$ for the hydrogen atom. On the other hand, the coupling constant in QCD exhibits completely different anomalous behavior, which did not correspond to observed in the beginning of 70's. To put it simply, the smaller distances are probed in the experiment, the weaker is the interaction between particles. Asymptotically the coupling constant goes to zero. For this discovery in 1973 [7, 8], known as asymptotic freedom, Gross, Wilczek and Politzer, were awarded the Nobel Prize in 2004. QCD is the only known (and confirmed) theory of "string"-like behavior in microworld, which at the larger scales leads to the confinement of quarks and gluons. A byproduct of all the subsequent discoveries was further development of quantum field theory and related mathematical methods.

About the same time, an unrelated branch of physics devoted to the study of continuous phase transitions in classical systems experienced many new developments. Let us recall that usually the critical behavior of a system is understood as its behavior near the critical point of a continuous phase transition, where a scaling with nontrivial exponents appears. For instance,

the critical exponent β characterizes the dependence on temperature of the magnetization of a ferromagnet in zero external field near critical temperature T_c ($T < T_c$)

$$M \sim (T_c - T)^{\beta} .$$

Calculation of the critical exponents, explanation of their universality and finding relations between them is the subject of study of the theory of critical behavior. The RG represents a powerful tool for solution of these problems. The RG approach was first applied by K. G. Wilson [9] (Wilson renormalization group) to static critical phenomena and soon after used to study critical dynamics as well [10,11]. Typical examples are the λ transition from normal to superfluid in ^{4}He and the transition from paramagnetic to ferromagnetic phase in uniaxial magnets (Ising model). The distinguishing physical feature is the presence of universal behavior, i.e. independence of the microscopic parameters of the model and importance of only gross properties such as space dimension, symmetry and nature of the order parameter. Moreover, in contrast to the typical problems in physics it was clear that the procedure of decoupling of scales is not possible - a characteristic scale was lacking in the critical system and all length scales must be treated simultaneously. This is the simple reason why the ordinary perturbation theory was not very useful. The lack of a typical scale manifests itself in the powerlike behavior with universal exponents.

In order to explain critical behavior new theoretical approaches were needed. Kadanoff [12] proposed a block scheme, in which using a specific contraction procedure it was possible to derive a macroscopic model from a microscopic one. However, the approach led often to uncontrolled approximations and worked only for special cases. An important contribution was made by the Soviet school - Landau, Pokrovskii, Patashinkii - who can be regarded as founders of the fluctuation theory of phase transitions [13, 14]. The crucial idea is the construction of model Hamiltonians with proper symmetry and physical considerations taken into account. In the beginning of 70s Wilson formulated a renormalization technique based on the integrating out of large-momentum degrees of freedom [15, 16]. This approach resulted in a conceptual framework, which is still in use. Wilson was awarded the Nobel Prize in 1983. Practical analytical tools such as dimensional regularization [17, 18], the famous ε expansion [19] and others were found [20]. With them in hand theoretical physicists have a set of rules which in principle allows computation of critical exponents (e.g. the Fisher index η) in a controlled fashion. Values obtained in this way are in a reasonable agreement with the experimentally measured values. Needless to say, critical exponents very often assume values, which are different from those predicted by the mean-field theory.

Functional formulation of quantum field theory based on the use of the generating functional [21–23] made it possible to forget about quantum nature of the fields and treat them as classical objects. In this sense a quantum-field operator corresponds to a fluctuating classical field. In the current literature field-theoretic methods has become the common term. It has to be stressed that in this approach the basic ingredients of the theory are classical fields, Lagrange functions, the corresponding actions and generating functionals. There is also a change in the underlying geometry - in the case of classical systems coordinates and times are given in euclidean space rather than in the Minkowski spacetime. Since the models analyzed are invariably non-relativistic, this feature boils down to the substitution of the Schrödinger evolution by the diffusion evolution, which is some cases (e.g. directed percolation) has the interpretation of time playing a role of a singled out direction. Another important difference is that, contrary

to non-relativistic quantum field theory, in classical reaction models the particle number is not conserved.

The first task in solving a classical statistical problem is to find the corresponding action functional. Its construction requires deep understanding of the physical situation and identification of relevant physical parameters. The use of methods of quantum field theory has enabled quick solution of problems in statistical physics on the basis of Ginzburg-Landau fluctuation theory. In particular, it has become possible to calculate critical indices to high orders in perturbation theory for diverse universality classes [24, 25].

Later dynamical models were formulated in terms of time-dependent fluctuations of slow variables near equilibrium. A detailed classification – which soon became standard – was proposed by Hohenber and Halperin [26]. An up-to-date review of dynamic critical phenomena can be found in Ref. [27]. In this approach dynamics in the vicinity of a critical point are described by Langevin equations, which historically were first used for description of the Brownian motion.

In 1973 Martin, Siggia and Rose [28] put forward the idea that in dynamic models an important role is played by a conjugate field and suggested to use it as an additional basic element. Later, Janssen and De Dominicis [29,30] showed how the dynamic models of the Martin-Siggia-Rose (MSR) approach can be formulated in the language of path integrals. This has the great advantage of identifying correctly linear terms to construct in an effective manner the perturbation theory.

In critical dynamics the state of thermodynamic equilibrium must be the stationary solution of the stochastic problem generated by the Langevin equation. This requirement imposes certain restrictions on the structure of the Langevin equation and the statistics of the noise (e.g. Onsager relations and fluctuation-dissipation theorems). However, if these restrictions are lifted, then it is possible to use the Langevin equation to describe stochastic problems with steady states which are not states of thermodynamic equilibrium. The problem of fully developed turbulence [31], stochastic magnetohydrodynamics, transport phenomena in turbulent environment and various advection-diffusion problems [32] belong to this class of stochastic problems. There are also situations, in which a physically reasonable stochastic problem cannot be formulated in terms of the Langevin equation. This is the case, when changes in the relevant variables cannot be treated as continuous functions of time. This happens, for instance, in models of kinetics of chemical reactions [33, 34], ecological models, econophysical models for financial markets or social systems including combat models of operations research. The stochastic problem must then be formulated with the use of the master equation. The latter allows for a representation very similar to second quantization in quantum mechanics. Second quantization in quantum mechanics gives rise to non-relativistic quantum field theory, which may be formulated in terms of generating functionals. The result of this outflanking is the formulation of the original stochastic problem in field-theoretic terms starting from the construction of the action functional.

We review generalization of the approach based on the Langevin equation with additive noise to the case with multiplicative noise. We also discuss in detail the choice of the functional representation of the perturbation expansion for the solution of the Langevin equation. We show that the ambiguity in the Feynman rules is an inherent property of the functional representation which is present both in case of deterministic and stochastic problems. This ambiguity resembles the Ito-Stratonovich ambiguity of stochastic differential equations (SDE) with multiplicative noise, for which we give a detailed account of construction of the field theory as well.

We review the construction of the functional representation for the solution of stochastic

problems based on the master equation. The popular way of construction of this representation with the use of the interpolation approach to the functional integral suffers from problems in taking into account the initial and final conditions which are often solved by hand-waving arguments. We recall an alternative way based on the operator formulation of evolution, which is free from ambiguities in dealing with the boundary conditions of evolution. Although the evolution in these systems is given by first order time derivative, they are similar to relativistic quantum systems in the sense that the number of particles is not conserved. A typical example is the directed bond percolation process [35], where a continuous creation and annihilation of interacting agents takes place. Using the method of "second quantization" of Doi the entire set of master equations can be cast into a "Schrödinger" equation with a given Liouville operator for a state vector in a Fock space. The large-scale properties of the solution can be then analyzed with the further use of Doi formalism and functional methods, which are presented in this article.

The final aim of the theory (either in stochastic dynamics or developed turbulence) is to find the time-space dependence of statistical correlations - mainly those who can be experimentally measured. It turns out that use of quantum field theory methods (RG included) allows to derive a linear differential equation, which contains stable solutions in the asymptotic region of large macroscopic scales.

The solutions have a form of a product of a power-like term with a nontrivial exponent and scaling function of dimensionless variables (not determined by the RG). In order to compute critical exponents in the form of asymptotic series one must resort to a certain scheme (we often employ variants of dimensional renormalization). Asymptotic properties of the scaling functions are analyzed by the operator product expansion, which is another theoretical tool developed mainly by Wilson, Wegner and Kadanoff. In the stochastic theory of fully developed turbulence scaling functions may be singular functions of their dimensionless arguments and this can drastically change the critical exponents. All these features of models turbulent transport are discussed here in a detail on concrete calculations for models describing diffusion and advection of a passive scalar quantity in a turbulent environment. The results demonstrate intermittent (multifractal) behavior of statistical correlations of the random fields of concentration of advected particles.

In contrast to critical dynamics, where the expansion parameter in the renormalized theory is – as a rule – the deviation of the space dimension from the upper critical dimension, in the field theory of fully developed turbulence the basic expansion parameter is the regulator of analytic renormalization, i.e. the deviation of the exponent of a powerlike correlation function of a random source field from its critical value. There may be several such regulators (as in stochastic magnetohydrodynamics, for instance) which makes it possible to construct expansions in several regulators. Moreover, at certain space dimensions additional divergences appear, for which the regulator of dimensional renormalization is usually introduced giving rise to even more diverse multi-parameter expansions in the renormalized theory. Combining information from different expansions allows to improve significantly the numerical accuracy of model calculations, as the calculation of the Kolmogorov constant with the use of the double expansion in isotropic turbulence demonstrates.

A typical approach to stochastic dynamics starts from the analysis of ideal systems - homogeneous in spatial and time, isotropic, incompressible (in case of fluids), possessing mirror symmetry etc. In the present review the corresponding results for fully developed turbulence are summarized. However, real systems almost always exhibit some form of anisotropy, compressibility or violated mirror symmetry. The effect of such deviations from the ideal system on

fluctuating random fields has been an object of intensive research activity, whose arguments and conclusions are described. The results have led to a general conclusion that such effects play a very important rôle. They can drastically change the large-scale behavior predicted by models of ideal systems.

In Sec. 1 a detailed account of the two different approaches to the formulation of field-theoretical models of stochastic problems is presented. The first approach based on a transition from a generic stochastic differential equation with additive or multiplicative noise to model with effective actions generated by the SDE. The second approach is based on the use of the master equation, which – with the use of the Doi approach and functional methods – leads to action functionals, where the effect of initial conditions can be explicitly taken into account. In Sec. 2 the method of RG is briefly described and the double expansion scheme is presented.

Discussion of stochastic models of developed turbulence and the results of the RG analysis thereof is presented in Sec. 3. Sec. 4 is devoted to an analysis of the effect of violated symmetries on the critical behavior. In particular, critical amplitudes, values of critical exponents and the stability of fixed points of the RG that govern the macroscopic (infrared) asymptotic behavior of different statistical mean values is of interest. In Sec. 5 results of the analysis of the effects of turbulent flow on certain reaction-diffusion problems are overviewed.

1.1 Hydrodynamic kinetic equation

In the Landau-Khalatnikov approach to kinetic phenomena in phase transitions [36] relaxation to equilibrium is described by the kinetic equation for the time evolution of the order parameter φ

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = -\gamma \frac{\partial F}{\partial \varphi}\,, \tag{1.1}$$

where F is a thermodynamic potential having a minimum at equilibrium. In the microcanonical ensemble it would be the entropy with the sign minus, in the canonical ensemble the Helmholtz free energy etc. In an inhomogeneous system the kinetic equation (1.1) is generalized to the form

$$\frac{\partial \varphi}{\partial t} = -\gamma \frac{\delta F}{\delta \varphi}\,, \tag{1.2}$$

where F is a functional of the order parameter. In this case the order parameter may be a conserved quantity itself, in which case the kinetic coefficient γ will be wave-number dependent and vanishing at $k = 0$. In the case of the paradigmatic Ginzburg-Landau theory of the ferromagnetic phase transition the thermodynamic potential is of the form [37]

$$F(\varphi) = F_0 + \int \mathrm{d}^3\boldsymbol{r}\ \left[g(\boldsymbol{\nabla}\varphi)^2 + a(T - T_c)\varphi^2 + B\varphi^4\right]\,. \tag{1.3}$$

The generic structure of the kinetic equation (1.2) is basically the same as in hydrodynamic transport equations, in which the right side is not necessarily a (functional) derivative of some thermodynamic potential or the like, but a more general functional of the slow parameter φ, which may, of course, have several components.

The kinetic equation (1.2) provides a deterministic macroscopic description, which does not take into account microscopic fluctuations. To describe effects of fluctuations without resorting to microscopic theory, some kind of randomness may be used in the problem brought about by the kinetic equation. The consistent construction of renormalizable field theories corresponding to nonlinear kinetic equations with noise was initially proposed in terms of quantum field theory by Martin, Siggia and Rose [28]. It should be noted that in the original paper the nature of the randomness was not specified. The operator approach of Martin, Siggia and Rose (MSR) was soon replaced by the equivalent functional-integral representation for the solution of the Langevin equation, in which a white-noise term is added to the right side of the kinetic equation [29, 30]. The functional-integral approach avoids any connection to the operator formalism of quantum field theory. It is ambiguous, however, in a way which makes it difficult to choose the most convenient scheme for calculations and confusing from the point of view of the mathematical ambiguity of stochastic differential equations with multiplicative white noise.

In the MSR approach the starting point is a system of nonlinear equations for field operators. An iterative solution of these equations is the tree-graph solution of a nonlinear differential equation, iterative construction which we shall shortly describe. This solution explicitly depends on random initial conditions or random coefficient functions and implicitly on random boundary conditions through the Green function of the linear problem. Expectation values of products of these tree-graph solutions over sources of randomness yield the solution of the stochastic problem.

This iterative solution may be compactly expressed in the functional-differential form of the quantum-field perturbation theory, in which the ambiguity of the representation is explicit in the interaction functional. It should be emphasized that the iterative solution is unambiguous, but the form of the functional representation, either integral or differential, is not. In this section we shall describe both ways to represent the solution of a nonlinear differential equation and stress once more, that for the time being the differential equations are completely deterministic. As a specific example we shall use the time-dependent Ginzburg-Landau model.

The generic kinetic equation for a near-equilibrium system in case of the Ginzburg-Landau free-energy functional [37] gives rise to a nonlinear partial differential equation in the form

$$\partial_t \varphi = -\gamma \left[-2g\nabla^2 \varphi + 2a(T - T_c)\varphi + 4B\varphi^3 - h \right], \tag{1.4}$$

where $\partial_t = \partial/\partial t$ is the time derivative, parameter γ sets the time scale. Description of dynamic phenomena in the critical region based on the solution of this equation is the *time-dependent Ginzburg-Landau model* (TDGL model). It is widely used especially in the theory of superconductivity, where inhomogeneous solutions are important.

Here, however, the TDGL equation (1.4) is needed for the description of the effect of small-scale fluctuations on the behavior of the system in the critical region. To this end, the external field h will eventually be regarded as a random noise thus giving rise to a stochastic differential equation (SDE). For the construction of perturbative solution of the SDE it is instructive, however, to introduce first the iterative solution of the deterministic TDGL equation (1.4). With the use of this approach a perturbative solution of any polynomially nonlinear differential equation may be constructed, provided a linear differential equation may be singled out as the starting point of the iteration.

In case of the TDGL equation the starting point of the iteration is the linear equation (for simplicity, the symmetric phase with vanishing equilibrium value of the field φ is considered

here)

$$\partial_t \varphi^{(0)} = -\gamma \left[-2g\nabla^2 \varphi^{(0)} + 2a(T - T_c)\varphi^{(0)} - h \right] . \tag{1.5}$$

Solution of this problem may be expressed in the form

$$\varphi^{(0)}(t, \boldsymbol{x}) = \int \mathrm{d}^d \boldsymbol{x}' \int \mathrm{d}t' \Delta_{12}(t, \boldsymbol{x}, t', \boldsymbol{x}') \gamma h(t', \boldsymbol{x}') \equiv \int \mathrm{d}x' \Delta_{12}(x, x') \gamma h(x') \,, \tag{1.6}$$

where here and henceforth $x = (t, \boldsymbol{x})$, $\mathrm{d}x \equiv \mathrm{d}^d \boldsymbol{x} \mathrm{d}t$, d is the dimension of the space and Δ_{12} is the *Green function* of the linear differential operator of equation (1.5), i.e.

$$\left[\partial_t - 2\gamma g \nabla^2 + 2a(T - T_c) \right] \Delta_{12}(t, \boldsymbol{x}, t', \boldsymbol{x}') = \delta_+(t - t') \delta(\boldsymbol{x} - \boldsymbol{x}') \,. \tag{1.7}$$

Here, δ_+ is the asymmetric δ function defined by

$$\int_{a+0}^{b} f(t') \delta_+(t - t') \, \mathrm{d}x' = \begin{cases} 0 \,, & t < a \vee t > b \,, \\ f(t + 0) & a \le t < b \end{cases} \tag{1.8}$$

Usually, vanishing boundary conditions are assumed. On time axis it is often technically simpler to consider "initial condition" $\varphi \to 0$, $t \to -\infty$, in which case the explicit expression for Green function in the time-wave-vector representation is

$$\Delta_{12}(t - t', \boldsymbol{k}) = \theta(t - t') \exp \left\{ -2\gamma \left[gk^2 + a(T - T_c) \right] (t - t') \right\} . \tag{1.9}$$

Should the Cauchy problem at a finite initial time instant t_0 be considered, it would be the simplest technically to introduce an "external field" concentrated at the initial time instant for the initial condition $\varphi(0, \boldsymbol{x}) = \varphi_0(\boldsymbol{x})$:

$$h(t, \boldsymbol{x}) \to h(t, \boldsymbol{x}) + \delta_+(t - t_0) \frac{\varphi_0(\boldsymbol{x})}{\gamma} \,.$$

It should be noted that the retardedness property of the Green function (propagator) (1.9) turns out to be extremely important in the explicit construction of the perturbation expansion of the solution, especially when the small-scale fluctuations will be taken into account.

To construct the iterative solution, it is convenient to cast the TDGL equation (1.4) in the form of an integral equation

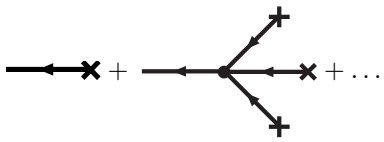
$$\varphi(x) = \int \mathrm{d}x' \Delta_{12}(x, x') \gamma h(x') - 4\gamma B \int \mathrm{d}x' \Delta_{12}(x, x') \varphi^3(x') \,. \tag{1.10}$$

The zeroth-order contribution $\varphi^{(0)}$ is obtained by putting the coupling constant equal to zero $B = 0$ in (1.10). To obtain the first-order contribution, the zeroth-order solution is substituted in the right side of (1.10) to yield

$$\varphi^{(1)}(x) = -4\gamma B \int \mathrm{d}x' \Delta_{12}(x, x') \left[\varphi^{(0)}(x') \right]^3 , \tag{1.11}$$

which then is substituted in the right side of (1.10) to obtain the second-order contribution etc.

It is convenient to express the iterative solution in a graphical form, the two leading terms of which may be depicted as

$$\varphi = \varphi^{(0)} + \varphi^{(1)} + \ldots =$$  $$+ \ldots \tag{1.12}$$

Here, the directed line corresponds to the propagator Δ_{12}, the cross to the external field term field γh and the full dot to the vertex factor obtained as the third order derivative with respect to φ of the right side of the kinetic equation (TDGL equation) (1.4).

From the iterative solution it is clearly seen that the propagator Δ_{12} is essentially the dynamic (linear) response function of the field φ:

$$\chi(x,x') = \left.\frac{\delta\varphi(x)}{\delta h(x')}\right|_{h=0} = \gamma\Delta_{12}(x,x')\,. \tag{1.13}$$

Little reflection shows that in the graphical representation of the solution of the kinetic equation there are only connected graphs without closed loops of propagators, i.e. the graphical solution of the TDGL equation consists of connected *tree graphs* only.

The tree-graph solution of the kinetic equation may be expressed in a compact functional form, which is useful to work out the effect of fluctuations. Consider the generic kinetic equation

$$\partial_t\varphi = V(\varphi) = -K\varphi + U(\varphi) + f \tag{1.14}$$

in which the right side is not necessarily proportional to a functional derivative. In the last expression of (1.14) we have singled out the source term for calculation of response functions and the first-order term giving rise to the propagator Δ_{12}

$$(\partial_t\varphi + K)\,\Delta_{12}(t,\boldsymbol{x},t',\boldsymbol{x}') = \delta_+(t-t')\delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{1.15}$$

The *generating function* of solutions $\varphi[f]$ of the kinetic equation (1.14) is defined as

$$\mathcal{G}(A) = \sum_{n=0}^{\infty}\frac{1}{n!}\left\{\int \mathrm{d}x A(x)\varphi[x,f]\right\}^n \equiv \exp\left\{A\varphi[f]\right\}\,, \tag{1.16}$$

where a *universal* notation is introduced, i.e. integrals and sums over parameters of the solution $\varphi[f]$ are implied.

Derivatives of the functional $\mathcal{G}(A)$ (1.16) at $A = 0$ are simply products of the solution $\varphi[f]$. With the use of functional δ function (whose application rules coincide with those of the δ function in a finite-dimensional Euclidean space) the generating function may be expressed as a functional integral [38]

$$\mathcal{G}(A) = \int \mathcal{D}\varphi\,\delta\left(\varphi - \varphi[f]\right)\,\exp\left(A\varphi\right)\,. \tag{1.17}$$

A change of variables in the δ function brings about the kinetic equation explicitly:

$$\mathcal{G}(A) = \int \mathcal{D}\varphi\, \delta\left[-\partial_t\varphi + V(\varphi) + f\right] |\det M|\, \exp\left(A\varphi\right) . \tag{1.18}$$

The functional $U(\varphi)$ contains the nonlinear terms of the functional $V(\varphi)$. We stress that the right-side functional $V(\varphi)$ in the kinetic equation need not be a functional derivative of some other functional, although this is often the case in near-equilibrium kinetics.

Representation the functional Jacobi determinant in (1.18) in the loop expansion of the expression $\exp\left(\mathrm{Tr}\ln M\right)$ yields

$$\det M = \det\left[\left(\partial_t + K\right)\delta(x-x') - \frac{\delta U(\boldsymbol{x},\varphi)}{\delta\varphi(x')}\right] = \det\left(\partial_t + K\right)\exp\left[-\Delta_{12}(0)U'(\varphi)\right] , \tag{1.19}$$

in which the following shorthand notation has been introduced

$$\Delta_{12}(0)U'(\varphi) \equiv \int \mathrm{d}x \int \mathrm{d}x' \Delta_{12}(t,\boldsymbol{x},t,\boldsymbol{x}')\frac{\delta U(\boldsymbol{x},\varphi)}{\delta\varphi(x')} . \tag{1.20}$$

Recall that the propagator is the retarded Green function of the linear kinetic equation, whose value at $t = t'$ is not determined. It is often convenient to use the time-wave-vector representation for the diagonal value of the propagator, in which case a wave-vector integral appears

$$\Delta_{12}(t,\boldsymbol{x},t'\boldsymbol{x}')\Big|_{\substack{t=t'\\ \boldsymbol{x}=\boldsymbol{x}'}} = \int \frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^d}\, \Delta_{12}(t-t',\boldsymbol{k})\big|_{t=t'} ,$$

for convergence of which an UV cutoff is usually needed. For instance, in case of the TDGL model the diagonal value of the propagator may be expressed as

$$\Delta_{12}(t,\boldsymbol{x},t'\boldsymbol{x}')\Big|_{\substack{t=t'\\ \boldsymbol{x}=\boldsymbol{x}'}} = \theta(t-t')\big|_{t=t'} \int \frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^d}\, \frac{\theta(\Lambda-k)}{2\gamma\left[gk^2 + a(T-T_c)\right]} ,$$

where the sharp UV cutoff has been used. Thus, not only are measures of caution to be taken to make the diagonal value of the propagator finite, but also value of the step function of the retarded propagator needs to be defined at the origin.

With the use of the Fourier-integral representation of the δ function in (1.18) we arrive at the functional integral over two fields (in the Fourier integral for the functional δ function the imaginary unit is usually omitted and the auxiliary field $\tilde{\varphi}$ taken as imaginary-number quantity, if necessary)

$$\begin{aligned}\mathcal{G}(A) = \int \mathcal{D}\varphi \int \mathcal{D}\tilde{\varphi}\, \left|\det\left(\partial_t + K\right)\right| \exp\left[-\Delta_{12}(0)U'(\varphi)\right] \exp\left\{\tilde{\varphi}\left[-\partial_t\varphi + V(\varphi) + f\right]\right\} \\ \times \exp\left(A\varphi\right) .\end{aligned} \tag{1.21}$$

Henceforth, the field-independent determinant of the differential operator $\det\left(\partial_t + K\right)$ will be included in the measure of integration and therefore not expressed explicitly.

In contrast with the usual approach [38], there is no Gaussian integral over the random noise in (1.21), which would render the resulting functional integral convergent as an iterated integral. In fact, the formal functional integral (1.21) is hardly convergent in any reasonable sense. However, by the standard transformation rules [39] it generates a functional-differential representation (S-matrix functional) for the generating function (1.16)

$$\mathcal{G}(A) = \exp\left(\frac{\delta}{\delta\varphi}\Delta_{12}\frac{\delta}{\delta\tilde{\varphi}}\right)\exp\left[\tilde{\varphi}U(\varphi) - \Delta_{12}(0)U'(\varphi) + \tilde{\varphi}f + A\varphi\right]\Big|_0 \tag{1.22}$$

where $|_0$ stands for $|_{\tilde{\varphi}=\varphi=0}$ and this notation will be employed in what follows. Further, the exponential functional differential operator in (1.22) is the *reduction operator*

$$\mathcal{P} = \exp\left(\frac{\delta}{\delta\varphi}\Delta_{12}\frac{\delta}{\delta\tilde{\varphi}}\right)$$

spanning propagator lines between *vertex factors*

$$V_{n,m}(x_1,\ldots,x_n;y_1,\ldots,y_m) = \frac{\delta^{n+m}V(\varphi,\tilde{\varphi})}{\delta\varphi(x_1)\cdots\delta\varphi(x_n)\delta\tilde{\varphi}(y_1)\cdots\delta\tilde{\varphi}(y_n)} \tag{1.23}$$

where, in the present case,

$$V(\varphi,\tilde{\varphi}) = \tilde{\varphi}U(\varphi) - \Delta_{12}(0)U'(\varphi) + \tilde{\varphi}f + A\varphi\,.$$

In the S-matrix functional (1.22)) all building blocks are mathematically well defined. Problems may, of course, arise with the convergence of the series. Since this representation is obtained by a heuristic functional-integral argument, in what follows it will separately demonstrated that the representation (1.22) yields the iterative solution of the kinetic equation (1.14). It should be noted that representation (1.22) may be obtained directly – without any functional integrals and determinants – from the MSR operator equations of motion with the use of standard rules of construction of perturbation theory for Green functions [40].

It is a generic property of the functional representation (1.22) that $\ln\mathcal{G}(A)$ consists of the connected graphs of $\mathcal{G}(A)$. The interaction term $\tilde{\varphi}U(\varphi)$ in (1.22) is linear in the auxiliary field φ, therefore only one directed propagator line comes out of the corresponding vertex in the graphical representation, whereas the number of incoming lines is equal to the power of the field φ in each monomial of $U(\varphi)$ (which is three in the TDGL model). Propagator lines form continuous directed chains, which either form closed loops and vanish due to retardation of the propagator (except the single-propagator closed loop $\Delta_{12}(0)$), or go through the graph starting form the external field f and ending at the source field A.

At the vertex corresponding to $\tilde{\varphi}U(\varphi)$ directed chains with a start at the external field f merge to produce a directed chain ending at A, whereas at the vertex corresponding to $-\Delta_{12}(0)U'(\varphi)$ directed incoming chains end without producing any outcoming line. Therefore, connected graphs of $\mathcal{G}(A)$ either contain one source vertex corresponding to $A\varphi$ or do not contain such a vertex at all. On the contrary, the number of vertices corresponding to external field $\tilde{\varphi}f$ in a connected graph is limited only by the order of perturbation theory. It is not difficult to see by direct inspection that all connected graphs of $\mathcal{G}(A)$ without A actually vanish, because the vertex

factors produced by terms $\tilde{\varphi}V(\varphi)$ and $-\Delta_{12}(0)U'(\varphi)$ cancel each other in them, therefore we end up at the result

$$\mathcal{G}(A) = \exp\left[A\mathcal{W}_1(f)\right] , \tag{1.24}$$

which, of course, coincides with (1.16), but is expressed in terms of standard generating functions.

The point of introducing the functional representation (1.22) is that the explicit linear exponential of the external field f is convenient in working out generating functions in case of random external field (noise). The basic equation of the generating functional $\mathcal{G}(A)$ is the Schwinger equation with respect to the source A. It is most straightforwardly obtained from the Gauss law for the functional integral (1.21) with respect to the auxiliary field $\tilde{\varphi}$, which yields

$$\begin{aligned} 0 &= \int \mathcal{D}\varphi \int \mathcal{D}\tilde{\varphi} \frac{\delta}{\delta\tilde{\varphi}} \exp\{\tilde{\varphi}\left[-\partial_t\varphi + V(\varphi) + f\right] - \Delta_{12}(0)U'(\varphi) + A\varphi\} \\ &= \int \mathcal{D}\varphi \int \mathcal{D}\tilde{\varphi}[-\partial_t\varphi + V(\varphi) + f] \exp\{\tilde{\varphi}\left[-\partial_t\varphi + V(\varphi) + f\right] - \Delta_{12}(0)U'(\varphi) + A\varphi\}. \end{aligned}$$

Pulling out the field variables as derivatives with respect to the source we obtain

$$\mathcal{G}(A)^{-1}\left[-\partial_t\frac{\delta}{\delta A} + V\left(\frac{\delta}{\delta A}\right) + f\right]\mathcal{G}(A) = \left[-\partial_t\mathcal{W}_1 + V\left(\mathcal{W}_1\right) + f\right] = 0\,, \tag{1.25}$$

which, of course, coincides with the original kinetic equation for the field φ.

The reader already familiar with the functional integral would probably like to ask, how the Schwinger equation obtained from the Gauss law with respect to the basic field φ fits here. The answer is that it actually reproduces the equation for the "response function"

$$\chi_{A\tilde{A}}(t,\boldsymbol{x};t'\boldsymbol{x}') = \frac{\delta\mathcal{W}_1(t,\boldsymbol{x})}{\delta\tilde{A}(t',\boldsymbol{x}')}\,. \tag{1.26}$$

Here, $\tilde{A} = f$ and quotation marks are used here, because the physical response function is obtained at vanishing source fields.

From the original kinetic equation with an additional source term

$$\partial_t\varphi = V(\varphi) = -K\varphi + U(\varphi) + \tilde{A} \tag{1.27}$$

an equation for $\chi_{A\tilde{A}}$ is obtained by differentiation with respect to $\tilde{A}$ with the result

$$\partial_t\chi_{A\tilde{A}} = V'(\varphi)\chi_{A\tilde{A}} + 1\,. \tag{1.28}$$

The Schwinger equation from the Gauss law with respect to the basic field φ for the functional (1.21) yields (with the account of expression (1.24))

$$-\Delta_{12}(0)U''(\mathcal{W}_1) + \frac{\delta V'(\mathcal{W}_1)}{\delta\tilde{A}} + A\partial_t\chi_{A\tilde{A}} + V'(\mathcal{W}_1)A\chi_{A\tilde{A}} + A = 0 \tag{1.29}$$

The dependence on the source field A is explicit here ($\chi_{A\tilde{A}}$ does not depend on A), therefore the independent of A terms and the sum of those proportional to A must vanish independently.

Careful tracing of the arguments reveals that the condition of vanishing of the coefficient of A in (1.29) is actually the equation (1.28), thereafter it is seen that the sum of two first terms in (1.29) vanishes identically on the solution W_1 of the kinetic equation (1.25).

It should be noted that so far a functional representation (1.22) has been constructed for generating functional of solutions of a *deterministic* nonlinear partial differential equation. There is no randomness here at all and the tree-graph solution of the equation is unique. It is an intrinsic property of the functional representation that it contains an ambiguity in the form of the diagonal value of the propagator $\Delta_{12}(0)$, but this has nothing to do with any stochastic problems, as it is sometimes presented in the literature [41].

It is instructive to analyze in a more formal fashion the independence of the generating functional (1.22) of the diagonal value of the propagator. To this end, use the following trick [39]. Consider a product of N functionals $F_n(\varphi_n, \tilde{\varphi}_n)$ of a set of N fields φ_n and $\tilde{\varphi}_n$ acted upon by a reduction operator containing all pairs of fields:

$$\exp\left(\sum_{i,j=1}^{N} \frac{\delta}{\delta\varphi_i}\Delta_{12}\frac{\delta}{\delta\tilde{\varphi}_j}\right)\prod_{n=1}^{N} F_n(\varphi_n, \tilde{\varphi}_n)\,. \tag{1.30}$$

Here, the reduction operator is a "complete square" of differential operators, i.e.

$$\sum_{i,j=1}^{N} \frac{\delta}{\delta\varphi_i}\Delta_{12}\frac{\delta}{\delta\tilde{\varphi}_j} = \left(\sum_{i=1}^{N}\frac{\delta}{\delta\varphi_i}\right)\Delta_{12}\left(\sum_{j=1}^{N}\frac{\delta}{\delta\tilde{\varphi}_j}\right)\,. \tag{1.31}$$

With the aid of the change of variables to "center of mass" φ defined as

$$\varphi \equiv \frac{\varphi_1 + \ldots + \varphi_N}{N} \tag{1.32}$$

and relative coordinates $\varphi_i - \varphi_{i+1}$ for $\{\varphi_n\}_{n=1}^{N}$ (together with a similar change for $\{\tilde{\varphi}_n\}_{n=1}^{N}$) we arrive at the conclusion [39]

$$\exp\left(\frac{\delta}{\delta\varphi}\Delta_{12}\frac{\delta}{\delta\tilde{\varphi}}\right)\prod_{n=1}^{N} F_n(\varphi, \tilde{\varphi}) = \exp\left(\sum_{i,j=1}^{N} \frac{\delta}{\delta\varphi_i}\Delta_{12}\frac{\delta}{\delta\tilde{\varphi}_j}\right)\prod_{n=1}^{N} F_n(\varphi_n, \tilde{\varphi}_n)\Big|_{\substack{\varphi_i=\varphi\\ \tilde{\varphi}_i=\tilde{\varphi}}}\,. \tag{1.33}$$

This relation may be used to separate generation of lines attached to the single vertex described by any individual functional from spanning of lines between different vertices. Extract the diagonal terms in the reduction operator on the right side of (1.33) to obtain

$$\exp\left(\sum_{i\neq j=1}^{N} \frac{\delta}{\delta\varphi_i}\Delta_{12}\frac{\delta}{\delta\tilde{\varphi}_j}\right)\prod_{n=1}^{N}\exp\left(\frac{\delta}{\delta\varphi_n}\Delta_{12}\frac{\delta}{\delta\tilde{\varphi}_n}\right)F_n(\varphi_n, \tilde{\varphi}_n)\,. \tag{1.34}$$

An effective vertex functional – *the normal form* of the interaction functional (or *reduced vertex* functional) – defined as

$$F'(\varphi, \tilde{\varphi}) \equiv \exp\left(\frac{\delta}{\delta\varphi}\Delta_{12}\frac{\delta}{\delta\tilde{\varphi}}\right)F(\varphi, \tilde{\varphi}) \tag{1.35}$$

appears and we see that

$$\exp\left(\frac{\delta}{\delta\varphi}\Delta_{12}\frac{\delta}{\delta\tilde{\varphi}}\right)\prod_{n=1}^{N}F_n(\varphi,\tilde{\varphi})=\exp\left(\frac{\delta}{\delta\varphi}\Delta'_{12}\frac{\delta}{\delta\tilde{\varphi}}\right)\prod_{n=1}^{N}F'_n(\varphi,\tilde{\varphi})\,, \tag{1.36}$$

where the reduction operator $\exp\left(\frac{\delta}{\delta\varphi}\Delta'_{12}\frac{\delta}{\delta\tilde{\varphi}}\right)$ generates lines between different functionals (vertices) F'_n only. An alert reader might notice a correspondence with the normal-product form of interaction operators in quantum field theory.

It should be emphasized that this procedure is a rearrangement of the perturbation expansion. However, if the vertex functional is local, i.e. a one-fold integral of a function of fields and their derivatives at a single space-time point, then the statement (1.36) is tantamount to saying that the diagonal value of the propagator Δ'_{12} is equal to zero: $\Delta'_{12}(x,x)=0$.

It follows from here, in particular, that the generating function (1.22) may be expressed in the simple form

$$\mathcal{G}(A)=\exp\left(\frac{\delta}{\delta\varphi}\Delta'_{12}\frac{\delta}{\delta\tilde{\varphi}}\right)\exp\left[\tilde{\varphi}U(\varphi)+\tilde{\varphi}f+A\varphi\right]\Big|_0 \tag{1.37}$$

completely independent of the variable $\Delta_{12}(0)$.

1.2 Langevin equation

Hydrodynamic kinetic equations are written as mean-field equations for averages of quantities, which intrinsically are random processes to some extent. To take fluctuations around the averages into account, a straightforward way to proceed is to introduce a source of randomness directly in the mean-field equation. Then the quantities solved from the mean-field equations become *stochastic processes* depending on coordinate variables, i.e. *stochastic fields*. This approach has been widely used in the description of dynamic critical phenomena (see, e.g., [27] for a fairly recent review) as well as in the analysis of fluctuations in reaction kinetics and transport phenomena [25, 42–44].

1.2.1 Models A and B of critical dynamics.

Although we intend to concentrate on the field theory of reactions and transport phenomena, let us remind the simplest paradigmatic models of critical dynamics.

In the Landau theory of phase transitions the dynamics of the order parameter φ near equilibrium are described time-dependent Ginzburg-Landau (TDGL) equation. In the theory of critical phenomena the effect fluctuations is crucial. It is customary to take fluctuations into account with the use of the Langevin approach, in which the external field term(s) of the TDGL equations are random quantities. For instance, in case of wave-number independent kinetic coefficient (i.e in case of non-conserved order parameter) the corresponding Langevin equation

$$\partial_t\varphi=-\gamma\frac{\delta F[\varphi]}{\delta\varphi}+f=-\gamma\left[-2g\nabla^2\varphi+2a(T-T_c)\varphi+4B\varphi^3\right]+\gamma h\,. \tag{1.38}$$

gives rise to the model A of critical dynamics. Here, the random noise field $f=\gamma h$ is assumed to have Gaussian distribution law with zero mean and the white-in-time-noise correlation function

$$\langle f(t,\boldsymbol{x})f(t',\boldsymbol{x}')\rangle=\delta(t-t')D(\boldsymbol{x}-\boldsymbol{x}')\,, \tag{1.39}$$

where the spatial correlation function is chosen from the condition that the long-time asymptotic state is described by the Gibbs ensemble with the functional $F[\varphi]$, i.e. the probability density function in this limit $\rho[\varphi] \to \exp\{-F[\varphi]/T\}/\mathrm{Tr}\,\exp\{-F[\varphi]/T\}$. This means that in case of model A the position-dependent part of the correlation function (1.39) is

$$D_A(\boldsymbol{x}-\boldsymbol{x}') = 2\gamma T\delta(\boldsymbol{x}-\boldsymbol{x}')\,. \tag{1.40}$$

In model B the order parameter is a conserved quantity and the kinetic coefficient vanishes at zero wave number: $\gamma \to -\gamma\nabla^2$. Correspondingly the Langevin equation is

$$\partial_t\varphi = \gamma\nabla^2\left[-2g\nabla^2\varphi + 2a(T-T_c)\varphi + 4B\varphi^3\right] + \gamma h \tag{1.41}$$

and the correlation function contains the Laplace operator as well:

$$D_B(\boldsymbol{x}-\boldsymbol{x}') = -2\gamma T\nabla^2\delta(\boldsymbol{x}-\boldsymbol{x}')\,. \tag{1.42}$$

1.2.2 Diffusion-limited reactions.

In reaction kinetics and population dynamics the simplest kinetic description of the dynamics of the average particle numbers is given by the *rate equation*. The rate equation is a deterministic differential equation for average particle numbers in a homogeneous system, therefore it does not take into account boundary conditions, spatial inhomogeneities and randomness in the individual reaction events. Spatial dependence is often accounted for by a diffusion term, which gives rise to models of *diffusion-limited reactions* (DLR).

As a simple example, consider the coagulation reaction $A + A \to A$. The diffusion-limited rate equation for the concentration φ of the compound A is

$$\partial_t\varphi = D\nabla^2\varphi - k\varphi^2\,, \tag{1.43}$$

where k is the *rate constant*.

The most straightforward way to take into account various effects of randomness is to add a random source and sink term to the rate equation:

$$\partial_t\varphi = D\nabla^2\varphi - k\varphi^2 + f\,. \tag{1.44}$$

This is a nonlinear Langevin equation for the field φ. Physically, in the case of concentration $\varphi \geq 0$.

There is an important difference between the reaction models and the critical dynamics: in the latter, deviations of the fluctuating order parameter from the (usually zero) mean may physically be of any sign (or direction). In particular, deviations from the equilibrium value are always allowed. In the reaction there is often an absorbing steady state, which does not permit fluctuations therefrom: once the system arrives at the absorbing state, it stays there forever. In particular, if the empty state is an absorbing state of the reaction, the random source should be multiplied by a factor vanishing in the limit $\varphi \to 0$ to prevent the system returning from the absorbing state by the noise. The simplest choice yields

$$\partial_t\varphi = D\nabla^2\varphi - k\varphi^2 + f\varphi \tag{1.45}$$

instead of (1.44). This is an equation with a *multiplicative noise*.

1.2.3 Multiplicative noise in diffusion-advection models

The issue of multiplicative noise practically does not arise in the field theory of critical phenomena. As seen from the previous example, it appears as consequence of boundary conditions in reaction problems. In transport equations with advection by random field it is customary to introduce the coupling of, say, a scalar quantity φ to the advecting field $\boldsymbol{v}$ through the substantial (or covariant) time derivative

$$\partial_t\varphi + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\varphi\,, \tag{1.46}$$

which, in case of random advection field $\boldsymbol{v}$, gives rise to a multiplicative noise. A stochastic differential equation (SDE) with multiplicative noise is not well-defined mathematically. Therefore, below we shall analyze construction of the field theory on the basis of a Langevin equation (SDE) in detail to sort out ambiguities appearing within this process.

1.2.4 Functional representation.

In a more generic setup [45], standard models of fluctuations in critical dynamics and stochastic reaction and transport equations are based on nonlinear Langevin equations

$$\partial_t\varphi = V(\varphi) + f\,, \tag{1.47}$$

For the random source ("force") a Gaussian white-in-time distribution is assumed:

$$\langle f(t,\boldsymbol{x})f(t',\boldsymbol{x}')\rangle = \delta(t-t')D(\boldsymbol{x},\boldsymbol{x}')\,,\quad \langle f(t,\boldsymbol{x})\rangle = 0\,. \tag{1.48}$$

The static correlation function $D(\boldsymbol{x},\boldsymbol{x}')$ is determined through the connection to the static equilibrium (fluctuation-dissipation theorem) in case of near-equilibrium fluctuations. In case of reaction and transport models far from equilibrium physical properties of a steady state are used to establish properties of the random source.

The Langevin equation with white-in-time noise f is mathematically inconsistent, because the time integral of the noise $\int f\mathrm{d}t$ is a Wiener process which is not differentiable anywhere as a function of time. The Langevin equation (1.47) is written mathematically correctly in the integral form

$$\mathrm{d}\varphi = V(\varphi)\mathrm{d}t + \mathrm{d}W\,, \tag{1.49}$$

where $\mathrm{d}W$ is an increment of the Wiener process (see, e.g. [46]).

To describe solution of the stochastic problem in terms of standard perturbation theory it is convenient to use a set of correlation functions consisting of a δ sequence in time, i.e.

$$\langle f(t,\boldsymbol{x})f(t',\boldsymbol{x}')\rangle = \overline{D}(t,\boldsymbol{x};t',\boldsymbol{x}') \to \delta(t-t')D(\boldsymbol{x},\boldsymbol{x}') \tag{1.50}$$

and passing to the white-noise limit at a later stage. The point here is that the noise may be regarded as a smooth function of time and the differential form of the stochastic differential equation (SDE) (1.47) may be used literally. From the mathematical point of view, this treatment gives rise to the solution of the stochastic differential equation (1.47) in the Stratonovich sense [47, 48]. Physically, this is often the most natural way to approach the white-noise case.

Averaging of the generating functional of solutions of the Langevin equation (1.47) over the Gaussian noise determined by the correlation function (1.39) is readily carried out with the use of the functional representation (1.22). The result of calculation of the Gaussian integral is (new notation for the generating functional of solutions of the stochastic differential equation is not introduced)

$$\mathcal{G}(A) = \exp\left(\frac{\delta}{\delta\varphi}\Delta_{12}\frac{\delta}{\delta\tilde{\varphi}}\right)\exp\left[\tilde{\varphi}V(\varphi) - \Delta_{12}(0)U'(\varphi) + \frac{1}{2}\,\tilde{\varphi}\overline{D}\tilde{\varphi} + A\varphi\right]\Bigg|_0 . \tag{1.51}$$

The functional integral accompanying relation (1.51) obtained with the use of (1.21)

$$\mathcal{G}(A) = \int\mathcal{D}\varphi\int\mathcal{D}\tilde{\varphi}\,\exp\left\{-\Delta_{12}(0)U'(\varphi) + \frac{1}{2}\,\tilde{\varphi}\overline{D}\tilde{\varphi} + \tilde{\varphi}\left[-\partial_t\varphi + V(\varphi)\right] + A\varphi\right\} \tag{1.52}$$

is formally convergent as an iterated integral, in which the auxiliary field $\tilde{\varphi}$ – which should be considered having *imaginary* values – is integrated first [49]. It should be noted that with the "colored-noise" correlation function $\overline{D}$ (inter)action functionals in these representations of the generating function are not time local.

Functional derivatives of the generating functional averaged over noise do not factorize any more, of course, and give correlation functions of the random fields φ:

$$\mathcal{G}(A) = \sum_{n=0}^{\infty}\frac{1}{n!}\int \mathrm{d}x_1\cdots\int \mathrm{d}x_n G_n(x_1,\ldots,x_n)A(x_1)\cdots A(x_n) \tag{1.53}$$

where

$$G_n(x_1,\ldots,x_n) = \langle\varphi(x_1)\cdots\varphi(x_n)\rangle_f = \frac{\delta^n\mathcal{G}(A)}{\delta A(x_1)\cdots\delta A(x_n)}\bigg|_{A=0} . \tag{1.54}$$

With the use of the normal form of the interaction functional the absence of $\Delta_{12}(0)$ in the perturbation expansion is made explicit

$$\mathcal{G}(A) = \exp\left(\frac{\delta}{\delta\varphi}\Delta'_{12}\frac{\delta}{\delta\tilde{\varphi}}\right)\exp\left[\tilde{\varphi}V(\varphi) + \frac{1}{2}\,\tilde{\varphi}\overline{D}\tilde{\varphi} + A\varphi\right]\Bigg|_0 \tag{1.55}$$

The corresponding functional integral is also simplified

$$\mathcal{G}(A) = \int\mathcal{D}'\varphi\int\mathcal{D}'\tilde{\varphi}\,\exp\left\{\frac{1}{2}\,\tilde{\varphi}\overline{D}\tilde{\varphi} + \tilde{\varphi}\left[-\partial_t\varphi + V(\varphi)\right] + A\varphi\right\} . \tag{1.56}$$

The prime in the measure denotes the additional rule $\Delta_{12}(0) = 0$ (which is tantamount to amending the definition of the temporal step function as $\theta(0) = 0$). Since appearances of integrands in (1.52) and (1.56) are different, it is obviously quite essential to bear the differences in the measure in mind, if the functional integral is calculated by any other means than perturbation theory.

The δ sequence of correlation functions $\overline{D}$ appears in all representations (1.51), (1.52) and (1.55) only in the quadratic form with the auxiliary field: $\tilde{\varphi}\overline{D}\tilde{\varphi}$ and passing to the white-noise

limit in this expression does not cause any problems either in the S-matrix functional or in the functional integral. Therefore, functional representations for the solution of the Langevin equation (1.47), (1.48) are obtained from (1.51), (1.52) and (1.55) simply by the replacement

$$\int \mathrm{d}x \int \mathrm{d}x' \tilde{\varphi}(x) \overline{D}(x;x') \tilde{\varphi}(x') \to \int \mathrm{d}t \int \mathrm{d}^d \boldsymbol{x} \int \mathrm{d}^d \boldsymbol{x}' \tilde{\varphi}(t,\boldsymbol{x}) D(\boldsymbol{x},\boldsymbol{x}') \tilde{\varphi}(t,\boldsymbol{x}') \,. \quad (1.57)$$

Explicit time-dependence is written here to emphasize that in the white-noise limit the action functional is time local.

Let us construct an iterative solution of the simplest multiplicative SDE of the *multiplicative linear white-noise* process

$$\partial_t \varphi = -K\varphi + f\varphi \,, \quad (1.58)$$

where K is a time-independent operator acting on the field φ (e.g. $K = -\nabla^2 + a$, $a > 0$) and f a Gaussian random field with zero mean and the "colored" correlation function $\overline{D}$ (1.50).

The iterative solution of the SDE (1.58) may expressed as the series

$$\varphi = u\chi + \Delta_{12} f u\chi + \Delta_{12} f \Delta_{12} f u\chi + \ldots \quad (1.59)$$

where χ is the initial condition of the solution

$$u\chi = \int \mathrm{d}^d \boldsymbol{x}' u(t, \boldsymbol{x} - \boldsymbol{x}') \chi(\boldsymbol{x}')$$

of the homogeneous equation

$$(\partial_t + K)\, u\chi = 0 \,. \quad (1.60)$$

Here, $u(t,\boldsymbol{x})$ is the singular solution of the homogeneous equation , i.e. a solution with the initial condition $u(0,\boldsymbol{x}) = \delta(\boldsymbol{x})$, whereas Δ_{12} in (1.59) is the (retarded) Green function of the same equation, i.e.

$$(\partial_t + K)\, \Delta_{12}(t,\boldsymbol{x},t',\boldsymbol{x}') = \delta_+(t-t')\delta(\boldsymbol{x}-\boldsymbol{x}') \,.$$

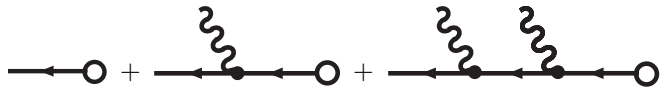
In (1.59) a shorthand notation has been used in which all time and space convolution integrals in the nonlinear terms are implied.

The solution (1.59) may be conveniently expressed graphically as a sum of chains of identically oriented lines corresponding to retarded propagators Δ_{12}. To this end, it is customary to start from the equation with the additive source field $\delta(t)\chi(\boldsymbol{x})$

$$\partial_t \varphi(t) = -K\varphi(t) + f(t)\varphi(t) + \delta(t)\chi \,, \quad (1.61)$$

instead of the original SDE (1.58).

The graphical representation of the solution of (1.61) is of the form

$\varphi =$  $+ \ldots$ (1.62)

where the circle stands for the initial condition term $\delta(t)\chi$, the wavy line corresponds to the random field f and the full dot represents the vertex factor (equal to one here) brought about

by the last term of the Langevin equation (1.58). From the iterative construction it is seen that to each vertex (crossing of lines and the open circle) of any graph a separate time variable (and position vector variable in case of fields) is prescribed and integrated over. Solution (1.62) is the *tree-graph* representation of the solution of the stochastic differential equation (1.61).

The perturbative solution of the SDE (1.58) is given by Wick's theorem for the Gaussian distribution of f, which graphically amounts to replacing any pair of f by the correlation function $\overline{D}$ depicted by an unoriented line in all possible ways and – in case of zero mean – discarding all graphs with an odd number of wavy lines. For instance, the graphical expression (1.62) gives rise to the representation

$\langle\varphi\rangle =$  $+ \ldots$ (1.63)

Since the free-field differential operator in (1.58) is of first order in time, the physically interesting Green function is the retarded propagator. In the limit of white-in-time correlations this leads to an enormous truncation of the averaged iterative series (1.59), because it brings about temporal δ functions contracting the ends of chains of the retarded propagators. Any graph containing at least one such closed loop of at least two propagators vanishes. Only those terms, in which the correlation function is multiplied by a single retarded propagator do not vanish automatically.

For instance, the one-loop graph in (1.63) in case of white-in-time noise gives rise to an ambiguity, which is directly related to that in the interpretation of the stochastic differential equation (1.58). A straightforward substitution of the white-noise correlation function in this graph gives rise to the expression


$$= \int \mathrm{d}t_1 \int \mathrm{d}^d\boldsymbol{x}_1 \int \mathrm{d}^d\boldsymbol{x}_2 \int \mathrm{d}^d\boldsymbol{x}_3\, \Delta_{12}(t,\boldsymbol{x},t_1,\boldsymbol{x}_1) \times \Delta_{12}(t_1,\boldsymbol{x}_1,t_1\boldsymbol{x}_2) D(\boldsymbol{x}_1,\boldsymbol{x}_2) \Delta_{12}(t_1,\boldsymbol{x}_2,0,\boldsymbol{x}_3)\chi(\boldsymbol{x}_3)\,, \quad (1.64)$$

where the value of the propagator at coinciding time arguments $\Delta_{12}(t_1,\boldsymbol{x}_1,t_1\boldsymbol{x}_2) = \theta(0)\delta(\boldsymbol{x}_1 - \boldsymbol{x}_2)$ is ambiguous.

With the use of the δ-sequence of correlation functions (1.50) this ambiguity is readily resolved and gives rise to the expression


$$= \frac{1}{2}\int \mathrm{d}t_1 \int d\boldsymbol{x}_1 \int \mathrm{d}\boldsymbol{x}_3\, \Delta_{12}(t,\boldsymbol{x},t_1,\boldsymbol{x}_1) D(\boldsymbol{x}_1,\boldsymbol{x}_1) \times \Delta_{12}(t_1,\boldsymbol{x}_1,0,\boldsymbol{x}_3)\chi(\boldsymbol{x}_3)\,, \quad (1.65)$$

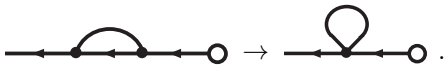
with the coefficient $\frac{1}{2}$ in front of the spatial δ function. As noted before, this procedure corresponds to the interpretation of the SDE (1.58) in the Stratonovich sense. Formally, this result may be obtained by amending the definition of the propagator according to the rule

$$\Delta_{12}(t,\boldsymbol{x},t,\boldsymbol{x}') = \frac{1}{2}\delta(\boldsymbol{x}-\boldsymbol{x}'). \quad (1.66)$$

Within this choice the graphical expression in (1.65) appears excessive, because it hints to twice the number of integrations than actually is carried out.

However, the white-noise limit may graphically be depicted as replacement of the one-loop graph with the noise correlation function by a new vertex factor with the coefficient $\frac{1}{2}D(\boldsymbol{x},\boldsymbol{x}) \equiv$

$\frac{1}{2}D(0)$:



(1.67)

The iterative solution of (1.58) with the eventual δ-function limit in the correlation function of the noise may be obtained also as the solution of a slightly different SDE, but with the convention $\Delta_{12}(0) = 0$ for the equal-time value of the propagator. In this case, passing to the white-noise limit is carried out solely with the use of the properties of the propagator and, therefore, the contribution of the graph (1.64) and the like vanishes. To restore this contribution in the solution, a term corresponding to the vertex factor of the effective vertex brought about by shrinking to point of the one-loop graph in (1.67) must be added to the SDE. From inspection of (1.67) we see that in this approach the SDE generating the solution constructed above is of the form

$$\partial_t \varphi = -K\varphi + \frac{1}{2}D(0)\varphi + f\varphi \tag{1.68}$$

and coincides with the SDE in the Ito interpretation for the present linear multiplicative noise problem. It should be emphasized that the rules of construction of the solution are different in (1.58) and (1.68), but the solution is the same. If the two sets of rules are applied to the same equation, different solutions are, of course, produced. It is interesting to note that the choice of interpretation of SDE in the iterative solution seems to boil down to the choice of the value of the propagator at coinciding times.

For fixed function f the generating function of solutions of (1.58) may be expressed in the form of an S-matrix functional

$$\mathcal{G}(A) = \exp\left(\frac{\delta}{\delta\varphi}\Delta_{12}\frac{\delta}{\delta\tilde{\varphi}}\right)\exp\left[\tilde{\varphi}f\varphi - \Delta_{12}(0)f + \tilde{\varphi}\tilde{A} + A\varphi\right]\Big|_0 . \tag{1.69}$$

Calculating expected value with respect to random f we obtain (no new notation for $\mathcal{G}(A)$ is introduced, field-independent determinant is included in the functional measure)

$$\mathcal{G}(A) = \exp\left(\frac{\delta}{\delta\varphi}\Delta_{12}\frac{\delta}{\delta\tilde{\varphi}}\right)\exp\left\{\frac{1}{2}\,[\tilde{\varphi}\varphi - \Delta_{12}(0)]\,\overline{D}\,[\tilde{\varphi}\varphi - \Delta_{12}(0)] + \tilde{\varphi}\tilde{A} + A\varphi\right\}\Big|_0 . \tag{1.70}$$

With the colored-noise correlation function the interaction functional here is not time local. The graphical analysis has revealed that technical problems arise from graphs in which the propagator Δ_{12} is spanned between fields φ and $\tilde{\varphi}$ belonging to the same compound vertex functional $\frac{1}{2}\,[\tilde{\varphi}\varphi - \Delta_{12}(0)]\,\overline{D}\,[\tilde{\varphi}\varphi - \Delta_{12}(0)]$. This situation is conveniently analyzed with the use of the normal form of the interaction functional, in which the effect of these "diagonal" terms is expressed explicitly and once for all in the normal form of the interaction functional. Calculation yields [45]

$$\exp\left(\frac{\delta}{\delta\varphi}\Delta_{12}\frac{\delta}{\delta\tilde{\varphi}}\right)\frac{1}{2}\,[\tilde{\varphi}\varphi - \Delta_{12}(0)]\,\overline{D}\,[\tilde{\varphi}\varphi - \Delta_{12}(0)] = \frac{1}{2}\,\tilde{\varphi}\varphi\overline{D}\tilde{\varphi}\varphi + \tilde{\varphi}\Delta_{12}\overline{D}\varphi + \frac{1}{2}\Delta_{21}\overline{D}\Delta_{21}\,. \tag{1.71}$$

Here, the white-noise limit in the first term of right side yields the time-local contribution $\frac{1}{2}\,\tilde{\varphi}\varphi D\tilde{\varphi}\varphi$ in the fashion (1.57), whereas the limit of the second term is familiar from the analysis of graphs

$$\int \mathrm{d}x \int \mathrm{d}x' \tilde{\varphi}(x)\Delta_{12}(x,x')\overline{D}(x,x')\varphi(x') \to \frac{1}{2}\int \mathrm{d}t \int \mathrm{d}^d\boldsymbol{x}\; \tilde{\varphi}(t,\boldsymbol{x})D(\boldsymbol{x},\boldsymbol{x})\varphi(t,\boldsymbol{x}) \equiv \frac{1}{2}D(0)\tilde{\varphi}\varphi\,. \tag{1.72}$$

The explicit expression for the third term ($\Delta_{21} = \Delta_{12}^{\top}$) contains a closed loop of two retarded propagators and therefore vanishes:

$$\int \mathrm{d}t \int \mathrm{d}t' \int \mathrm{d}^d\boldsymbol{x} \int \mathrm{d}^d\boldsymbol{x}' \Delta_{12}(t,\boldsymbol{x};t',\boldsymbol{x}')\overline{D}(t,\boldsymbol{x};t',\boldsymbol{x}')\Delta_{12}(t',\boldsymbol{x}';t,\boldsymbol{x}) = 0\,. \tag{1.73}$$

Therefore, the white-noise limit of the generating functional (1.70) is

$$\mathcal{G}(A) = \exp\left(\frac{\delta}{\delta\varphi}\Delta'_{12}\frac{\delta}{\delta\tilde{\varphi}}\right)\exp\left[\frac{1}{2}\tilde{\varphi}\varphi D\tilde{\varphi}\varphi + \frac{1}{2}D(0)\tilde{\varphi}\varphi + \tilde{\varphi}\tilde{A} + A\varphi\right]\Bigg|_0\,. \tag{1.74}$$

In the corresponding functional integral

$$\mathcal{G}(A) = \int \mathcal{D}'\varphi \int \mathcal{D}'\tilde{\varphi}\,\exp\left\{\frac{1}{2}\tilde{\varphi}\varphi D\tilde{\varphi}\varphi + \tilde{\varphi}\left[-\partial_t\varphi - K\varphi + \frac{1}{2}D(0)\varphi\right] + A\varphi\right\} \tag{1.75}$$

the linear in $\tilde{\varphi}$ term of the dynamic action

$$\mathcal{S}'[\varphi,\tilde{\varphi}] = \frac{1}{2}\tilde{\varphi}\varphi D\tilde{\varphi}\varphi + \tilde{\varphi}\left[-\partial_t\varphi - K\varphi + \frac{1}{2}D(0)\varphi\right] \tag{1.76}$$

apart from the deterministic part of the right side of the Langevin equation (1.58) contains an additional term corresponding to the Langevin equation in the Ito interpretation (1.68). It should be borne in mind that the generating function has (1.75) has been constructed to yield the solution of the SDE in the Stratonovich interpretation. However, this solution is most conveniently expressed with the use of the normal form of the dynamic action, whose form and rules of calculation remind of the SDE in Ito interpretation.

Consider the Langevin equation with the multiplicative noise of generic form

$$\partial_t\varphi = V(\varphi) + fB(\varphi) := -K\varphi + U(\varphi) + fB(\varphi)\,. \tag{1.77}$$

Here, $B(\varphi)$ is a functional of φ and $U(\varphi)$ is a nonlinear functional of φ. Both functionals are time-local, i.e. depend only on the current times instant of the SDE. Again the starting point of the construction of the generating function of solutions is the δ sequence of "colored-noise" correlation functions (1.50).

Generating function of solutions may be expressed in the functional-differential form in the same fashion as preceding cases of additive noise and linear multiplicative noise with the result problem

$$\mathcal{G}(A) = \exp\left(\frac{\delta}{\delta\varphi}\Delta_{12}\frac{\delta}{\delta\tilde{\varphi}}\right)\int \mathcal{D}f\,\exp\left(-\frac{1}{2}f\overline{D}^{-1}f\right)$$

$$\times \exp\left\{\tilde{\varphi}\left[U(\varphi)+fB(\varphi)\right]-\Delta_{12}(0)\left[U'(\varphi)+fB'(\varphi)\right]+A\varphi\right\}\Big|_0$$
$$=\exp\left(\frac{\delta}{\delta\varphi}\Delta_{12}\frac{\delta}{\delta\tilde{\varphi}}\right)\exp\Big\{\tilde{\varphi}U(\varphi)-\Delta_{12}(0)U'(\varphi)\frac{1}{2}\left[\tilde{\varphi}B-\Delta_{12}(0)B'\right]\overline{D}$$
$$\times\left[\tilde{\varphi}B-\Delta_{12}(0)B'\right]+A\varphi\Big\}\Big|_0 . \tag{1.78}$$

Here, the term corresponding to closed loop of two propagators Δ_{12} (cf. (1.73)) has already been omitted.

With the aid of the normal form the diagonal terms of the reduction operator are singled out to give rise to representation

$$\mathcal{G}(A)=\exp\left(\frac{\delta}{\delta\varphi}\Delta'_{12}\frac{\delta}{\delta\tilde{\varphi}}\right)\exp\Big\{\tilde{\varphi}U(\varphi)+\frac{1}{2}\tilde{\varphi}B\overline{D}\tilde{\varphi}B+\Delta_{12}(1,2)B(2,\varphi)$$
$$\times\overline{D}(2,3)\frac{\delta B(3,\varphi)}{\delta\varphi(1)}\tilde{\varphi}(3)+A\varphi\Big\}\Big|_0 . \tag{1.79}$$

Here, contractions of variables are no more obvious and they have been expressed explicitly with the use of condensed notation, e.g. $\varphi(1)\equiv\varphi(t_1,\boldsymbol{x}_1)$.

The white-noise limit is obtained by the same argument as in the case of linear multiplicative noise and yields the generating function in the relatively simple form [45]

$$\mathcal{G}(A)=\exp\left(\frac{\delta}{\delta\varphi}\Delta'_{12}\frac{\delta}{\delta\tilde{\varphi}}\right)\exp\left[\tilde{\varphi}U+\frac{\tilde{\varphi}BD\tilde{\varphi}B}{2}+\frac{B(2,\varphi)D(2,1)}{2}\frac{\delta B(1,\varphi)}{\delta\varphi(2)}\tilde{\varphi}(1)+A\varphi\right]\Big|_0 .$$

Here, the dynamic action is already time local

$$S'[\varphi,\tilde{\varphi}]=\frac{1}{2}\tilde{\varphi}BD\tilde{\varphi}B+\tilde{\varphi}\left[-\partial_t\varphi+V+\frac{1}{2}BDB'\right] \tag{1.80}$$

and the absence of diagonal loops is thus tantamount to definition $\Delta_{12}(0)=0$. We see that the linear in $\tilde{\varphi}$ term corresponds to the deterministic part of the SDE in the Ito form also in the generic case.

Let us remind that the dynamic action (1.80) corresponds to the Stratonovich interpretation of the SDE (1.77). Henceforth, we shall refer to the additional term – on top of those brought about by the deterministic part of the SDE – in the dynamic action as the white-noise contraction term. Due to the straightforward connection between the white-noise and colored-noise cases we shall stick to the Stratonovich interpretation in the following, if not stated otherwise. However, for completeness of representation we will describe the dynamic action corresponding to the Ito interpretation of the stochastic problem posed by the SDE (1.77) in the next section.

1.3 Fokker-Planck equation

Account of fluctuations in kinetic problems with the use of Langevin equations is straightforward, because the starting point is the hydrodynamic kinetic equation. The solution is given in terms of a generating functional for correlation and response functions (Green functions). The

disadvantage of this approach lies in its mathematical ambiguity in the case of multiplicative noise. Therefore, it is instructive to infer the generating functional from the mathematically well-defined setup of the stochastic problem with the use of evolution equations for probability density functions of the relevant random quantities. In case of continuous stochastic processes the usual starting point is the Fokker-Planck equation and in case of jump processes the master equation – both consequences of the fundamental Chapman-Kolmogorov equation of Markov processes, which is the most important class of stochastic processes from the point of view of fluctuation kinetics.

The basic tool to describe stochastic processes are the joint probability density functions (PDF) $p\left(\varphi_1, t_1; \varphi_2, t_2; \ldots; \varphi_n, t_n\right)$, which give the probability to observe the values $\varphi_1, \varphi_2, \ldots \varphi_n$ of the random variable (field) at time instances $t_1, t_2, \ldots, t_n$ in a suitable volume element of the space of values of $\{\varphi_i\}_{i=1}^n$. For simplicity, only zero-dimensional fields (i.e. functions of time only) will be discussed in these notes. Spatial dependence of fields may be taken into account in most cases simply by replacing partial derivatives by functional derivatives. Some attention has to be paid to contractions of arguments, though.

The most important stochastic processes in physics are the Markov processes, in which the joint probability density function is completely expressed in terms of the conditional PDF $p\left(\varphi, t|\varphi_0, t_0\right)$ and the single-event PDF $p\left(\varphi, t\right)$:

$$p\left(\varphi_1, t_1; \ldots; \varphi_n, t_n\right) = p\left(\varphi_1, t_1|\varphi_2, t_2\right) p\left(\varphi_2, t_2|\varphi_3, t_3\right) \cdots p\left(\varphi_{n-1}, t_{n-1}|\varphi_n, t_n\right) p\left(\varphi_n, t_n\right)$$

provided $t_1 \geq t_2 \geq t_3 \geq \ldots \geq t_{n-1} \geq t_n$. Both these functions can be found as solutions of the Fokker-Planck equation.

Consider evolution generated by the generic Fokker-Planck equation

$$\frac{\partial}{\partial t} p\left(\varphi, t|\varphi_0, t_0\right) = -\frac{\partial}{\partial \varphi}\left\{\left[-K\varphi + U(\varphi)\right] p\left(\varphi, t|\varphi_0, t_0\right)\right\} + \frac{1}{2}\frac{\partial^2}{\partial \varphi^2}\left[b(\varphi) D b(\varphi) p\left(\varphi, t|\varphi_0, t_0\right)\right] . \tag{1.81}$$

The conditional probability density is the fundamental solution of this equation, i.e.

$$p\left(\varphi, t_0|\varphi_0, t_0\right) = \delta(\varphi - \varphi_0)$$

and is also properly normalized

$$\int \mathrm{d}\varphi\, p\left(\varphi, t|\varphi_0, t_0\right) = 1 .$$

The probability density function $p(\varphi, t)$ is the solution of the FPE (1.81) as well, but with the initial condition $p(\varphi, t_0) = p_0(\varphi)$.

The Fokker-Planck equation (1.81) is similar to the Schrödinger equation of quantum mechanics. Solution of both equations is a function related to probability density function of dynamic variables. This analogy allows to use the the quantum-mechanical operator approach to construct a (formal) solution of the Fokker-Planck equation [50, 51].

Therefore, let us introduce – in analogy with quantum mechanics – the state vector $|\, p_t \,\rangle$ according to representation

$$p(\varphi, t) := \langle\, \varphi \,|\, p_t \,\rangle .$$

To construct the evolution operator for the state vector, introduce the (nearly) quantum-mechanical momentum and coordinate operators by

$$\hat{\pi} f(\varphi) = -\frac{\partial}{\partial \varphi} f(\varphi)\,, \quad \hat{\varphi} f(\varphi) = \varphi f(\varphi)\,.$$

The non-trivial commutation relation is

$$[\hat{\varphi}, \hat{\pi}] = 1\,.$$

In these terms, the FPE for the PDF gives rise to the evolution equation for the state vector in the form

$$\frac{\partial}{\partial t} |\, p_t \,\rangle = \hat{L} |\, p_t \,\rangle\,,$$

where the Liouville operator, according to (1.81), assumes the form

$$\hat{L} = \hat{\pi}\,[-K\hat{\varphi} + U(\hat{\varphi})] + \frac{1}{2}\hat{\pi}^2 b(\hat{\varphi}) D b(\hat{\varphi})\,. \tag{1.82}$$

In this notation, the conditional PDF may be expressed as the matrix element

$$p\,(\varphi, t|\varphi_0, t_0) = \left\langle\, \varphi \,\middle|\, \mathrm{e}^{\hat{L}(t-t_0)} \,\middle|\, \varphi_0 \,\right\rangle\,. \tag{1.83}$$

Introduce time-dependent operators $\hat{\varphi}(t)$ in the Heisenberg picture of imaginary time quantum mechanics (i.e. Euclidean quantum mechanics):

$$\hat{\varphi}_H(t) = \mathrm{e}^{-\hat{L}t}\,\hat{\varphi}\,\mathrm{e}^{\hat{L}t}\,, \tag{1.84}$$

and define the time-ordered product (chronological product, T product) of time-dependent operators

$$T\,[\hat{\varphi}_H(t_1)\cdots\hat{\varphi}_H(t_n)] = \sum_{P(1,\ldots,n)} P\,[\theta\,(t_1 \ldots t_n)\,\hat{\varphi}_H(t_1)\cdots\hat{\varphi}_H(t_n)]\,, \tag{1.85}$$

where

$$\theta\,(t_1 \ldots t_n) \equiv \theta\,(t_1 - t_2)\,\theta\,(t_2 - t_3)\cdots\theta\,(t_{n-1} - t_n)\,.$$

In definition (1.85) the sum is taken over all permutations of the labels of the time arguments $\{t_i\}_{i=1}^n$ and the operators in each term are put in the order of growing time arguments from the right to the left. Thus, under the T-product sign operators commute. It should be noted that the definition of the time-ordered product should be amended for coinciding time arguments. We shall return to this point later and exclude this case for the time being.

Introduce then the n-point Green function of the Heisenberg fields (1.84)

$$G_n(t_1, t_2, \ldots t_n) := \mathrm{Tr}\,\{\hat{\rho}\,T\,[\hat{\varphi}_H(t_1)\hat{\varphi}_H(t_2)\cdots\hat{\varphi}_H(t_n)]\} \tag{1.86}$$

with the density operator

$$\hat{\rho} := \int \mathrm{d}\varphi |\, p_0 \,\rangle\langle\, \varphi \,|\,. \tag{1.87}$$

Choosing, for definiteness, the time sequence $t_1 > t_2 > t_3 > \ldots > t_{n-1} > t_n > t_0$ it is readily seen by direct substitution of relations (1.83), (1.84) and (1.87) in (1.86) with the aid of the normalization conditions of the PDF and insertions of the resolution of the identity $\int d\varphi \,|\, \varphi \,\rangle\langle\, \varphi \,|\, = 1$ that

$$\int \mathrm{d}\varphi_1 \ldots \int \mathrm{d}\varphi_n \, \varphi_1 \cdots \varphi_n p\,(\varphi_1, t_1; \ldots; \varphi_n, t_n) = G_n(t_1, \ldots t_n)\,, \tag{1.88}$$

i.e. the Green function (1.86) is equal to the moment function (1.88). This relation connects the operator approach to evaluation of moments of the random process: moments and correlation functions of the random field φ may be calculated as derivatives of the *generating function* of Green functions

$$\begin{aligned}\mathcal{G}(A) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}t_1 \cdots \int \mathrm{d}t_n \, G_n(t_1, \ldots, t_n) A(t_1) \cdots A(t_n) \\ &= \mathrm{Tr} \left\{ \hat{\rho}\, T \left[\exp \left(\int dt \, A(t) \hat{\varphi}_H(t) \right) \right] \right\} . \end{aligned} \tag{1.89}$$

To construct perturbation expansion for the generating function it is convenient to introduce field operators in the interaction (or Dirac) representation, in which the time evolution is generated by the free Liouville equation

$$\hat{\varphi}(t) = \mathrm{e}^{-\hat{L}_0 t} \, \hat{\varphi} \, \mathrm{e}^{\hat{L}_0 t}\,, \qquad \hat{L}_0 = -\hat{\pi} K \hat{\varphi}\,. \tag{1.90}$$

There is some freedom in the choice of the free Liouvillean L_0. The expression adopted here on the basis of the expression (1.82) is quite convenient for practical calculations.

In the interaction representation the *evolution operator* $\hat{U}$ is of the form

$$\hat{U}(t, t') = \mathrm{e}^{-\hat{L}_0 t} \, \mathrm{e}^{\hat{L}(t-t')} \, \mathrm{e}^{\hat{L} t'}\,. \tag{1.91}$$

It is a fundamental theorem of quantum field theory that the time-ordered product of Heisenberg operators (1.84) may be expressed in terms of Dirac operators (1.90) as

$$T \exp \left(\int_{t_i}^{t_f} \hat{\varphi}_H(t) A(t) \mathrm{d}t \right) = \hat{U}(0, t_f) T \exp \left[\int_{t_i}^{t_f} \hat{L}_I(t) \, \mathrm{d}t + \int_{t_i}^{t_f} \hat{\varphi}(t) A(t) \, \mathrm{d}t \right] \hat{U}(t_i, 0)\,. \tag{1.92}$$

In quantum field theory calculation of various expectation values of operators in the interaction representation is most conveniently carried out, when the operators are expressed in the form of a *normal product*. In case of quantum-mechanical momentum and position operators in the normal product of an operator monomial all momentum operators stand to the left of all position operators. When an arbitrary operator is cast in the form of a normal product, the result is a linear combination of normal products of operator monomials, whose coefficients are determined by the commutation rules of operators, e.g.

$$\hat{\varphi}\hat{\pi}^2 = \hat{\pi}^2 \hat{\varphi} + 2\hat{\pi}\,.$$

Wick's theorems state results of representation of operator products in normal form. For the time-ordered exponential of generating function (1.92) it follows from Wick's theorems [39] that

$$T\exp\left\{\int_{t_i}^{t_f}[\hat{L}_I(t)+\hat{\varphi}(t)A(t)]\,\mathrm{d}t\right\}=N\left\{\exp\left(\frac{\delta}{\delta\varphi}\Delta_{12}\frac{\delta}{\delta\pi}\right)\exp\left[\int_{t_i}^{t_f}L_I(\varphi(t),\pi(t))\,\mathrm{d}t+\int_{t_i}^{t_f}\varphi(t)A(t)\,\mathrm{d}t\right]\right\}\Bigg|_{\substack{\varphi=\hat{\varphi}\\ \pi=\hat{\pi}}}. \tag{1.93}$$

On the right side $L_I(\varphi,\pi)$ is the interaction functional, in which the operators $\hat{\varphi}$ and $\hat{\pi}$ in $\hat{L}_I=\hat{L}-\hat{L}_0$ are replaced by functions φ and π. Then the *reduction operator*

$$\mathcal{P}=\exp\left(\frac{\delta}{\delta\varphi}\Delta_{12}\frac{\delta}{\delta\pi}\right)$$

replaces pairs of functions φ and π by the *propagator*

$$\Delta_{12}(t,t')=T\left[\hat{\varphi}(t)\hat{\pi}(t')\right]-N\left[\hat{\varphi}(t)\hat{\pi}(t')\right]$$

after which all remaining functions φ and π are replaced by corresponding operators in the normal order, because by definition

$$N\left\{P\left[\hat{\pi}(t_1)\cdots\hat{\pi}(t_m)\hat{\varphi}(t'_1)\cdots\hat{\varphi}(t'_n)\right]\right\}=\hat{\pi}(t_1)\cdots\hat{\pi}(t_m)\hat{\varphi}(t'_1)\cdots\hat{\varphi}(t'_n)\,,$$

where P stands for any ordering of operators $\hat{\pi}$ and $\hat{\varphi}$.

Thus, we arrive at the representation

$$\mathcal{G}(A)=\mathrm{Tr}\left(\hat{U}(t_i,0)\,\hat{\rho}\,\hat{U}(0,t_f)\,N\left\{\exp\left(\frac{\delta}{\delta\varphi}\Delta_{12}\frac{\delta}{\delta\pi}\right)\exp\left[\int_{t_i}^{t_f}L_I(\varphi(t),\pi(t))\,dt+\int_{t_i}^{t_f}\varphi(t)A(t)\,dt\right]\right\}\Bigg|_{\substack{\varphi=\hat{\varphi}\\ \pi=\hat{\pi}}}\right). \tag{1.94}$$

for the generating function. At this point the ambiguity in the definition of the T product should be fixed, because it affects the explicit form of the interaction functional. The point is that contributions to perturbation expansion produced by the action of the reduction operator on the interaction functional $L_I(\varphi,\pi)$ give rise to a *reduced* interaction functional (or to the *normal form* of the interaction functional)

$$L'_I(\varphi,\pi)=\exp\left(\frac{\delta}{\delta\varphi}\Delta_{12}\frac{\delta}{\delta\pi}\right)L_I(\varphi,\pi)\,,$$

which, in general, is different from $L_I(\varphi,\pi)$. The form of the interaction functional generated by the Fokker-Planck is preserved, if the T product at coinciding time arguments is defined as the N product, in which case, in particular, $\Delta_{12}(t,t)=0$ and the normal form of the time-local interaction functional coincides with the original interaction functional.

In (1.94) under the sign of normal product N there stands some *operator functional* $F[\hat{\varphi},\hat{\pi}]$, i.e. a Taylor series of operators $\hat{\varphi}$ and $\hat{\pi}$. To calculate the expectation value (1.94) of an arbitrary

operator functional, it is sufficient to calculate the expectation value of a linear exponential, because

$$\mathrm{Tr}\,\left\{N\,[F(\hat{\varphi},\hat{\pi})]\,\hat{U}(t_i,0)\,\hat{\rho}\,\hat{U}(0,t_f)\right\} = F\left(\frac{\delta}{\delta A},\frac{\delta}{\delta B}\right)\mathrm{Tr}\left\{\hat{U}(t_i,0)\,\hat{\rho}\,\hat{U}(0,t_f)\right.$$
$$\left.\times\, N\,\exp\left(\hat{\varphi}A+\hat{\pi}B\right)\right\}\Big|_{A=B=0}. \tag{1.95}$$

Writing out explicitly the trace in the coordinate representation, the density operator $\hat{\rho} = \int \mathrm{d}\varphi|\,p_0\,\rangle\langle\,\varphi\,|$, and an identity resolution $\int \mathrm{d}\zeta\,|\,\zeta\,\rangle\langle\,\zeta\,| = 1$ to produce matrix elements of evolution operators we arrive at the representation

$$\begin{aligned}
&\mathrm{Tr}\,\left\{\hat{U}(t_i,0)\,\hat{\rho}\,\hat{U}(0,t_f)N\ \exp\left(\hat{\varphi}A+\hat{\pi}B\right)\right\}\\
&\quad=\int \mathrm{d}\chi\,\langle\,\chi\,|\hat{U}(t_i,0)\int \mathrm{d}\varphi\,|\,p_0\,\rangle\langle\,\varphi\,|\hat{U}(0,t_f)N\,\exp\left(\hat{\varphi}A+\hat{\pi}B\right)|\,\chi\,\rangle\\
&\quad=\int \mathrm{d}\chi\int \mathrm{d}\varphi\int \mathrm{d}\zeta\,\langle\,\chi\,|\hat{U}(t_i,0)|\,p_0\,\rangle\langle\,\varphi\,|\hat{U}(0,t_f)|\,\zeta\,\rangle\langle\,\zeta\,|N\,\exp\left(\hat{\varphi}A+\hat{\pi}B\right)|\,\chi\,\rangle\,.
\end{aligned} \tag{1.96}$$

With the choice of the free Liouvillean in the form $\hat{L}_0 = -\hat{\pi}K\hat{\varphi}$ the time-dependent operators are

$$\hat{\pi}(t) = \hat{\pi}\,\mathrm{e}^{Kt}, \quad \hat{\varphi}(t) = \hat{\varphi}\,\mathrm{e}^{-Kt}. \tag{1.97}$$

Calculate first the matrix element of the exponential in the coordinate basis:

$$\langle\,\zeta\,|N\,\exp\left(\hat{\varphi}A+\hat{\pi}B\right)|\,\chi\,\rangle = \langle\,\zeta\,|\exp\left(\hat{\pi}\tilde{B}\right)\exp\left(\hat{\varphi}\tilde{A}\right)|\,\chi\,\rangle\,. \tag{1.98}$$

where $\tilde{A} = \int e^{-Kt}A(t)\,\mathrm{d}t$ and $\tilde{B} = \int e^{Kt}B(t)\,\mathrm{d}t$. In the coordinate basis the operator $\hat{\varphi}$ is the multiplication operator, whereas the exponential of the momentum operator is the translation operator. Therefore (1.98) immediately yields

$$\langle\,\zeta\,|N\,\exp\left(\hat{\varphi}A+\hat{\pi}B\right)|\,\chi\,\rangle = \delta\left(\zeta-\chi-\tilde{B}\right)\exp\left(\chi\tilde{A}\right). \tag{1.99}$$

The simple and sufficient choice for the Cauchy problem is to put $t_i = 0$, after which the matrix element $\langle\,\chi\,|\hat{U}(t_i,0)|\,p_0\,\rangle$ in (1.96) reduces to $\langle\,\chi\,|\,p_0\,\rangle = p_0(\chi)$. The matrix element of the other evolution operator in (1.96) yields the unity due to probability conservation:

$$\int \mathrm{d}\varphi\langle\,\varphi\,|\hat{U}(0,t_f)|\,\zeta\,\rangle = 1\,.$$

Thus, the average of the linear exponential (1.96) is reduced to

$$\begin{aligned}
&\mathrm{Tr}\,\left\{\hat{U}(t_i,0)\,\hat{\rho}\,\hat{U}(0,t_f)N\,\exp\left(\hat{\varphi}A+\hat{\pi}B\right)\right\}\\
&\quad=\int d\chi\int d\zeta\,\delta\left(\zeta-\chi-\tilde{B}\right)\exp\left(\chi\tilde{A}\right)p_0(\chi)=\int d\chi\,\exp\left(\chi\tilde{A}\right)p_0(\chi)\,.
\end{aligned} \tag{1.100}$$

Therefore, the expectation value of the normal product of any operator function(al) $F\left[\hat{\varphi}(t),\hat{\pi}(t)\right]$ is (here, $t_i=0$ at the outset) [52]

$$\begin{aligned}&\int \mathrm{d}\chi \int \mathrm{d}\varphi \,\langle\,\chi\,|\,p_0\,\rangle\langle\,\varphi\,|\,\hat{U}(0,t_f)N\left[F(\hat{\varphi}(t),\hat{\pi}(t))\right]|\,\chi\,\rangle\\&=F\left[\frac{\delta}{\delta A(t)},\frac{\delta}{\delta B(t)}\right]\int \mathrm{d}\chi\int \mathrm{d}\varphi\,\langle\,\chi\,|\,p_0\,\rangle\langle\,\varphi\,|\,\hat{U}(0,t_f)N\,\exp\left(\hat{\varphi}A+\hat{\pi}B\right)|\,\chi\,\rangle\big|_{A=B=0}\\&=\int \mathrm{d}\chi F\left[\frac{\delta}{\delta A(t)},0\right]p_0(\chi)\exp\left(\chi\tilde{A}\right)\big|_{A=0}=\int \mathrm{d}\chi F\left[\varphi_\chi(t),0\right]p_0(\chi)\,,\end{aligned}\tag{1.101}$$

where φ_χ is the solution of the free-field equation

$$\left(\partial_t+K\right)\varphi_\chi(t)=0$$

with the initial condition $\varphi_\chi(0)=\chi$. It should be borne in mind that functional variables here may be fields, in which case K usually is a second-order differential operator in space.

Introducing the Liouville operator $\hat{L}=\hat{L}_0+\hat{L}_I$ and corresponding functionals explicitly, we obtain the generating function of Green functions of the Cauchy problem of the Fokker-Planck equation in the form

$$\begin{aligned}\mathcal{G}(A)&=\int \mathrm{d}\chi\int \mathrm{d}\varphi\,p_0(\chi)\,\langle\,\varphi\,|\,T\exp\left(\hat{\varphi}_H A\right)|\,\chi\,\rangle\\&=\int \mathrm{d}\chi\,p_0(\chi)\left\{\exp\left(\frac{\delta}{\delta\varphi}\Delta\frac{\delta}{\delta\pi}\right)\exp\left[\int_0^{t_f}\left[L_I(\varphi(t),\pi(t))+\varphi(t)A(t)\right]\mathrm{d}t\right]\right\}_{\substack{\varphi=\varphi_\chi\\ \pi=0}}.\end{aligned}\tag{1.102}$$

It should be noted that $\hat{\varphi}_H$ here refers to the position operator of the Fokker-Planck equation, not to the generic field operator and that the propagator

$$\Delta(t,t')=T\left[\hat{\varphi}(t)\hat{\pi}(t')\right]-N\left[\hat{\varphi}(t)\hat{\pi}(t')\right]=\theta(t-t')\mathrm{e}^{-K(t-t')}\,,\tag{1.103}$$

is also written for fields of this particular problem.

Representation in the form of a functional integral may be introduced by the trick using a formal functional integral for the reduction operator:

$$\exp\left(\frac{\delta}{\delta\varphi}\Delta_{12}\frac{\delta}{\delta\pi}\right)=\int\mathcal{D}\varphi\int\mathcal{D}\tilde{\varphi}\,\exp\left[\tilde{\varphi}\left(-\partial_t-K\right)\varphi+\varphi\frac{\delta}{\delta\varphi}+\tilde{\varphi}\frac{\delta}{\delta\pi}\right].\tag{1.104}$$

Here, the determinant of the differential operator has been included in the integration measure and for integration variables a notation has been introduced which will be used in the analysis of the Langevin equation.

Functional shift operators on the right side of (1.104) generate argument changes $\varphi\to\varphi+\varphi$ and $\pi\to\pi+\tilde{\varphi}$ in the interaction functional in (1.102), after which the substitutions $\varphi=\varphi_\chi$ and $\pi=0$ may be carried out leading to expression

$$\mathcal{G}(A)=\int \mathrm{d}\chi\,p_0(\chi)\int\mathcal{D}\varphi\int\mathcal{D}\tilde{\varphi}\,\exp\left\{\mathcal{S}[\varphi+\varphi_\chi,\tilde{\varphi}]+\int_0^{t_f}\left[\varphi(t)+\varphi_\chi(t)\right]A(t)\mathrm{d}t\right\},\tag{1.105}$$

where the *De Dominicis-Janssen dynamic action* $\mathcal{S}[\varphi,\tilde{\varphi}]$ is

$$\mathcal{S}[\varphi,\tilde{\varphi}] = \frac{1}{2}\tilde{\varphi}^2 B(\varphi) DB(\varphi) + \tilde{\varphi}\left[-\partial_t\varphi - K\varphi + U(\varphi)\right] . \tag{1.106}$$

The form of the dynamic action is unambiguously determined by the appearance of the Fokker-Planck equation. It has to be kept in mind that functional representations (1.102) and (1.105) have been derived with the use of the convention that the T product at coinciding time arguments is defined as the N product. In particular, this means that the propagator vanishes at coinciding time arguments $\Delta_{12}(t,t) = 0$.

The explicit shift of the field $\varphi + \varphi_\chi$ in the functional integral (1.105) may be absorbed in the definition of the space of integration. The point is that formal convergence of the Gaussian integral (1.104) requires that solutions of the homogeneous equation $(\partial_t + K)\varphi_0 = 0$ must be excluded from the space of integration of the Gaussian integral. The space of integration may be constructed as $\varphi(t) = \int_0^t \mathrm{d}t' \Delta(t-t')\eta(t')$, where $\eta(t)$ is a function vanishing at all borders of the spacetime. Then, by definition, $\varphi(0) = 0$ and the combination $\varphi(t) = \varphi(t) + \varphi_\chi(t)$ is the solution of the inhomogeneous equation $(\partial_t + K)\varphi(t) = \eta(t)$ satisfying the initial condition $\varphi(0) = \chi$. Defining the space of integration as a manifold consisting of such functions we may write

$$\mathcal{G}(A) = \int \mathrm{d}\chi\, p_0(\chi) \int \mathcal{D}_\chi\varphi \int \mathcal{D}\tilde{\varphi}\, \exp\left\{\mathcal{S}[\varphi,\tilde{\varphi}] + \int_0^{t_f} \varphi(t) A(t) \mathrm{d}t\right\} , \tag{1.107}$$

where the subscript χ in the measure reminds on the dependence on the initial condition for φ.

The dynamic action (1.106) obtained for the functional representation of the solution of the Fokker-Planck equation (1.81) is different from the dynamic action (1.80) obtained for the Langevin equation (1.47). This difference is connected with different interpretations of the stochastic differential equation (1.47). Dynamic action (1.80) corresponds to the Stratonovich interpretation of the SDE, whereas dynamic action (1.106) corresponds to the Ito interpretation of the same SDE. Which interpretation is chosen as the basis of the field theory is largely a matter of model construction. In those cases, however, when the SDE is considered as white-noise limit of a sequence of colored noises, the Stratonovich interpretation must be used.

1.4 Master equation

Markov processes described in terms of the Fokker-Planck equation have continuous sample paths. Not all interesting stochastic processes belong to this category. A wide class of such processes describe changes in occupation numbers (e.g. individuals of some population, molecules in chemical reaction) which cannot be naturally described by continuous paths. This kind of processes are described by *master equations* – a special case of (differential) Kolmogorov equations.

The generic form of a master equation written for the conditional probability density $p\left(\varphi,t|\varphi_0,t_0\right)$ of a Markov process is

$$\frac{\partial}{\partial t} p\left(\varphi,t|\varphi_0,t_0\right) = \int \mathrm{d}\chi \left[W(\varphi|\chi,t) p\left(\chi,t|\varphi_0,t_0\right) - W(\chi|\varphi,t) p\left(\varphi,t|\varphi_0,t_0\right)\right] , \tag{1.108}$$

where $W(\varphi|\chi,t)$ is the *transition probability* per unit time, whose formal definition from the differential Kolmogorov equation is (for all $\varepsilon > 0$)

$$W(\varphi|\chi,t) = \lim_{\Delta t\to 0} \frac{p\,(\varphi, t+\Delta t|\chi,t)}{\Delta t}\,,$$

uniformly in φ, χ and t for all $|\varphi - \chi| \geq \varepsilon$.

We shall be using the master equation for discrete variables, in this case the discontinuous character of the paths of the *jump processes* described by the master equation is especially transparent:

$$\frac{\partial}{\partial t} p\,(n,t|m,t_0) = \sum_l \left[W(n|l,t)p\,(l,t|m,t_0) - W(l|n,t)p\,(n,t|m,t_0)\right]\,. \tag{1.109}$$

This set of master equations shall also be cast in the form of an evolution equation of the type of Schrödinger equation, but the representation is not as straightforward as in the case of the Fokker-Planck equation.

In the occupation-number basis and in the stationary field operators there is no explicit Planck constant. The ideas of the Fock-space representation and creation/annihilation operators may therefore as well be used in classical problems with a variable number of particles or some other entities.

Various processes in biology and chemistry are described in terms of variable numbers of some agents or representatives of species (”particle numbers” or ”occupation numbers”). In many cases changes in the particle number are caused by interactions between colliding particles, i.e. *reactions*.

Description of a particular reaction may usually be given by the *reaction equation*. For instance, for a two species process with the *rate constants* k_+ and k_- the reaction equation is

$$s_A A + s_B B \underset{k_-}{\overset{k_+}{\rightleftharpoons}} r_A A + r_B B\,, \tag{1.110}$$

where A and B denote the two species and s_A, s_B, r_A and r_B are coefficients (usually integers) describing in which proportions the agents react.

The simplest kinetic description of the dynamics of the average particle numbers is given by the *rate equation*. For the binary reaction (1.110), e.g.

$$\frac{\mathrm{d}c_A}{\mathrm{d}t} = k_+(r_A - s_A)c_A^{s_A}c_B^{s_B} + k_-(s_A - r_A)c_A^{r_A}c_B^{r_B}\,, \tag{1.111}$$

where c_X is the concentration of the species X.

The rate equation is a deterministic differential equation for average particle numbers in a homogeneous system, therefore it does not take into account boundary conditions, spatial inhomogeneities and randomness in the individual reaction events.

To keep things simple, consider a system with just one variable. A classic example is the *Verhulst* (*logistic*) *model* of population growth. The rate equation for the particle number n (the number of individuals in the population) may be written as

$$\frac{\mathrm{d}n}{\mathrm{d}t} = -\beta n + \lambda n - \gamma n^2\,, \tag{1.112}$$

where β is the death rate, λ the birth rate and γ the damping coefficient necessary to bring about a saturation for the population.

A complete (microscopic) description of a stochastic problem of random particle number is given by the master equations written for the probabilities $P(t,n)$ to find exactly n particles in the system at the time instant t. Transition rates (transition probabilities per unit time) of master equations are related to coefficients of the rate equation, but the correspondence is not unambiguous: given rate equations may generate different master equations, therefore a verbal description of the process is often necessary.

Regarding birth and death as reactions

$$A \underset{\gamma}{\overset{\lambda}{\leftrightharpoons}} 2A\,, \qquad A \xrightarrow{\beta} \varnothing$$

master equations for the *stochastic Verhulst model* may be written as (this is slightly different from the original formulation by Feller [53])

$$\begin{aligned}
\frac{\mathrm{d}P(t,N)}{\mathrm{d}t} &= \lambda(N-1)P(t,N-1) - \left(\beta N + \gamma N^2\right) P(t,N)\,, \\
\frac{\mathrm{d}P(t,n)}{\mathrm{d}t} &= [\beta(n+1) + \gamma(n+1)^2]P(t,n+1) + \lambda(n-1)P(t,n-1) \\
&\quad - \left(\beta n + \lambda n + \gamma n^2\right) P(t,n)\,, \quad 0 < n < N\,, \\
\frac{\mathrm{d}P(t,0)}{\mathrm{d}t} &= (\beta+\gamma)P(t,1)\,.
\end{aligned} \tag{1.113}$$

Here, boundary conditions for the particle number are made explicit in the equations for the probabilities of the boundary values of the particle number. Usually, the empty state ($n = 0$) is an *absorbing state* (once the system occurs in an absorbing state, it will stay there forever) and the state with the maximum sustainable population ($n = N$) is a *reflecting state* (there are no available states beyond a reflecting state, but the system does not get stuck to that state) as here.

Using master equations (1.113) a coupled set of evolution equations for the moments of the particle number $\langle n^m \rangle$ may be written. The evolution equation for the average particle number $\langle n \rangle$ coincides with the rate equation (1.112), when all correlations are neglected, i.e. moments replaced by corresponding powers of the average particle number $\langle n^m \rangle \to \langle n \rangle^m$. This, however, is not always the case, but only *if* the transition rates vanish with n.

The set of master equations for the probabilities $P(t,n)$ may be cast into a single kinetic equation by the "second quantization" of Doi [54, 55]. Recall that the conditional probabilities $P(t,n|t_0,m)$ obey the same set of equations, but with a different initial condition:

$$P(t_0,n|t_0,m) = \delta_{nm}. \tag{1.114}$$

Let us first construct a Fock space spanned by the annihilation and creation operators $\hat{a}$, $\hat{a}^\dagger$

$$[\,\hat{a},\hat{a}^\dagger] = 1\,, \quad [\,\hat{a},\hat{a}] = [\,\hat{a}^\dagger,\hat{a}^\dagger] = 0 \tag{1.115}$$

and the vacuum vector $|\,0\,\rangle$

$$\hat{a}|\,0\,\rangle = 0 \tag{1.116}$$

so that the basis vectors are

$$|\, n \,\rangle = \left(\hat{a}^\dagger\right)^n |\, 0 \,\rangle \,. \tag{1.117}$$

From these relations it follows that

$$\hat{a}|\, n \,\rangle = n|\, n-1 \,\rangle \,, \quad \hat{a}^\dagger|\, n \,\rangle = |\, n+1 \,\rangle \,, \quad \langle\, n \,|\, m \,\rangle = n!\delta_{nm} \,. \tag{1.118}$$

Note that the action of creation and annihilation operators on the basis vector as well as the normalization of the latter are different from those used in quantum mechanics. Connection between Fock space and occupation numbers is given by the occupation number operator $\hat{n} = \hat{a}^\dagger\hat{a}$, which obeys

$$\hat{n}|\, n \,\rangle = \hat{a}^\dagger\hat{a}|\, n \,\rangle = n|\, n \,\rangle \,. \tag{1.119}$$

To cast the set of master equations into a single kinetic equation, define the state vector

$$|\, P_t \,\rangle = \sum_{n=0}^{\infty} P(t,n)|\, n \,\rangle \,. \tag{1.120}$$

The probability of occupation number n may by extracted from the state vector through the following scalar product

$$P(t,n) = \frac{1}{n!} \langle\, n \,|\, P_t \,\rangle \,.$$

The set of master equations yields for the state vector (1.120) a single kinetic equation without any explicit dependence on the occupation number:

$$\frac{\mathrm{d}|\, P_t \,\rangle}{\mathrm{d}t} = \hat{L}(\hat{a}^\dagger,\, \hat{a})|\, P_t \,\rangle \,, \tag{1.121}$$

where the Liouville operator $\hat{L}(\hat{a}^\dagger,\, \hat{a})$ is constructed from the set of master equations according to rules:

$$\begin{aligned} nP(t,n)|\, n \,\rangle &= \hat{a}^\dagger\hat{a}P(t,n)|\, n \,\rangle \,, \\ nP(t,n)|\, n-1 \,\rangle &= \hat{a}P(t,n)|\, n \,\rangle \,, \\ nP(t,n)|\, n+1 \,\rangle &= \hat{a}^\dagger\hat{a}^\dagger\hat{a}P(t,n)|\, n \,\rangle \,. \end{aligned}$$

For instance, the Liouville operator for the stochastic Verhulst model is

$$\hat{L}(\hat{a}^\dagger, \hat{a}) = \beta(I - \hat{a}^\dagger)\hat{a} + \gamma(I - \hat{a}^\dagger)\hat{a}\hat{a}^\dagger\hat{a} + \lambda(\hat{a}^\dagger - I)\hat{a}^\dagger\hat{a} \,. \tag{1.122}$$

The formal solution of (1.121) is

$$|\, P_t \,\rangle = \mathrm{e}^{t\hat{L}(\hat{a}^\dagger,\, \hat{a})}|\, P_0 \,\rangle \,, \tag{1.123}$$

where the initial state vector is determined by the initial condition for the probabilities:

$$|\, P_0 \,\rangle = \sum_{n=0}^{\infty} P(0,n) |\, n \,\rangle \,.$$

In principle, the relation (1.123) may be used for a compact representation of the solution of the set of master equations, particularly if it is possible to calculate the matrix exponential in a closed form.

To represent expectation values of occupation-number dependent quantities in the operator formalism use the projection vector $\langle\, P \,|$:

$$\langle\, P \,| = \sum_{n=0}^{\infty} \frac{1}{n!} \langle\, n \,| = \sum_{n=0}^{\infty} \frac{1}{n!} \langle\, 0 \,| \hat{a}^n = \langle\, 0 \,| \mathrm{e}^{\hat{a}} \,. \tag{1.124}$$

The projection vector is a left eigenvector of the creation operator with the eigenvalue equal to unity

$$\langle\, P \,| \hat{a}^\dagger = \langle\, P \,| \,. \tag{1.125}$$

From here it follows that

$$\langle\, P \,| \, n \,\rangle = 1 \,, \quad \forall\, n \,. \tag{1.126}$$

Basis states are eigenstates of the number operator $\hat{n} = \hat{a}^\dagger \hat{a}$, therefore from relation (1.126) it follows

$$\langle\, P \,| (\hat{a}^\dagger \hat{a})^m | \, n \,\rangle = n^m \langle\, P \,| \, n \,\rangle = n^m \,. \tag{1.127}$$

Equation (1.121) gives rise to the Heisenberg evolution of operators

$$\hat{a}^\dagger_H(t) = \mathrm{e}^{-\hat{L}t} \hat{a}^\dagger \mathrm{e}^{\hat{L}t} \,, \qquad \hat{a}_H(t) = \mathrm{e}^{-\hat{L}t} \hat{a} \mathrm{e}^{\hat{L}t} \,. \tag{1.128}$$

To construct interaction representation, we need Dirac operators as well:

$$\hat{a}^\dagger(t) = \mathrm{e}^{-\hat{L}_0 t} \hat{a}^\dagger \mathrm{e}^{\hat{L}_0 t} \,, \qquad \hat{a}(t) = \mathrm{e}^{-\hat{L}_0 t} \hat{a} \mathrm{e}^{\hat{L}_0 t} \,. \tag{1.129}$$

It is convenient to choose the free Liouville operator in the form

$$\hat{L}_0 = -(\hat{a}^\dagger - I) K \hat{a} \,, \tag{1.130}$$

because it has the property of "probability conservation"

$$\langle\, P \,| \hat{L}_0 = 0 \,. \tag{1.131}$$

From the structure of master equations it follows that the Liouville operator always has the structure

$$\hat{L}(\hat{a}^\dagger, \hat{a}) = (\hat{a}^\dagger - I) K(\hat{a}^\dagger, \hat{a})$$

and therefore obeys the probability conservation

$$\langle P|\hat{L} = 0 \tag{1.132}$$

as well.

The basic relation for calculation of expectation values follows from the definition in terms of probabilities, which may be cast in the form of a matrix element between the projection vector $\langle P|$ and the state vector $|P_t\rangle$:

$$\langle Q(n)\rangle = \sum_{n=0}^{\infty} Q(n)P(t,n) = \langle P|Q(\hat{a}^\dagger\hat{a})|P_t\rangle\,. \tag{1.133}$$

The last equality, again without explicit occupation-number dependence, comes from the the relation (1.127). Here and henceforth, it is assumed that $Q(n)$ is a polynomial function or Taylor series of n.

Thus, with the use of the solution (1.123) the expectation value of an occupation-number dependent quantity $Q(n)$ may be written as

$$\langle Q(n)\rangle = \langle P|Q(\hat{a}^\dagger\hat{a})\,\mathrm{e}^{t\hat{L}(\hat{a}^\dagger,\,\hat{a})}|P_0\rangle\,. \tag{1.134}$$

We are dealing with non-commuting operators, therefore the question of operator ordering arises here as well. Define the normal product of creation and annihilation operators as a product, in which all creation operators stand to the left of all annihilation operators. Define further the normal form of the operator $Q(\hat{a}^\dagger\hat{a})$ as

$$Q(\hat{a}^\dagger\hat{a}) = N\left[Q_N(\hat{a}^\dagger,\hat{a})\right]\,. \tag{1.135}$$

Then we see that the expectation value of $Q(n)$ is equal to

$$\langle Q(n)\rangle = \langle P|Q_N(1,\hat{a})\,\mathrm{e}^{t\hat{L}(\hat{a}^\dagger,\,\hat{a})}|P_0\rangle\,. \tag{1.136}$$

Introducing identity resolutions $1 = \exp(\hat{L}t)\exp(-\hat{L}t)$ between operators $\hat{a}$ and using probability conservation to write

$$\langle P| = \langle P|\exp(-\hat{L}t)$$

we obtain

$$\langle Q(n)\rangle = \langle P|Q_N(1,\hat{a}_H(t))\,|P_0\rangle\,. \tag{1.137}$$

Here, the right side is a linear combination of equal-time *Green functions* of the annihilation operators.

Consider the Green function of creation and annihilation operators $\hat{a}_H^\dagger(t)$ and $\hat{a}_H(t)$:

$$G_{m,n}(t_1,\ldots t_m;t'_1,\ldots t'_n) = \mathrm{Tr}\left\{\hat{\rho}\,T\left[\hat{a}_H^\dagger(t_1)\cdots\hat{a}_H^\dagger(t_m)\hat{a}_H(t'_1)\cdots\hat{a}_H(t'_n)\right]\right\}\,, \tag{1.138}$$

with the density operator

$$\hat{\rho} = |P_0\rangle\langle P|\,. \tag{1.139}$$

From definitions it follows that the conditional probability density function for the master equation may be written as (the factorial in front of the matrix element is due to the unusual normalization of the basis states)

$$P\left(n,t|n_0,t_0\right)=\frac{1}{n!}\left\langle\, n\,\left|\,\mathrm{e}^{\hat{L}(t-t_0)}\,\right|\,n_0\,\right\rangle\,. \tag{1.140}$$

Choosing, for definiteness, the time sequence $t_1>t_2>t_3>\ldots>t_{n-1}>t_n>t_0$ it is readily seen by direct substitution of relations (1.140) and (1.139) in (1.138) with the aid of the normalization conditions of the PDF and insertions of the resolution of the identity $\sum_n \frac{1}{n!}\,|\,n\,\rangle\langle\,n\,|=1$ that

$$\sum_{n_1}\ldots\sum_{n_m} n_1\cdots n_m P\left(n_1,t_1;\ldots;n_m,t_m\right)=G_{m,m}(t_1,,\ldots t_m;t_1,\ldots t_m)\,, \tag{1.141}$$

i.e. the Green function (1.138) is equal to the moment function (1.141). This relation connects the operator approach to evaluation of moments of the random process described by a master equation.

Reduction to the interaction representation and Wick's theorem for the normal product are – up to notation – the same as in the case of the Fokker-Planck equation. Calculation of the expectation value of the linear exponential is, however, different.

In case of master equation the form of the evolution operator is just the same as in the case of Fokker-Planck equation, but the density operator is different. Operators $\hat{a}^\dagger$ and $\hat{a}$ specific for this problem shall be used in what follows.

$$\begin{aligned}&\mathrm{Tr}\,\left\{\hat{U}(t_i,0)\,\hat{\rho}\,\hat{U}(0,t_f)N\;\exp\left(\hat{a}B^\dagger+\hat{a}^\dagger B\right)\right\}\\&\quad=\sum_n\frac{1}{n!}\langle\,n\,|\hat{U}(t_i,0)|\,P_0\,\rangle\langle\,P\,|\hat{U}(0,t_f)N\,\exp\left(\hat{a}B^\dagger+\hat{a}^\dagger B\right)|\,n\,\rangle\\&\quad=\langle\,P\,|\hat{U}(0,t_f)N\,\exp\left(\hat{a}B^\dagger+\hat{a}^\dagger B\right)\hat{U}(t_i,0)|\,P_0\,\rangle\,.\end{aligned} \tag{1.142}$$

Again, to keep things simple, choose $t_i\to 0$, which yields

$$\hat{U}(t_i,0)|\,P_0\,\rangle\to|\,P_0\,\rangle\,.$$

Choose the free Liouville operator in the form

$$\hat{L}_0=-\left(\hat{a}^\dagger-I\right)K\hat{a} \tag{1.143}$$

for which the "conservation of probability" holds:

$$\langle\,P\,|\mathrm{e}^{\hat{L}_0 t}=\langle\,P\,|$$

because the projection vector is the left eigenstate of the creation operator

$$\langle\,P\,|\hat{a}^\dagger=\langle\,P\,|\,.$$

The evolution operator $\hat{U}(0,t_f)$ is a product of operator exponentials of $\hat{L}_0$ and $\hat{L}$. The "conservation of probability" holds also for the latter by the very construction, thus

$$\langle\, P\,|\,\hat{L}_0 = 0\,, \qquad \langle\, P\,|\,\hat{L} = 0\,.$$

Therefore, we obtain

$$\begin{aligned}\langle\, P\,|\hat{U}(0,t_f)N\,\exp\left(\hat{a}B^\dagger + \hat{a}^\dagger B\right)|\,P_0\,\rangle &= \langle\, P\,|N\,\exp\left(\hat{a}B^\dagger + \hat{a}^\dagger B\right)|\,P_0\,\rangle \\ &= \langle\, P\,|\,\exp\left(\hat{a}^\dagger B\right)\exp\left(\hat{a}B^\dagger\right)|\,P_0\,\rangle\,. \end{aligned} \tag{1.144}$$

The free Liouvillean (1.143) gives rise to time-dependent operators

$$\hat{a}^\dagger(t) = \hat{a}^\dagger\,\mathrm{e}^{Kt} + \left(1-\mathrm{e}^{Kt}\right)\,, \quad \hat{a}(t) = \hat{a}\,\mathrm{e}^{-Kt}\,. \tag{1.145}$$

Therefore, writing the time integrals in (1.144) explicitly, we arrive at representation

$$\langle\, P\,|\,\mathrm{e}^{\hat{a}^\dagger B}\,\mathrm{e}^{\hat{a}B^\dagger}\,|\,P_0\,\rangle = \exp\left[\int_0^{t_f}\left(1-\mathrm{e}^{Kt}\right)B(t)\,\mathrm{d}t\right]\langle\, P\,|\,\mathrm{e}^{\hat{a}^\dagger\tilde{B}}\,\mathrm{e}^{\hat{a}\tilde{B}^\dagger}\,|\,P_0\,\rangle\,, \tag{1.146}$$

where

$$\tilde{B} = \int \mathrm{e}^{Kt}B(t)\mathrm{d}t\,, \qquad \tilde{B^\dagger} = \int \mathrm{e}^{-Kt}B^\dagger(t)\mathrm{d}t\,.$$

Pull now the operator exponential $\exp\hat{a}$ from $\langle\, P\,|$ to the right by the rule

$$(\exp\hat{a})\,\hat{a}^\dagger = (\hat{a}^\dagger + I)\exp\hat{a}\,,$$

which boils down to the shift $\hat{a}^\dagger \to \hat{a}^\dagger + I$ in operators, through which the coherent-state exponential is pulled:

$$\langle\, P\,|\,\exp\!\left(\hat{a}^\dagger\tilde{B}\right)\exp\!\left(\hat{a}\tilde{B^\dagger}\right)|\,n\,\rangle = \langle\, 0\,|\,\exp\!\left(\left(\hat{a}^\dagger + I\right)\tilde{B}\right)\exp\left(\hat{a}\tilde{B^\dagger}\right)\left(\hat{a}^\dagger + I\right)^n|\,0\,\rangle \tag{1.147}$$

Combining (1.146) and (1.147) we obtain

$$\left\langle\, P\,\left|\exp\left(\hat{a}^\dagger B\right)\,\exp\left(\hat{a}B^\dagger\right)\right|\,P_0\,\right\rangle = P(0,n)\exp\left[\int_0^{t_f}B(t)\mathrm{d}t\right]\,\left[\int_0^{t_f}\mathrm{e}^{-Kt}B^\dagger(t)\mathrm{d}t + 1\right]^n\,. \tag{1.148}$$

Thus, the expectation value of the linear exponential (1.142) is [52]

$$\begin{aligned}\left\langle\, P\,\left|\hat{U}(0,t_f)N\,\exp\left(\hat{a}B^\dagger + \hat{a}^\dagger B\right)\right|\,P_0\,\right\rangle &= \sum_n P(0,n)\exp\left[\int B(t)dt\right] \\ &\times\left[\int_0^\infty \mathrm{e}^{-Kt}B^\dagger(t)\mathrm{d}t + 1\right]^n\,. \end{aligned} \tag{1.149}$$

The result for the expectation value of the normal form of an operator functional $F\left[\hat{a}^\dagger(t),\hat{a}(t)\right]$ may be written in a compact form with the use of the identity

$$x^n = \frac{n!}{2\pi i}\oint_C \frac{\mathrm{e}^{xz}}{z^{n+1}}\,\mathrm{d}z\,,$$

where C is a closed Jordan contour encircling the origin of the complex z plane. We obtain

$$\begin{aligned}\left\langle P \left| N\, F\left[\hat{a}^{\dagger}(t),\hat{a}(t)\right] \right| P_0 \right\rangle &= F\left[\frac{\delta}{\delta B(t)},\frac{\delta}{\delta B^{\dagger}(t)}\right]\left\langle P \left| N\,\mathrm{e}^{\hat{a}B^{\dagger}+\hat{a}^{\dagger}B} \right| P_0 \right\rangle\Big|_{B=B^{\dagger}=0} \\ &= F\left[\frac{\delta}{\delta B(t)},\frac{\delta}{\delta B^{\dagger}(t)}\right]\exp\left(\int B(t)\mathrm{d}t\right) \end{aligned} \tag{1.150}$$

$$\begin{aligned}&\sum_n P(0,n)\frac{n!}{2\pi i}\oint_C \frac{\exp\left[z\left(\int_0^{t_f} e^{-Kt}B^{\dagger}(t)dt+1\right)\right]}{z^{n+1}}\,\mathrm{d}z\Big|_{B=B^{\dagger}=0} \\ &\quad= \sum_n P(0,n)\frac{n!}{2\pi i}\oint_C \frac{\mathrm{e}^z}{z^{n+1}}\,F\left[1,nz\right]\,\mathrm{d}z \\ &\quad= \frac{1}{2\pi i}\oint_C \mathrm{e}^z\widetilde{G}(z)\,F\left[1,nz\right]\,\mathrm{d}z\,, \end{aligned} \tag{1.151}$$

where $nz=\mathrm{e}^{-Kt}z$. In (1.150) the shorthand notation

$$\widetilde{G}(z)=\sum_n \frac{P(0,n)n!}{z^{n+1}}$$

has been used on the right side.

If the initial PDF is that of the *Poisson distribution*, which is the assumption usually made [42],

$$P(0,n)=\frac{n_0^n\mathrm{e}^{-n_0}}{n!}\,, \tag{1.152}$$

then the expression for the expected value of linear exponential is tremendously simplified, since

$$\begin{aligned}&\left\langle P \left| \hat{U}(0,t_f)N\,\exp\left(\hat{a}B^{\dagger}+\hat{a}^{\dagger}B\right) \right| P_0 \right\rangle \\ &= \sum_n P(0,n)\exp\left[\int B(t)\,\mathrm{d}t\right]\left[\int_0^{\infty}\mathrm{e}^{-Kt}B^{\dagger}(t)\mathrm{d}t+1\right]^n \\ &= \exp\left[\int B(t)\mathrm{d}t+n_0\int_0^{\infty}\mathrm{e}^{-Kt}B^{\dagger}(t)\mathrm{d}t\right]. \end{aligned} \tag{1.153}$$

Therefore, we arrive at the representation

$$\left\langle P \left| \hat{U}(0,t_f)N\,F(\hat{a}^{\dagger},\hat{a}) \right| P_0 \right\rangle = F(1,\mathrm{e}^{-Kt}n_0)\,. \tag{1.154}$$

Introducing again the Liouville operator $\hat{L}=\hat{L}_0+\hat{L}_I$ and corresponding functionals explicitly, we obtain the generating function of Green functions of the Cauchy problem of the master equation equation in the form:

$$\mathcal{G}(A)=\langle\, P\,|\hat{U}(0,t_f)T\exp\left(\hat{a}_H A^{\dagger}+\hat{a}_H^{\dagger}A\right)|\,P_0\,\rangle$$

$$= \frac{1}{2\pi i} \oint_C \mathrm{e}^z \widetilde{G}(z) \left[\exp\left(\frac{\delta}{\delta a} \Delta \frac{\delta}{\delta a^\dagger} \right) \exp\left\{ \int_0^{t_f} L_I(a(t), a^\dagger(t))\, \mathrm{d}t \right.\right.$$
$$\left.\left. + \int_0^{t_f} \left[a(t) A^\dagger(t) + a^\dagger(t) A(t) \right] \mathrm{d}t \right\} \right]_{\substack{a = nz \\ a^\dagger = 1}} . \tag{1.155}$$

In case of initial Poisson distribution the generating function is

$$\mathcal{G}(A) = \exp\left(\frac{\delta}{\delta a} \Delta \frac{\delta}{\delta a^\dagger} \right)$$
$$\times \exp\left\{ \int_0^{t_f} L_I(a(t), a^\dagger(t))\, \mathrm{d}t + \int_0^{t_f} \left[a(t) A^\dagger(t) + a^\dagger(t) A(t) \right] \mathrm{d}t \right\} \Bigg|_{\substack{a = \mathrm{e}^{-Kt} n_0 \\ a^\dagger = 1}} . \tag{1.156}$$

The functional-differential representation may again be transformed to a functional integral by a trick similar to that used in the Fokker-Planck case with the result

$$\mathcal{G}(A) = \int \mathcal{D}\varphi \int \mathcal{D}\tilde{\varphi}\, \exp\left\{ \mathcal{S}[\varphi, \varphi^\dagger] + \int_0^{t_f} \left[\varphi(t) A^\dagger(t) + \varphi^\dagger(t) A(t) \right] \right\} , \tag{1.157}$$

where the *Peliti dynamic action* [56] is

$$\mathcal{S}[\varphi, \varphi^\dagger] = \varphi^\dagger \left(-\partial_t \varphi - K\varphi \right) + L_I(\varphi^\dagger + 1, \varphi + \mathrm{e}^{-Kt} n_0) . \tag{1.158}$$

The explicit shift of the field argument by the solution of the homogeneous free-field equation $\varphi + e^{-Kt} n_0$ is the same as in the Fokker-Planck case. The shift $\varphi^\dagger + 1$ may be carried out explicitly and gives rise to the "shifted" action discussed in the literature.

For the Verhulst model the Peliti action after this procedure is

$$\mathcal{S}[\varphi, \varphi^\dagger] = \varphi^\dagger \left[-\partial_t \varphi + (\lambda - \beta - \gamma)\varphi \right] - \gamma \varphi^\dagger \varphi^2 + \lambda {\varphi^\dagger}^2 \varphi - \gamma {\varphi^\dagger}^2 \varphi^2 . \tag{1.159}$$

The structure of the dynamic action here is similar to that of the Fokker-Planck problem. We have seen that a very similar expression arises in the case, when the account of randomness is introduced by constructing a Langevin equation instead of the master equation. Therefore, functional representation leads also to a possibility make conclusions on this basis about connections between different approaches to stochastic problems.

1.5 Random sources and sinks in the master equation

Apart from reactions and transport of particles effect of random sources and sinks may sometimes be of interest, e.g., to maintain a steady state in the system. In most cases this is carried out by including an additive noise term in the Langevin equation of the stochastic process. When the analysis is based on the master equation, this is not quite appropriate and a more elaborated approach is called for [57]. Unfortunately, there is no unique way to introduce random sources in the master equation corresponding to the random noise of the mean-field (Langevin) description.

We use the simplest choice, described in detail, e.g., in [58], which is tantamount to amending the problem by reactions $A \to X$ and $Y \to A$, where X and Y stand for particle baths of the sink and the source, respectively. In a homogeneous system these reactions give rise to the master equations

$$\frac{\mathrm{d}P(t,n)}{\mathrm{d}t} = \mu_+ V\left[P(t,n-1) - P(t,n)\right] + \mu_-\left[(n+1)P(t,n+1) - nP(t,n)\right] + \dots \quad (1.160)$$

where $P(t,n)$ is the probability to find n particles at the time instant t in the system. The ellipsis in (1.160) stands for the terms describing the annihilation reaction, diffusion and advection in the system. In (1.160) μ_+ and μ_- are the reaction constants of the creation and annihilation reactions, respectively. The transition rate has been chosen proportional to the particle number n, which seems the quite natural and also preserves the empty state as an absorbing state. In the transition rate for creation process V is the volume of the (for the time being) homogeneous system and will be important in passing to the continuum limit of the inhomogeneous system. We recall that the master equation (1.160) gives rise to the reaction-rate equation

$$\frac{\mathrm{d}\langle n\rangle}{\mathrm{d}t} = \mu_+ V - \mu_-\langle n\rangle + \dots \quad (1.161)$$

Master equations (1.160) give rise to the following terms in the Liouville operator

$$\hat{L}_g(\hat{a}^\dagger, \hat{a}) = \mu_+ V\left(\hat{a}^\dagger - I\right) + \mu_-\left(I - \hat{a}^\dagger\right)\hat{a}\,, \quad (1.162)$$

where I is the identity operator. The corresponding contribution to the dynamic action is

$$\mathcal{S}_g = \int_0^\infty \mathrm{d}t\,\left[\mu_+ V a^+(t) - \mu_- a^+(t)a(t)\right]\dots \quad (1.163)$$

Only the generic time-derivative term and terms brought about by the random source model are expressed here explicitly, while the ellipsis stands for terms corresponding to other reactions and initial conditions.

Let the transition rates $\mu_\pm$ be random functions uncorrelated in time with a probability distribution given in terms of the moments $\langle \mu_\pm^n\rangle = E_{\pm,n}$. At this point we also generalize the treatment to the case of a spatially inhomogeneous system and introduce a lattice subscript as the spatial argument, e.g. $a(t) \to a_i(t)$. In this case the volume V becomes the volume element attached to the lattice site. For simplicity, we replace the time integral with the integral sum $\int_0^\infty \mathrm{d}t \to \sum_\alpha \Delta t$ and assume that the transition rates at each time instant and lattice site $\mu_{\pm,\alpha,i}$ are independent random variables. Then the average over the distribution of random sources reduces to the calculation of the expectation value

$$\prod_{\alpha,i}\langle \mathrm{e}^{\mu_{+,\alpha,i} V \tilde{a}_{\alpha,i}\Delta t - \mu_{-,\alpha,i}\tilde{a}_{\alpha,i}a_{\alpha,i}\Delta t}\rangle\,. \quad (1.164)$$

For each particular time instant and lattice this yields (we assume that the moments of $\mu_\pm$ are the same for all α and i and omit labels for brevity) this gives rise to the usual cumulant expansion

$$\langle \mathrm{e}^{\mu b\Delta t}\rangle = 1 + b\Delta t E_1 + \frac{1}{2}E_2(b\Delta t)^2 + \frac{1}{6}E_3(b\Delta t)^3 + \cdots$$

$$= \mathrm{e}^{b\Delta t E_1 + \frac{1}{2}\left(E_2 - E_1^2\right)(b\Delta t)^2 + \frac{1}{6}\left(E_3 - 3E_1E_2 + E_1^3\right)(b\Delta t)^3 + \cdots} \tag{1.165}$$

Here, b stands for either $V\tilde{a}$ or $-\tilde{a}a$. Thus, for instance the average over μ_+ assumes the form

$$\prod_{\alpha,i} \langle \mathrm{e}^{\mu_{+,\alpha,i} V \tilde{a}_{\alpha,i} \Delta t} \rangle = \mathrm{e}^{\sum_\alpha \sum_i \left[\Delta t E_{+1} V \tilde{a}_{\alpha,i} + \frac{1}{2}\left(E_{+2} - E_{+1}^2\right)(V\tilde{a}_{\alpha,i}\Delta t)^2\right]} \times \mathrm{e}^{\sum_\alpha \sum_i \left[\frac{1}{6}\left(E_{+3} - 3E_{+1}E_{+2} + E_{+1}^3\right)(V\tilde{a}_{\alpha,i}\Delta t)^3 + \cdots\right]} . \tag{1.166}$$

In the continuum limit the function $\tilde{a}_{\alpha,i}$ is replaced by the field $\varphi^\dagger(t,\boldsymbol{x})$, whereas in the limit $V \to 0$ the expression $a_{\alpha,i}/V$ gives rise to the field $\varphi(t,\boldsymbol{x})$. The sum over α together with Δt gives rise to the time integral and the sum over i together with the volume element gives rise to the spatial integral $\sum_i V \to \int \mathrm{d}^d\boldsymbol{x}$. In the first term of the exponential in (1.166) we thus obtain

$$\sum_\alpha \sum_i \Delta t E_{+1} V \tilde{a}_{\alpha,i} \to E_{+1} \int \mathrm{d}t \int \mathrm{d}^d\boldsymbol{x}\, \varphi^\dagger(t,\boldsymbol{x}) .$$

In the cumulants of second and higher order the continuum limit is not so obvious. We assume the simplest nontrivial distribution for $\mu_\pm$, in which only the variance term has a finite limit, when $\Delta t \to 0$ and $V \to 0$, whereas the contributions of higher-order cumulants vanish, for instance

$$\left(E_{+2} - E_{+1}^2\right) V \Delta t \to \sigma_+ , \quad \Delta t \to 0 , \; V \to 0 , \tag{1.167}$$

$$\left(E_{+3} - 3E_{+1}E_{+2} + E_{+1}^3\right)(V\Delta t)^2 \to 0 , \quad \Delta t \to 0 , \; V \to 0 . \tag{1.168}$$

Therefore, the contribution of the average over μ_+ to the effective dynamic action assumes the form

$$\mathcal{S}_+ = \int \mathrm{d}t \int \mathrm{d}^d\boldsymbol{x} \left\{ E_{+1}\varphi^\dagger(t,\boldsymbol{x}) + \frac{1}{2}\sigma_+ \left[\varphi^\dagger(t,\boldsymbol{x})\right]^2 \right\} . \tag{1.169}$$

For the average over μ_- a similar argument yields

$$\mathcal{S}_- = \int \mathrm{d}t \int \mathrm{d}^d\boldsymbol{x} \left\{ -E_{-1}\varphi^\dagger(t,\boldsymbol{x})\varphi(t,\boldsymbol{x}) + \frac{1}{2}\sigma_- \left[\varphi^\dagger(t,\boldsymbol{x})\varphi(t,\boldsymbol{x})\right]^2 \right\} . \tag{1.170}$$

These contributions to the effective dynamic action may, of course, be generated by suitably chosen normal distributions of $\mu_\pm$.

This way of introduction of random sources and sinks has the annoying feature that it does not conserve the number of particles in the system. For a comparison with the treatment of this problem in the Langevin approach the random sources and sinks should be introduced in such a way that the particle number is conserved.

The simplest way to effect this is to add to the random source a term proportional to the particle number, i.e. use the "reaction constant" $\mu_+ V + \mu_{1+} n$ instead of $\mu_+ V$ in the master equation. The source terms on the right-hand side of the master equation (1.160) in this case the assume the form

$$\frac{\mathrm{d}P(t,n)}{\mathrm{d}t} = \mu_+ V \left[P(t,n-1) - P(t,n)\right] + \mu_{1+} \left[(n-1)P(t,n-1) - nP(t,n)\right] \ldots$$

The new part of the master equation corresponds to a branching process [58].

The added term gives rise to the following contribution to the Liouville operator

$$\hat{L}_{g_2}(\hat{a}^\dagger, \hat{a}) = \mu_{1+}\left(\hat{a}^\dagger - I\right)\hat{a}^\dagger\hat{a}\,. \tag{1.171}$$

Performing the steps described above we arrive at the contribution to the dynamic action in the form

$$\mathcal{S}_{1+} = \int \mathrm{d}t \int \mathrm{d}^d\boldsymbol{x} \left\{ E_{1+1}\varphi^\dagger\left(\varphi^\dagger + 1\right)\varphi + \frac{1}{2}\sigma_{1+}{\varphi^\dagger}^2\left(\varphi^\dagger + 1\right)^2\varphi^2 \right\}. \tag{1.172}$$

It is easy to see now that if we exclude the plain source (i.e. put $E_{+1} = \sigma_+ = 0$) and choose $E_{1+1} = E_{-1}$, then the empty state remains absorbing and the "mass term" $\propto \varphi^+\varphi$ disappears in the dynamic action and we arrive at the dynamic action of random sources and sinks

$$\mathcal{S}_{gc} = \int \mathrm{d}t \int \mathrm{d}^d\boldsymbol{x} \left\{ E_{1+1}{\varphi^\dagger}^2\varphi + \frac{1}{2}\sigma_-\left(\varphi^\dagger\varphi\right)^2 + \frac{1}{2}\sigma_{1+}{\varphi^\dagger}^2\left(\varphi^\dagger + 1\right)^2\varphi^2 \right\}$$

which conserves the average number of particles.

2 Renormalization group in field theory

2.1 Renormalization procedure and renormalization group

Usually, the Feynman graphs of Green functions, which are the graphical representation of some integrals, contain divergences in the range of large and small scales (wave vectors.) Therefore it is necessary to find an effective procedure to eliminate these divergences step by step in each order of concrete perturbation scheme. Below we will demonstrate renormalization methods in the framework of the stochastic model of developed turbulence and related applications.

The method of renormalization group (RG) has been proposed in the framework of the quantum field theory in the fifties of the previous century [2–5, 59, 60]. From the practical point of view RG method represents an effective way to determine non-trivial asymptotic behavior of Green functions in the range of large (ultraviolet) or small (infrared) wave vectors (scales). The asymptotic behavior is non-trivial if in a given order of a perturbative calculation the divergences in a certain range of wave vectors appear (e.g. so called large logarithms) which compensate the smallness of coupling constant g. In such a case one needs to sum all terms of perturbation series. This summation can be carried out by means of RG approach. Technically, one obtains linear partial differential RG equations for the Green functions. The coefficient functions (RG-functions) in the differential operator (see below) are calculated at a given order of the perturbation scheme. However, the solution of the RG equation represents the sum of the infinite series. For example, if the RG-functions are calculated at the lowest non-trivial order of the perturbation theory and after corresponding RG-equation is solved, obtained result is a sum of leading logarithms of all the perturbation series. Moreover if the RG-functions will be calculated with an improved precision the solution of the RG equation will include corrections to the leading logarithms.

A simple criterion how to determine the true asymptotic range exist in the framework of RG. One of the RG-functions is the β-function, which is a coefficient at the operation ∂_g in the RG equation. The β-function is calculated perturbatively as infinite series of powers of the coupling constant g and for relativistic models has form: $\beta(g) = \beta_2 g^2 + \beta_3 g^3 + ...$ Non-trivial asymptotic behavior is governed by *RG fixed points* g^*, which are roots of β-functions (solutions of equation $\beta(g) = 0$). A fixed point can be IR or UV stable depending on behavior of the β-function in th vicinity g^*. If the coefficient $\beta_2 > 0$ then g^* is an IR stable fixed point. In the opposite case it is UV stable fixed point. In the time when the RG technique appeared no physical models with non-trivial UV asymptotic behavior were known. Moreover, as non-trivial IR behavior is possible only for massless models, which also were not known in this time, RG method remained unused up to the seventies of the previous century.

In 1977 D. Forster, D. R. Nelson and M. J. Stephen applied the RG method to calculate the correlations of velocity field [61] governed by stochastic Navier-Stokes equation with external random forcing. Later it was shown by C. De Dominicis and P. C. Martin [62] that in the range of small wave numbers the correlations of the velocity field manifest a scaling behavior with the celebrated Kolmogorov exponents. The basic idea of application of RG in the theory of developed turbulence consists in elimination of the direct influence of the modes with high frequencies and wave numbers on observed quantities. Effectively their influence is included to some effective variables, e.g. to the turbulent viscosity.

Numerous versions of RG methods exist. Constricted all they are equivalent but technically can be quite different. The most formalized is RG developed in the framework of the quantum

field theory (field-theoretic RG), which has been used in papers [30, 63].

The field theoretic RG is based on non-trivial techniques of UV renormalization. The basic procedure consists of calculation of the RG-functions in the framework of a prescribed scheme of regularization [24]. To find and analyze all possible UV divergences in concrete field-theoretic models a counting of canonical scaling dimensions of fields and parameters of the model is used. Let us briefly remind the essence of such a power counting which is closely connected with the existence of a scale invariance in the model.

2.2 Extended homogeneity

This conception is introduced for the formulation of the hypothesis of similarity (critical scaling, scale invariance) and it is useful for classification of all UV divergences in field-theoretical models: critical statics and dynamics, developed turbulence and so on [24, 25, 44]. We give its definition and basic properties.

A function $F(e)$ which depends on variables $e \equiv \{e_1 \dots e_N\}$ is termed extended homogeneous (or simply speaking dimensional) if for a set of numbers Δ and arbitrary $\lambda > 0$ the following equation is valid:

$$F(\lambda^{\Delta_1} e_1, ..., \lambda^{\Delta_N} e_N) = \lambda^{\Delta_W} F(e_1, ..., e_N)\,, \tag{2.1}$$

or shortly

$$F(e_\lambda) = \lambda^{\Delta_F} F(e), \tag{2.2}$$

where $e_\lambda = \lambda^{\Delta_i} e_i$ and $i = 1, ..., N$. The parameters $\Delta_i \equiv \Delta_{e_i} \equiv \Delta[e_i]$ are (canonical or critical or anomalous) scaling dimensions (exponents) of corresponding variables e_i and $\Delta_F \equiv \Delta[F]$ represents dimension (exponent) of function F. If $\Delta_F = 0$ then the function F is scale invariant (dimensionless). A dimensional function $F(e)$ depending on one variable e is proportional to the power $|e|^\beta$ with the exponent $\beta = \Delta_F/\Delta_e$. A function F depending on N variables can be expressed in the form of product of a power function and *scaling function* R which is a function of $N-1$ dimensionless combinations of its arguments, e.g.

$$F(e_1, \dots, e_N) = |e_N|^\beta R\left(\frac{e_1}{|e_N|^{\beta_1}}, \dots, \frac{e_{N-1}}{|e_N|^{\beta_{N-1}}}\right), \tag{2.3}$$

where $\beta = \Delta_F/\Delta_n$ and $\beta_i = \Delta_i/\Delta_n$ $i = 1, 2, \dots n-1$. If one differentiates the equation (2.2) with respect to λ and afterwards puts $\lambda = 1$, then will obtain Euler differential equation, which represents another equivalent formulation of the extended homogeneity (2.2):

$$\left[\sum_e \Delta_e e \partial_e - \Delta_F\right] F(e) = 0. \tag{2.4}$$

The substitution $\lambda \to \lambda^a$ in the equation (2.2) is equivalent to multiplying of all exponents Δ by parameter a, therefore the exponent of one of the variables can be fixed. Usually the dimension of the wave number k is selected to be the unity (or, equivalently, dimension of the space coordinate is equal to -1):

$$\Delta[k] = 1\,. \tag{2.5}$$

This definition is standard and we use it everywhere. We note that in dynamical models (1.106) also the dimension of frequency can be fixed (see below).

2.3 UV-renormalization. RG equations

Let us summarize main ideas of the quantum-field theory of renormalization and RG technique; a detailed account can be found in monographs [24, 25, 44, 64, 65].

We will be mainly concerned with models whose diagrams are calculated without UV-cutoff Λ and UV-divergences manifest themselves as poles in a certain dimensionless "parameter of deviation from logarithmic theory ε". Historically, this term appeared in connection with infinite summation of main logarithms (see aforementioned discussion)), which is necessary to make in the case when the bare coupling constant g_0 (or constants) is canonically dimensionless ($\Delta_g \equiv d_g = 0$.) The procedure of multiplicative renormalization removing UV-divergences (in the given case, poles in in a parameter ε) consists in the following: the original action $\mathcal{S}(\varphi, e_0)$ is declared to be unrenormalized; its parameters e_0 (letter e_0 stands for the whole set of parameters) are the bare parameters, and they are considered to be some functions of the new renormalized parameters e, whereas a new renormalized action is assumed to be the functional $\mathcal{S}_R[\varphi] = \mathcal{S}[\varphi Z_\varphi]$ with certain (also to be determined) renormalization constants of fields Z_φ (one per each independent component of the field). In unrenormalized full Green functions $G_N = \langle \varphi \dots \varphi \rangle$ the functional averaging $\langle \dots \rangle$ is performed with the "weight" $\exp \mathcal{S}[\varphi]$; while in renormalized functions , G_N^R with the "weight" $\exp \mathcal{S}_R[\varphi]$. The connection between the functionals $\mathcal{S}[\varphi]$ and $\mathcal{S}_R[\varphi]$ leads to the relation between the corresponding Green functions $G_N^R = Z_\varphi^{-N} G_N$, where by definition $G_N = G_N(e_0, \varepsilon \dots)$(ellipsis denotes other arguments like coordinates or wave numbers), and, by convention, the quantities G_N^R and Z_φ are expressed in terms of the parameters e. The correspondence $e_0 \leftrightarrow e$ within perturbation theory is assumed to be one-to-one, therefore either of the sets e_0, e can be taken as the independent variables.

For translation invariant theories it is convenient to deal not with the full Green functions G_N, but either with their connected parts W_N (their generating functional being $\mathcal{W}(A) = \ln \mathcal{G}(A)$) or with 1-irreducible functions Γ_N (also called one particle irreducible functions), which generating functional is defined by the functional Legendre transform [39]

$$\Gamma(\alpha) = W(A) - A\alpha, \quad \alpha = \frac{\delta \mathcal{W}(A)}{\delta A(x)}\,. \tag{2.6}$$

These unrenormalized and renormalized Green functions satisfy relation:

$$W_N^R(e, \varepsilon, \dots) = Z_\varphi^{-N}(e, \varepsilon) W_N(e_0(e, \varepsilon), \varepsilon, \dots)\,, \tag{2.7}$$

$$\Gamma_N^R(e, \varepsilon, \dots) = Z_\varphi^{N}(e, \varepsilon) \Gamma_N(e_0(e, \varepsilon), \varepsilon, \dots)\,, - \tag{2.8}$$

where the functions $e_0(e, \varepsilon)$, $Z_\varphi^N(e, \varepsilon)$ can be chosen arbitrarily, which implies an arbitrary choice of normalization of the fields and parameters e at given e_0. In the present text we also interchangeably use the following notation for the connected Green functions

$$W_{\varphi_1 \dots \varphi_N} \equiv \langle \varphi_1 \dots \varphi_N \rangle_{\text{conn}}. \tag{2.9}$$

and for the one particle irreducible (1PI) Green functions

$$\Gamma_{\varphi_1 \dots \varphi_N} \equiv \langle \varphi_1 \dots \varphi_N \rangle_{\text{1-ir}}. \tag{2.10}$$

In the present section it is superfluous, because for simplicity we are assuming one field theory. However, later on we will discuss theories with more than one field and then it becomes necessary to have more general notation at hand.

The basic statement of the theory of renormalization is that for the multiplicatively renormalizable models these functions can be chosen to provide UV-finiteness of Green functions as $\varepsilon \to 0$. With this choice, all UV-divergences (poles in ε) contained in the functions $e_0(e,\varepsilon)$, $Z_\varphi^N(e,\varepsilon)$ are absent in renormalized Green functions $W_N^R(e,\varepsilon)$. We note that the UV-finiteness in this sense of any one set of Green functions (full, connected, 1-irreducible) automatically leads to the UV-finiteness of any other. The RG equations are written for the renormalized functions W_N^R which differ from the original unrenormalized functions W_N only by normalization, and therefore, can be used equally well to analyze the critical scaling. Let us demonstrate an elementary derivation of the RG equations [25, 64]. The requirement of elimination of divergences does not uniquely determine the functions $e_0(e,\varepsilon)$ and $Z_\varphi(e,\varepsilon)$. An arbitrariness remains which allows to introduce in these functions (and via them also into W_N^R) an additional dimensional parameter - scale setting parameter (renormalization mass) μ:

$$W_N^R(e,\mu,\varepsilon,\dots) = Z_\varphi^{-N}(e,\mu,\varepsilon) W_N(e_0(e,\mu,\varepsilon),\varepsilon,\dots)\,. \tag{2.11}$$

A change of μ at fixed e_0 leads to a change of e, Z_φ and W_R for unchanged $W_N(e_0,\varepsilon,\dots)$. We use $\tilde{\mathcal{D}}_\mu$ to denote the differential operator $\mu\partial_\mu$ for fixed e_0 and operate on both sides of the equation $Z_\varphi W_N^R = W_N$ with it. This gives the basic RG differential equation :

$$\left[\mu\partial_\mu + \sum_e \tilde{\mathcal{D}}_\mu e \partial_e + N\gamma_\varphi\right] W_N^R(e,\mu,\varepsilon,\dots) = 0,\ \gamma_\varphi \equiv \tilde{\mathcal{D}}_\mu \ln Z_\varphi \tag{2.12}$$

where the operator $\tilde{\mathcal{D}}_\mu$ is expressed in the variables μ, e. The coefficients $\tilde{\mathcal{D}}_\mu e$ and γ_φ are called the RG functions and are calculated in terms of various renormalization constants Z. Coupling constants (charges) g are those parameters e, on which the renormalization constants $Z = Z(g)$ depend. Logarithmic derivatives of charges in (2.12) are β functions

$$\beta_g = \tilde{\mathcal{D}}_\mu g\,. \tag{2.13}$$

All the RG-functions are UV-finite, i.e. have no poles in ε, which is a consequence of the functions W_N^R being UV-finite in (2.12).

The general theory of renormalization [25,65] distinguishes unrenormalized $\mathcal{S}$, renormalized $\mathcal{S}_R$, and base $\mathcal{S}_B$ actions; the last is obtained from $\mathcal{S}$ by replacement of all the bare parameters e_0 by their renormalized counterparts e. The UV-divergences are removed by adding to the base action $\mathcal{S}_B$ all necessary counterterms $\Delta\mathcal{S}$ which are determined by the known rules (see below). If the renormalized action thus obtained $\mathcal{S}_R = \mathcal{S}_B + \Delta\mathcal{S}$ can be reproduced by the above procedure of redefinition of fields and parameters in the original unrenormalized action $\mathcal{S}$, the model is multiplicatively renormalizable. Therefore, the first step in the RG analysis of any model is to explicitly determine all counter-terms required for the removal of UV-divergences and to verify its multiplicative renormalizability.

The form of the required counterterms is determined by the analysis of canonical (engineering) dimensions of the 1PI Green functions of model with the action $\mathcal{S}_B$, which satisfy the

equation of extended homogeneity (2.1) (or, equivalently, (2.4)) with definite canonical exponents $\Delta_e \equiv d_e, d_\mu = 1, d_\varphi$ of parameters e_0, e, μ and fields φ respectively. Dynamical models (1.106), in contrast to static ones, are two-scale models, i.e. two independent canonical wave-number (momentum) d_k^Q and frequency d_ω^Q dimensions can be assigned to every quantity Q (fields and parameters in the action). They are easily determined from the natural normalization conditions

$$d_k^k = -d_x^k = 1, \qquad d_\omega^\omega = -d_t^\omega = 1 \tag{2.14}$$

and from the requirement that every term T of the actions S, S_B is dimensionless ($d_k^T = d_\omega^T = 0$). After that summarized full canonical (engineering) dimension d^Q can be determined by means of d_k^Q and d_ω^Q. Formally one can write

$$d_F = d_k^Q + d_\omega d_\omega^Q, \tag{2.15}$$

where a value of d_ω depends on the particular model [25]. For example, for stochastic developed turbulence and relaxational models A and B we have $d_\omega = 2$, whereas for models D and H $d_\omega = 4$.

Of course, the existence of the two aforementioned wave-number and frequency scale invariance can be expressed by means of two differential equations similar to the Eq. (2.4) with corresponding exponents d_k^e, d_ω^e of parameters e.

In the scheme of renormalization of dynamical models (1.106) the full dimension d^Q plays the same role as the conventional (momentum) dimension does in static problems. Canonical dimensions of an arbitrary 1PI Green function Γ with n_φ (multiple index) fields φ', φ for a $d+1$-dimensional problem are given by the relations

$$d_\Gamma^k = d - \sum_\varphi n_\varphi d_\varphi^k\,, \quad d_\Gamma^\omega = d - \sum_\varphi n_\varphi d_\varphi^\omega\,, \quad d_\Gamma = d + 2 - \sum_\varphi n_\varphi d_\varphi \tag{2.16}$$

with summation over all the fields φ entering into given function Γ. In a logarithmic theory, which corresponds to $\varepsilon = 0$ when bare coupling constant(s) g of concrete model is (are) dimensionless ($d^g = d_k^g = d_\omega^g = 0$), full canonical dimension of Γ_N is equal to a formal index of UV-divergence δ. The UV-divergences which must be removed by suitable counterterms are allowable only in those functions Γ_N for which index δ is nonnegative and integer [25]. All counterterms are polynomial functions of wave vector $\boldsymbol{k}$ and frequency ω.

For models considered in the present work the analysis of divergences should be based on the following auxiliary considerations:

a) For any dynamic model (1.106) all 1PI Green functions containing only the original fields φ are proportional to the closed circles of step functions hence they vanish, and thus do not generate counterterms.

b) If for some reason several external momenta or frequencies occur as an overall factor in all the graphs of a particular Green function, the real degree of divergence δ' is less than $\delta \equiv d_\Gamma(\varepsilon = 0)$ by the corresponding number of units.

c) Sometimes the divergences formally allowed dimensionally are absent due to symmetry requirements, for instance, the Galilean invariance of the fully developed turbulence [32] restricts the form of possible counterterms.

d) Nonlocal terms of the model are not renormalized.

These general considerations and formula (2.16) permit us to determine all superficially divergent functions and to explicitly obtain the corresponding counter-terms for any concrete dynamic model.

2.4 Solution of RG equations. Invariant variables. RG-representations of correlation functions.

In this section we demonstrate general mathematical methods how to find solutions of RG equations of type (2.12) which are typical of models under consideration. Consider linear differential equation which is typical of RG approach:

$$LF(u) = \gamma(u)\,, \quad L = -s\partial_s + \sum_{i=2}^{n} Q_i(u)\partial_{e_i}\,, \tag{2.17}$$

where $s = u_1$ is the scaling parameter, $e_i = u_i, i = 2, \ldots, n$, Q_i are given functions of the parameters u and F is a sought function. The general solution of this inhomogeneous equation is the sum of its particular solution and a solution of the homogeneous equation. The latter is an arbitrary function of the full set of independent first integrals, which represent arbitrary solutions of the homogeneous equation. The number of independent first integrals is equal to the number of parameters e. It is convenient to choose first integrals $\bar{e}_i = \bar{e}_i(s, e)$, which are defined as follows:

$$L\bar{e}_i(s, e) = 0\,, \quad \bar{e}_i(s, e)|_{s=1} = e_i\,. \tag{2.18}$$

These quantities are usually called *invariant (running) variables (charges)*.

The differential operator in the RG-equation (2.12) belongs to an important type of operators $\mathcal{D}_{\mathcal{RG}}$ defined by the equation

$$\mathcal{D}_{\mathcal{RG}}\Upsilon(s, g, a) = 0\,, \quad \mathcal{D}_{\mathcal{RG}} \equiv \left[-s\partial_s + \beta(g)\partial_g - \sum_a \gamma_a(g)a\partial_a + \gamma(g)\right]\,, \tag{2.19}$$

where g is the charge which defines β-function, a are other parameters ($e = g, a$) and the functions $\beta(g)$, $\gamma_a(g)$ and $\gamma(g)$ are independent of s. It is possible to show, that in this special case the invariant charge $\bar{g} = \bar{g}(s, g)$ is independent of the parameters a and satisfies the differential equation known as the Gell-Mann-Low equation:

$$s\partial_s\bar{g} = \beta(\bar{g})\,, \quad \bar{g}|_{s=1} = g\,. \tag{2.20}$$

This equation is easily integrable:

$$\ln s = \int_g^{\bar{g}} \frac{\mathrm{d}x}{\beta(x)}\,. \tag{2.21}$$

The last expression implicitly defines $\bar{g} = \bar{g}(s, g)$ as a function of the scale parameter s and the charge g. For models with n charges g_i $(i = 1, 2, \ldots n)$ one obtains a set of n equations

$$s\frac{\mathrm{d}\bar{g}_i}{\mathrm{d}s} = \beta_{g_i}(\bar{g})\,, \quad \bar{g} \equiv (\bar{g}_1, \bar{g}_2, \ldots \bar{g}_n)\,, \tag{2.22}$$

where $\bar{g}_i$ is a set of invariant charges with initial values equal to g_i ($\bar{g}_i|_{s=1} = g_i$). A straightforward integration (at least numerically) of these equations gives a way to find their fixed points. Instead, very often one solves the set of equations

$$\beta_{g_i^*} = 0 \tag{2.23}$$

which defines so-called fixed points $g^* \equiv (g_1^*, \dots g_n^*)$ of the model. To determine the type of a fixed point one calculates the matrix $\Omega \equiv \Omega_{ik}$ of the first derivatives of β functions at a given fixed point g^*:

$$\Omega_{ik} = \left.\frac{\partial \beta_{g_i}(g)}{\partial g_k}\right|_*, \quad i,k = 1,2,\dots n, \quad \dots \Big|_* \equiv \dots \Big|_{g=g^*}. \tag{2.24}$$

If the matrix Ω is positive (negative) definite, then the fixed point is IR (UV) stable. Technically one needs to determine the eigenvalues $\{\lambda_j\}, j = 1, \dots, n$ of the Ω matrix. A given fixed point is infrared (or ultraviolet) if all real parts of the eigenvalues are positive (or negative). In other words for $\overline{g}(s)$ close to g^* we obtain a system of linearized equations

$$\left(I \frac{\mathrm{d}}{\mathrm{d}\ln s} - \Omega\right) (\overline{g} - g^*) = 0, \tag{2.25}$$

where I is the unit matrix of the size n. Solutions of this system behave like $\overline{g} = g^* + \mathcal{O}(s^{\lambda_j})$, when $s \to 0$.

The other invariant variables $\bar{a}$ satisfy equations

$$s\partial_s \bar{a} = -\bar{a}\gamma_a(\bar{g}), \quad \bar{a}|_{s=1} = a. \tag{2.26}$$

In models with one charge (coupling constant) g these equations are easily integrable and the solution can be written [24, 64] the form

$$\bar{a} = \bar{a}(s,g,a) = a \exp\left[-\int_g^{\bar{g}} \mathrm{d}x \frac{\gamma_a(x)}{\beta(x)}\right]. \tag{2.27}$$

Finally, the general solution of the equation (2.19) has the form

$$\Upsilon(s,g,a) = \Upsilon(1,\bar{g},\bar{a}) \exp\left[\int_g^{\bar{g}} \mathrm{d}x \frac{\gamma(x)}{\beta(x)}\right], \tag{2.28}$$

where $\Upsilon(1,\bar{g},\bar{a})$ is an arbitrary (scaling) function of the first integrals.

2.5 Dimensional renormalization and the scheme of minimal subtractions

Parameters e, on which the renormalization constant $Z_\varphi(g,\varepsilon)$ depends, are *coupling constants* or *charges* of the model. It is customary to choose renormalized charges dimensionless, therefore in concrete models renormalized and unrenormalized values of charges g obey relations of the type

$$g_0 = \mu^{2\varepsilon} g Z_g(g,\varepsilon), \tag{2.29}$$

where the renormalization constant Z_g is dimensionless and the canonical dimension of the unrenormalized coupling constant in this example is 2ε. RG functions corresponding to charges are β functions (single charge is assumed here for simplicity). From (2.29) it follows that

$$\beta_g = \tilde{\mathcal{D}}_\mu g = g\left[-2\varepsilon - \gamma_g(g,\varepsilon)\right] . \tag{2.30}$$

According to the main statement of renormalization theory, renormalization constants Z can be chosen so as to eliminate all the UV divergences in the Green functions, in this case, poles in ε. This is the main requirement on the functions Z, but it does not determine them uniquely. The remaining arbitrariness is fixed by imposing some auxiliary conditions. This is referred to as the choice of subtraction scheme. Various schemes are used in which the renormalized Green functions differ only by a UV-finite renormalization and from the viewpoint of physics are equivalent. Therefore, the scheme is chosen on the basis of convenience.

The most convenient scheme for analytic calculations is the minimal subtraction (MS) scheme proposed in [66], in which all the constants Z have the following form in perturbation theory:

$$Z^{\mathrm{MS}}(g,\varepsilon) = 1 + \sum_{n=1}^{\infty} g^n \sum_{k=1}^{n} \varepsilon^{-k} c_{n,k} . \tag{2.31}$$

In dimensional renormalization the contribution to the coefficient of g^n in (2.31) may be expressed as a Laurent series in ε. In the MS scheme only the singular part of the Laurent expansion of each coefficient is retained. In any other renormalization scheme the renormalization constant is of the form

$$Z(g,\varepsilon) = 1 + \sum_{n=1}^{\infty} g^n \sum_{k=-n}^{\infty} \varepsilon^k c_{n,-k} , \tag{2.32}$$

where the regular part of each coefficient $\sum_{k=0}^{\infty} \varepsilon^k c_{n,-k}$ is, by and large, an arbitrary regular function of ε at the origin.

It should be emphasized that even in the MS scheme the contribution of a graph to the renormalization constant is not determined solely by the singular part of the Laurent expansion of the graph itself. Calculations in perturbation theory are usually carried out order by order in the number of loops. For example, in the φ^4-theory the singular part of a one-loop graph is a constant (i.e. independent of external wave vectors and frequencies), which is taken as the contribution of the graph to the renormalization constant at one-loop order. When a superficially divergent one-irreducible graph contains two or more loops, its singular part is a function of external wave vectors and frequencies. To extract its contribution to the renormalization constant, renormalization of various subgraphs must be taken into account according to the rules of the consistent renormalization procedure (R operation) [65]. In the course of this operation the Laurent series of each one-loop subgraph up to the constant term is included in the calculation of the contribution of a two-loop graph to the renormalization constant in the MS scheme. For consistent calculation of the contribution of a superficially divergent three-loop graph to the renormalization constant the Laurent expansion of each one-loop subgraph is needed to the linear order in ε etc. Therefore, even in the MS scheme the Laurent series in ε of each superficially divergent graph must eventually be calculated. The difference is, so to speak, in the order of appearance of the terms of this series.

In the theory of critical phenomena and stochastic dynamics there are several models including long-range correlations or interactions described – in the Fourier space – by fractional powers of wave numbers of the type k^{-2a}. When the exponent a is treated as a parameter of the model, the model may become logarithmic at some critical value a_c. In such a case the difference $\varepsilon = a_c - a$ may be used as a regulator of the model and we arrive at a particular case of analytic renormalization [67] with a single regulator. In this particular case all relations about the analytic behavior of graphs, renormalization constants and RG functions are the same as in dimensional renormalization and henceforth the terminology of dimensional renormalization will be used both in the case of genuine dimensional renormalization and in the case of analytic renormalization with a single regulator.

2.6 Scheme dependence of critical exponents in dimensional renormalization

It is one of the basic properties and the main reason of success of the RG approach to asymptotic analysis that values of relevant physical quantities are independent of the finite renormalization, i.e. independent of the renormalization scheme (see, e.g. [64]). The basic scheme-independent quantities are the anomalous dimensions (i.e. values of the coefficient functions γ at the fixed point) and the eigenvalues of the Jacobi matrix of the set of β functions at the fixed point. This, however, is a global statement which does not take into account the approximation method used in the calculation of these quantities. In particular, in dimensional renormalization of perturbation theory the RG functions are calculated as power series in coupling constants at fixed ε. The leading term in the β function in this expansion is proportional to ε, whereas coefficient of high-order terms have – as a rule – a finite limit, when $\varepsilon \to 0$. Therefore, power counting of RG-functions in the coupling constant and power counting in ε of fixed-point values thereof do not coincide. As a consequence, at any finite order of perturbation theory the condition of the scheme independence is not fulfilled and anomalous dimensions as well as fixed-point eigenvalues of the Jacobi matrix exhibit heavy dependence of the renormalization scheme, as will be shown below. Calculations within the dimensional renormalization have been customarily carried out with the use of the MS (minimal subtractions) scheme, therefore the issue of scheme dependence of critical exponents practically has not appeared.

Consider a dimensionally renormalized model with a single dimensionless renormalized charge g and a single coefficient function γ_a related to renormalization of a field, temperature (mass) or transport coefficient. In the dimensional renormalization the generic form of the RG functions is

$$\beta(g,\varepsilon) = \tilde{\mathcal{D}}_\mu g = g\left[-\varepsilon - \gamma_g(g,\varepsilon)\right] , \tag{2.33}$$

$$\gamma_\varphi(g,\varepsilon) = \tilde{\mathcal{D}}_\mu \ln Z_\varphi(g,\varepsilon) . \tag{2.34}$$

In perturbation theory the γ functions of the RG are series expansions in the renormalized charge by construction, whose coefficient are regular functions of the parameter ε at the origin, i.e.

$$\gamma_(g,\varepsilon) = \sum_{n=1}^{\infty} a_{g\,n}(\varepsilon) g^n , \tag{2.35}$$

$$\gamma_\varphi(g,\varepsilon) = \sum_{n=1}^{\infty} a_{\varphi\,n}(\varepsilon) g^n . \tag{2.36}$$

The usual argument leading to scheme independence of an anomalous dimension (i.e. the value of a γ function at a fixed point of the RG) goes as follows [64]. In two different schemes the renormalized charges g and g' are connected by a relation $g' = G(g)$ in the form of series expansion in g. Renormalization constants giving rise to γ functions are connected – due to the group property – by the multiplicative relation

$$Z'_i(g', \varepsilon) = F_i(g, \varepsilon) Z_i(g, \varepsilon)\,, \qquad i = g, \varphi\,, \tag{2.37}$$

where the function $F(g, \varepsilon)$ is also expressed as a series in g with regular in ε coefficients. Therefore,

$$\gamma'_i(g', \varepsilon) = \gamma_i(g, \varepsilon) + \beta(g) \frac{\partial}{\partial g} \ln F_i(g, \varepsilon)\,, \qquad i = g, \varphi\,. \tag{2.38}$$

At a fixed point g^* of the RG $\beta(g^*) = 0$ and $\beta'(G(g^*)) = 0$. Therefore, the second term on the right side of (2.38) vanishes rendering the anomalous dimensions equal in the two renormalization schemes.

This is a global argument assuming that all functions in relation (2.38) are known completely. This is not the case, however, in perturbation theory. Renormalization constants and the RG functions are calculated order by order as power series in the charge g. Typically, expansions of the coefficient functions start with a linear term. In that case the linear term on the right side of (2.38) is produced by the function $\gamma(g, \varepsilon)$ and the term $g\varepsilon$ multiplied by the coefficient of the linear term of $\partial_g \ln F(g, \varepsilon)$. The second contribution to the β function (2.33) is $\mathcal{O}(g^2)$, should not be included in the linear contribution to the right side (2.38) and the vanishing at the fixed point factor is lost! Obviously, the same property holds at every finite order of perturbation theory and we arrive at the conclusion that in the perturbative dimensional renormalization the value of the anomalous dimension at a non-trivial ($g^* \neq 0$) fixed point heavily depends on the renormalization scheme!

To analyze this scheme dependence in more detail, consider renormalization constants and RG functions calculated to some finite order N of perturbation theory. In this case relation (2.38) gives rise to

$$\sum_{n=1}^{N} a'_{i\,n}(\varepsilon) g'^n = \sum_{n=1}^{N} a_{i\,n}(\varepsilon) g^n + \sum_{m=1}^{N} \left[g\varepsilon - g \sum_{n=1}^{m} a_{g\,n}(\varepsilon) g^n \right] \sum_{n=1}^{N-m-1} f_{i\,n}(\varepsilon) g^n\,, \tag{2.39}$$

where terms to the order g^N are taken into account on the right side. In (2.39) $f_{i\,n}(\varepsilon)$ are the coefficients of the perturbation expansion of $\ln F(g, \varepsilon)$. They are regular functions of ε by definition. The perturbative non-trivial ($g^* \neq 0$) fixed point is sought in the form of an ε expansion

$$g^* = \sum_{n=1} g^*_n \varepsilon^n \tag{2.40}$$

for the solution of the equation $\varepsilon + \gamma_g(g, \varepsilon) = 0$. By direct substitution it is seen that at such a fixed point

$$g^* \varepsilon - g^* \sum_{n=1}^{m} a_{g\,n}(\varepsilon) g^{*n} = \mathcal{O}(\varepsilon^{m+2})\,. \tag{2.41}$$

At the fixed point (2.40) we therefore see that the second term on the right side of (2.39) is of the order $\mathcal{O}(\varepsilon^{N+1})$ and the statement of the scheme independence is actually that the ε expansion of a critical dimension is scheme independent up to the order given by a consistent perturbative calculation. The ε expansion is given by the MS scheme, therefore in any other renormalization scheme anomalous dimensions contain ε dependent contributions which are not controlled by the perturbation theory beyond the order of calculation of the renormalization constants.

In case of the stability exponent the renormalization invariance is based on the connection between β functions

$$\beta'(g') = \frac{\mathrm{d}G(g)}{\mathrm{d}g}\beta(g) \tag{2.42}$$

from which it follows that

$$\frac{\mathrm{d}\beta'(g')}{\mathrm{d}g'} = \frac{\mathrm{d}\beta(g)}{\mathrm{d}g} + \left[\frac{\mathrm{d}G(g)}{\mathrm{d}g}\right]^{-1} \frac{\mathrm{d}^2G(g)}{\mathrm{d}g^2}\beta(g)\,. \tag{2.43}$$

At a fixed point the rightmost term on the right side of (2.43) vanishes, if complete functions are known and the value of the derivative of the β function at the fixed point (the critical exponent ω) is the same in both renormalization schemes. Order by order in perturbation theory this is again not true. Again, the β function of the additional term on the right side of (2.43) at a fixed point in the ε expansion produces an excess factor $\mathcal{O}(\varepsilon)$ and the exponent ω is renormalization invariant at the leading order of the ε expansion, which is easily verified by direct calculation with the use of representations (2.35) and (2.36). It should also be recalled that in the usual single-charge case (i.e. when the leading term in γ_g is linear in g) all anomalous dimensions and the stability exponent ω are expressed in regular at the origin power expansions in ε. This is not always the case in multi-charge problems.

Critical dimensions and stability conditions of asymptotic patterns should be independent of the renormalization scheme and the different values obtained for them signal that approximations used for their calculation are different as well. In dimensional renormalization differences in renormalization schemes show in the ε dependence of various quantities in the renormalized model. It is customary to carry out calculations in the form of ε expansions. Practical evaluation is usually performed loop by loop in graphs of perturbation theory with each consecutive loop improving the accuracy of results of the ε expansions by some fixed order in ε. In the prevailing MS scheme of the field-theoretic approach the contribution of each superficially divergent graph to a renormalization constant at a given order of the loop expansion is restricted to the singular part of its Laurent expansion in ε which in practice means that within this scheme the calculation of critical exponents and other relevant quantities yields directly the ε expansions thereof without any excessive ε on top of that. If any other renormalization scheme is used, then finite renormalization means, in principle, that to the singular part of the Laurent expansion of the graph an arbitrary regular function of ε is added. This finite renormalization introduces ε dependence in coefficients of the perturbation expansions of RG functions which destroys the connection between the order of the ε expansion and the loop expansion. This in turn means that values of critical exponents and other relevant quantities calculated in an arbitrary dimensional renormalization scheme at each finite order in loops contain terms of the ε expansion, whose coefficients are subject to corrections in the subsequent orders of the loop expansion. Reliable in

this sense information is provided by expanding everything in the results of the arbitrary renormalization scheme in ε and retaining only terms whose coefficients are consistently given by the current order of loop expansion. This returns us to the calculation in the MS scheme. It is quite possible that evaluation of a power series in some other way than simply adding terms order by order provides advantages in numerical accuracy. However, to make such a conclusion, the functions expanded in the power series should be known in some other, preferably closed form. In perturbation expansion of a field theory this kind of information is not available directly and usually it is quite difficult to obtain, e.g. by instanton analysis. The bottom line here is that although different renormalization schemes provide different information about critical exponents and the like, to make conclusions going beyond the strict ε expansion some additional arguments are required.

2.7 Composite operators and operator product expansion

In this section we recall the basic information about renormalization and critical exponents (dimensions) of composite operators. A composite operator F is any monomial of the basic fields of model and their derivatives. In models we are interested they are constructed from the velocity field $\boldsymbol{v}$, scalar field θ or magnetic field $\boldsymbol{b}$ at the single space-time point $x \equiv (t, \boldsymbol{x})$. Examples are $\boldsymbol{v}^n$, $\boldsymbol{b}^n$, θ^n, $\partial_t \boldsymbol{v}^n$, $\boldsymbol{v}\Delta\boldsymbol{v}$, $(\boldsymbol{\nabla}\theta \cdot \boldsymbol{\nabla}\theta)^n$ and so on. The term is borrowed from the quantum field theory, where fields and anything constructed from them are actually operators in a Hilbert space. In statistical field theory they are just random fluctuating quantities.

Study of composite operators and their renormalization is important at least for two reasons. First, their critical dimensions and correlation functions can be measured experimentally and for some operators such data are available [68, 69]. In the developed turbulence the mean of the energy dissipation proportional to the statistical average of the composite operator $\boldsymbol{v}\Delta\boldsymbol{v}$ enters the equation of energy balance, hence playing a crucial role in redistribution of the energy of the turbulent motion and its dissipation. Moreover, strong statistical fluctuations of the operator of energy dissipation seem to be responsible for deviations from Kolmogorov exponents and lead to the intermittency (multifractality) of the turbulent processes [70]. Second, the general solution of the RG equation (2.28) contains an unknown scaling function depending on dimensionless effective variables (coupling constants, viscosity etc.). This function can be calculated in the framework of usual perturbation scheme in an expansion parameter but as was already mentioned above, in certain asymptotic ranges of scales this calculation fails. In theory of turbulence dependence of correlation functions on outer (integral) scale L is of interest. In particular, in turbulence one is interested in the scaling function $R(1, g^*, kL)$ (compare with Eq. (2.3)) in the inertial interval $kL \gg 1$. In the theory of critical phenomena, the asymptotic form of scaling functions for $kL \gg 1$ (formally $L \to \infty$) is studied using the well known the Wilson operator product expansion (OPE), see e.g. [65], [24]; the analog of L is there the correlation length r_c. It has turned out that this technique can be used also in the theory of turbulence and simplified (toy) models associated with the genuine turbulence, see e.g. [25, 32, 71–73].

The generating functional of the correlation functions of the field φ with one insertion of the composite operator $F(\varphi)$ has the form (compare with the generating functional (1.56) for the usual correlation functions of φ)

$$\mathcal{G}(A, F) = \int \mathcal{D}\varphi \, F(\varphi) \exp\left[\mathcal{S}[\varphi] + A\varphi\right], \tag{2.44}$$

where all normalization factors are included in the functional measure. Since the arguments of the fields in F coincide, correlation functions with these operators contain additional UV divergences, which are removed by an additional renormalization procedure, see e.g. [24, 25, 65]. For the renormalized correlation functions the standard RG equations are obtained, which describe IR scaling with definite critical dimensions $\Delta_F \equiv \Delta[F]$ of certain "basis" operators F. Owing to the renormalization, $\Delta[F]$ does not coincide in general with the naive sum of critical dimensions of the fields and derivatives entering into F. Detailed description of the renormalization of composite operators for the stochastic Navier–Stokes equation is given in the review paper [71], below we confine ourselves to only the necessary information.

In general, composite operators are mixed in renormalization, i.e., an UV finite renormalized operator F^R (the correlation functions with one insertion of F_R don't possess the UV divergences) has the form $F^R = F+$ counterterms, where the contribution of the counterterms is a linear combination of F itself and, possibly, other unrenormalized operators which "admix" to F in renormalization. Let $F \equiv \{F_\alpha\}$ be a closed set, all of whose monomials mix only with each other in renormalization. The renormalization matrix $Z_F \equiv \{Z_{\alpha\beta}\}$ and the matrix of anomalous dimensions $\gamma_F \equiv \{\gamma_{\alpha\beta}\}$ for this set are given by

$$F_\alpha = \sum_\beta Z_{\alpha\beta} F^R_\beta, \qquad \gamma_F = Z_F^{-1} \tilde{\mathcal{D}}_\mu Z_F, \tag{2.45}$$

and the corresponding matrix of critical dimensions $\Delta_F \equiv \{\Delta_{\alpha\beta}\}$ is given by

$$\Delta[F] \equiv \Delta_F = d_F^k + \Delta_\omega d_F^\omega + \gamma_F^*, \tag{2.46}$$

in which d_F^k, d_F^ω, and d_F are understood as the diagonal matrices of canonical dimensions of the operators under consideration (with the diagonal elements equal to sums of corresponding dimensions of all fields, their derivatives and renormalized parameters constituting F) and $\gamma_F^* \equiv \gamma_F(g^*)$ is the matrix (2.45) at the fixed point.

Critical dimensions of the set $F \equiv \{F_\alpha\}$ are given by the eigenvalues of the matrix Δ_F. The "basis" operators that possess definite critical dimensions have the form

$$\bar{F}^R_\alpha = \sum_\beta U_{\alpha\beta} F^R_\beta \tag{2.47}$$

where the matrix $U_F = \{U_{\alpha\beta}\}$ is such that $\Delta'_F = U_F \Delta_F U_F^{-1}$ is diagonal.

In general, counterterms to a given operator F are determined by all possible 1PI Green functions with one operator F and arbitrary number of primary fields φ,

$$\Gamma_{N;F} = \langle F(t, \boldsymbol{x}) \varphi(t, \boldsymbol{x}_1) \dots \varphi(t, \boldsymbol{x}_N) \rangle. \tag{2.48}$$

The total canonical dimension (formal index of divergence) for such functions is given by

$$d_\Gamma = d_F - N_\Phi d_\Phi, \tag{2.49}$$

with the summation over all types of fields entering into the function. For the UV divergent diagrams, d_Γ is a nonnegative integer (cf. 2.16).

According to the OPE, the single-time product $F_1(t,\boldsymbol{x}_1)F_2(t,\boldsymbol{x}_2)$ of two renormalized operators at $\boldsymbol{x} \equiv (\boldsymbol{x}_1+\boldsymbol{x}_2)/2 = const$, and $\boldsymbol{r} \equiv \boldsymbol{x}_1 - \boldsymbol{x}_2 \to 0$ has the representation

$$F_1(t,\boldsymbol{x}_1)F_2(t,\boldsymbol{x}_2) = \sum_\alpha A_\alpha(\boldsymbol{r})\bar{F}^R_\alpha(t,\boldsymbol{x}), \tag{2.50}$$

in which the functions A_α are the Wilson coefficients regular in L and $\bar{F}^R_\alpha$ are all possible renormalized local composite operators of the type (2.47) allowed by symmetry, with definite critical dimensions $\Delta_{\bar{F}^R_\alpha}$.

The renormalized correlator $\langle F_1(t,\boldsymbol{x}_1)F_2(t,\boldsymbol{x}_2)\rangle$ is obtained by averaging (2.50) with the weight $\exp S_R$, the quantities $\langle \bar{F}^R_\alpha\rangle \propto L^{-d_\alpha} f_\alpha(g, L\mu)$ involving dimensionless (scaling) functions $f_\alpha(g,L\mu)$ appear on the right hand side. Their asymptotic behavior for $L\mu \to 0$ is found from the corresponding RG equations (see [74] for the case of Kraichnan model) and has the form

$$\langle \bar{F}^R_\alpha\rangle \propto L^{\Delta_{\bar{F}^R_\alpha}}. \tag{2.51}$$

From the operator product expansion (2.50) we therefore

$$\langle F_1(t,\boldsymbol{x}_1)F_2(t,\boldsymbol{x}_2)\rangle = \sum_{\bar{F}^R} C_{\bar{F}^R}(r/L)^{\Delta_{\bar{F}^R}}, \quad r/L \to 0, \tag{2.52}$$

Here $C_{\bar{F}^R}$ generated by the Wilson coefficients A_α in (2.50) are regular in L, the summation is implied over all possible composite basic renormalized operators $\bar{F}^R$ allowed by the symmetry of the left-hand side, and $\Delta_{\bar{F}^R}$ are their critical dimensions. The leading contributions for $r/L \to 0$ are those with the smallest dimension $\Delta_{\bar{F}^R}$. In the theory of critical phenomena all the nontrivial composite operators have positive critical dimensions $\Delta_{\bar{F}^R} > 0$ for small ε and the leading term in (2.52) is determined by the simplest operator $\bar{F}^R = 1$ with $\Delta_{\bar{F}^R} = 0$, i.e., the function $R(r/L)$ is finite as $L \equiv r_c \to 0$, see [24]. However, as has been observed in [72] in the model of developed turbulence composite operators with *negative* critical dimensions exist and are responsible for possible singular behavior of the scaling functions like N-point correlation functions $W_N = \langle \varphi \dots \varphi\rangle$ as $r/L \to 0$. We shall term the operators with $\Delta_{\bar{F}^R} < 0$, if they exist, dangerous [71], as they correspond to contributions to (2.52) which diverge for $r/L \to 0$. The scaling functions (2.52) decomposed in dangerous operators exhibit *anomalous scaling behavior* which is a manifestation of a nontrivial multifractal (intermittent) nature of the statistical fluctuations of the random fields under consideration and globally all the physical system.

Dangerous composite operators in the stochastic model of turbulence occur only for finite values of the RG expansion parameter ε, and within the ε expansion it is impossible to decide whether or not a given operator is dangerous, provided its critical dimension is not found exactly using the Schwinger-type functional equations or the Galilean symmetry, see [71], [75]. Moreover, dangerous operators enter into the operator product expansion in the form of infinite families with the spectrum of critical dimensions unbounded from below, and the analysis of the large L behavior implies the summation of their contributions.

In view of the difficulties encountered by the RG approach to the model of developed turbulence it is reasonable to apply the formalism to simpler models, which exhibit some of the features of genuine turbulent flows, but are easier to study. Much attention has been attracted

by a simple model of the passive advection of a scalar quantity by a Gaussian velocity field, introduced by Obukhov [76] and Kraichnan [77]. Of special interest are structure functions S_N, which for the scalar field θ can be defined as follows

$$S_N(r) \equiv \left\langle [\theta(t, \boldsymbol{x} + \boldsymbol{r}) - \theta(t, \boldsymbol{x})]^N \right\rangle, \quad r \equiv |\boldsymbol{r}|, \tag{2.53}$$

where homogeneity and isotropy has been assumed. It turns out, that the structure functions of the scalar field in aforementioned Obukhov-Kraichan model exhibit anomalous scaling behavior and the corresponding anomalous exponents can be calculated explicitly within an expansion in certain small parameter, see [43, 73] and reference therein.

3 Stochastic models of developed turbulence

3.1 Stochastic version of Navier-Stokes equation

Systems with large number of degrees of freedom display similar behavior in certain asymptotic regimes independently of numerous microscopic details of the system. In the theory of strongly developed turbulence this *universality* is connected with long-distance asymptotics of velocity correlation functions. The main indication of the universality in the turbulence comes from the celebrated Kolmogorov scaling theory [78] describing the large-scale behavior of velocity structure functions.

During the last decade much attention has been paid to the inertial range of fully developed turbulence, which contains wave numbers larger then those that pump the energy into the system and smaller enough then those that are related to the dissipation processes [79, 80]. Foundations of theory of the inertial range turbulence were laid in the well known Kolmogorov–Obukhov (KO) phenomenological theory (see, e.g., [70, 79, 81]). One of the main problems in the modern theory of fully developed turbulence is to verify the validity of the basic principles of KO theory and their consequences within the framework of a microscopic model. Recent experimental and theoretical studies indicate possible deviations from the celebrated Kolmogorov scaling exponents. The scaling behavior of the velocity fluctuations with exponents, which values are different from Kolmogorov ones, is called as anomalous and usually is associated with phenomenon of intermittency. Roughly speaking, intermittency means that statistical properties (for example, correlation or structure functions of the turbulent velocity field) are dominated by rare spatiotemporal configurations, in which the regions with strong turbulent activity have exotic (fractal) geometry and are embedded into the vast regions with regular (laminar) flow. In the turbulence such phenomenon is believed to be related to the strong fluctuations of the energy flux which, therefore leads to deviations from the predictions of the aforementioned KO theory. Such deviations, referred to as "anomalous" or "non-dimensional" scaling, manifest themselves in singular (arguably power-like) dependence of correlation or structure functions on the distances and the integral (external) turbulence scale L. The corresponding exponents are certain nontrivial and nonlinear functions of the order of the correlation function, the phenomenon referred to as "multiscaling".

Although great progress in the understanding of intermittency and anomalous scaling in turbulence has been achieved as a result of intensive studies, their investigation in fully developed turbulence still remains a major theoretical problem.

Although the theoretical description of the fluid turbulence on the basis of the "first principles", i.e., on the stochastic Navier-Stokes (NS) equation [79] remains essentially an open problem, considerable progress has been achieved in understanding simplified model systems that share some important properties with the real problem: shell models [82], stochastic Burgers equation [83] and passive advection by random "synthetic" velocity fields [73].

A crucial role in these studies is played by models of advected passive scalar field [76]. A simple model of a passive scalar quantity advected by a random Gaussian velocity field, white in time and self-similar in space (the latter property mimics some features of a real turbulent velocity ensemble), the so-called Kraichnan's rapid-change model [84], is an example. The interest to these models is based on two important facts: first, as were shown by both natural and numerical experimental investigations, the deviations from the predictions of the classical

Kolmogorov-Obukhov phenomenological theory [70, 79, 81, 85] is even more strongly displayed for a passively advected scalar field than for the velocity field itself (see, e.g., [86–96] and references cited therein), and second, the problem of passive advection is much more easier to be consider from theoretical point of view. There, for the first time, the anomalous scaling was established on the basis of a microscopic model [77], and corresponding anomalous exponents was calculated within controlled approximations [89–91, 97–102] (see also reviews [43, 73] and references therein).

The greatest stimulation to study the simple models of passive advection not only of scalar fields but also of vector fields (e.g., weak magnetic field) is related to the fact that even simplified models with given Gaussian statistics of so-called "synthetic" velocity field describes a lot of features of anomalous behavior of genuine turbulent transport of some quantities (as heat or mass) observed in experiments, see, e.g., [77, 84, 88–108].

An effective method for investigation of a self-similar scaling behavior is the renormalization group (RG) technique [24, 25, 65]. It was widely used in the theory of critical phenomena to explain the origin of the critical scaling and also to calculate corresponding universal quantities (e.g., critical dimensions). This method can be also directly used in the theory of turbulence [25,32,62,63,71], as well as in related models like a simpler stochastic problem of a passive scalar advected by prescribed stochastic flow. In these investigations, the diagram technique of Wyld [109] for the stochastically forced Navier-Stokes equation with powerlike correlation function of the random force $\boldsymbol{f}$. The exponent of this correlation function gives rise to an expansion parameter similar to that of the famous $\varepsilon = d_c - d$ expansion in the theory of critical phenomena. In the wave-vector space the spectrum of force correlations of the form

$$\langle f_i(\boldsymbol{k}) f_j(-\boldsymbol{k})\rangle \propto k^{4-d-2\varepsilon} \tag{3.1}$$

allows to obtain a regular expansion of scaling exponents and amplitude coefficients in the small parameter ε. In what follows the conventional ("quantum field theory" or field-theoretic) RG will be use which is based on the standard renormalization procedure, i.e., on the elimination of the UV divergences in the logarithmic model.

In work [110] the field theoretic RG and operator-product expansion (OPE) were used in the systematic investigation of the rapid-change model. It was shown that within the field theoretic approach the anomalous scaling is related to the very existence of so-called "dangerous" composite operators with negative critical dimensions in OPE (see, e.g., [25, 32] for details). In the subsequent papers [111] the anomalous exponents of the model were calculated within the ε expansion to order ε^3 (three-loop approximation). Here ε is a parameter which describes a given equal-time pair correlation function of the velocity field (see subsequent section).

Afterwards, various generalized descendants of the Kraichnan model, namely, models with inclusion of large and small scale anisotropy [112], compressibility [113] and finite correlation time of the velocity field [74, 114] were studied by the field theoretic approach. Moreover, advection of a passive vector field by the Gaussian self-similar velocity field (with and without large and small scale anisotropy, pressure, compressibility, and finite correlation time) has been also investigated and all possible asymptotic scaling regimes and cross-over among them have been classified [115–119]. General conclusion is: the anomalous scaling, which is the most important feature of the Kraichnan rapid change model, remains valid for all generalized models.

Let us describe briefly the solution of the problem in the framework of the field theoretic approach. It can be divided into two main stages. On the first stage the multiplicative renor-

malizability of the corresponding field theoretic model is demonstrated and the differential RG equations for its correlation functions are obtained (See Sec. 2.3). The asymptotic behavior of the latter on their ultraviolet argument (r/ℓ) for $r \gg \ell$ and any fixed (r/L) is given by infrared stable fixed points of those equations. Here ℓ and L are an inner (ultraviolet) and an outer (infrared) scales. It involves some "scaling functions" of the infrared argument (r/L), whose form is not determined by the RG equations. On the second stage, their behavior at $r \ll L$ is found from the OPE (See Sec. 2.7) within the framework of the general solution of the RG equations. There, the crucial role is played by the critical dimensions of various composite operators, which give rise to an infinite family of independent aforementioned scaling exponents (and hence to multiscaling). Of course, these both stages (and thus the phenomenon of multiscaling) have long been known in the RG theory of critical behavior.

In Ref. [74] the problem of a passive scalar advected by the Gaussian self-similar velocity field with finite correlation time [101, 120] was studied by the field theoretic RG method. There, the systematic study of the possible scaling regimes and anomalous behavior was present at one-loop level. The two-loop corrections to the anomalous exponents were obtained in work [121]. It was shown that the anomalous exponents are non-universal as a result of their dependence on a dimensionless parameter, the ratio of the velocity correlation time, and turnover time of a scalar field.

The Navier-Stokes equations conserve kinetic energy and helicity in inviscid limit. Presence of two quadratic invariants leads to the possibility of appearance of double cascade. It means that cascades of energy and helicity take place in different ranges of wave numbers analogously to the two-dimensional turbulence and/or the helicity cascade appears concurrently to the energy one in the direction of small scales [122, 123]. Particularly, helicity cascade is closely connected with the existence of exact relation between triple and double correlations of velocity known as "2/15" law analogously to the "4/5" Kolmogorov law [124]. According to [122] aforementioned scenarios of turbulent cascades differ from each other by spectral scaling. Theoretical arguments given by Kraichnan [125] and results of numerical calculations of Navier-Stokes equations [126–128] support the scenario of concurrent cascades. The appearance of helicity in turbulent system leads to constraint of non-linear cascade to the small scales. This phenomenon was firstly demonstrated by Kraichnan [125] within the modeling problem of statistically equilibrium spectra and later in numerical experiments.

3.2 Double expansion and the ray scheme

RG calculations with two (or even more) small parameters which may serve as regulators in dimensional or analytic renormalization have been widely used in the analysis of static critical phenomena [129–131], dynamic critical phenomena [132–136], diffusion in random environment [137–140], interface growth [141] and in stochastic hydrodynamics [142–150]. Critical exponents and other relevant quantities may be expressed in a double expansion in these parameters. The two parameters may both be regulators of analytic renormalization or one of them is the regulator of dimensional renormalization. In the following, this pair of parameters will be denoted ε and Δ.

Analytic renormalization would be a natural renormalization scheme to use to construct a double expansion in the two regulators, since it yields the RG functions as analytic functions of the two parameters at the origin. The genuine analytic renormalization involves rather tedious calculations [151]. Moreover, in analytic renormalization there is no analog of the MS scheme

to simplify practical calculations. Therefore, it is invariably assumed (implicitly or explicitly) that both parameters are of the same order of magnitude. This is made explicit by putting them proportional to each other in the ray scheme [152,153]): $\Delta = \zeta\varepsilon$, where ζ is fixed and finite. This assumption effectively restores the dimensional renormalization with a single small parameter and the MS scheme may be used.

In any case, critical exponents turn out to be scheme dependent in the same sense as in the dimensional renormalization. Typically there are at least two charges in these models and therefore a rather generic case of two charges and a single anomalous dimension γ (corresponding to a field renormalization) will be analyzed here. It should be emphasized that we are considering coupling constants which serve as expansion parameters of the perturbation theory. When there are several coupling constants, it is customary to classify the order of perturbation theory by the number of loops. In multi-charge problems there are coupling constants, which should be calculated in closed form at each such order of perturbation theory (e.g. ratios of coefficients of viscosity, diffusion and thermal conductivity). We do not discuss such coupling constants here.

Two different structures of β functions are met. In stochastic hydrodynamics two (or more) random sources with different powerlike falloff of correlation functions are often introduced [142, 143, 145, 147, 148]: always random force for the stochastic momentum equation (Navier-Stokes equation) and the random source for either the stochastic diffusion or heat conduction equation (the passive scalar problem) or for Faraday's law (magnetohydrodynamics). Similar constructions have been used in critical dynamics [43, 133–136] and the interface growth problem [141]. Thus, two analytic regulators are used: deviations of exponents of these powerlike correlation functions from their critical values. The regulators are invariably put explicitly proportional to each other and renormalization is treated in the framework of the usual dimensional renormalization. Nevertheless, a double expansion in the regulators is implied, if not always worked out explicitly. In models of this type the structure of the β functions is similar to the single-charge case, i.e. the renormalized coupling constant is a common factor in the expression for the corresponding β function (for brevity, parameters ε and Δ are omitted in the list of arguments):

$$\beta_1(g_1, g_2) = \mu\frac{\partial}{\partial\mu}\Big|_0 g_1 = g_1\left[-\varepsilon - \gamma_1(g_1, g_2)\right], \tag{3.2}$$

$$\beta_2(g_1, g_2) = \mu\frac{\partial}{\partial\mu}\Big|_0 g_2 = g_2\left[-\Delta - \gamma_2(g_1, g_2)\right], \tag{3.3}$$

$$\gamma_\varphi(g_1, g_2) = \mu\frac{\partial}{\partial\mu}\Big|_0 \ln Z_\varphi(g_1, g_2) \tag{3.4}$$

and the coefficient functions γ_1, γ_2 and γ_φ are regular expansions in powers of g_1 and g_2, whose coefficients depend on the regulators ε and Δ. We shall refer to this situation as the regular multi-charge case.

Connections between renormalization constants and the corresponding RG functions in different schemes in this case are

$$Z_i'(g_1', g_2') = F_i(g_1, g_2) Z_i(g_1, g_2), \qquad i = 1, 2, \varphi, \tag{3.5}$$

$$\gamma_i'(g_1', g_2') = \gamma_i(g_1, g_2) + \sum_{j=1}^{2} \beta_j(g_1, g_2)\frac{\partial}{\partial g_j}\ln F_i(g_1, g_2). \tag{3.6}$$

When the problem is treated in the framework of analytic renormalization, coefficients of the perturbation expansion of the RG functions are regular functions of the two parameters ε and Δ at the origin by construction of the renormalization scheme. The argument about the scheme dependence of the anomalous dimensions then goes in analogy with the dimensional renormalization argument above. Due to the analytic properties of the RG functions the perturbative non-trivial fixed point may be found in the form of a double expansion in ε and Δ. Here and henceforth only fixed points with both non-vanishing charges ($g_1^* \neq 0$, $g_2^* \neq 0$) will be considered, if not stated otherwise.

Regularity of the fixed points and RG functions imply that anomalous dimensions are obtained in the form of regular expansions in ε and Δ. Little reflection shows that the second term in relation (3.6) at a fixed point gives rise to a contribution which is of higher order by $\mathcal{O}(\varepsilon)$ or $\mathcal{O}(\Delta)$ in comparison with the double expansion of the anomalous dimension in the two renormalization schemes.

However, in practical calculations instead of the analytic renormalization the ray scheme is used, in which the regulators are proportional to each other and the renormalization is carried out as in dimensional renormalization with an additional finite and fixed parameter $\zeta = \Delta/\varepsilon$. At one-loop order the γ's are linear functions of the charges vanishing at the origin with coefficients which are regular functions of ε and Δ in the ray scheme as well. This is because the UV divergences show in the form of meromorphic functions with simple poles $A(\varepsilon, \Delta)/\varepsilon$, $B(\varepsilon, \Delta)/\Delta$ and $C(\varepsilon, \Delta)/(\varepsilon + \Delta)$, where A, B and C are regular functions of ε and Δ which determine the RG functions at the one-loop order. At higher orders, however, subtraction of divergent subgraphs in the renormalization gives rise to expressions containing products of terms of the type $[\varepsilon D(\varepsilon, \Delta) + \Delta E(\varepsilon, \Delta)]\,/(m\varepsilon + n\Delta)$, where – in analytic renormalization – integers m and n are the numbers of correlations functions with exponents ε and Δ, respectively, in a (sub)graph and D and E are regular functions of ε and Δ at the origin. In case of combined dimensional and analytic renormalization m is the number of loops in the (sub)graph. It should be noted that in the ray scheme such expressions are finite quantities, but in analytic renormalization they are singular functions which should not appear in the RG functions. In the ray scheme the common power of ε is extracted by the rule $\Delta = \zeta\varepsilon$, which gives rise to meromorphic functions of ζ as coefficients of the perturbative expansion. From the point of view of dimensional renormalization these meromorphic functions produce contributions to finite renormalization. Thus, in the ray scheme the coefficient functions of the perturbation expansion of RG functions are not analytic in the variable $\zeta = \Delta/\varepsilon$, i.e. they are not analytic functions of the regulators ε and Δ.

At the leading one-loop order in both schemes the equations for the fixed point with both $g_1^* \neq 0$ and $g_2^* \neq 0$ are linear equations for the fixed point values of the charges, whose solution is a unique linear function of ε and Δ. In the analytic renormalization all γ's are power series in charges with analytic in ε and Δ coefficients. Therefore, it is immediately seen that in analytic renormalization all anomalous dimensions are regular functions of ε and Δ. Notwithstanding the non-analytic coefficient functions, the result for the anomalous dimensions is the same up to the order in ε and Δ guaranteed by the loop expansion within the ray scheme. In the ray scheme minimal subtractions may be used leading to much simpler calculation of graphs.

The other possibility is dimensional regularization amended by analytic regularization (only one analytic regulator will be considered here, although several have been introduced). In this case either in propagators or interactions the wave-number dependence contains the combination $a + bk^{-2\alpha}$, in which $\alpha > 0$ (in propagators this combination is usually multiplied by the factor

k^2). For small α a non-trivial problem of renormalization of field operators with this structure arises [43, 129–131, 133–136, 140, 141, 146], since in the limit $\alpha \to 0$ the terms in $a + bk^{-2\alpha}$ become indistinguishable and it is not clear, which of them should be renormalized. The problem is solved by the prescription of the counter terms to renormalization of the local (analytic in k^2) contribution [146, 153]. The basic idea is that renormalization produces only local counterterms. Construction of renormalization constants is carried out in the regularized model, in which the local and non-local term are clearly distinguishable ($\alpha > 0$ although small) and the counterterms have the structure of the local term and thus contribute to the renormalization of that term only. If the original model did not contain the local term at the outset (which is often the case when models with long-range effects are constructed), then it is usually brought about by the renormalization procedure [129]. In the field-theoretic approach such "generation terms" are to the original model to make it multiplicatively renormalizable, which is very convenient from the technical point of view.

In many cases the analytic properties of RG functions in problems with combined dimensional and analytic regulators are analogous to those of the case with two analytic regulators. A different situation takes place, for instance, in critical systems with quenched disorder [129–131, 140] and in stochastic hydrodynamics with competing long-range and short-range correlations [146]. The interplay of long-range and short-range correlations is accompanied by the appearance of generation terms. Generation terms are contributions to renormalization of a charge produced by other charges only. Generation terms produce contributions to renormalization constant of the corresponding charges in which the charge corresponding to the generation term stands in the denominator of a polynomial functions of other charges. This introduces significant changes to conclusions obtained from connections between renormalization constants and charges in different schemes. First, contrary to the regular multi-charge case the fixed-point values of charges in the analytic renormalization are not regular functions of the regulators (although the RG functions are). Therefore, critical exponents may not be regular functions of regulators either. Another feature of this class of models is that the very number of the fixed points becomes scheme dependent. This may be seen in the example of stochastic hydrodynamics near two dimensions, in which one-loop calculations in four different schemes are available [146, 149, 153, 154]. In the MS scheme in the ray approach the one-loop solution for the two charges $g_1^* \neq 0$ and $g_2^* \neq 0$ is obtained from a system of equations which is essentially linear and the solution is unique [146], whereas in the other schemes the one-loop equation for charges is quadratic [149, 153] with two different solutions corresponding to different choices of the sign of the quadratic root in the solution. In most cases only the stable fixed point with a regular expansion in regulators has been discussed, however, with modifications taking into account the additional solution [43, 149]. The explicit root solutions have been used in a random walk problem [140].

In the multi-charge case the stability exponents are eigenvalues of the Jacobi matrix of the set of β functions. From connections between charges

$$g_i' = G_i(g_1, g_2)\,, \qquad i = 1, 2\,, \tag{3.7}$$

it follows that

$$\frac{\partial \beta_i'}{\partial g_j'} = [J(G)]^{-1}_{ni} \frac{\partial \beta_m(g)}{\partial g_n} J(G)_{jm} + [J(G)]^{-1}_{ni} \frac{\partial^2 G_j(g)}{\partial g_n g_m} \beta_m(g)\,, \tag{3.8}$$

where

$$J(G)_{ij} = \frac{\partial G_i}{\partial g_j}$$

and summation over repeated indices is implied. The matrix transformation of the Jacobi matrix on the right side leaves eigenvalues intact, but the β functions in the second term produce again non-vanishing terms of higher order than the matrix transformation of the first term in the double expansion of the Jacobi matrix and thus to its eigenvalues. Eigenvalues are solutions of algebraic equations containing fractional powers, therefore at the outset there is no expectation of regularity of the stability exponents, but the very equations are scheme independent only up to the consistently calculated order of the double expansion. The scheme dependence of perturbation expansion has been used in field-theoretic RG approach to critical dynamics to catch qualitative features absent at the leading order of a double expansion [141, 150, 155] as well as to improve numerical results [152–154]. It appears that in the momentum-shell RG approach this is not considered an issue at all. However, conclusions made on the basis of scheme-dependent behavior should be corroborated by independent arguments to be reliable.

3.3 Randomly stirred fluid near two dimensions

Let us now analyze the large-distance long-time behavior of randomly stirred fluid with powerlike correlation (3.1) of the random force near two dimensions. This problem may not be directly related to the problem of two-dimensional turbulence due to significant physical differences between turbulence in two and three dimensions. Nevertheless, this analysis allows to infer useful information about the behavior of the perturbation expansion in the most generic case which may then be used to improve numerical accuracy of calculation of the amplitudes – which, in general, are scheme dependent – of the powerlike asymptotics of correlation and response functions. In the stochastic model of fully developed turbulence a double expansion in dimensional and analytic regulators may be constructed using the two-dimensional system as the formal starting point. Two is the critical dimension of the model of randomly stirred fluid [61], for which an expansion in the parameter $2\Delta = d - 2$, the deviation of the space dimensionality from the critical value, has been constructed in full analogy with the theory of the critical phenomena. Near two dimensions both deviation parameters ε and Δ are small, and a double expansion may be established. The renormalization is carried out at the critical values of the parameters, which leads to the following problem. The long-range correlation function of the random force is a powerlike function of the momentum $\propto k^{4-d-2\varepsilon}$ and thus, in general, a singular function of the momentum at the origin. Renormalization gives rise to regular in the momentum terms only, therefore singular terms are not renormalized. When $d = 2$ and $\varepsilon = 0$, however, the correlation function becomes a regular function of the momentum $\propto k^2$. It is not obvious, how the model should be renormalized in this case. This problem was originally discussed for this model in [144], but due to an incorrect renormalization procedure with false conjectures about the asymptotic behavior of the forced Navier-Stokes equation. Similar inconsistencies have occurred in the renormalization of magnetohydrodynamics with long-range correlations of the random force [142, 143] and in the renormalization-group approach to two-dimensional turbulence [156, 157].

We consider the stochastic Navier - Stokes equation for the homogeneous flow of incom-

pressible fluid

$$\nabla_t v_i = \nu_0 \boldsymbol{\nabla}^2 v_i - \partial_i p + f_i^v, \quad \nabla_t \equiv \partial_t + \boldsymbol{v}\cdot\boldsymbol{\nabla}, \tag{3.9}$$

where $\boldsymbol{v}(t,\boldsymbol{x})$ is the transverse velocity field, ν_0 is the kinematic viscosity. Here and henceforth because of the future use of renormalization group we distinguish between unrenormalized (with the subscript "0") quantities and renormalized terms (without the subscript "0"). The renormalized fields will be denoted by the subscript R. Further, P_{ij}, is the transverse projection operator, in the momentum space given as follows

$$P_{ij} = \delta_{ij} - k_i k_j/k^2. \tag{3.10}$$

Further, p is the pressure and f_i is the random force. Here, and henceforth summation over repeated indices is implied. As usual [62, 63, 158, 159], the random force is assumed to have a gaussian distribution with zero mean and the correlation function in the momentum space of the form

$$\langle f_i^v(t,\boldsymbol{x}) f_j^v(t',\boldsymbol{x}')\rangle \equiv D_{ij}(t,\boldsymbol{x};t'\boldsymbol{x}') = \frac{\delta(t-t')}{(2\pi)^d}\int \mathrm{d}^d\boldsymbol{k}\, P_{ij}(\boldsymbol{k}) d_f(k) \mathrm{e}^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}, \tag{3.11}$$

$$d_f(k) = D_0 k^{4-d-2\varepsilon}, \tag{3.12}$$

$$\langle f_i^v(t,\boldsymbol{k}) f_j^v(t',\boldsymbol{k}')\rangle = d_f(k) P_{ij}(\boldsymbol{k})\delta(\boldsymbol{k}+\boldsymbol{k}')\delta(t-t'). \tag{3.13}$$

Here, ε is an arbitrary parameter, the "physical" value of which is determined by the condition that the parameter $g_0\nu_0^3$ has the dimension of the energy injection rate [62,63]. Moreover as $\varepsilon \to 2$, the amplitude $D_0 \sim (2-\varepsilon)$ [32], from which it follows that $d_f(k) \sim \delta(\boldsymbol{k})$, which corresponds to the energy injection by infinitely large eddies.

The connection between D_0 and $\overline{\mathcal{E}}$ is determined by an exact relation expressing $\overline{\mathcal{E}}$ in terms of the function $d_f(k)$ in the correlation function (3.11)

$$\overline{\mathcal{E}} = \frac{(d-1)}{2(2\pi)^d}\int \mathrm{d}^d\boldsymbol{k}\, d_f(k). \tag{3.14}$$

Substituting here function (3.12) and introducing the UV cutoff $k \le \Lambda = (\overline{\mathcal{E}}/\nu_0^3)^{1/4}$ (the inverse dissipation length), we obtain the following connection between the parameters $\overline{\mathcal{E}}$ and D_0

$$D_0 = \frac{4(2-\varepsilon)\,\Lambda^{2\varepsilon-4}}{\overline{S}_d(d-1)}\,\overline{\mathcal{E}}\,. \tag{3.15}$$

Idealized injection by infinitely large eddies corresponds to $d_f(k) \propto \delta(\boldsymbol{k})$. More precisely, according to Eq. (3.14)

$$d_f(k) = \frac{2(2\pi)^d\,\mathcal{E}\,\delta(\boldsymbol{k})}{d-1}. \tag{3.16}$$

In view of the relation

$$\delta(\boldsymbol{k}) = \lim_{\varepsilon\to 2}(2\pi)^{-d}\int \mathrm{d}^d\boldsymbol{x}(\Lambda x)^{2\varepsilon-4}\exp(i\boldsymbol{k}\cdot\boldsymbol{x}) \;= S_d^{-1}k^{-d}\lim_{\varepsilon\to 2}\left[(4-2\varepsilon)(k/\Lambda)^{4-2\varepsilon}\right],$$

the powerlike injection with $d_f = D_0 k^{4-d-2\varepsilon}$ and the amplitude D_0 from Eq. (3.15) in the limit $\varepsilon \to 2$ from the the region $0 < \varepsilon < 2$ gives rise to the δ sequence (3.16).

The stochastic problem (3.9) and (3.11) may be cast [62, 63] to a field theory with the generating functional

$$\mathcal{G}(A) = \int \mathcal{D}v \int \mathcal{D}v' \; \mathrm{e}^{\mathcal{S}+Av+\tilde{A}v'} \tag{3.17}$$

where v' is an auxiliary field, $A, \tilde{A}$ are the source fields, and the action

$$\begin{aligned}\mathcal{S}[\boldsymbol{v},\boldsymbol{v}'] = {}& \frac{1}{2}\int \mathrm{d}t \int \mathrm{d}^d\boldsymbol{x} \int \mathrm{d}t' \int \mathrm{d}^d\boldsymbol{x}' \; v_i'(t,\boldsymbol{x})D_{ij}(t,\boldsymbol{x};t',\boldsymbol{x}')v_j(t',\boldsymbol{x}') \\ & + \int \mathrm{d}t \int \mathrm{d}^d\boldsymbol{x}\; v_i'(t,\boldsymbol{x})\left[-\partial_t v_i(t,\boldsymbol{x}) + \nu_0\boldsymbol{\nabla}^2 v_i(t,\boldsymbol{x}) - v_j(t,\boldsymbol{x})\partial_j v_i(t,\boldsymbol{x})\right] .\end{aligned} \tag{3.18}$$

The canonical scaling dimensions of fields and variables are summarized in Tab. 3.1. The model

Q	v	v'	ν_0	g_0
d_Q^k	-1	$d+1$	-2	2ε
d_Q^ω	1	-1	1	0
d_Q	1	$d-1$	0	2

Table 3.1. Canonical dimensions of the bare fields and bare parameters for the model (3.18).

(3.18) gives rise to the standard Feynman diagrammatic technique; the bare propagators (lines in the diagrams) in the time-wave-vector $(t, \boldsymbol{k})$ representation are

$$\begin{aligned}&\left\langle v_i(t)v_j'(t')\right\rangle_0 = \theta(t-t')\exp\left\{-\nu_0 k^2(t-t')\right\}\; P_{ij}(\boldsymbol{k}) = \text{———+},\\ &\left\langle v_i(t)v_j(t')\right\rangle_0 = \frac{d_f(k)}{2\nu_0 k^2}\;\exp\left\{-\nu_0 k^2|t-t'|\right\}\; P_{ij}(\boldsymbol{k}) = \text{———},\\ &\left\langle v_i'(t)v_j'(t')\right\rangle_0 = 0,\end{aligned} \tag{3.19}$$

with $d_f(k)$ from (3.12) and the Heaviside step function $\theta(\dots)$. The interaction in (3.18) corresponds to the three-point vertex $-v'(v\partial)v = v_i' V_{ijs} v_j v_s/2$ with the vertex factor

$$V_{ijs} = \mathrm{i}(k_j\delta_{is} + k_s\delta_{ij}) = \quad , \tag{3.20}$$

where $\boldsymbol{k}$ is the wave-vector argument of the field v'. The coupling constant $g_0 \equiv D_0/\nu_0^3$ is the expansion parameter of the perturbation theory.

The model (3.18) is logarithmic, i.e. $d_{g_0} = 0$, when $\varepsilon = 0$ independently of the dimension of the space d. Due to the Galilei invariance of the action (3.18), the one-particle irreducible (1PI) Green function $\Gamma_{v'vv}$, which is superficially divergent by power counting, is actually convergent [61–63]. Therefore, above two dimensions only the graphs of the 1PI Green function $\Gamma_{v'v}$, yield divergent contributions to the renormalization of the model, which leads to the renormalization of the parameter ν. At $d = 2$, however, a new set of graphs corresponding to the 1PI Green function $\Gamma_{v'v'}$ becomes divergent, and they must be taken into account in the renormalization of the model.

In the stochastic Navier-Stokes problem [63] critical dimensions are expressed through γ_ν, whose value at the fixed point is determined to all orders in perturbation theory by the fixed-point equation. Thus, critical dimensions do not depend on d and thus for them the problem of singularities in the limit $d \to 2$ is not relevant. There are, however, other important physical quantities such as the skewness factor, Kolmogorov constant, critical dimensions of various composite operators to which this problem persists. It is important that for these quantities the problem of anomalous scaling is absent, which cannot be treated in the framework of the model with massless injection (3.12) lacking a dimensional parameter to account for the external scale of turbulence.

For such quantities, the solutions contain full series of the form

$$R(\varepsilon, d) = \sum_{k=0}^{\infty} R_k(d)\varepsilon^k, \tag{3.21}$$

and the coefficients $R_k(d)$ in the limit $d \to 2$ reveal singular behavior of the type $\sim (d-2)^{-k} \sim \Delta^{-k}$ ($2\Delta \equiv d-2$) giving rise to the growth of the relative part of the correction terms at $d \to 2$. The effect of these is fairly discernable also at the real value $d = 3$, hence the natural desire to sum up contributions of the form $(\varepsilon/\Delta)^k$ at all orders of the ε expansion (3.21). This may be done with the aid of the double (ε, Δ) expansion [144, 146].

In a fixed dimension $d > 2$ the value $\varepsilon = 2$ corresponds to the "real problem". Calculations in the framework of the ε expansion have a rigorous meaning only in the vicinity of $\varepsilon = 0$, whereas continuation of the results to the "real" value $\varepsilon = 2$ is always understood as an extrapolation. In the scheme applicable for $d > 2$ this extrapolation corresponds to the continuation along the vertical ray from the point (d, $\varepsilon = 0$) to the point (d, $\varepsilon = 2$) in the (d, ε) plane. The same final point may be reached along a ray from any starting point ($d_0 \neq d$, $\varepsilon = 0$) at which the model is logarithmic as well. The extrapolation along the ray starting from the origin ($d_0 = 2$, $\varepsilon = 0$) is, however, singled out, because at $d = 2$ in model (3.18) an additional UV divergence (absent at $d > 2$) occurs in the 1PI function $\Gamma_{v'v'}$. On such a ray we put

$$d = 2 + 2\Delta, \qquad \Delta/\varepsilon = \zeta = const. \tag{3.22}$$

The parameters ε and Δ are considered small of the same order and their ratio $\Delta/\varepsilon = \zeta$ a fixed constant [$\zeta = 1/4$ in the extrapolation to the point ($d = 3$, $\varepsilon = 2$)].

Extraction of contributions of the order ε^m with $\Delta/\varepsilon = const$ corresponds to the account of all contributions of the form $\varepsilon^m(\varepsilon/\Delta)^n$ with any $n = 0, 1, 2...$ and $m + n = k$ in Eq. (3.21). Thus the use of the (ε, Δ) expansion in such a form is directly related to the problem of the account of the singularities at $\Delta \to 0$ pointed out in the discussion of relation (3.21).

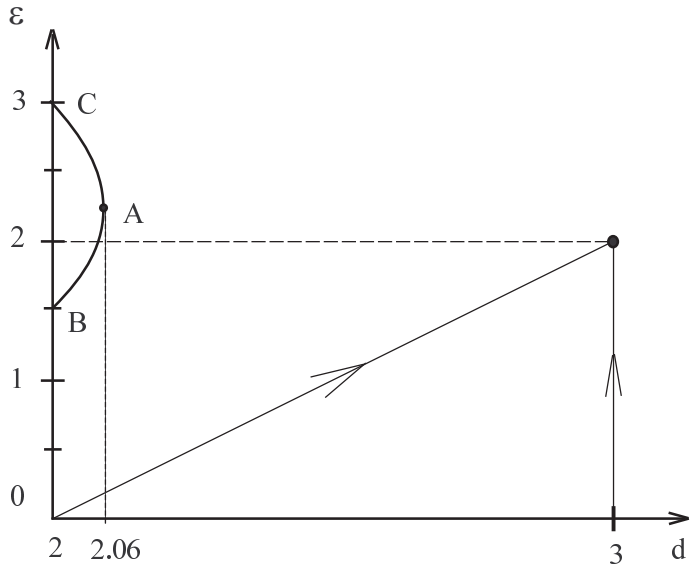


Figure 3.1. The borderline BAC between the regions of parameter space d, ε corresponding to direct (to the right from the curve BAC) and inverse (to the left) energy cascades.

It is worth emphasizing that the very process of extrapolation along a ray from the starting point $(d = 2, \varepsilon = 0)$ is inapplicable to description of two-dimensional turbulence in which the physics is totally different from the three-dimensional problem due to the appearance of the inverse energy cascade [70]. In Fig. 3.1 we have plotted the borderline curve BAC between the direct (normal) and inverse energy cascades obtained in Ref. [61]. The starting point of the extrapolation for the two-dimensional case $(d = 2, \varepsilon = 0)$ lies in the region of the direct cascade, whereas the final point $(d = 2, \varepsilon = 2)$ in the region of the inverse cascade. Thus the ray connecting these points intersects the borderline – the curve BAC – so that the extrapolation becomes impossible. However, the ray connecting the starting point $(d = 2, \varepsilon = 0)$ and a final point like $(d = 3, \varepsilon = 2)$ lies completely in the region of the direct cascade, therefore on such a ray the problem of the change of the cascade pattern does not arise. The rightmost point of the region of the inverse cascade (point A on Fig. 3.1) has the coordinate $d_A \simeq 2.06$ [160]. In the preceding discussion of the extrapolation along the vertical ray from the point $(d, \varepsilon = 0)$ to the point $(d, \varepsilon = 2)$ at $d > 2$, it should have been noted that the condition is not simply $d > 2$, but $d > d_A = 2.06$. From the practical point of view this is irrelevant, because we are interested in the space dimension $d = 3$.

The idea of the double (ε,Δ) expansion together with the extrapolation along the ray $\Delta \sim \varepsilon$ of relation (3.22) in the context of the present problem was first put forward in Ref. [144]. The UV divergences are present not only in the 1PI function $\Gamma_{v'v}$ but also in $\Gamma_{v'v'}$ and appear in the form of poles in the parameters ε and Δ and linear combinations thereof, or, equivalently, as poles in ε with the fixed ratio $\Delta/\varepsilon \equiv \zeta = const$. To remove the additional divergences from the graphs of the 1PI function $\Gamma_{v'v'}$ renormalization of the amplitude D_0 in the nonlocal correlation function of the random force (3.11) and (3.12) was used in Ref. [144]. The renormalization

scheme of Ref. [144] is not internally consistent, however. This is not obvious in the one-loop approximation, but becomes apparent already in the two-loop approximation [153].

In the (ε ,Δ) scheme (3.22) the multiplicative renormalization [144] of the amplitude D_0 in Eq. (3.12) is not acceptable. The reason is that the counterterm with structure (3.12) is nonlocal $\sim k^{4-d-2\varepsilon} = k^{2-2\Delta-2\varepsilon}$ on rays (3.22).

Guided by the general theory of the UV renormalization, the authors of Ref. [146] put forward another scheme, in which a local counterterm $\sim k^2$ instead of the nonlocal one $\sim k^{2-2\Delta-2\varepsilon}$ is used to absorb singularities from the graphs of the 1PI function $\Gamma_{v'v'}$. This corresponds to addition of the term $\sim v'\boldsymbol{\nabla}^2 v'$ to the action functional. In functional (3.18) with the correlation function D from Eqs. (3.11) and (3.12) there is no such term, so that upon the addition of the term $\sim v'\boldsymbol{\nabla}^2 v'$ the renormalization ceases to be multiplicative. This would be unessential, if our only goal was the elimination of divergences from Green's functions which is quite possible by a non-multiplicative renormalization. For the use of the standard technique of the RG multiplicative renormalization is, however, necessary. This is why the authors of Ref. [146] proposed to consider a two-charge model in which to function (3.12) $\sim k^{4-d-2\varepsilon} = k^{2-2\Delta-2\varepsilon}$ the term $\sim k^2$ is added at the outset with an independent coefficient:

$$d_f(k) = D_{10}k^{2-2\Delta-2\varepsilon} + D_{20}k^2 = g_{10}\nu_0^3\, k^{2-2\Delta-2\varepsilon} + g_{20}\nu_0^3\, k^2\,. \tag{3.23}$$

Here, the amplitude D_0 of Eq. (3.12) is denoted by D_{10}. The parameters g_{10} and g_{20} introduced in Eq. (3.23) play the role of two independent bare charges.

We investigate the model in $d = 2 + 2\Delta$ dimensions regarding ε and Δ as small parameters of a regular expansion. The renormalized action is

$$\mathcal{S}_R = \frac{1}{2}g_1\nu^3\mu^{2\varepsilon}v'Dv' + \frac{1}{2}Z_{D_2}g_2\nu^3\mu^{-2\Delta}\partial_i v'_j\partial_i v'_j + \boldsymbol{v}'\cdot[-\partial_t\boldsymbol{v} + Z_\nu\nu\nabla^2\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}], \tag{3.24}$$

where μ is the scaling-setting parameter (or renormalization mass). Note that in this work we always interpret μ in this way. Further, the renormalized parameters ν and g are defined by $\nu_0 = \nu Z_\nu$ and $g_{10} = \mu^{2\varepsilon}g_1 Z_{g_1}$, $g_{20} = \mu^{-2\Delta}g_2 Z_{g_2}$. As usual, the renormalized coupling constants g are chosen to be both spatially and temporally dimensionless. The non-local term of the action (3.24) is not renormalized, therefore the renormalization constants Z_{g_1} and Z_ν, are related as

$$Z_{g_1} = Z_\nu^{-3}. \tag{3.25}$$

We have introduced a new parameter g_2 with the canonical scaling dimensions $d^k_{g_{20}} = 2 - d = -2\Delta$, $d^\omega_{g_{20}} = 0$, $d_{g_{20}} = -2\Delta$ to account for the new divergences at two dimensions. This is necessary, because the counterterms from the graphs are always local in space and time, and therefore cannot be taken into account by renormalization of the parameters of the initial action, in which the $v'v'$ term has a non-local kernel. This amendment is crucial to the correct renormalization of the model. This subtle point has caused confusion on several other occasions [142, 143, 156, 157]. The anomalous asymptotic behavior in the present problem arises from the singularities of the perturbation expansion at small momenta and frequencies. The region of large momenta is assumed to have a physical ultraviolet cutoff parameter Λ inversely proportional to the typical microscopic length of the problem. Here, the cutoff is most conveniently implemented

in the correlation function of the random force, which therefore is a rapidly decaying function at large momenta. For instance, the substitution

$$k^{4-d-2\varepsilon} \to k^{4-d-2\varepsilon} \mathrm{e}^{-k^2/\Lambda^2} \tag{3.26}$$

would suffice. It is not difficult to see by power counting that there are no singularities in the infrared limit, when $\varepsilon < 0$ and $\Delta > 0$ and the ultraviolet-regularized perturbation expansion may be used as it stands. If $\varepsilon \geq 0$ or $\Delta \leq 0$ small-momentum singularities do occur. At present, there is no way to treat these infrared singularities of the perturbation expansion in a consistent manner, except for the logarithmic case $\varepsilon = \Delta = 0$, in which the analysis can be entirely transferred to the analysis of ultraviolet divergences. By the scale transformation $\boldsymbol{k} \to s\boldsymbol{k}$, $\omega \to s^2\omega$ of all the momenta and frequencies the analysis of the behavior of the perturbation expansion at small momenta and frequencies, i.e. in the limit $s \to 0$, may be transferred to the large momentum limit of the momentum integrals of the perturbation expansion, since the cutoff becomes Λ/s and there is no other s-dependence left in these integrals. However, the coupling constants scale as $g_{10} \to g_{10}s^{-2\varepsilon}, g_{20} \to g_{20}s^{2\Delta}$, and the effective expansion parameters $g_{10}s^{-2\varepsilon}$, $g_{20}s^{2\Delta}$ become large in the limit $s \to 0$, when ($\varepsilon > 0$ or $\Delta < 0$. In general, the effect of these singular terms cannot be estimated in any consistent manner. However, in the logarithmic case $\varepsilon = \Delta = 0$, and the coupling constants remain fixed. Then, by the standard procedure of the field-theoretic renormalization [65, 75, 86, 130] the ultraviolet singularities of the integrals may be absorbed in a redefinition of the parameters of the model. The renormalization relevant to the anomalous asymptotic behavior must therefore be carried out at the critical values of the parameters $\varepsilon = 0$, $d = 2$, which correspond to the logarithmic model. The model cannot be consistently renormalized at any other values of $\varepsilon > 0$ and $d < 2$. This is a somewhat dangerous point, since the inconsistencies do not show in any way in the one-loop calculations, which are usually performed, although they become obvious already in two-loop order [153]. When the model is renormalized at $d = 2$, the two terms $\propto v'v'$ of the action (3.24) are indistinguishable: both are of the form $const \times \int \mathrm{d}\omega \int \mathrm{d}^d\boldsymbol{k}\, v_i'(\omega, \boldsymbol{k})k^2 v_i'(-\omega, -\boldsymbol{k})$ and the question arises, which one of them should be renormalized? To answer this question, let us look at the renormalization procedure at $d > 2, \varepsilon < 0$. In this case the terms of the ultraviolet-regularized perturbation expansion do not contain any infrared divergences and there is no anomalous asymptotic behavior brought about by the higher order terms. We may, of course, carry out renormalization of the model also in this case in the usual way, which means that we perform the first steps of the gradient expansion of the perturbation expansion, and add all the higher order perturbative contributions to the coefficients of the corresponding terms of the action (3.24). The renormalization constants obtained are finite and do not give rise to any anomalous asymptotic behavior of the Green functions. The important feature here is that the gradient expansion produces only terms, the Fourier transforms of which are polynomial functions of the momenta, at least if the large-momentum cutoff is chosen wisely to be smooth enough.

Therefore, the non-local term $v'k^{4-d-2\varepsilon}v'$ is not renormalized at all in this case, when it is not a polynomial of the momentum and thus it is clearly distinguishable from the local term $v_i'k^2v_i'$. When we go to the limit $\varepsilon \to 0$, $d \to 2$, the renormalization constants become singular reflecting the infrared divergences of the integrals discussed above. Therefore, it is reasonable to prescribe all the contributions $\propto v'v'$ produced by the renormalization to the local term $\partial_i v'\partial_i v'$ also in the limit $\varepsilon \to 0$, $d \to 2$. Another argument to support this renormalization prescription is

obtained, when dimensional and analytic regularization are used. When the renormalization constants are calculated with ε and Δ as regulators, the long-range term stands in the non-analytic form $v'k^{2-2\varepsilon-2\Delta}v'$, therefore the polynomial in k^2 counterterms do not renormalize it, but the short-range term $v'k^2v'$. It should be noted that this local term is always produced by the renormalization, but it is irrelevant for $d > 2$. To keep the model multiplicatively renormalizable, the local term with an independent coupling constant should be added to the original action at the outset.

3.3.1 Renormalization-group equations and fixed points

From the connection between the renormalized and unrenormalized generating functionals $\mathcal{G}_R(g_1, g_2, \nu, \mu) = \mathcal{G}(g_{10}, g_{20}, \nu_0)$ (there are enough parameters available, no field renormalization is needed here) we obtain the basic RG equation

$$\mathcal{D}_{RG}\mathcal{W}_R \equiv [\mathcal{D}_\mu + \beta_1\partial_{g_1} + \beta_2\partial_{g_2} - \gamma_\nu\mathcal{D}_\nu]\mathcal{W}_R = 0 \tag{3.27}$$

for the generating functional $\mathcal{W}_R = \ln \mathcal{G}_R$ of the connected renormalized Green functions.

To illustrate the idea of asymptotic analysis, we consider the equal-time velocity-velocity correlation function $G_{ij}(\boldsymbol{x} - \boldsymbol{x}') = \langle v_i(t, \boldsymbol{x})v_j(t, \boldsymbol{x}')\rangle$. It is convenient to express the Fourier transform of the correlation function

$$\langle v_i(t, \boldsymbol{x})v_j(t, \boldsymbol{x}')\rangle \equiv G_{ij}(\boldsymbol{r}), \qquad \boldsymbol{r} \equiv \boldsymbol{x} - \boldsymbol{x}' \tag{3.28}$$

in the form

$$G_{ij}(\boldsymbol{p}) = P_{ij}(\boldsymbol{p})G(p), \tag{3.29}$$

where $P_{ij}(\boldsymbol{p})$ is the transverse projection operator and $p \equiv |\boldsymbol{p}|$. By dimensional arguments the scalar function $G(p)$ can be expressed as

$$G(p) = \nu^2 p^{-d+2}R(s, g), \quad s \equiv p/\mu, \quad g \equiv (g_1, g_2), \tag{3.30}$$

where R is a scaling function of dimensionless arguments. Introduce a set of invariant parameters $\bar{e}(s) = (\bar{\nu}(s), \bar{g}_1(s), \bar{g}_2(s))$ corresponding to the set of renormalized parameters $e = (\nu, g_1, g_2)$ as solutions fixed bare parameters e_0. In terms of invariant parameters the correlation function assumes the form

$$G(p) = \nu^2 p^{2-d}R(s, g) = \bar{\nu}^2 p^{2-d}R(1, \bar{g}). \tag{3.31}$$

Equation (3.31) is valid because both sides of it satisfy the RG equation and coincide at $s = 1$ owing to the normalization of the invariant parameters. The right-hand side of (3.31) depends on s through the invariant parameters $\bar{e}(s, e)$. They have simple asymptotic behavior as $s \to 0$, which is governed by the infrared-stable fixed point: the invariant charges $\bar{g}$ tend to the fixed-point values $g^* = \mathcal{O}(\varepsilon)$ and the invariant coefficient of viscosity $\bar{\nu}$ exhibits simple power-law behavior. To determine the latter it is convenient to express the invariant parameters $\bar{e} = (\bar{\nu}, \bar{g}_1, \bar{g}_2)$ in

terms of the bare variables $e_0 = (\nu_0, g_{10}, g_{20})$ and the wave number p. Due to definition the bare variables e_0 also satisfy the RG equation by relations

$$\nu_0 = \bar{\nu} Z_\nu(\bar{g}), \qquad g_{10} = \bar{g}_1 p^{2\varepsilon} Z_{g_1}(\bar{g}), \quad g_{20} = \bar{g}_2 p^{-2\Delta} Z_{g_2}(\bar{g}). \tag{3.32}$$

Relations (3.32) are valid because both sides of them satisfy the RG equation, and because relations (3.32) at $s \equiv p/\mu = 1$ coincide with their counterparts in (3.55) owing to the normalization conditions. Using the connection $Z_g Z_\nu^3 = 1$ between the renormalization constants defined in (3.55), and eliminating these constants from the first two expressions in (3.32) we find $g_{10}\nu_0^3 = D_{10} = \bar{g}_1 p^{2\varepsilon}\, \bar{\nu}^{\,3}$, from which it follows that

$$\bar{\nu} = (D_{10} p^{-2\varepsilon} / \,\bar{g}_1)^{1/3}\,. \tag{3.33}$$

In the limit $\bar{g}_1 \to g_1^*$ the sought asymptotic behavior of the invariant coefficient of viscosity as $s \to 0$ thus assumes the form

$$\bar{\nu} \to \bar{\nu}^* = (D_{10}/g_1^*)^{1/3} p^{-2\varepsilon/3}, \qquad s \to 0. \tag{3.34}$$

Substituting this result into (3.31) we obtain the relation

$$G(p) \simeq (D_{10}/g_1^*)^{2/3} p^{2-d-4\varepsilon/3} R(1, g^*), \qquad s \to 0 \tag{3.35}$$

describing the large-scale asymptotic behavior of the pair correlation function.

For the physical values of the parameters $\Delta = 1/2, \varepsilon = 2$, chosen from the condition that the dimensional parameters of the model are viscosity and energy injection rate, the scaling behavior of the equal-time correlation function G in the three-dimensional space corresponds to the Kolmogorov scaling $G(p) \sim p^{-11/3}$ [62, 63]. The scaling form (3.35) yields the large-scale asymptotic behavior of the original correlation function, if the fixed point is infrared stable, i.e. if $\overline{g}_1 \to g_1^*$, $\overline{g}_2 \to g_2^*$, when $p \to 0$.

One-loop calculation in the ray scheme yields the following expressions for the β functions:

$$\beta_1 = g_1' \left(2\Delta - 3g_2' - 3g_1'\right), \quad \beta_2 = g_2' \left(2\varepsilon - \frac{{g_1'}^2}{g_2'} + g_1' + 2g_2'\right), \tag{3.36}$$

where $g_2' = g_2/(32\pi)$, $g_1' = g_1/(32\pi)$. Three fixed points are determined by the system of equations $\beta_1(g_1^*, g_2^*) = \beta_2(g_1^*, g_2^*) = 0$.

From relation (3.31) near a fixed point it follows that the fixed point is infrared stable, when the matrix Ω_{ij} defined in Eq. (2.24) is positively definite at the fixed point. The trivial fixed point: ${g_1'}^* = {g_2'}^* = 0$ is infrared stable only if $\Delta > 0$, $\varepsilon < 0$. For the nontrivial fixed point ${g_1'}^* = 0$, ${g_2'}^* = -\Delta$ the region of stability is determined by the inequalities $\Delta < 0, 2\varepsilon < -3\Delta$. This fixed point corresponds to the fixed point for the model A of Forster, Nelson and Stephen [61] and yields the asymptotic behavior of the solution of the Navier-Stokes equation with short-range correlated random force. Physically, this corresponds to the effect of thermal fluctuations.

The anomalous asymptotic behavior of the long-range model above two dimensions is governed by the third fixed point

$${g_1'}^* = \frac{2}{9}\,\frac{\varepsilon(3\Delta + 2\varepsilon)}{\Delta + \varepsilon}\,, \qquad {g_2'}^* = \frac{2}{9}\,\frac{\varepsilon^2}{\Delta + \varepsilon}\,, \tag{3.37}$$

at which the Ω matrix is

$$\Omega = \begin{pmatrix} \frac{2(2\varepsilon^2+3\varepsilon\Delta)}{3(\varepsilon+\Delta)} & \frac{2(2\varepsilon^2+3\varepsilon\Delta)}{3(\varepsilon+\Delta)} \\ -\frac{2(2\varepsilon^2+3\varepsilon\Delta)}{3(\varepsilon+\Delta)} & \frac{2(3\Delta^2+4\varepsilon\Delta+2\varepsilon^2)}{3(\varepsilon+\Delta)} \end{pmatrix} \tag{3.38}$$

with the eigenvalues

$$\Omega_{1,2} = \frac{1}{3}[3\Delta + 4\varepsilon \pm \sqrt{9\Delta^2 - 12\Delta\varepsilon - 8\varepsilon^2}] \tag{3.39}$$

It should be noted that the fixed point ${g_1'}^* \neq 0$, ${g_2'}^* \neq 0$ is unique due to the degeneracy of the β functions in the ray scheme.

The inequalities $\Delta + (2/3)\varepsilon > 0, \varepsilon > 0$ determine the basin of attraction of the fixed point (3.37), in which the present results may be compared with those of the RG analysis above two dimensions [62, 63]. The anomalous dimension γ_ν^* is a continuous function of the parameters Δ and ε at the borderlines of the basins of attraction on the (Δ, ε)-plane.

3.4 Improved ε expansion in the RG analysis of turbulence

In the description of developed turbulence in the framework of the stochastic Navier-Stokes equation representation of the problem in the form of an effective field-theoretic model opens new possibilities for understanding of both the Kolmogorov scaling and the deviation therefrom as well as for calculation of principal physical quantities. The RG approach with its various perturbation schemes (e.g. the famous ε expansion [9]), with effective analytical and numerical algorithms for evaluation of quantities involved forms a robust method, which allows to put into practice this scenario.

A specific feature of the renormalization-group approach and the ε expansion in the theory of developed turbulence is that the formal small parameter ε is not connected with the space dimension and it is determined only by the noise correlator of random forcing in the stochastic Navier-Stokes equation [62, 63]. Its physical value $\varepsilon = 2$ is not small [32, 71], hence reasonable doubts arise about the effectiveness of such an expansion. For some paramount physical quantities like the critical dimensions of the velocity field and the coefficient of viscosity the ε expansion terminates at the first term due to the Galilei invariance of the theory [62, 63]. Therefore, exact values are predicted for these quantities. However, there are other physically important quantities, viz. the skewness factor, the Kolmogorov constant and critical dimensions of various composite operators, for which the ε series do not terminate [152, 153, 161], therefore the question about the effectiveness of the expansion remains open.

Consider a quantity A calculated at the fixed point of the RG in the renormalized field theory of developed turbulence. In d dimensions it is a function of the parameters ε and d: $A = A(\varepsilon, d)$. In practice, calculations are often carried out in the ε expansion, whose coefficients for the quantity $A(\varepsilon, d)$ depend on the space dimension d

$$A(\varepsilon, d) = \sum_{k=0}^{\infty} A_k(d)\varepsilon^k. \tag{3.40}$$

Analysis shows that these coefficients $A_k(d)$ have singularities at small dimension $d \leq 2$. The singularity at $d = 2$ – the nearest to the physical value $d = 3$ – gives rise to new divergences

as $d \to 2$ not eliminated by the renormalization of the d-dimensional theory [144, 146, 153]. These divergences manifest themselves in the form of poles in the parameter $(d-2) \equiv 2\Delta$ in the coefficients of the ε expansion $A_k(d)$, which therefore may be expressed as Laurent series of the form

$$A_k(d) = \sum_{l=0}^{\infty} a_{kl}\, \Delta^{l-k}. \tag{3.41}$$

A two-loop calculation of the Kolmogorov constant and skewness factor at various values of space dimension d carried out in [161] has shown that at $d = 3$ the relative part of the two-loop contribution is just large – it is of order 100% in comparison with the one-loop contribution. The two-loop contribution, however, rapidly decreases as d increases, and at $d = 5$ it gives only 30 %, and at $d \to \infty$ decreases to 10 %. On the contrary, when the space dimension decreases from $d = 3$ to $d = 2$ rapid growth of the two-loop correction term is observed. This growth is due to diagrams which contain singularities at $d = 2$. Analysis has shown that it is just these diagrams which form the main part of two-loop contribution at $d = 3$. Therefore, the nearest singularity strongly manifests itself at the realistic value $d = 3$ and allows to improve the ε expansion by means of summation of singular contributions in all orders of this expansion [152, 153].

Divergences in Δ may be absorbed to suitable additional counterterms, which gives rise to a different renormalized field theory (the physical unrenormalized field theory is, of course, the same in both cases). Henceforth, we therefore consider the theory with two formal small parameters ε and Δ, which satisfy the relation $\zeta \equiv \Delta/\varepsilon = const$, and then construct a new ε expansion proposed by Honkonen and Nalimov [146], an alternative to the expansion (3.40)

$$A(\varepsilon, \zeta) = \sum_{k=0}^{\infty} b_k(\zeta)\varepsilon^k\,, \qquad \zeta \equiv \Delta/\varepsilon\,. \tag{3.42}$$

Relation between the two expansions (3.40) and (3.42) of the quantity A becomes evident upon substitution of the Laurent series (3.41) into the ε expansion (3.40), which gives rise to the double series

$$A(\varepsilon, d) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl}\, \frac{\Delta^l}{\zeta^k}. \tag{3.43}$$

To clarify the connection between representations (3.40), (3.42) and (3.43), consider Figure 3.2. The point (k, l) corresponds to a term with the coefficient a_{kl} in the double sum (3.43). Thus, all points of the first quadrant in the k, l plane correspond to the full double sum (3.43). In the usual ε expansion (3.40) the index of summation labels columns in the graphical representation of the double sum (3.43). In the alternative expansion (3.42) the index of summation labels rows of points in Figure 3.2 (see [153] and also [154]). Therefore, the coefficients b_k are expressed as

$$b_k(\zeta) = \sum_{l=0}^{\infty} a_{lk}\, \zeta^{k-l}. \tag{3.44}$$

An n-loop calculation in the usual scheme leads to the approximate expression

$$A^{(n)}_{\varepsilon,\, d} \equiv \sum_{k=0}^{n-1} A_k(d)\varepsilon^k \tag{3.45}$$

for the quantity sought, which corresponds to the inclusion of all terms of the double sum in the first n vertical bands shown in Figure 3.2. Calculation in the alternative scheme gives rise to the approximation

$$A^{(n)}_{\varepsilon,\,\zeta} \equiv \sum_{k=0}^{n-1} b_k(\zeta)\varepsilon^k, \tag{3.46}$$

which includes all terms in n horizontal bands.

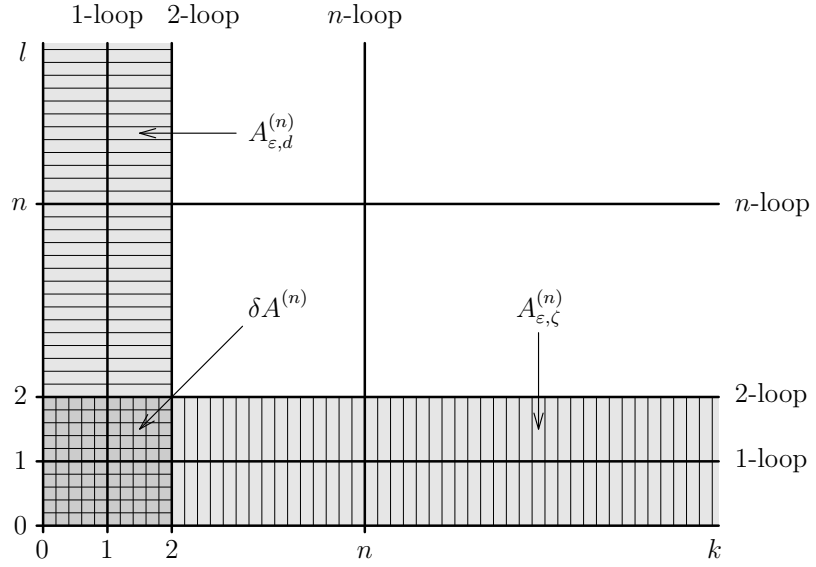


Figure 3.2. Illustration of the subsequences of the double sum (3.43) summed in the calculation in the usual ε expansion (3.40) and in the alternative expansion (3.42) Terms in the double sum (3.43) taken into account in the approximations $A^{(n)}_{\varepsilon,\,d}$ (3.45) and $A^{(n)}_{\varepsilon,\,\zeta}$ (3.46) at the two-loop order correspond to the shaded horizontal and vertical stripes, respectively. The correction term $\delta A^{(n)}$ corresponds to the double sum in (3.47) over the double-shaded square.

The ε expansion may be improved by the use of the complementary information about the quantity A contained in the finite sums (3.45) and (3.46). This is carried out by means of the approximation

$$A^{(n)}_{eff} = A^{(n)}_{\varepsilon,\,d} + A^{(n)}_{\varepsilon,\,\zeta} - \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} \zeta^{-k} a_{kl}\,\Delta^l\,, \tag{3.47}$$

which includes all terms in n vertical and horizontal bands simultaneously ("n region") and also eliminates the double counting in the overlap region of these bands. These calculations have led to notable improvement of the agreement of calculated values of the Kolmogorov constant and skewness factor with their experimental values [152, 153] in comparison with the results of

the usual ε expansion [161]. We recall that calculations in [152, 153, 161] were carried out in two-loop order.

The MS ray scheme was used in [152, 153]. It turns out that even better numerical performance may be obtained with the use of the scheme with a normalization point, in which the renormalization constants are normalized by prescribing to suitable renormalized correlation functions finite values at given wave numbers. An n-loop calculation in such a scheme (*without* expansion in ε) guarantees true reproduction of terms from the n region and leads to good agreement with experiment already in the one-loop approximation. From the point of view of renormalization of the d-dimensional theory this approach corresponds to the inclusion in the action (see below) of an infrared irrelevant at $d > 2$ operator, which becomes relevant, when $d \to 2$.

3.4.1 Renormalization with inclusion of divergences at $d = 2$

We shall not dwell on the detailed analysis of the rather cumbersome results of the two-loop calculation of the amended double expansion in the MS scheme [152, 153]. Instead, we present main steps of this analysis in the case of an NP scheme following [154]. The only difference between the two cases is in the renormalization schemes, the rest of the analysis is completely analogous in both cases.

We start the analysis from the stochastic Navier-Stokes model in d dimensions (3.18), in which, however, to the energy pumping kernel (3.12) we prefer the more realistic function

$$d_f(k) = D_0\, k^{4-d-2\varepsilon}\, h(m/k), \qquad h(0) = 1, \tag{3.48}$$

where $m = 1/L$ is the reciprocal to the integral turbulence scale L and $h(m/k)$ is some well-behaved function that provides the infrared regularization. Its specific form is inessential and we shall always use the sharp cutoff

$$h(m/k) = \theta(k - m), \; . \tag{3.49}$$

Choice (3.49) is the most convenient from the point of view of calculation of graphs of perturbation theory.

Model (3.18) is logarithmic (i.e. the coupling constant g_0 is dimensionless) at $\varepsilon = 0$, and the ultraviolet (UV) divergences appear in the form of the poles in ε in the correlation functions of the fields $\{v, v'\}$. The standard analysis of canonical dimensions supplemented by arguments of Galilei invariance shows that for $d > 2$ superficial UV divergences in the model (3.18) are present only in the one-irreducible function $\langle v'v\rangle_{1-\mathrm{ir}}$ (notation introduced in (2.10) is employed) and the compensating counterterm enters in the form $v'\partial^2 v$. In the special case $d = 2$ a new UV divergence appears in the one-irreducible function $\langle v'v'\rangle_{1-\mathrm{ir}}$.

The inclusion of the counterterm $v'\boldsymbol{\nabla}^2 v$ in the action (3.18) is tantamount to multiplicative renormalization of the parameters ν_0 and g_0 :

$$\nu_0 = \nu Z_\nu, \qquad D_0 = g_0\nu_0^3 = g\mu^{2\varepsilon}\nu^3, \qquad g_0 = g\mu^{2\varepsilon}Z_g, \qquad Z_g = Z_\nu^{-3}. \tag{3.50}$$

The renormalization constant of viscosity Z_ν is the only independent renormalization constant in (3.50). Relation between the renormalization constants of the coupling constant and coefficient

of viscosity shown in (3.50) follows from the fact that the amplitude of the correlator of the random force D_0 is not renormalized [no renormalization constant is needed for the term $v'v'$ in action (3.18)]. The quantity μ in (3.50) is the scale setting parameter, ν is the renormalized viscosity and g is the dimensionless renormalized coupling constant (charge).

An RG analysis of the model (3.18) shows the presence of an IR-stable fixed point $g^*{\sim}\varepsilon$ at small $\varepsilon > 0$, which governs the infrared scaling. The parameters of this scaling – critical dimensions and universal scaling functions – are calculated in the form of series in ε. Due to the Galilei invariance of the theory the series for critical dimensions of fields Δ_v , $\Delta_{v'}$ and the frequency Δ_ω are terminated at the first order

$$\Delta_v = 1 - 2\varepsilon/3, \qquad \Delta_{v'} = d - 1 + 2\varepsilon/3, \qquad \Delta_\omega = 2 - 2\varepsilon/3. \tag{3.51}$$

These formulas are exact without higher order corrections with respect to ε. They follow from relation (3.50) between the renormalization constants Z_g and Z_ν, which, in turn, is a consequence of the absence of renormalization of the non-local term in (3.18). At the realistic value $\varepsilon = 2$ quantities (3.51) assume the Kolmogorov values

$$\Delta_v = -1/3, \qquad \Delta_\omega = 2/3. \tag{3.52}$$

Critical dimensions (3.51) are free from singularities at $d \to 2$ referred to in section 3.2. However, other physical quantities like the skewness factor, the Kolmogorov constant and critical dimensions of various composite operators strongly depend on the space dimension. Contrary to (3.51), these quantities are expressed in the form of infinite series of types (3.40), (3.41), (3.42). The following sections will be devoted to the analysis of the renormalization of the model at the exceptional space dimension $d = 2$ and to the construction of a renormalization scheme which provides for improved ε expansions of the aforementioned quantities.

As it was already emphasized, in model (3.18) additional UV divergences appear in the one-irreducible function $\left\langle v'v' \right\rangle_{1-\mathrm{ir}}$ as $d \to 2$. In particular, these divergences manifest themselves in the form of singularities in $\Delta = (d-2)/2$ in coefficients of the usual ε-expansion (3.40). They can be eliminated by addition of a counterterm of the form $v'\boldsymbol{\nabla}^2 v'$ to the action. This counterterm is local and quite different from the nonlocal contribution $v'Dv'/2$ to action (3.18). To restore the (technically convenient) multiplicative renormalization of the model, the authors of [146] have proposed to pass to the two-charge model (i.e. with two coupling constants) in which to the function (3.12) ${\sim}k^{4-d-2\varepsilon} = k^{2-2\Delta-2\varepsilon}$ the term ${\sim}k^2$ with an independent coefficient is added:

$$d_f(k) = D_{10}k^{2-2\Delta-2\varepsilon} + D_{20}k^2, \quad D_{i0} = g_{i0}\nu_0^3, \;\; i = 1, 2\,. \tag{3.53}$$

Here, the amplitude D_0 from (3.12) is now denoted by D_{10}. The parameters g_{10} and g_{20} introduced in (3.53) play the rôle of two independent bare charges. The noise correlation function chosen for the velocity field reflects the detailed intrinsic statistical definition of forcing, whose consequences are thoroughly discussed in Ref. [162]. Correlation functions of noise with long-range and short-range terms include two principal — low and high wave number scale — kinetic forcings separated by a transition region at the vicinity of the characteristic wavenumber of order $\mathcal{O}([g_{v10}/g_{v20}]^{\frac{1}{5}})$. In the language of classical hydrodynamics the forcing contribution $\propto k^2$ corresponds to the appearance of large eddies convected by small and active eddies.

As before, the unrenormalized action is taken in the form (3.18), but now instead of the injection function (3.12) the function (3.53) is used in function (3.48); thus the unrenormalized action of the extended model is

$$\mathcal{S} = \frac{1}{2} v'(D_{10} k^{2-2\Delta-2\varepsilon} + D_{20} k^2) v' + \boldsymbol{v}' \cdot [-\partial_t \boldsymbol{v} + \nu_0 \boldsymbol{\nabla}^2 \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{\nabla}) \boldsymbol{v}] \,. \tag{3.54}$$

Relations between renormalized and bare parameters are expressed by the formulas

$$\begin{aligned} &D_{10} = g_{10} \nu_0^3 = g_1 \mu^{2\varepsilon} \nu^3, \qquad && D_{20} = g_{20} \nu_0^3 = g_2 \mu^{-2\Delta} \nu^3 Z_{D_2}, \qquad && \nu_0 = \nu Z_\nu, \\ &g_{10} = g_1 \mu^{2\varepsilon} Z_{g_1}, && g_{20} = g_2 \mu^{-2\Delta} Z_{g_2}, \end{aligned} \tag{3.55}$$

with two independent renormalization constants for the coefficient of viscosity ν_0 and for the amplitude D_{20}. The amplitude D_{10} of the non-local term of the correlator of the random force is not renormalized. Therefore two additional relations follow

$$Z_{g_1} Z_\nu^3 = 1, \qquad Z_{g_2} Z_\nu^3 = Z_{D_2}. \tag{3.56}$$

The independent renormalization constants Z_ν and Z_{D_2} are found from the condition that the one-irreducible functions $\langle v'v \rangle_{1-\mathrm{ir}}\big|_{\omega=0}$ and $\langle v'v' \rangle_{1-\mathrm{ir}}\big|_{\omega=0}$ are UV finite (are free from poles with respect to ε at $\Delta/\varepsilon = const$). In [146] the renormalization constants are calculated in the minimal subtraction scheme (MS) [only poles with respect to ε, $\Delta = \mathcal{O}(\varepsilon)$ and their linear combinations are subtracted]. To reach our goal a different subtraction scheme with a normalization point (NP) is more suitable. To this end, introduce normalized scalar one-irreducible functions as

$$\Gamma_{v'v} = \frac{\langle v_i' v_i \rangle_{1-\mathrm{ir}}\big|_{\omega=0}}{\nu p^2 (1-d)}, \quad \Gamma_{v'v'} = \frac{\langle v_i' v_i' \rangle_{1-\mathrm{ir}}\big|_{\omega=0}}{\nu^3 p^2 (d-1)} - g_1 (\mu/p)^{2\Delta+2\varepsilon} - g_2 \tag{3.57}$$

and determine the renormalization constants from the conditions

$$\Gamma_{v'v}\Big|_{\substack{p=0,\\ \mu=m}} = 1 \,, \qquad \Gamma_{v'v'}\Big|_{\substack{p=0,\\ \mu=m}} = 0 \,. \tag{3.58}$$

Note that these normalization conditions are different from those used in [152, 153].

Application of the RG approach to the renormalized model specified by the condition (3.58) leads to the same ε, Δ expansions in the ray scheme as in [146]. Instead, to exploit the scheme dependence of perturbation theory, we propose to use the renormalization scheme (3.58) *without* the ε, Δ expansion. Such a scheme reproduces correctly the leading terms of expansion in both regimes $\varepsilon \to 0 \,, \Delta = const$ and $\varepsilon \sim \Delta \to 0$ simultaneously. For all that, in the first case, the additional term $v' \boldsymbol{\nabla}^2 v'$ is an infrared irrelevant operator in the action and the renormalization constant Z_{D_2} describes the renormalization of this additional operator.

One-loop calculation of these renormalization constants yields

$$Z_\nu = 1 + \frac{d-1}{4(d+2)} \Big(- \frac{u_1}{2\varepsilon} + \frac{u_2}{2\Delta} \Big) , \tag{3.59}$$

$$Z_{D_2} = 1 + \frac{d^2-2}{4d(d+2)} \Big(- \frac{u_1^2}{2(2\varepsilon+\Delta) u_2} - \frac{u_1}{\varepsilon} + \frac{u_2}{2\Delta} \Big) , \tag{3.60}$$

where instead of g_1 and g_2 more convenient charges u_1 and u_2 are used:

$$u_1 \equiv \bar{S}_d g_1\,, \quad u_2 \equiv \bar{S}_d g_2\,, \tag{3.61}$$

where

$$\bar{S}_d \equiv S_d/(2\pi)^d, \quad S_d \equiv 2\pi^{d/2}/\Gamma(d/2) \tag{3.62}$$

are often encountered geometrical factors. Here, S_d is the surface area of the unit sphere in d-dimensional space and Γ is Euler's Gamma function. From (3.55), (3.59) and (3.60) the renormalization constants of the charges u_1 and u_2 are determined as

$$Z_{u_1} = 1 + \frac{3(d-1)}{4(d+2)}\Big(\frac{u_1}{2\varepsilon} - \frac{u_2}{2\Delta}\Big)\,, \tag{3.63}$$

$$Z_{u_2} = 1 + \frac{d^2-2}{4d(d+2)}\Big(-\frac{u_1^2}{2(2\varepsilon+\Delta)\,u_2} - \frac{u_1}{\varepsilon} + \frac{u_2}{2\Delta}\Big) + \frac{3(d-1)}{4(d+2)}\Big(\frac{u_1}{2\varepsilon} - \frac{u_2}{2\Delta}\Big)\,. \tag{3.64}$$

The corresponding RG functions are then found straightforwardly

$$\gamma_i = (\beta_1\partial_{u_1} + \beta_2\partial_{u_2})\ln Z_{u_i}, \qquad i = 1,\,2\,, \tag{3.65}$$

$$\beta_1 = -u_1(2\varepsilon + \gamma_1)\,, \qquad \beta_2 = -u_2(-2\Delta + \gamma_2)\,. \tag{3.66}$$

In the linear approximation for γ's with respect to charges u_1 and u_2 it is enough to use $\beta_1 \simeq -2u_1\varepsilon$, $\beta_2 \simeq 2u_2\Delta$, because $\gamma_1 = \mathcal{O}(u)$ and $\gamma_2 = \mathcal{O}(u)$ in (3.65). We find

$$\gamma_1 = -\frac{3(d-1)}{4(d+2)}(u_1 + u_2)\,, \tag{3.67}$$

$$\gamma_2 = \frac{(d^2-2)(u_1+u_2)^2}{4d(d+2)\,u_2} - \frac{3(d-1)}{4(d+2)}(u_1+u_2). \tag{3.68}$$

With the use of (3.66), (3.67) and (3.68), the coordinates of the nontrivial fixed point $u_1^* > 0$, $u_2^* > 0$ are found as the solution of the equations $\beta_1(u^*) = 0$, $\beta_2(u^*) = 0$ in the form

$$u_1^* + u_2^* = \frac{8\varepsilon(d+2)}{3(d-1)}\,, \tag{3.69}$$

$$u_2^* = \frac{\varepsilon^2}{\varepsilon+\Delta}\,\frac{8(d^2-2)(d+2)}{9d(d-1)^2}\,. \tag{3.70}$$

Stability of the fixed point is determined by the sign of the real part of the eigenvalues $\Omega_\pm$ of the matrix Ω:

$$\Omega_\pm = \Delta + \frac{2\varepsilon(2d^2-3d+2)}{3d(d-1)} \pm \sqrt{\Delta^2 - \frac{4(d^2-2)}{3d(d-1)}\,\varepsilon\,\Delta - \frac{4(d^2-2)(2d^2-3d+2)}{9d^2(d-1)^2}\,\varepsilon^2}\,. \tag{3.71}$$

We are interested in the region $\varepsilon > 0$, $\Delta > 0$, in which the eigenvalues $\Omega_\pm$ are positive. Therefore, the fixed point (3.69), (3.70) is infrared stable. In the regime $\varepsilon{\sim}\Delta \to 0$ with the use of (3.71) we obtain

$$\Omega_\pm = \Delta + \frac{4\varepsilon}{3} \pm \sqrt{\Delta^2 - \frac{4}{3}\varepsilon\,\Delta - \frac{8}{9}\,\varepsilon^2} + \mathcal{O}(\varepsilon^2) \tag{3.72}$$

which is consistent with the result of [146]. At $\varepsilon \to 0$, $\Delta = const$ it follows that

$$\Omega_- = 2\varepsilon + \frac{2(d^2-2)}{3d(d-1)} \cdot \frac{\varepsilon^2}{\Delta} + \mathcal{O}(\varepsilon^3)\,, \tag{3.73}$$

$$\Omega_+ = 2\Delta + \frac{2(d^2-3d+4)}{3d(d-1)}\varepsilon - \frac{2(d^2-2)}{3d(d-1)} \cdot \frac{\varepsilon^2}{\Delta} + \mathcal{O}(\varepsilon^3)\,. \tag{3.74}$$

The quantity Ω_- plays the rôle of the correction index ω in the framework of the prevailing ε expansion in the theory of developed turbulence whereas Ω_+ determines the critical exponent of the infrared-irrelevant composite operator $v'\partial^2 v'$.

The terms $\sim\varepsilon$ in (3.73) and (3.74) are reliable: for Ω_- relation (3.73 reproduces the known one-loop expression [62] for the exponent ω; for Ω_+ we have checked the result by a direct calculation of the critical dimension of the composite operator $v'\partial^2 v'$ in the usual ε expansion. Moreover, the terms $\sim\varepsilon^2/\Delta$ yield the true singular part with respect to Δ in the coefficients of ε^2: for Ω_- it was confirmed by the pioneering two-loop calculation [161], for Ω_+ we have checked it by calculation of the critical dimension of the composite operator $v'\partial^2 v'$ in two-loop approximation. Expression (3.71), in fact, correctly reproduces main singular terms of the form $\varepsilon(\varepsilon/\Delta)^k$ and all leading terms of the ε expansion, i.e. the first terms of the corresponding Laurent series (3.41).

Calculation of graphs with increasing number of loops in our renormalization scheme guarantees that results become more precise step by step in the sense that the number of true terms of the ε expansion and the number of singular in Δ contributions is increased: an n-loop calculation correctly reproduces first n of all coefficients $A_k(d)$ in (3.40), while simultaneously of the coefficients $A_k(d)$ with $k > n$ the first n terms of their Laurent series (3.41) will be reproduced correctly.

3.4.2 Calculation of the Kolmogorov constant through the skewness factor

Even in the field-theoretic RG approach several ways have been proposed [72, 148, 149, 153, 154, 161, 163] to calculate the (non-universal) amplitude factor – the Kolmogorov constant – in Kolmogorov's 5/3 law for the turbulent energy spectrum

$$E(k) \sim C'_K \overline{\mathcal{E}}^{2/3} k^{-5/3}\,, \tag{3.75}$$

where $\overline{\mathcal{E}}$ is the average energy injection rate per unit mass. The notation in (3.75) follows that of [161].

Different approximations for the connection of model parameters and the average energy injection rate lead have resulted in different values for the Kolmogorov constant. At present the most reliable approach appears to be that based on the connection between the Kolmogorov constant and the skewness factor [161]. For the latter consistent expansions both in ε and ε, Δ may be constructed and actually have been calculated in two-loop approximation. As in case of critical exponents, scheme-dependence or perturbative calculation may be used to improve the performance of regulator expansions. The renormalization scheme presented in preceding sections following [154] allows to obtain best match with experimental data up to date.

The Kolmogorov constant is not determined uniquely in the ε expansion in the model with power-law injection (3.12) (for details, see [153]). On the other hand physical quantities independent of the amplitude D_{10} (3.12) (universal quantities) are determined unambiguously in the framework of the ε expansion. The skewness factor

$$\mathcal{S} \equiv S_3/S_2^{3/2}, \tag{3.76}$$

is an example of such a quantity. In (3.76) S_n are structure functions defined by relations

$$S_n(r) \equiv \big\langle [v_r(t,\boldsymbol{x}+\boldsymbol{r}) - v_r(t,\boldsymbol{x})]^n \big\rangle, \qquad v_r \equiv (\boldsymbol{v}\cdot\boldsymbol{r})/r, \quad r \equiv |\boldsymbol{r}|. \tag{3.77}$$

According to the Kolmogorov theory, the second-order structure function $S_2(r)$ in the inertial range is of the form

$$S_2(r) = C_K \mathcal{E}^{2/3} r^{2/3}, \tag{3.78}$$

where $\mathcal{E}$ is the average energy dissipation rate per unit mass (in the steady state it coincides with the mean energy injection rate $\overline{\mathcal{E}}$, see Eq. (3.75)) and C_K is the Kolmogorov constant, the value of which is not determined in the framework of the phenomenological approach. Although there is strong experimental evidence that the Kolmogorov scaling $S_n(r) \sim r^{n/3}$ does not hold in the inertial range for the structure functions of order $n \geq 4$, for the second-order structure function $S_2(r)$ the experimental situation about anomalous scaling [i.e., deviation of the power of r from the Kolmogorov value $2/3$ in (3.35)] in the inertial range is still controversial and in any case this deviation is small [164,165]. Therefore, we shall use the Kolmogorov asymptotic expression (3.78) for the second-order structure function $S_2(r)$ in the following analysis.

The amplitude of the third-order structure function $S_3(r)$ is determined in the Kolmogorov theory exactly [70,79]:

$$S_3(r) = -\frac{12}{d(d+2)}\,\mathcal{E}\,r. \tag{3.79}$$

All these expressions together with (3.76), (3.78) allow to connect the Kolmogorov constant with the skewness factor:

$$C_K = \Big[-\frac{12}{d(d+2)\mathcal{S}} \Big]^{2/3}. \tag{3.80}$$

Among the three quantities $S_2(r)$, $S_3(r)$ and $\mathcal{S}$ only the last one has a unique well-defined ε expansion. Thus, relation (3.80) (valid only at the physical value $\varepsilon = 2$) may be used to determine C_K by means of the calculated value $\mathcal{S}(\varepsilon = 2)$.

To find the RG representation of the skewness factor (3.76) the RG representations of the functions $S_2(r)$ and $S_3(r)$ have to be determined. The function $S_2(r)$ is connected with the Fourier transform of the pair correlation function $G(p)$ by relation

$$S_2(r) = 2 \int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} G(k) \left[1 - (\boldsymbol{k}\cdot\boldsymbol{r})^2/(kr)^2\right] \{1 - \exp\left[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r})\right]\}. \tag{3.81}$$

Therefore, the RG representation of $S_2(r)$ can be specified with the aid of the RG representation (3.35). A similar RG representation can be written for the function $S_3(r)$. It is, however, more convenient to use the exact result analogous to expression (3.79)

$$S_3(r) = -\frac{3(d-1)\,\Gamma(2-\varepsilon)\,(r/2)^{2\varepsilon-3} D_{10}}{(4\pi)^{d/2}\,\Gamma(d/2+\varepsilon)}. \tag{3.82}$$

This relation clearly demonstrates that the amplitude of the structure function expressed through D_{10} has a singularity at $\varepsilon \to 2$. In this case the singularity is in the form $\sim(2-\varepsilon)^{-1}$. After the substitution of the amplitude $D_{10}\sim(2-\varepsilon)$ into (3.82) the singularity on the right-hand side of (3.82) is canceled by the node of D_{10}. This leads to a finite expression for $S_3(r)$ at $\varepsilon = 2$ which coincides with (3.79).

Relations (3.35), (3.81) and (3.82) could be used as the basis for the construction of the ε expansion of the skewness factor (3.76), but on this way there is an additional complication. The point is that the behavior $S_2(r)\sim r^{2-2\varepsilon/3}$, which is determined by power counting from (3.81) and (3.35), is valid only at $\varepsilon > 3/2$, because at $\varepsilon < 3/2$ the integral (3.81) diverges as $k \to \infty$ [it means that in this case the leading contribution to $S_2(r)$ is given by the term $\left\langle v_r^2(t,\boldsymbol{x})\right\rangle$ independent of r]. The derivative $r\partial_r S_2(r)$, however, is free from this flaw and according to (3.81) it assumes the form

$$r\partial_r S_2(r) = 2 \int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} G(k) \left[1 - (\boldsymbol{k}\cdot\boldsymbol{r})^2/(kr)^2\right] (\boldsymbol{k}\cdot\boldsymbol{r}) \sin(\boldsymbol{k}\cdot\boldsymbol{r}). \tag{3.83}$$

The integral in (3.83) converges at all values $0 < \varepsilon < 2$. On the other hand, at the physical value $\varepsilon = 2$ the amplitudes in $S_2(r)$ and $r\partial_r S_2(r)$ differ from each other only by the factor 2/3, therefore the ε expansion has been constructed for the following analogue of the skewness factor [152, 161, 166]

$$Q(\varepsilon) \equiv \frac{r\partial_r S_2(r)}{|S_3(r)|^{2/3}} = \frac{r\partial_r S_2(r)}{(-S_3(r))^{2/3}}. \tag{3.84}$$

The Kolmogorov constant and the skewness factor are expressed through the value $Q(\varepsilon = 2)$ according to (3.76), (3.78) and (3.79) by relations:

$$C_K = [3Q(2)/2]\,[12/d(d+2)]^{2/3}, \quad \mathcal{S} = -\,[3Q(2)/2]^{-3/2}. \tag{3.85}$$

Substituting expressions (3.83), (3.35) and (3.82) into (3.84) we obtain

$$Q(\varepsilon) = \left[4(d-1)/9/u_{1*}^2\right]^{1/3} A(\varepsilon)\, R(1, u_{1*}, u_{2*}), \tag{3.86}$$

where

$$A(\varepsilon) = \frac{\Gamma(2-2\varepsilon/3)\Gamma^{1/3}(d/2)\Gamma^{2/3}(d/2+\varepsilon)}{\Gamma(d/2+2\varepsilon/3)\Gamma^{2/3}(2-\varepsilon)} = 1 + \mathcal{O}(\varepsilon^2). \tag{3.87}$$

Consecutive loop calculations of the coordinates of the fixed point u_1^*, u_2^* and the scaling function $R(1, u_1^*, u_2^*)$ in (3.86) in the framework of the scheme used increase the number of true terms of the usual ε expansion in these quantities. In all higher coefficients of the ε expansion the number of true terms of their Laurent series (3.41) increases as well. The quantity $A(\varepsilon)$ is not singular at $d = 2$ therefore it is sufficient to use the corresponding part of its ε-expansion.

At the leading order the scaling function $R(1, u_1^*, u_2^*)$ in (3.86) is

$$R(1, u_1^*, u_2^*) \approx \frac{u_1^* + u_2^*}{2}. \tag{3.88}$$

Putting $A(\varepsilon) \approx 1$ we obtain in one-loop approximation

$$Q(\varepsilon) = \left[4(d-1)/9/(u_1^*)^2\right]^{1/3} \cdot \frac{u_1^* + u_2^*}{2}, \tag{3.89}$$

where for the coordinates u_1^*, u_2^* values (3.69) and (3.70) are implied. Calculating Q in (3.89) at $d = 3$ and $\varepsilon = 2$ we obtain $Q(2) \approx 1.461$. Further, using this value for calculation of the Kolmogorov constant and skewness factor (3.85) we arrive at the values $C_K \approx 1.889$ and $\mathcal{S} \approx -0.308$. The values $C_K \approx 2.01$ and $\mathcal{S} \approx -0.28$ are considered the most reliable experimental values of these quantities [167]. Therefore, the scheme which we have suggested for calculations by means of improvement of the ε expansion provides quite reasonable agreement with the experiment. It should be recalled that the prevailing ε expansion at one-loop order gives the values $C_K \approx 1.47$ and $\mathcal{S} \approx -0.45$.

The version of the RG approach used in the present paper bears certain resemblance with the well known RG method in the real space, which is widely used in the theory of critical phenomena (it is also called the "g expansion"). In the theory of phase transitions the parameter ε has the meaning of deviation from the critical space dimension (e.g, $\varepsilon = 4 - d$ for the φ^4-model). In the framework of the g expansion renormalization constants – adopted from the logarithmic theory – are calculated in the form of power series in the coupling constant g directly at $\varepsilon = 1$, i.e. at real value of the space dimension $d = 3$. The use of term "RG in real space" is just explained by this feature. In the theory of turbulence the parameter ε has different meaning and it is more pragmatic to use the term g expansion. In the framework of the g expansion calculations are notably simplified because it is much easier to calculate finite integrals in three dimensions than to calculate integrals with singularities at $d \to 4$. This is the reason why the use of the g expansion in the theory of critical phenomena has allowed to achieve better accuracy in perturbative calculations than in the usual ε expansion. However, we must keep in mind a drawback of the g expansion. While the results of the ε expansion are unambiguous and independent of the renormalization scheme, the g expansion approximates higher terms of the ε expansion, and the result of such an approximation depends, in particular, on the choice of the form of the IR regularization. In the theory of critical phenomena a natural regularization present in the model at $T - T_c \neq 0$ is usually used, but it does not eliminate the problem of ambiguity.

Here, an approach akin to the g expansion has been used to achieve a different aim. The choice of the renormalization constants from natural normalization conditions for the response

and correlation functions together with the additional renormalization of random forcing allowed to include singular in $d-2$ contributions to the coefficients in all orders of the ε expansion. This summation led to a remarkable improvement of agreement of the theoretical prediction with the experimental value of the Kolmogorov constant already in the one-loop approximation.

3.5 Advection of passive scalar

One of the most studied models related to the turbulence is an advection of passive scalar quantity [32, 73]. The main object of study are statistical properties of a field, which is coupled to the velocity field $\boldsymbol{v}$ through advective term (See Sec. 1.2.3). From general point of view there are two main approaches depending on the way $\boldsymbol{v}$ is generated:

a) field $\boldsymbol{v}$ is governed by the stochastic Navier-Stokes equation.

b) field $\boldsymbol{v}$ is considered as an random variable with prescribed properties.

In what follows, we apply the first approach, whereas the latter will be analyzed in Sec. 4.

3.5.1 Functional formulation of the passive scalar problem near two dimensions

Here, we study the problem of the advection of the passive scalar using a random velocity field generated by the stochastically forced Navier-Stokes equation [109], which has been widely used to produce stochastic velocity field with the Kolmogorov scaling behavior obtained by the use of the field-theoretic renormalization group [62, 63]. The passive scalar problem has already been treated within the RG approach of the randomly forced Navier-Stokes equation for both the local [61] and long-range [168] correlations of the random force, but without random pumping of the passive scalar, due to which the behavior of the correlation functions of the passive scalar was not addressed at all.

The velocity field in the functional-integral approach is generated by the dynamic action of the randomly forced Navier-Stokes equation (3.54), to which a contribution corresponding to the advection of the passive scalar is added. The statistical model of the advection of the passive scalar characterized by the concentration field $\theta(x)$ in the turbulent environment (see e.g. [168], [145]) is given by the stochastic differential equation

$$\partial_t\theta + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\theta - \nu_0 u_0 \boldsymbol{\nabla}^2\theta = f^\theta\,, \tag{3.90}$$

where u_0 is the inverse Prandtl number. Physically it represents the ratio between diffusion and viscosity in a liquid. The random source field f^θ is assumed to be Gaussian with zero mean and the correlation function'

$$\begin{aligned}\langle\, f^\theta(x_1) f^\theta(x_2)\,\rangle &\equiv D^\theta(x_1-x_2)\\ &= \frac{u_0^3\nu_0^3}{d-1}\delta(t_1-t_2)\int\frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^d}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\left[g_{\theta 10}\,k^{2-2\Delta-2a\varepsilon}+g_{\theta 20}k^2\right].\end{aligned} \tag{3.91}$$

Since near two dimensions a local term is generated to the correlation function of the scalar noise (describing thermal noise), we have taken the correlation function with both the long-range and local term at the outset.

A detailed analysis [168] of the renormalization of the passive scalar model has shown that there are superficial divergences in the graphs corresponding to the 1PI Green functions $\Gamma_{v'v}$, and $\Gamma_{\theta'\theta}$ in the renormalization scheme of Refs. [62, 63] applicable for space dimensions $d > 2$. We will refer to the approach of [62,63] as the standard scheme. The 1PI Green functions $\Gamma_{v'vvv}$, and $\Gamma_{\theta'\theta v}$, which could, by standard power counting, give rise to the renormalization of the nonlinear terms in the Navier-Stokes and advection-diffusion equation, are actually finite due to the Galilei invariance of the stochastic equations with temporally white noise.

The stochastic problem of the passive scalar (3.90), (3.91) gives rise to the field-theoretic action

$$\mathcal{S}_{\rm PS}[\theta,\theta',\boldsymbol{v}] = \frac{1}{2}\theta' D^{\theta}\theta' + \theta' \left[-\partial_t\theta + u_0\nu_0\boldsymbol{\nabla}^2\theta - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\theta\right] . \tag{3.92}$$

As explained above, this action containing non-analytic terms (proportional to the coupling constants g_{v10} and $g_{\theta 10}$) requires also the analytic terms (proportional to g_{v20} and $g_{\theta 20}$) in order to be multiplicatively renormalizable. All dimensional constants g_{v10}, g_{v20}, $g_{\theta 10}$ and $g_{\theta 20}$, which control the amount of randomly injected energy and mass play the role of the expansion parameters of the perturbation theory.

For convenience of further calculations the factors ν_0^3 and $\nu_0^3 u_0^3$ including the "bare" (molecular) viscosity ν_0 and the "bare" (molecular or microscopic) diffusion coefficient $\nu_0 u_0$ have been extracted.

3.5.2 Calculation of the fixed points of the renormalization group

The model given by the sum of dynamic actions (3.54) and (3.92) is renormalizable by the standard power-counting rules for $\Delta = 0$ (see (3.22) for the definition of Δ) and $\varepsilon = 0$. The divergent 1PI Green functions are $\Gamma_{vv'}$, $\Gamma_{\theta\theta'}$ as in the standard case [63] and also $\Gamma_{v'v'}$, $\Gamma_{\theta'\theta'}$ typical of $d = 2$, so that the model is multiplicatively renormalizable. The standard Feynman diagrammatic expansion can be used in a straightforward fashion. The inverse matrix of the quadratic part in the actions determines a form of the bare propagators. The propagators are presented in the wave-number-time representation, which is for the translationally invariant systems the most convenient way for doing explicit calculations. Graphical representation of propagators is depicted in Fig. 3.3 and the corresponding algebraic expressions are

$$\begin{aligned}
\Delta_{ij}^{vv'}(\boldsymbol{k},t) &= \Delta_{ij}^{v'v}(-\boldsymbol{k},-t) = \theta(t)\, P_{ij}(\boldsymbol{k})\, \mathrm{e}^{-\nu_0 k^2 t},\\
\Delta^{\theta\theta'}(\boldsymbol{k},t) &= \Delta^{\theta'\theta}(-\boldsymbol{k},-t) = \theta(t)\mathrm{e}^{-u_0\nu_0 k^2 t},\\
\Delta_{ij}^{vv}(\boldsymbol{k},t) &= \frac{1}{2}\nu_0^2\, P_{ij}(\boldsymbol{k})\left(g_{v10}\, k^{-2\varepsilon-2\Delta} + g_{v20}\right)\mathrm{e}^{-\nu_0 k^2|t|},\\
\Delta^{\theta\theta}(\boldsymbol{k},t) &= \frac{1}{2}\, u_0^2\,\nu_0^2\left(\, g_{\theta 10} k^{-2\,a\varepsilon-2\Delta} + g_{\theta 20}\right)\mathrm{e}^{-u_0\nu_0 k^2|t|}.
\end{aligned} \tag{3.93}$$

In our model there are two different interaction vertices, which are graphically depicted in Fig. 3.4, where we have included also the corresponding vertex factors. The corresponding expressions in momentum representation are

$$V_{ijl} = i(p_j\delta_{il} + p_l\delta_{ij}) \tag{3.94}$$

and

$$V_j = ik_j. \tag{3.95}$$

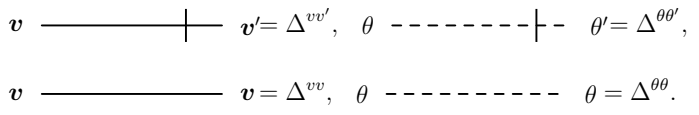


Figure 3.3. Diagrammatic representation of the bare propagators. The time flows from right to left.

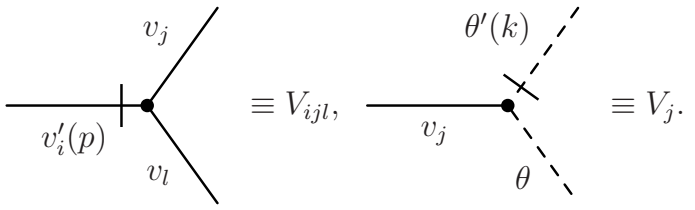


Figure 3.4. Diagrammatic representation of the interaction vertices describing an interactions between velocity components (left) and advection interaction (right).

Note that the momentum is always carried by the slashed field.

Graphical representation of the one-loop contributions to diverging 1PI functions is depicted in Fig. 3.5.

The divergences show in the form of poles in Δ, ε and their linear combinations and the ray scheme is used (which implies that Δ and ε are treated as small parameters of same order).

Renormalized Green functions are expressed in terms of the renormalized parameters

$$\begin{aligned} &g_{v1} = g_{v10}\mu^{-2\varepsilon}\, Z_1^3, && g_{v2} = g_{v20}\,\mu^{2\Delta}\, Z_1^3 Z_3^{-1}, && \nu = \nu_0 Z_1^{-1}, \\ &g_{\theta 1} = g_{\theta 10}\mu^{-2a\varepsilon-2\Delta} Z_2^3, && g_{\theta 2} = g_{\theta 20} Z_2^3 Z_4^{-1}, && u = u_0 Z_1 Z_2^{-1} \end{aligned} \tag{3.96}$$

appearing in the renormalized action. From connections (3.96) it follows that the β functions are

$$\begin{aligned} &\beta_{v1} = g_{v1}\left(-2\varepsilon + 3\gamma_1\right), && \beta_{v2} = g_{v2}\left(2\Delta + 3\gamma_1 - \gamma_3\right), && \beta_u = u\left(\gamma_1 - \gamma_2\right), \\ &\beta_{\theta 1} = g_{\theta 1}\left(-2a\varepsilon - 2\Delta + 3\gamma_2\right), && \beta_{\theta 2} = g_{\theta 2}\left(3\gamma_2 - \gamma_4\right). \end{aligned} \tag{3.97}$$

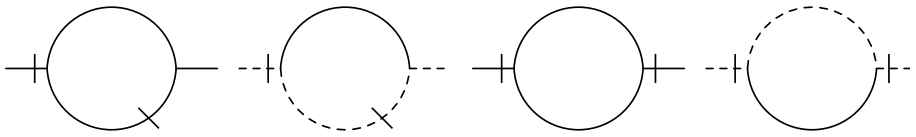


Figure 3.5. One-loop graphs giving rise to the divergent terms in the perturbative expansion of the one-particle irreducible Green functions $\Gamma_{vv'}$, $\Gamma_{\theta\theta'}$, $\Gamma_{\theta'\theta'}$, and $\Gamma_{\theta'\theta'}$, respectively. The slashes on the lines denote the derivatives appearing in the cubic interaction terms.

The function γ_4 may be expressed in the following form

$$\gamma_4 = \gamma_4'(g_{v1}, g_{v2}, u) + \frac{g_{\theta 1}}{g_{\theta 2}}\gamma_4''(g_{v1}, g_{v2}, u). \tag{3.98}$$

As a consequence, the RG functions $\beta_{\theta 1}$ and $\beta_{\theta 2}$ are linear functions of the coupling constants $g_{\theta 1}$ and $g_{\theta 2}$

$$\beta_{\theta 1} = g_{\theta 1}(-2a\varepsilon - 2\Delta + 3\gamma_2)\,, \quad \beta_{\theta 2} = g_{\theta 2}(3\gamma_2 - \gamma_4') - g_{c1}\gamma_4''\,. \tag{3.99}$$

Since γ_1, γ_2 and γ_3 functions depend on g_{v1}, g_{v2} and u only, vanishing of the first three β functions in (3.97) already yields a closed system of equations for the fixed point values of g_{v1}, g_{v2} and u.

Thus, fixed points in the model with scalar pumping are determined by the system of equations

$$g_{v1}\,(-2\varepsilon + 3\gamma_1) = 0, \quad g_{v2}\,(2\Delta + 3\,\gamma_1 - \gamma_3) = 0, \quad u\,(\gamma_1 - \gamma_2) = 0, \tag{3.100}$$

of the passive scalar model without the random scalar source.

Apart from the Gaussian fixed point $g_{v1}^* = g_{v2}^* = 0$, which is stable for $\Delta > 0$, $\varepsilon < 0$, there are two nontrivial fixed points of the RG: the fixed point corresponding to short-range correlations of the random force [61] with

$$g_{v1}^* = 0, \quad g_{v2}^* = -32\pi\Delta, \tag{3.101}$$

and the inverse Prandtl number

$$u^* = \frac{\sqrt{17} - 1}{2} \simeq 1.562. \tag{3.102}$$

The region of stability of this short-range fixed point $2\Delta + 3\varepsilon < 0$, $\Delta < 0$ is determined by the positivity of the eigenvalues $\Omega_i, i = 1, 2, 3$ of the Ω matrix

$$\Omega_1 = -2\varepsilon - 3\Delta, \quad \Omega_2 = -2\Delta, \quad \Omega_3 = -2\Delta\frac{\sqrt{17}}{\sqrt{17} + 1}. \tag{3.103}$$

The third fixed point is the *kinetic* fixed point, which is the fixed point relevant to the description of turbulent diffusion. At the kinetic fixed point the value of the renormalized inverse Prandtl number u is given by (3.102) and the values of the other relevant coupling constants

$$g_{v1}^* = \frac{64\,\pi}{9}\,\frac{\varepsilon\,(2\varepsilon + 3\Delta)}{\varepsilon + \Delta}, \quad g_{v2}^* = \frac{64\pi}{9}\,\frac{\varepsilon^2}{\Delta + \varepsilon}\,, \tag{3.104}$$

of the double expansion of the stochastic Navier-Stokes equation [146].

The calculation of the Ω matrix (up to the order $\mathcal{O}(\varepsilon, \Delta)$) at this fixed point yields the eigenvalues

$$\Omega_{1,2} = \frac{1}{3}\left[(3\Delta + 4\varepsilon) \pm \sqrt{9\Delta^2 - 12\Delta\varepsilon - 8\varepsilon^2}\right], \quad \Omega_3 = \frac{4\sqrt{17}}{\sqrt{17} + 1}\varepsilon. \tag{3.105}$$

From these expressions we see that the region of stability of the kinetic fixed point is $\varepsilon > 0$, $2\varepsilon + 3\Delta > 0$. Further analysis of this model can be found in [169].

3.6 Stochastic magnetohydrodynamics at $d \geq 2$

During the past four decades asymptotic analysis of stochastic transport equations [Navier-Stokes equation, magnetohydrodynamic (MHD) equations, advection-diffusion equation and the like] has attracted increasing attention. Somewhat less effort has been devoted to the asymptotic analysis of stochastic magnetohydrodynamics since the pioneering work of Fournier et al [142] and Adzhemyan et al [143]. In particular, in these papers the existence of two different anomalous scaling regimes (kinetic and magnetic) in three dimensions was established corresponding to two non-trivial infrared-stable fixed points of the renormalization group. It was also conjectured that in two dimensions the magnetic scaling regime does not exist due to instability of the magnetic fixed point. However, in both papers there were flaws in the renormalization of the model in two dimensions [32,146]. Even more serious shortcomings are present in investigations of MHD turbulence [170, 171], in which a specifically two-dimensional setup has been applied with the use of the stream function and magnetic potential. Therefore, results obtained for the two-dimensional case in these papers cannot be considered completely conclusive.

Here, we present a field-theoretic RG analysis of the stochastically forced equations of magnetohydrodynamics with the proper account of additional divergences which arise in two dimensions. This gives rise to a double expansion in the analytic and dimensional regulators ε and Δ, respectively. In the ray scheme of the double expansion the standard procedure of minimal subtractions was used. The RG analysis of the large-scale asymptotic behavior of the model confirms the basic conclusions of the previous analyses [142, 143] that near two dimensions a scaling regime driven by the velocity fluctuations may exist, but no magnetically driven scaling regime can occur.

Second, a renormalization of the model with a different choice of finite renormalization is performed in order to find at which non-integer dimension the magnetic fixed point ceases to be stable. This borderline dimension was found in Ref. [142] with the use of the momentum-shell RG in a setup valid in a fixed space dimension $d > 2$. In the double expansion in ε and $2\Delta = d - 2$ this effect cannot be traced at one-loop order, but higher-order calculation is required. Therefore, also a description based on the RG analysis according to the "principle of maximum divergences" is included. This procedure gives rise to RG functions such that in the limit of small Δ, ε they reproduce the results of the two-parameter expansion, and in the limit of small ε (but finite Δ) they yield the results of the usual ε expansion [142, 143]. These properties are similar to those of the normalization point schemes in the stochastic Navier-Stokes problem.

3.6.1 Field theory for stochastic magnetohydrodynamics

The model of stochastically forced conducting fluid is described by the system of magnetohydrodynamic equations for the fluctuating velocity field $\boldsymbol{v}(t, \boldsymbol{x}) \equiv \boldsymbol{v}(x)$ of an incompressible conducting fluid and the magnetic induction $\boldsymbol{B} = \sqrt{\rho\mu}\boldsymbol{b}$ (ρ is the density and μ the permeability of the fluid) in the form [142, 143]

$$\partial_t \boldsymbol{b} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{\nabla})\boldsymbol{v} - \nu_0 u_0 \nabla^2 \boldsymbol{b} = \boldsymbol{f}^b\,, \quad \boldsymbol{\nabla} \cdot \boldsymbol{f}^b = 0 \tag{3.106}$$

In (3.106) parameter $1/u_0$ is the unrenormalized magnetic Prandtl number.

The statistics of $\boldsymbol{v}$ and $\boldsymbol{b}$ are completely determined by the non-linear equations (3.9), (3.106) and the probability distribution of the external large-scale random forces $\boldsymbol{f}^v$, $\boldsymbol{f}^b$.

To analyze renormalization near two dimensions the SDE (3.106) is supplemented by the forcing statistics

$$\langle\, f_i^b(x_1) f_j^b(x_2)\,\rangle = u_0^2\,\nu_0^3\,D_{ij}\delta(t)\int \frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^d}\; P_{ij}(\boldsymbol{k})\,\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}_1-\boldsymbol{x}_2} \times \left[g_{b10}\,k^{2-2\Delta-2\,a\,\varepsilon}+g_{b20}\,k^2\,\right] \equiv D_{ij}^b(x_1-x_2)\,, \tag{3.107}$$

$$\langle\, f_i^v(x_1) f_j^b(x_2)\,\rangle = 0\,. \tag{3.108}$$

We choose uncorrelated kinematic and magnetic driving [$\langle\, f_i^v f_j^b\,\rangle = 0$], because we are considering arbitrary space dimension $d \geq 2$ and it is not possible to define a non-vanishing correlation function of a vector field and a pseudovector field in this case. This can be done separately for integer dimensions of space, but, contrary to claims of some authors [142, 172], is no obstacle for application of the RG [143]. As usual the prefactors $u_0\,\nu_0^3$ and $u_0^2\,\nu_0^3$ in (3.108) have been extracted for the dimensional reasons.

We are working in an arbitrary dimension, but the renormalization will be carried out within the two-dimensional model. In magnetohydrodynamic turbulence, in contrast to fluid turbulence, there are direct energy cascades in both two and three dimensions. Therefore it is natural to expect that the scaling behavior is rather similar in both cases, and we apply the same forcing spectrum in all space dimensions $d \geq 2$.

The system of the stochastic MHD equations (3.9), (3.106) and (3.108) gives rise to the following De Dominicis-Janssen action:

$$\begin{aligned}\mathcal{S}[\boldsymbol{v},\boldsymbol{v}',\boldsymbol{b},\boldsymbol{b}'] = {}& \frac{1}{2}\boldsymbol{b}' D^b \boldsymbol{b}' + \boldsymbol{b}'\cdot[-\partial_t\boldsymbol{b}+u_0\nu_0\nabla^2\boldsymbol{b}+(\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{v}-(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{b}] \\ & + \frac{1}{2}\boldsymbol{v}' D^v \boldsymbol{v}' + \boldsymbol{v}'\cdot[-\partial_t\boldsymbol{v}-(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}+\nu_0\nabla^2\boldsymbol{v}+(\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{b}].\end{aligned} \tag{3.109}$$

The dimensional constants g_{v10}, g_{b10}, g_{v20}, and g_{b20}, which control the amount of randomly injected energy through (3.108), play the role of expansion parameters of the perturbation theory.

3.6.2 Double expansion of the model

The action (3.109) gives rise to four three-point interaction vertices defined by the standard rules [25], and the following set of propagators

$$\Delta_{ij}^{vv'}(\boldsymbol{k},t) = \Delta_{ij}^{v'v}(-\boldsymbol{k},-t) = \theta(t)\,P_{ij}(\boldsymbol{k})\,\mathrm{e}^{-\nu_0\,k^2 t}\,, \tag{3.110}$$

$$\Delta_{ij}^{bb'}(\boldsymbol{k},t) = \Delta_{ij}^{b'b}(-\boldsymbol{k},-t) = \theta(t)\,P_{ij}(\boldsymbol{k})\,\mathrm{e}^{-u_0\nu_0 k^2 t}\,, \tag{3.111}$$

$$\Delta_{ij}^{vv}(\boldsymbol{k},t) = \frac{1}{2}\,u_0\,\nu_0^2\,P_{ij}(\boldsymbol{k})\,\mathrm{e}^{-\nu_0 k^2|t|}\left(\,g_{v10}\,k^{-2\varepsilon-2\delta}+g_{v20}\,\right)\,, \tag{3.112}$$

$$\Delta_{ij}^{bb}(\boldsymbol{k},t) = \frac{1}{2}\,u_0\,\nu_0^2\,P_{ij}(\boldsymbol{k})\,\mathrm{e}^{-u_0\nu_0 k^2|t|}\left(\,g_{b10}\,k^{-2\,a\varepsilon-2\delta}+g_{b20}\,\right) \tag{3.113}$$

in the time-wave-number representation. With due account of Galilei invariance of the action (3.109), and careful analysis of the structure of the perturbation expansion it can be shown

[143] that for any fixed space dimension $d > 2$ only three one-particle irreducible (1PI) Green functions $\Gamma_{v'v}$, $\Gamma_{b'b}$ and $\Gamma_{v'bb}$ with superficial UV divergences are generated by the action. They give rise to counterterms of the form already present in the action, which thus is multiplicatively renormalizable by power counting for space dimensions $d > 2$.

We would like to emphasize that the structure of renormalization should always be analyzed separately and it is not at all obvious that the nonlinear terms are not renormalized in the solution of the stochastic MHD equations. In fact, direct calculation shows that the Lorentz-force term is renormalized. There seems to be a certain amount of confusion about this point in the literature. For instance, in Refs. [170, 172] the authors erroneously neglect renormalization of nonlinear terms as high-order effect. The approach adopted in Ref. [170] for two-dimensional MHD turbulence was criticized by Kim and Young [171], who, alas, in their field-theoretic treatment of the same problem ignore renormalization of the Lorentz force without any justification. They also neglect renormalization of the forcing correlations by effectively considering renormalization of the model at $d > 2$, which does not seem to be appropriate in a setup in which the strictly two-dimensional quantities, the stream function and magnetic potential, are used for the description of the problem.

The analysis of the correlation functions of the velocity field and magnetic induction is essential near two dimensions, since in two dimensions additional divergences in the graphs of the 1PI Green functions $\Gamma_{v'v'}$ and $\Gamma_{b'b'}$ occur. The simplest way to include the corresponding local counter terms $\boldsymbol{v}'\boldsymbol{\nabla}^2\boldsymbol{v}'$ and $\boldsymbol{b}'\boldsymbol{\nabla}^2\boldsymbol{b}'$ in the renormalization is to add corresponding *local* terms to the force correlation function at the outset in order to keep the model multiplicatively renormalizable. This is why we have used the force correlation functions (3.108) and (3.13) with the relation (3.23) with both long-range and short-range correlations taken into account. As a result, the action (3.109) is multiplicatively renormalizable and allows for a standard RG asymptotic analysis [25, 173].

The model is regularized using a combination of analytic and dimensional regularization with the parameters ε and Δ, the latter was introduced in Eq. (3.22). As a consequence, the UV divergences appear as poles in the following linear combinations of the regularizing parameters: ε, Δ, $2\varepsilon + \Delta$, and $(a + 1)\varepsilon + \Delta$. In principle, the exponent of magnetic forcing correlations $2a\varepsilon$ may be treated as the second analytic regulator to construct a triple expansion of renormalized quantities at a suitable fixed point. However, in the ray scheme used here all regulators are assumed to be of same order and the discussion of a triple expansion in this case would be rather formal.

The UV divergences can be removed by adding suitable counterterms to the basic action S_B obtained from the unrenormalized one (3.109) by the substitution of the renormalized parameters for the bare ones: $g_{v10} \to \mu^{2\varepsilon} g_{v1}$, $g_{v20} \to \mu^{-2\Delta} g_{v2}$, $g_{b10} \to \mu^{2a\varepsilon} g_{b1}$, $g_{b20} \to \mu^{-2\Delta} g_{b2}$, $\nu_0 \to \nu$, $u_0 \to u$, where μ is a scale setting parameter having the same canonical dimension as the wave number.

For the actual calculations the ray scheme with minimal subtractions is convenient. The counter terms for the basic action corresponding to the unrenormalized action (3.109) are

$$\Delta\mathcal{S} = \nu\,(Z_1 - 1)\,\boldsymbol{v}'\boldsymbol{\nabla}^2\,\boldsymbol{v} + u\nu\,(Z_2 - 1)\,\boldsymbol{b}'\boldsymbol{\nabla}^2\,\boldsymbol{b} + \frac{1}{2}(1 - Z_4)u\nu^3 g_{v2}\mu^{-2\delta}\boldsymbol{v}'\boldsymbol{\nabla}^2\boldsymbol{v}' + \frac{1}{2}(1 - Z_5)u^2\nu^3 g_{b2}\mu^{-2\Delta}\boldsymbol{b}'\boldsymbol{\nabla}^2\boldsymbol{b}' + (Z_3 - 1)\boldsymbol{v}'(\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{b}, \tag{3.114}$$

where the renormalization constants Z_1, Z_2, Z_4, Z_5 renormalizing the unrenormalized (bare) parameters $e_0 = \{g_{v10}, g_{v20}, g_{b10}, g_{b20}, u_0, \nu_0\}$ and the constant Z_3 renormalizing the fields $\boldsymbol{b}$, and $\boldsymbol{b}'$, are chosen to cancel the UV divergences appearing in the Green functions constructed using the basic action. Due to the Galilean invariance of the action the fields $\boldsymbol{v}'$, and $\boldsymbol{v}$ are not renormalized.

In a multiplicatively renormalizable model, such as (3.109), the counter terms (3.114) can be chosen in a form containing a finite number of terms of the same algebraic structure as the terms of the original action (3.109). Thus, all UV divergences of the graphs of the perturbation expansion may be eliminated by a redefinition of the parameters of the original model.

Renormalized Green functions are expressed in terms of the renormalized parameters

$$\begin{aligned} &g_{v1} = g_{v10}\mu^{-2\varepsilon} Z_1^2 Z_2, \qquad && g_{v2} = g_{v20}\mu^{2\Delta} Z_1^2 Z_2 Z_4^{-1}, \qquad && \nu = \nu_0 Z_1^{-1}, \\ &g_{b1} = g_{b10}\mu^{-2\,a\varepsilon} Z_1 Z_2^2 Z_3^{-1}, && g_{b2} = g_{b20}\mu^{2\Delta} Z_1 Z_2^2 Z_3^{-1} Z_5^{-1}, && u = u_0\, Z_2^{-1} Z_1\,. \end{aligned} \tag{3.115}$$

The definitions (2.30) and the relations (3.115) yield β functions of the form

$$\begin{aligned} &\beta_{g_{v1}} = g_{v1}(-2\varepsilon + 2\gamma_1 + \gamma_2), \qquad && \beta_{g_{v2}} = g_{v2}(2\Delta + 2\gamma_1 + \gamma_2 - \gamma_4), \\ &\beta_{g_{b1}} = g_{b1}(-2a\varepsilon + \gamma_1 + 2\gamma_2 - \gamma_3), && \beta_{g_{b2}} = g_{b2}(2\Delta + \gamma_1 + 2\gamma_2 - \gamma_3 - \gamma_5), \\ &\beta_u = u(\gamma_1 - \gamma_2). \end{aligned} \tag{3.116}$$

Apart from the Gaussian fixed point $g^*_{v1} = g^*_{v2} = g^*_{b1} = g^*_{b2} = 0$ with no fluctuation effects on the large-scale asymptotics, which is IR stable for $\Delta > 0$, $\varepsilon < 0$, $a > 0$, there are two nontrivial IR stable fixed points of the RG with nonnegative g^*_{v1}, g^*_{v2}, g^*_{b1},g^*_{b2}, and u^*.

The thermal fixed point is generated by short-range correlations of the random force with

$$g^*_{v1} = 0, \quad g^*_{v2} = -4\pi(1 + \sqrt{17})\Delta, \quad g^*_{b1} = 0, \quad g^*_{b2} = 0, \tag{3.117}$$

and the inverse magnetic Prandtl number

$$u^* = \frac{\sqrt{17} - 1}{2} \simeq 1.562\,. \tag{3.118}$$

Physically, the asymptotic behavior described by this fixed point is brought about by thermal fluctuations of the velocity field [61]. The region of stability of the thermal fixed point (3.117), (3.118) is $2\varepsilon + 3\Delta < 0$, $\Delta < 0$ in the Δ, ε plane. For the magnetic forcing-decay parameter a the stability region is determined by the inequality $8a\varepsilon + (13 + \sqrt{17})\Delta < 0$.

The *kinetic* fixed point [142] generated by the forced fluctuations of the velocity field is given by the universal inverse magnetic Prandtl number (3.118), the parameters

$$g^*_{v1} = \frac{128\pi}{9(\sqrt{17} - 1)} \frac{\varepsilon(2\varepsilon + 3\Delta)}{\varepsilon + \Delta}, \quad g^*_{v2} = \frac{128\pi}{9(\sqrt{17} - 1)} \frac{\varepsilon^2}{\Delta + \varepsilon}, \tag{3.119}$$

and zero couplings of the magnetic forcing

$$g^*_{b1} = g^*_{b2} = 0\,, \tag{3.120}$$

and it may be associated with turbulent advection of the magnetic field. The values of g^*_{v1} and g^*_{v2} in (3.119) correspond to those find previously in Ref. [146]. The region of stability of the

kinetic fixed point in the Δ, ε plane is $\varepsilon > 0$, $2\varepsilon + 3\Delta > 0$. The stability of this fixed point also requires that the parameter $a < (13 + \sqrt{17})/12 \approx 1.427$ independent of the ratio Δ/ε. In spite of the absence of renormalization of the forcing correlation, the momentum-shell approach [142] yields the same condition.

The system of equations for the fixed points in this multi-charge problem is rather complicated and thus has several (in general complex-number) solutions, which we do note quote explicitly here, because they are not physically relevant: apart from the fixed points listed above there are eight IR unstable real-number fixed points in the physical region (all $g \geq 0$) of the parameter space, and several unphysical ones. Among the unstable fixed points are, in particular, all the possible candidates to *magnetic* fixed points, i.e. fixed points with a non-vanishing magnetic coupling constant. Therefore, the conclusion made in Refs. [142, 143] (although on inconsistent grounds) that the RG does not predict any magnetically driven scaling regime at and near two dimensions, is confirmed in the double-expansion approach.

An analysis of beta functions reveals that at the leading order of the ray scheme there is no fixed point with both g_{b1} and g_{b2} non-vanishing: at least one of them must be zero [149]. This, of course, severely reduces the set of possible magnetic fixed points at the outset.

In three dimensions there are stable magnetic fixed points, whose stability as a function of space dimension may be followed in the usual ε expansion. Stability of the magnetic fixed point disappears at a borderline dimension, whose leading-order value is $d_c = (3 + \sqrt{649})/10 \approx 2.848$ [142]. As seen from the results presented above, to follow the crossover between the two regimes from below within the ray scheme of triple expansion calculation beyond the one-loop order is needed.

3.6.3 Renormalization with maximum divergences above two dimensions

All the renormalization constants and the RG functions quoted above may be calculated also at finite Δ. The resulting system of fixed-point equations allows for a solution in a form of an ε expansion (with finite Δ) and yields the same result as the usual ε expansion at the leading order. However, this approach is not self-consistent in the sense that the field theory is not renormalizable at finite $\Delta > 0$, but only in the form of a simultaneous expansion in the coupling constants and Δ [25].

The aim is now to maintain the model UV finite for $\Delta > 0$ and simultaneously keep track of the effect of the additional divergences near two dimensions. To this end we introduce an additional UV cutoff in all propagators, i.e. instead of the set (3.113) we use the propagators

$$\Delta_{ij}^{vv'\Lambda}(\boldsymbol{k},t) = \theta(t)\,\theta(\Lambda - k)\,P_{ij}(\boldsymbol{k})\mathrm{e}^{-\nu_0 k^2 t}, \tag{3.121}$$

$$\Delta_{ij}^{bb'\Lambda}(\boldsymbol{k},t) = \theta(t)\theta(\Lambda - k)P_{ij}(\boldsymbol{k})\mathrm{e}^{-u_0\nu_0 k^2 t}, \tag{3.122}$$

$$\Delta_{ij}^{vv\Lambda}(\boldsymbol{k},t) = \frac{1}{2}\,\theta(\Lambda - k)u_0\nu_0^2\,P_{ij}(\boldsymbol{k})\mathrm{e}^{-\nu_0 k^2|t|}\left(g_{v10}k^{-2\varepsilon-2\Delta} + g_{v20}\right), \tag{3.123}$$

$$\Delta_{ij}^{bb\Lambda}(\boldsymbol{k},t) = \frac{1}{2}\theta(\Lambda - k)u_0\nu_0^2 P_{ij}(\boldsymbol{k})\mathrm{e}^{-u_0\nu_0 k^2|t|}\left(g_{b10}k^{-2a\varepsilon-2\Delta} + g_{b20}\right), \tag{3.124}$$

where Λ is the cutoff wave number. This change obviously does not affect the large-scale properties of the model. We would like to emphasize that the additional cutoff must be introduced uniformly in all lines in order to maintain the model multiplicatively renormalizable. An attempt

to introduce the cutoff, say, in the local part of the correlation functions only by the substitution $k^2 \to \theta(\Lambda - k)k^2$ would fail to renormalize the model multiplicatively, because loop contributions to the complete (dressed) correlation function would not reproduce such a structure in the wave-vector space.

In contrast with particle field theories we will keep the cutoff parameter Λ fixed, although large compared with the physically relevant wave-number scale. This introduces an explicit dependence on Λ in the coefficient functions of the RG, which has to be analyzed separately in the large-scale limit in the coordinate space. The setup is thus very similar to that of Polchinski [174].

The coefficient functions of the RG equations become, in general, functions of the parameters μ and Λ through the dimensionless ratio μ/Λ. Solution for the scalar coefficients of equal-time velocity and magnetic induction pair correlation functions is (cf. (3.29)):

$$G^{vv}(sk, \Lambda; g) = \overline{\nu}^2 k^{-2\Delta} R_v \left(1, \frac{\mu s}{\Lambda}; \overline{g}\right), \tag{3.125}$$

$$G^{bb}(sk, \Lambda; g) = \mathrm{e}^{\int_1^s \mathrm{d}x\, \gamma_3(x)/x} \overline{\nu}^2 k^{-2\Delta} R_b \left(1, \frac{\mu s}{\Lambda}; \overline{g}\right), \tag{3.126}$$

where $\overline{g}$ is now the solution of the Gell-Mann-Low equations:

$$\frac{\mathrm{d}\overline{g}(s)}{\mathrm{d}\ln s} = \beta_g \left[\overline{g}(s), \frac{\mu s}{\Lambda}\right], \tag{3.127}$$

with the β functions explicitly depending on s, the dimensionless wave number.

Above two dimensions an UV renormalization of the model would require and infinite number of counter terms and in this sense it is not renormalizable in the limit $\Lambda \to \infty$. To avoid dealing with these formal complications, we keep the additional cutoff Λ fixed (although large), and choose the renormalization procedure according to the principle of maximum divergences [130]: the same terms of the action are renormalized as in the two-dimensional case in the previous section (3.114), but the renormalization constants may have an explicit dependence on the scale-setting parameter through the ratio μ/Λ. The two counter terms

$$\int \mathrm{d}x \left[\frac{1}{2}(1 - Z_4) u \nu^3 g_{v2}\, \mu^{-2\Delta} \boldsymbol{v}' \nabla^2 \boldsymbol{v}' + \frac{1}{2}(1 - Z_5) u^2 \nu^3 g_{b2} \mu^{-2\Delta} \boldsymbol{b}' \nabla^2 \boldsymbol{b}'\right] \tag{3.128}$$

are superfluous in the sense that in the limit $\mu/\Lambda \to 0$ the contribution to the Green functions of the graphs containing the coupling constants g_{v2} and g_{b2} remains finite provided $2\Delta = d - 2$ is fixed and finite and the other counter terms are properly chosen. This is guaranteed by Polchinski's theorem [174]. We retain these counter terms in order to have a possibility to pass to the limit $\Delta \to 0$ in the RG equations.

The presence of the explicit cutoff implies some technical difficulties in the calculation of the renormalization constants in the traditional field-theoretic approach, which arise because we are dealing with massless vector fields. It turns out that the coefficient functions of the RG equation are simplest in a renormalization procedure, which is similar to the momentum-shell renormalization of Wilson [9]. If we were calculating over the whole wave-vector space without an explicit UV cutoff, there would not be much difference between the effort required in both approaches. The presence of the UV cutoff makes calculations with non-vanishing external wave vectors rather tedious.

Let us remind that the choice of a renormalization procedure basically is the choice of the rule according to which the counter-term contributions are extracted from the perturbation expansion of the Green functions of the model. The usual field-theoretic prescription goes as follows [65]: Consider a 1PI graph γ, let $R(\gamma)$ be the renormalized value of the graph, and let $\overline{R}(\gamma)$ the value of the graph with subtracted counter terms of all the subgraphs, then

$$R(\gamma) = \overline{R}(\gamma) - T\overline{R}(\gamma) \tag{3.129}$$

where the operator T extracts the counter-term contribution from $\overline{R}(\gamma)$. Usually $\overline{R}(\gamma) = \gamma$ on one-loop graphs, and the renormalization scheme is specified by the action of $\overline{R}(\gamma)$ on multi-loop graphs together with the definition of the operator T. The counter terms may then be constructed recursively with the use of (3.129) and the definitions of T and $\overline{R}(\gamma)$. There is rather large freedom in the choice of the counter-term operator, but to arrive at Green functions finite in the limit $\Lambda \to \infty$ in two dimensions – which we want to have a connection with the double expansion – the operation $\overline{R}$ must be chosen properly.

Here, we have used a renormalization procedure, in which the operation $\overline{R}(\gamma)$ is standard [65], and the subtraction operator T is defined as follows: let $F_\gamma(\omega, \boldsymbol{k}, \Lambda)$ be the function of external frequencies and wave-vectors (which also depends on the cutoff parameter Λ) corresponding to the expression $\overline{R}(\gamma)$ (this is not a 1PI graph, in general). The subtraction operator T extracts the same set of terms of the Maclaurin-expansion in the external wave-vectors, which generate the counter terms (3.114), from the *difference* $F_\gamma(\omega, \boldsymbol{k}, \Lambda) - F_\gamma(\omega, \boldsymbol{k}, \mu)$. These coefficients of the Maclaurin-expansion are calculated at vanishing external frequencies and wave-vectors. It should be noted that the coefficients of this Maclaurin expansion of the function $F_\gamma(\omega, \boldsymbol{k}, \Lambda)$ itself may not exist in the limit $\omega \to 0$ in this "massless" model, but the difference $F_\gamma(\omega, \boldsymbol{k}, \Lambda) - F_\gamma(\omega, \boldsymbol{k}, \mu)$ allows for a Maclaurin expansion finite in the limit $\omega \to 0$ to the order required for the renormalization. The counter-term operator T and the combinatorics of the renormalization procedure for higher-order graphs may then be constructed in the standard fashion. Although this is actually not needed in the present one-loop calculation, the very possibility of this extension is necessary to guarantee that the renormalization renders the model finite in the limit $\Lambda \to \infty$ in two dimensions.

Effectively, at one-loop order this prescription reduces the region of integration to the momentum shell $\mu < k < \Lambda$, which leads to the same calculation as in the momentum-shell renormalization. In higher orders, however, our renormalization scheme does not coincide with the momentum-shell renormalization. The point of the present renormalization procedure is that without some sort of IR cutoff the subtraction at zero momenta and frequencies is, in general, not possible in a massless model, whereas a subtraction at vanishing frequencies and external momenta of the order of μ leads to much more complicated calculations due the heavy index structure.

At one-loop accuracy in this scheme the γ functions are

$$\gamma_1 = \frac{1}{2B}\Big[\left(d^2 - d - 2\varepsilon\right) u\, g_{v1} + \left(d^2 + d - 4 + 2a\varepsilon\right) g_{b1} + \left(d^2 - 2\right)\left(u\, g_{v2} + g_{b2}\right)\Big],$$
$$\gamma_2 = \frac{1}{(1+u)B}\Big[(d-1)\,(d+2)\,(g_{v1} + g_{v2}) + (d+2)\,(d-3)\,(g_{b1} + g_{b2})\Big],$$
$$\gamma_3 = \frac{2}{B}\left[g_{b1} + g_{b2} - g_{v1} - g_{v2}\right],$$

$$\gamma_4 = \frac{d^2-2}{2g_{v2}B}\left[u\left(g_{v1}+g_{v2}\right)^2+\left(g_{b1}+g_{b2}\right)^2\right],$$
$$\gamma_5 = \frac{2(d-2)(d+2)}{g_{b2}(1+u)B}\left(g_{b1}+g_{b2}\right)\left(g_{v1}+g_{v2}\right), \tag{3.130}$$

where $B = d(d+2)\Gamma(d/2)(4\pi)^{d/2}$. These expressions reveal an additional advantage of this renormalization scheme: at one-loop order there is no explicit dependence on μ/Λ in the coefficient functions of the RG. At one-loop level a direct comparison with the expressions obtained in the framework of the Wilson RG is also possible: the dependence on g_{v1}, g_{b1}, and u of the β functions β_{gv1}, β_{gb1} and β_u corresponding to (3.130) coincides with that of their counterparts of Ref. [142] up to notation.

The set of β functions generated by (3.130) allows for a fixed-point solution in the form of an ε expansion. Little reflection shows that the fixed-point equations in this case have a self-consistent solution with the following leading-order behavior: $u = \mathcal{O}(1)$, $g_{v1} = \mathcal{O}(\varepsilon)$, $g_{b1} = \mathcal{O}(\varepsilon)$, $g_{v2} = \mathcal{O}(\varepsilon^2)$ and $g_{b2} = \mathcal{O}(\varepsilon^2)$. The actual fixed-point values of g_{v1}, g_{b1} and u in the ε expansion as well as the stability regions with respect to ε are determined by the same set of equations as in the earlier momentum-shell [142] and field-theoretic [143] calculation above two dimensions. The stability condition with respect to the dimension of space of these fixed points is, as expected, $d > 2$.

It should be noted that the function γ_5 is finite in the set (3.130), whereas in the double-expansion approach it was equal to zero This means that magnetic fixed points with both g_{b1} and g_{b2} may exist. In fact, there is one such fixed point stable at high dimensions of space which gives rise to a magnetically driven scaling regime. This fixed point may be found in the ε expansion, and we have also investigated its stability numerically. Technically speaking, the appearance of a magnetic fixed point with both magnetic couplings non-vanishing would be a completely expected thing to happen in the two-loop approximation, since we have not found any symmetry reasons or the like to prevent the renormalization of the magnetic forcing at higher orders. Thus, to investigate this effect consistently in the ε expansion would require a full two-loop renormalization of the model, which is beyond the scope of the present analysis.

The stability of the kinetic scaling regime is strongly affected by the behavior of magnetic fluctuations: from Fig. 3.6 it is seen that the steeper falloff of the correlations of the magnetic forcing in the wave-vector space compared with that of the kinetic forcing the lower is the space dimension, above which the kinetic fixed point is stable. In particular, when the parameter $a > 1.427$ the kinetic fixed point ceases to be stable even in two dimensions. In three dimensions the kinetic scaling regime is stable against magnetic forcing, when $a < 1.15$.

The monotonic growth of the kinetic-fixed-point value of the inverse magnetic Prandtl number u^* as a function of the kinetic forcing-decay parameter in a fixed space dimension is depicted in Fig. 3.7. The plot shows also that u^* is a monotonically decreasing function of the space dimension at fixed ε. The lowest-lying curve corresponds to the leading order of the ε expansion [142, 168]

$$u^* = \frac{1}{2}\left[-1+\sqrt{1+\frac{8\,(d+2)}{d}}\right]. \tag{3.131}$$

We are particularly interested in the stability of the magnetic fixed point, and have carried out

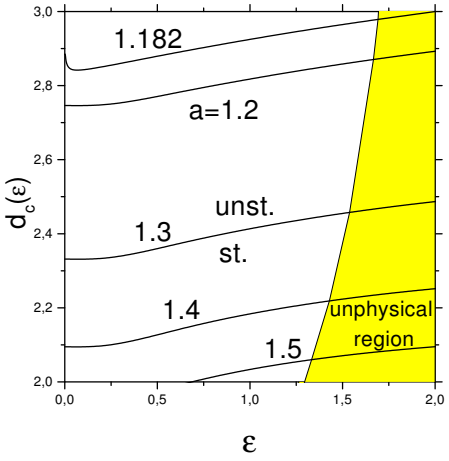


Figure 3.6. The borderline dimension d_c between the stability regions of the kinetic fixed point of the RG equations for (3.116) and (3.130) for magnetic forcing-decay parameter a near the double-expansion threshold $a = 1.427$. This plot reveals the strong dependence of the borderline dimension on the parameter a. The shaded region on the right corresponds to values $\varepsilon > 2/a$, for which the forcing correlation function in the powerlike form (3.12) leads to intractable IR divergences, and a corresponding IR cutoff (magnetic integral length scale) must be introduced.

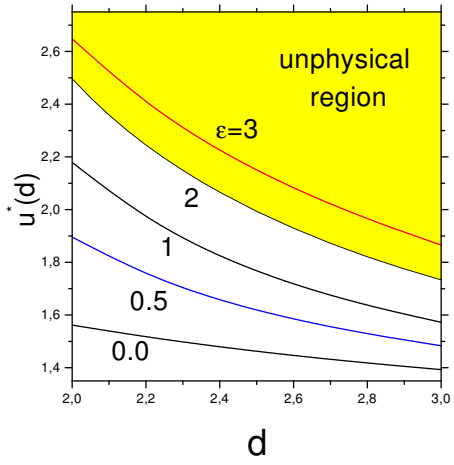


Figure 3.7. The fixed-point value of the inverse magnetic Prandtl number u^* as a function of the space dimension d and the decay parameter ε. The lowest curve corresponds to the leading order in the ε expansion, which is not affected by thermal fluctuations. The shaded region in the upper part of the plot corresponds to values $\varepsilon > 2$, for which an IR cutoff (kinetic integral length scale) must be introduced in the correlation function (3.12).

extensive numerical calculations of the stability of this fixed point as a function of ε and the space dimension d. The results are plotted in Figs. 3.8 and 3.9.

In Fig. 3.8 the magnetic forcing-decay parameter $a < 1$ (i.e. the kinetic-forcing correlations fall off steeper in the wave-vector space than those the magnetic forcing) and it is seen that for very small a a slowly enough decaying kinetic forcing renders the magnetic scaling regime unstable. In particular, this threshold is very small in three dimensions. With the growth of a a strip of stability of the magnetic fixed point in the ε, d plane appears such that the magnetic regime remains stable in three dimensions for all allowed kinetic forcing patterns. It is also seen that the magnetic fixed point is persistently unstable at $d \leq 2.46$ for all ε. This borderline dimension should be compared with that given by the ε expansion $d_c = 2.85$. From the solution it can be seen that this significant discrepancy is due to the appearance of a stable magnetic fixed point completely different from that found in the ε expansion: in the latter the magnetic fixed point is given by $g^*_{v1} = g^*_{v2} = g^*_{b2} = u^* = 0$ and $g^*_{b1} = 4d(d+2)\Gamma(d/2)(4\pi)^{d/2}a\varepsilon/(d^2 - 3d - 32)$, whereas at the magnetic fixed point, whose stability is plotted in Figs. 3.8 and 3.9, only $g^*_{v1} = u^* = 0$ with non-vanishing fixed-point values of the other couplings. Thus, the lowering of the borderline dimension of stability of the mag-

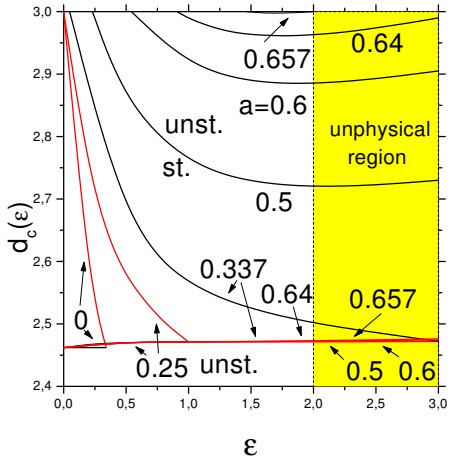


Figure 3.8. The borderline dimension d_c between the stability regions of the magnetic fixed point of the RG flow equations (3.115) and (3.130) for magnetic forcing-decay parameter $a < 1$. For sufficiently small values of a the magnetic fixed point is unstable for any finite value of ε, but the region of stability grows with the growth of a so that for $a > 0.658$ the magnetic point becomes stable even in three dimensions for finite values of ε. The shading shows the region, where $\varepsilon > 2$, in which the powerlike correlation function (3.12) cannot be consistently used.

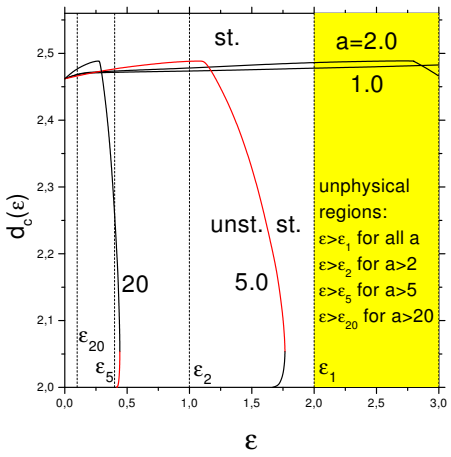


Figure 3.9. The borderline dimension d_c between the stability regions of the magnetic fixed point of the RG equations (3.116) and (3.130) for large values of the magnetic forcing-decay parameter $a > 1$. The shaded area and half-planes with vertical dashed border lines show regions, where $\varepsilon > 2/a$, in which the powerlike correlation function (3.12) cannot be consistently used.

netic scaling regime is a strong effect of the thermal fluctuations described by the short-range terms in the forcing correlation functions. Fig. 3.9 shows the lower boundary of the stability region of the magnetic fixed point for large values of a, when magnetic-forcing correlations fall off much faster that kinetic-forcing correlations in the wave-vector space. A remarkable feature of both plots is the insensitivity of the lower border of the stability strip to the value of magnetic forcing-decay parameter a.

4 Violated symmetries

Models that we have considered so far can be in some sense regarded as ideal. As has been mentioned in Sec. 3 a traditional approach to the description of fully developed turbulence is based on the stochastic Navier-Stokes equation [109]. The complexity of this equation leads to great difficulties which do not allow one to solve it even in the simplest case when one assumes the isotropy of the system under consideration. On the other hand, the isotropic turbulence is almost delusion and if exists is still rather rare. Therefore, if one wants to model more or less realistic developed turbulence, one is pushed to consider more general models of turbulence. This, of course, rapidly increases complexity of the corresponding model which itself has to involve the part responsible for description of the anisotropy.

In this section we give a brief overview of two possible deviation from ideality. First, in Sec. 4.1 employing the Kraichnan model, we study violation of mirror-symmetry on the advection of passive quantity. Later, in Sec. 4.2, the advection problem will be studied using velocity field with strong anisotropy taken into account and in Sec. 4.3 the effect of anisotropy on stochastic magnetohydrodynamics will be studied. In the last part, Sec. 4.4, the anisotropy will be introduced in the stochastic Navier-Stokes equation itself and corresponding influence on large-scale regimes will be determined.

4.1 Effect of helicity

Further topic we want to address is the effect of helicity or violation of mirror symmetry. Helicity is defined as the scalar product of velocity and vorticity and its non zero value expresses mirror symmetry breaking of turbulent flow. It plays significant role in the processes of magnetic field generation in electrically conductive fluid [175, 176] and represents one of the most important characteristics of large-scale motions as well [177–179]. The presence of helicity is observed in various natural (like large air vortices in atmosphere) and technical flows [180–182]. Despite of this fact the role of the helicity in hydrodynamical turbulence is not completely clarified up to now.

Turbulent viscosity and diffusivity, which characterize influence of small-scale motions on heat and momentum transport, are basic quantities investigated in the theoretic and applied models. The constraint of direct energy cascade in helical turbulence has to be accompanied by decrease of turbulent viscosity. However, no influence of helicity on turbulent viscosity was found in some works [183, 184]. Similar situation is observed for turbulent diffusivity in helical turbulence. Although the modeling calculations demonstrate intensification of turbulent transfer in the presence of helicity [185, 186] direct calculation of diffusivity does not confirm this effect [185, 187, 188]. Helicity is the pseudoscalar quantity hence it can be easily understood, that its influence appears only in quadratic and higher terms of perturbation theory or in the combination with another pseudoscalar quantities (e.g. large-scale helicity). Really, simultaneous consideration of memory effects and second order approximation indicate effective influence of helicity on turbulent viscosity [189, 190] and turbulent diffusivity [186, 191, 192] already in the limit of small or infinite correlation time.

Helicity, as we shall see below, does not affect known results in one-loop approximation and, therefore, it is necessary to turn to the second order (two-loop) approximation to be able to analyze possible consequences. It is also important to say that in the framework of classical Kraichnan model, i.e., model of passive advection by the Gaussian velocity field with δ-like

correlations in time, it is not possible to study the influence of the helicity because all diagrams with helical contribution are identically equal to zero at all orders in the perturbation theory. It is interesting and important to study the helicity effects because many turbulence phenomena are directly influenced by them (like large air vortices in atmosphere). For example, in stochastic magnetic hydrodynamics, which studies the turbulence in electrically conducting fluids, it leads to a nontrivial fact of the very existence of so called "turbulent dynamo" – the generation of a large-scale magnetic field by the energy of the turbulent motion [175, 176, 193–197]. This is an important effect in astrophysics.

4.1.1 The model

As we have already seen the advection of a passive scalar field $\theta \equiv \theta(x) \equiv \theta(t, \boldsymbol{x})$ by a Navier-Stokes ensemble is governed by Eq. (3.90). Although the theoretical description of the fluid turbulence on the basis of the "first principles", i.e., on the stochastic Navier-Stokes (NS) equation [79] remains an open problem, considerable progress has been achieved in understanding simplified model systems that share some important properties with the real problem: shell models [82], stochastic Burgers equation [83] and passive advection by random "synthetic" velocity fields [73].

The crucial role in these studies are played by models of advected passive scalar field [76]. We just note here that it is also possible to consider a vector formulation of this problem and his numerous variants [198–202]. A simple model of a passive scalar quantity advected by a random Gaussian velocity field, white in time and self-similar in space (the latter property mimics some features of a real turbulent velocity ensemble), the so-called Kraichnan's rapid-change model [84], is an example. The interest to these models is based on two important facts: first, as were shown by both natural and numerical experimental investigations, the deviations from the predictions of the classical Kolmogorov-Obukhov phenomenological theory [70, 79, 81, 85] is even more strongly displayed for a passively advected scalar field than for the velocity field itself (see, e.g., [86–96] and references cited therein), and second, the problem of passive advection is much more easier to be consider from theoretical point of view. There, for the first time, the anomalous scaling was established on the basis of a microscopic model [77], and corresponding anomalous exponents was calculated within controlled approximations [89–91,97–102] (see also review [73] and references therein). Within the "zero-mode approach," developed in [89, 91, 97–102], nontrivial anomalous exponents are related to the zero modes (unforced solutions) of the closed exact equations satisfied by the equal-time correlations. Within the approach based on the field theoretic RG and OPE (discussed in Sec. 2), the anomalous scaling emerges as a consequence of the existence in the model of composite operators with *negative* critical dimensions, which determine the anomalous exponents [74, 110, 113, 114, 203, 204].

The standard notation for advection problem using Kraichnan model slightly differs from the one using stochastic Navier-Stokes ensemble. Therefore in what follows we give a brief overview of basic physical ideas behind Kraichnan model and introduce corresponding notation.

The analog of Eq. (3.90) is now given by the following stochastic equation

$$\partial_t \theta + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\theta - \nu_0 \boldsymbol{\nabla}^2 \theta = f^\theta, \tag{4.1}$$

where ν_0 is the coefficient of molecular diffusivity and $f^\theta \equiv f^\theta(x)$ is a Gaussian random noise with zero mean and correlation function

$$\langle f^\theta(x) f^\theta(x') \rangle = \delta(t - t') C(\boldsymbol{r}/L), \;\; \boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}'. \tag{4.2}$$

The noise f^θ in (4.1) maintains the steady-state of the system but the concrete form of the correlator is not essential. The only condition which must be fulfilled by the function $C(\boldsymbol{r}/L)$ is that it must decrease rapidly for $r \equiv |\boldsymbol{r}| \gg L$, where L denotes an integral scale related to the stirring. In the case when C depends not only on the modulus of the vector $\boldsymbol{r}$ but also on its direction, it plays the role of a source of large-scale anisotropy, whereupon the noise can be replaced by a constant gradient of scalar field. Eq. (4.1) then reads

$$\partial_t\theta + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\theta - \nu_0\boldsymbol{\nabla}^2\theta = -\boldsymbol{h}\cdot\boldsymbol{v}. \tag{4.3}$$

Here, $\theta(x)$ is the fluctuation part of the total scalar field $\Theta(x) = \theta(x) + \boldsymbol{h}\cdot\boldsymbol{x}$, and $\boldsymbol{h}$ is a constant vector that determines distinguished direction. The direct formulation with a scalar gradient is even more realistic one; see, e.g. Refs. [74, 88–91, 101, 102, 114].

In accordance with the generalized Kraichnan model [74, 88] with finite correlation time taken into account we assume that the velocity field is driven by simple linear stochastic equation

$$\partial_t v_i + Rv_i = f_i^v, \tag{4.4}$$

where $R \equiv R(x)$ is a linear operation to be specified below and $f_i^v \equiv f_i^v(x)$ is an external random stirring force with zero mean and the correlator

$$\langle f_i^v(x) f_j^v(x')\rangle \equiv D_{ij}^f(x;x') = \frac{1}{(2\pi)^{d+1}}\int \mathrm{d}\omega \int \mathrm{d}^d\boldsymbol{k}\, P_{ij}^\rho(\boldsymbol{k})\tilde{D}^f(\omega,\boldsymbol{k})\mathrm{e}^{-i(t-t')+i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}. \tag{4.5}$$

It should be noted that in the SDE (4.1) the multiplicative noise due to random velocity is not a white noise in time as in the original Kraichnan model. Therefore there is no need to specify the interpretation of the SDE. However, in the analysis the white-noise limit will be considered and it should recalled that in this limit the results correspond to the Stratonovich interpretation of the SDE (4.1).

The transition to a helical fluid corresponds to the giving up of conservation of spatial parity, and technically, this is expressed by the fact that the correlation function is specified in the form of mixture of a true tensor and a pseudotensor. In our approach, it is represented by two parts of transverse projector

$$P_{ij}^\rho(\boldsymbol{k}) = P_{ij}(\boldsymbol{k}) + H_{ij}(\boldsymbol{k}), \tag{4.6}$$

which consists of non-helical standard transverse projector $P_{ij}(\boldsymbol{k})$ (See Eq.(3.10)) and presence of helicity is modeled by a term

$$H_{ij}(\boldsymbol{k}) = i\rho\,\varepsilon_{ijl}k_l/k, \tag{4.7}$$

where $k = |\boldsymbol{k}|$ is the wave number and ε_{ijl} is Levi-Civita's completely antisymmetric tensor of rank 3 (it is equal to 1 or -1 according to whether (i,j,l) is an even or odd permutation of $(1,2,3)$ and zero otherwise), and the real parameter of helicity, ρ, characterizes the amount of helicity. Due to the requirement of positive definiteness of the correlation function the absolute

value of ρ must be in the interval $|\rho| \in \langle 0, 1\rangle$ [194, 195]. Physically, non-zero helical part (proportional to ρ) expresses existence of non-zero correlations $\langle \boldsymbol{v} \cdot \mathrm{rot}\, \boldsymbol{v}\rangle$.

Of course, due to the presence of Levi-Civita's tensor, the dimension of the $\boldsymbol{x}$ space must be considered to be three. Nevertheless, in what follows, we shall remain the d-dimensionality of all results which are not related to helicity to be also able to study d-dependence of non-helical case of the model. The correlator $\tilde{D}^f$ is chosen [43, 74, 114] in the following form

$$\tilde{D}^f(\omega, \boldsymbol{k}) = g_0 \nu_0^3 (k^2 + m^2)^{2-d/2-\varepsilon-\eta/2}, \tag{4.8}$$

where

$$\tilde{R}(\boldsymbol{k}) = u_0 \nu_0 (k^2 + m^2)^{1-\eta/2}, \tag{4.9}$$

the wave-number representation of $R(x)$. Positive amplitude factors g_0 and u_0 play the role of the coupling constants of the model. In addition, g_0 is a formal small parameter of the ordinary perturbation theory. The positive exponents ε and η ($\varepsilon = \mathcal{O}(\eta)$) are small RG expansion parameters, the analogs of the parameter $\varepsilon = 4 - d$ in the φ^4- theory. Thus, from the point of view of perturbation theory we again have to deal with a double expansion model discussed in Sec. 3.2. Now in the (ε, η)-plane around the origin $\varepsilon = \eta = 0$.

Note the presence of two scales in the problem - integral scale L introduced in (4.2) and momentum scale m, which has appeared in (4.9). Clearly, they have different physical origin. However, these two scales can be related to each other and for technical purposes [43] it is reasonable to choose $L = 1/m$. When not explicitly stated, this relation is always assumed.

In the limit $k \gg m$ the functions (4.8) and (4.9) take on simple powerlike form

$$\tilde{D}^f(\omega, \boldsymbol{k}) = g_0 \nu_0^3 k^{4-d-2\varepsilon-\eta}, \quad \tilde{R}(\boldsymbol{k}) = u_0 \nu_0 k^{2-\eta}, \tag{4.10}$$

which are used in actual calculations. The needed IR regularization will be given by restrictions on the region of integrations.

From Eqs. (4.4), (4.5), and (4.10) the statistics of the velocity field $\boldsymbol{v}$ can be determined. It obeys Gaussian distribution with zero mean and correlator

$$\langle v_i(x) v_j(x')\rangle \equiv D^v_{ij}(x; x') = \frac{1}{(2\pi)^{d+1}} \int \mathrm{d}\omega \int \mathrm{d}^d \boldsymbol{k}\, P^\rho_{ij}(\boldsymbol{k}) \tilde{D}^v(\omega, \boldsymbol{k}) \mathrm{e}^{-i\omega(t-t') + i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \tag{4.11}$$

with

$$\tilde{D}^v(\omega, \boldsymbol{k}) = \frac{g_0 \nu_0^3 k^{4-d-2\varepsilon-\eta}}{(i\omega + u_0\nu_0 k^{2-\eta})(-i\omega + u_0\nu_0 k^{2-\eta})}. \tag{4.12}$$

The correlator (4.12) is directly related to the energy spectrum via the frequency integral [74, 103–107]

$$E(k) \simeq k^{d-1} \int \mathrm{d}\omega \tilde{D}^v(\omega, k) \simeq \frac{g_0 \nu_0^2}{u_0} k^{1-2\varepsilon}. \tag{4.13}$$

Therefore, the coupling constant g_0 and the exponent ε describe the equal-time velocity correlator or, equivalently, energy spectrum. On the other hand, the constant u_0 and the second exponent

η are related to the frequency $\omega \simeq u_0\nu_0 k^{2-\eta}$ (or to the function $\tilde{R}(k)$, the reciprocal of the correlation time at the wave number k) which characterizes the mode k [74, 103–107, 203, 205]. Thus, in our notation, the value $\varepsilon = 4/3$ corresponds to the well-known Kolmogorov "five-thirds law" for the spatial statistics of velocity field, and $\eta = 4/3$ corresponds to the Kolmogorov frequency. Simple dimensional analysis shows that the parameters (charges) g_0 and u_0 are related to the characteristic ultraviolet (UV) momentum scale Λ (of the order of inverse Kolmogorov length) by

$$g_0 \simeq \Lambda^{2\varepsilon+\eta}, \quad u_0 \simeq \Lambda^{\eta}. \tag{4.14}$$

In Ref. [88], it was shown that the linear model (4.4) (and therefore also the Gaussian model (4.11), (4.12)) is not Galilean invariant and, as a consequence, it does not take into account the self-advection of turbulent eddies. As a result of these so-called "sweeping effects" the different time correlations of the Eulerian velocity are not self-similar and depend strongly on the integral scale; see, e.g., Ref. [206, 206–208]. But, on the other hand, the results presented in Ref. [88] show that the Gaussian model gives reasonable description of the passive advection in the appropriate frame, where the mean velocity field vanishes. One more argument to justify the model (4.11), (4.12) is that, in what follows, we shall be interested in the equal-time, Galilean invariant quantities (structure functions), which are not affected by the sweeping, and therefore, as we expect (see, e.g., Ref. [74, 114, 121]), their absence in the Gaussian model (4.11), (4.12) is not essential.

In the end of this section, let us briefly discuss two important limits of the considered model (4.11), (4.12). First of them it is so-called rapid-change model limit when $u_0 \to \infty$ and $g_0' \equiv g_0/u_0^2 = \text{const}$,

$$\tilde{D}^v(\omega, \boldsymbol{k}) \to g_0'\nu_0 k^{-d-2\varepsilon+\eta}, \tag{4.15}$$

and the second one is so-called quenched (time-independent or frozen) velocity field limit which is defined by $u_0 \to 0$ and $g_0'' \equiv g_0/u_0 = \text{const}$,

$$\tilde{D}^v(\omega, \boldsymbol{k}) \to g_0''\nu_0^2\pi\delta(\omega)k^{-d+2-2\varepsilon}, \tag{4.16}$$

which is similar to the well-known models of the random walks in random environment with long range correlations; see, e.g., Refs. [138, 209–213]. The third order results of the composite operators for the former case (4.15) have been obtained in [214, 215] and belong to a few models for which high order calculations were possible.

4.1.2 Field theoretic formulation of the model

For completeness of our text in this and next section we shall present and discuss the principal moments of the RG theory of the model defined by Eqs. (4.3), (4.11), and (4.12).

Using Eq. (1.106) the stochastic problem (4.3)-(4.5) can be recast into the equivalent field theoretic model of the doubled set of fields $\Phi \equiv \{\theta, \theta', \boldsymbol{v}, \boldsymbol{v}'\}$ with the action functional

$$\mathcal{S}[\Phi] = \frac{1}{2}\boldsymbol{v}' D^f \boldsymbol{v}' + \theta' \left[-\partial_t\theta - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\theta + \nu_0\boldsymbol{\nabla}^2\theta - \boldsymbol{h}\cdot\boldsymbol{v}\right] + \boldsymbol{v}'\cdot\left[-\partial_t\boldsymbol{v} - R\boldsymbol{v}\right], \tag{4.17}$$

where D^f_{ij} is defined in Eq. (4.5), and as usual θ' and $\boldsymbol{v}'$ are auxiliary response fields.

Generating functionals of full Green functions $\mathcal{G}(A)$ and connected Green functions $\mathcal{W}(A)$ are defined by the Eq. (1.56), where now linear form $A\varphi$ is defined as

$$A\Phi = A^{\theta}\theta + A^{\theta'}\theta' + A_i^{v} v_i + A_i^{v'} v_i'(x). \tag{4.18}$$

Following the arguments in [74], we can put $A_i^{\boldsymbol{v}'} = 0$ in Eq. (4.18) and then perform the explicit Gaussian integration over the auxiliary vector field $\boldsymbol{v}'$ as a consequence of the fact that, in what follows, we shall not be interested in the Green functions involving field $\boldsymbol{v}'$. After this integration one is left with the field theoretic model described by the functional action

$$\mathcal{S}[\Phi] = -\frac{1}{2}\boldsymbol{v}(D^v)^{-1}\boldsymbol{v} + \theta'[-\partial_t\theta - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\theta + \nu_0\boldsymbol{\nabla}^2\theta - \boldsymbol{h}\cdot\boldsymbol{v}], \tag{4.19}$$

where the second term represents De Dominicis-Janssen action for the stochastic problem (4.3) at fixed velocity field $\boldsymbol{v}$, and the first term describe the Gaussian averaging over $\boldsymbol{v}$ defined by the correlator D^v in Eqs. (4.11) and (4.12). The latter explicitly reads

$$\mathcal{S}_{\text{vel}}[\boldsymbol{v}] = \frac{1}{2}\int \mathrm{d}t_1 \int \mathrm{d}t_2 \int \mathrm{d}^d\boldsymbol{x}_1 \int \mathrm{d}^d\boldsymbol{x}_2\, \boldsymbol{v}_i(t_1,x_1) D_{ij}^{-1}(t_1-t_2, \boldsymbol{x}_1-\boldsymbol{x}_2)\boldsymbol{v}_j(t_2,\boldsymbol{x}_2). \tag{4.20}$$

Action (4.19) is given in a form convenient for a realization of the field theoretic perturbation analysis with the standard Feynman diagrammatic technique. From the quadratic part of the action one obtains the matrix of bare propagators. The wave-number-frequency representation of, in what follows, important propagators are: a) the bare propagator $\langle\theta\theta'\rangle_0$ defined as

$$\langle\theta\theta'\rangle_0 = \langle\theta'\theta\rangle_0^* = \frac{1}{-i\omega + \nu_0 k^2}, \tag{4.21}$$

and b) the bare propagator for the velocity field $\langle vv\rangle_0$ given directly by Eq. (4.12), namely

$$\langle v_i v_j\rangle_0 = P_{ij}^{\rho}(\boldsymbol{k})\tilde{D}^v(\omega,\boldsymbol{k}), \tag{4.22}$$

where $P_{ij}^{\rho}(\boldsymbol{k})$ is the transverse projector defined in previous section by Eq. (4.6). Their graphical representation is given in a similar manner as has been presented in Fig.3.3. It should be noted that the choice of the action in the rather standard form (4.19) corresponds to the choice of the diagonal value of the propagator equal to zero: $\langle\theta\theta'\rangle_0(0,0) = 0$.

The triple (interaction) vertex $-\theta' v_j\partial_j\theta = \theta' v_j V_j\theta$ has been presented by the right picture in Fig. 3.4.

It is appropriate to eliminate the magnitude $h \equiv |\boldsymbol{h}|$ of the vector field $\boldsymbol{h}$ from the action (4.19) by rescaling of the scalar fields: $\theta \to h\theta$ and $\theta' \to \theta'/h$. This representation directly leads to the fact, which is important from the point of view of the renormalization of the model (see next section), namely, that the superficial divergences can be presented only in the Green functions $\langle\theta(x_1)\cdots\theta(x_n)\theta'(y_1)\cdots\theta'(y_p)\rangle$ with $n = p$, i.e., equal number of θ and θ' fields (see Ref. [74] for details).

4.1.3 UV renormalization and RG analysis

Detail analysis of divergences in the problem (4.19) was done in Ref. [216, 217] (see also Refs. [32, 71, 74]), therefore we shall present here only basic facts and conclusions rather than

F	$\boldsymbol{v}$	θ	θ'	m, Λ, μ	ν_0, ν	g_0	u_0	g, u, h
d_F^k	-1	-1	$d+1$	1	-2	$2\varepsilon+\eta$	η	0
d_F^ω	1	0	0	0	1	0	0	0
d_F	1	-1	$d+1$	1	0	$2\varepsilon+\eta$	η	0

Table 4.1. Canonical dimensions of the fields and parameters of the model under consideration.

to repeat all details. First of all, every the one-irreducible Green function with $N_{\theta'} < N_\theta$ vanish. On the other hand, dimension analysis based on the Table I leads to the conclusion that for any d, superficial divergences can be present only in the one-irreducible Green functions $\langle\theta'\theta\cdots\theta\rangle$ with only one field θ' ($N_{\theta'} = 1$) and arbitrary number N_θ of field θ. Therefore, in the model under investigation, the superficial divergences can be found only in the one-irreducible function $\langle\theta'\theta\rangle$. To remove them one needs to include into the action functional the counterterm of the form $\theta'\boldsymbol{\nabla}^2\theta$. Its inclusion is manifested by the multiplicative renormalization of the bare parameters g_0, u_0, and ν_0 in action functional (4.19)

$$\nu_0 = \nu Z_\nu, \quad g_0 = g\mu^{2\varepsilon+\eta} Z_g, \quad u_0 = u\mu^\eta Z_u. \tag{4.23}$$

The standard notation is employed, where the dimensionless parameters g, u, and ν are the renormalized counterparts of the corresponding bare ones, μ is the scale setting parameter, and $Z_i = Z_i(g, u)$ are renormalization constants.

The renormalized action functional has the following form

$$\mathcal{S}_R[\Phi] = -\frac{1}{2}\boldsymbol{v}[D^v]^{-1}\boldsymbol{v} + \theta'\left[-\partial_t\theta - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\theta + \nu Z_1\boldsymbol{\nabla}^2\theta - \boldsymbol{h}\cdot\mathbf{v}\right], \tag{4.24}$$

where the correlator D^v_{ij} is written in renormalized parameters (in wave-number-frequency representation)

$$\tilde{D}^v_{ij}(\omega, k) = \frac{P^\rho_{ij}(\boldsymbol{k}) g\nu^3\mu^{2\varepsilon+\eta}k^{4-d-2\varepsilon-\eta}}{(i\omega + u\nu\mu^\eta k^{2-\eta})(-i\omega + u\nu\mu^\eta k^{2-\eta})}. \tag{4.25}$$

By comparison of the renormalized action (4.24) with definitions of the renormalization constants Z_i, $i = g, u, \nu$ (4.23) we are coming to the relations among them:

$$Z_\nu = Z_1, \;\; Z_g = Z_\nu^{-3}, \;\; Z_u = Z_\nu^{-1}. \tag{4.26}$$

The second and third relations are consequences of the absence of the renormalization of the term with D^v in renormalized action (4.24). Renormalization of the fields, the mass parameter m, and the vector $\boldsymbol{h}$ is not needed, i.e., $Z_\Phi = 1$ for all fields, $Z_m = 1$, and also $Z_{\boldsymbol{h}} = 1$.

The issue of interest is, in particular, the behavior of the equal-time structure functions

$$S_n(r) \equiv \langle[\theta(t, \boldsymbol{x}) - \theta(t, \boldsymbol{x}')]^n\rangle \tag{4.27}$$

in the inertial range, specified by the inequalities $l \sim 1/\Lambda \ll r \ll L = 1/m$ (l is an internal length). Here parentheses $\langle\cdots\rangle$ mean functional average over fields $\Phi = \{\theta, \theta', \boldsymbol{v}\}$ with

weight $\exp S_R[\Phi]$. In the isotropic case, the odd functions S_{2n+1} vanish, while for S_{2n} simple dimensional considerations give

$$S_{2n}(r) = \nu_0^{-n}\, r^{2n}\, R_{2n}(r/l, r/L, g_0, u_0, \rho), \tag{4.28}$$

where R_{2n} are scaling functions of dimensionless variables (see Sec. 2.2). In principle, they can be calculated within the ordinary perturbation theory (i.e., as series in g_0), but this is not useful for studying inertial-range behavior: the coefficients are singular in the limits $r/l \to \infty$ and/or $r/L \to 0$, which compensate the smallness of g_0, and in order to find correct IR behavior we have to sum the entire series. The desired summation can be accomplished using the field theoretic RG and OPE; see Sec. 2 and Refs. [74, 110, 111].

The RG analysis consists of two main stages (Sec. 2). On the first stage, the multiplicative renormalizability of the model is demonstrated and the differential RG equations for its correlation (structure) functions are obtained. The asymptotic behavior of the functions like (4.27) for $r/l \gg 1$ and any fixed r/L is given by IR stable fixed points (see next section) of the RG equations and has the form

$$S_{2n}(r) = \nu_0^{-n}\, r^{2n}\, (r/l)^{-\gamma_n} R_{2n}(r/L, \rho), \quad r/l \gg 1 \tag{4.29}$$

with yet unknown scaling functions $R_{2n}(r/L, \rho)$. We remind the reader that the quantity $\Delta[S_{2n}] \equiv -2n + \gamma_n$ is termed the critical dimension, and the exponent γ_n, the difference between the critical dimension $\Delta[S_{2n}]$ and the canonical dimension $-2n$, is called the anomalous dimension. In the case at hand, the latter has an extremely simple form: $\gamma_n = n\epsilon$. Whatever be the functions $R_n(r/L, \rho)$, the representation (4.29) implies the existence of a scaling (scale invariance) in the IR region ($r/l \gg 1$, r/L fixed) with definite critical dimensions of all "IR relevant" parameters, $\Delta[S_{2n}] = -2n + n\epsilon$, $\Delta_r = -1$, $\Delta_{L^{-1}} = 1$ and fixed "irrelevant" parameters ν_0 and l.

On the second stage, the small r/L behavior of the functions $R_{2n}(r/L, \rho)$ is studied within the general representation (4.29) using the OPE technique (Sec. 2.7). It shows that, in the limit $r/L \to 0$, the functions $R_{2n}(r/L, \rho)$ have the asymptotic forms

$$R_{2n}(r/L) = \sum_F C_F(r/L)\, (r/L)^{\Delta_n}, \tag{4.30}$$

where C_F are coefficients regular in r/L. In general, the summation is implied over certain renormalized composite operators F with critical dimensions Δ_n. In case under consideration the leading operators F have the form $F_n = (\partial_i \theta \partial_i \theta)^n$. In Sec. 4.1.6 we shall consider them in detail where the complete two-loop calculation [217] of the critical dimensions of the composite operators F_n will be present for arbitrary values of n, d, u and ρ.

The actual calculation [216, 217] have been performed to the with two-loop approximation. The calculation of higher-order corrections is more difficult in the models with turbulent velocity field with finite correlation time than in the cases with δ-correlation in time. First of all, one has to calculate more relevant Feynman diagrams in the same order of perturbation theory (see below). Second, and more problematic distinction is related to the fact that the diagrams for the finite correlated case involve two different dispersion laws, namely, $\omega \propto k^2$ for the scalar field and $\omega \propto k^{2-\eta}$ for the velocity field. This leads to complicated expressions for renormalization constants even in the simplest (one-loop) approximation [74, 114]. But, as was discussed in

[74, 114, 121], this difficulty can be avoided by the calculation of all renormalization constants in an arbitrary specific choice of the exponents ε and η that guarantees UV finiteness of the Feynman diagrams. From the point of calculations the most suitable choice is to put $\eta = 0$ and leave ε arbitrary.

Thus, the knowledge of the renormalization constants for the special choice $\eta = 0$ is sufficient to obtain all important quantities as the γ-functions, β-functions, coordinates of fixed points, and the critical dimensions.

Such possibility is not automatic in general. In the model under consideration it is the consequence of an analysis which shows that in the MS scheme all the needed anomalous dimensions are independent of the exponents ε and η in the two-loop approximation. But in the three-loop approximation they can simply appear [121].

In Ref. [121] the two-loop corrections to the anomalous exponents of model (4.19) without helicity were studied. Now the effect of helicity can be discussed for comparison.

Now we can continue with renormalization of the model. The relation $\mathcal{S}[\theta, \theta', \boldsymbol{v}, e_0] = \mathcal{S}^R[\theta, \theta', \boldsymbol{v}, e, \mu]$, where e_0 stands for the complete set of bare parameters and e stands for renormalized one, leads to the relation $\mathcal{W}(A, e_0) = \mathcal{W}^R(A, e, \mu)$ for the generating functional of connected Green functions. By application of the operator $\tilde{\mathcal{D}}_\mu \equiv \mu\partial_\mu$ at fixed e_0 on both sides of the latest equation one obtains the basic RG differential equation

$$\mathcal{D}_{RG}\mathcal{W}^R(A, e, \mu) = 0, \tag{4.31}$$

where $\mathcal{D}_{RG}$ represents operation $\tilde{\mathcal{D}}_\mu$ written in the renormalized variables. Its explicit form is

$$\mathcal{D}_{RG} = \mathcal{D}_\mu + \beta_g(g, u)\partial_g + \beta_u(g, u)\partial_u - \gamma_\nu(g, u)\mathcal{D}_\nu, \tag{4.32}$$

where we denote $\mathcal{D}_x \equiv x\partial_x$ for any variable x and the RG functions (the β and γ functions) are given by well-known definitions and in our case, using relations (4.26) for renormalization constants, they have the following form

$$\gamma_\nu \equiv \tilde{\mathcal{D}}_\mu \ln Z_\nu, \tag{4.33}$$

$$\beta_g \equiv \tilde{\mathcal{D}}_\mu g = g(-2\varepsilon - \eta + 3\gamma_\nu), \tag{4.34}$$

$$\beta_u \equiv \tilde{\mathcal{D}}_\mu u = u(-\eta + \gamma_\nu). \tag{4.35}$$

The renormalization constant Z_ν is determined by the requirement that the one-irreducible Green function $\langle\theta'\theta\rangle_{1-ir}$ must be UV finite when is written in renormalized variables. In our case it means that they have no singularities in the limit $\varepsilon, \eta \to 0$. The one-irreducible Green function $\langle\theta'\theta\rangle_{1-ir}$ is related to the self-energy operator $\Sigma_{\theta'\theta}$ by the Dyson equation

$$\langle\theta'\theta\rangle_{1-ir} = -i\omega + \nu_0 p^2 - \Sigma_{\theta'\theta}(\omega, p). \tag{4.36}$$

Thus Z_ν is found from the requirement that the UV divergences are canceled in Eq. (4.36) after substitution $\nu_0 = \nu Z_\nu$. This determines Z_ν up to an UV finite contribution, which is fixed by the choice of the renormalization scheme. In the MS scheme all the renormalization constants have the form: 1 + *poles in* ε, η *and their linear combinations*. The self-energy operator $\Sigma_{\theta'\theta}$ is represented by the corresponding one-irreducible diagrams. In contrast to rapid-change model, where only one-loop diagram exists (it is related to the fact that all higher-loop diagrams contain at least

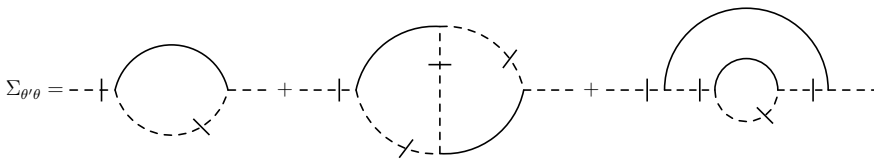


Figure 4.1. The one- and two-loop diagrams that contribute to their self-energy operator $\Sigma_{\theta'\theta}$.

one closed loop which is built on by only retarded propagators, thus are automatically equal to zero), in the case with finite correlations in time of the velocity field, higher-order corrections are non-zero. In two-loop approximation the self-energy operator $\Sigma_{\theta'\theta}$ is defined by diagrams which are shown in Fig. 4.1.

4.1.4 Fixed points and scaling regimes

Possible scaling regimes of a renormalizable model are directly given by the IR stable fixed points of the corresponding system of RG equations [24, 25]. In the considered model, the coordinates g^*, u^* of the fixed points are found from the system of two equations

$$\beta_g(g^*, u^*) = \beta_u(g^*, u^*) = 0. \tag{4.37}$$

The beta functions β_g and β_u are defined in Eqs. (4.34), and (4.35). To investigate the IR stability of a fixed point the eigenvalues of the matrix Ω

$$\Omega = \begin{pmatrix} \frac{\partial\beta_g}{\partial g} & \frac{\partial\beta_g}{\partial u} \\ \frac{\partial\beta_u}{\partial g} & \frac{\partial\beta_u}{\partial u} \end{pmatrix}_*. \tag{4.38}$$

have to be determined (see Eq. 2.24 for general case).

The possible scaling regimes of the model in one-loop approximation were investigated in Ref. [74]. Our first question is how the two-loop approximation change the picture of "phase" diagram of scaling regimes discussed in Ref. [74], and the second one is what restrictions on this picture are given by helicity (in two-loop approximation). The two-loop approximation in the model under our consideration without helicity was studied in Ref. [121] but the question of scaling regimes from two-loop approximation point of view was not discussed in details.

First of all, let us take a look at the rapid-change limit, which is obtained in the limit $u \to \infty$. In this regime, it is convenient to make transformation to new variables, namely, $w \equiv 1/u$, and $g' \equiv g/u^2$, with the corresponding changes in the β functions:

$$\beta_{g'} = g'(\eta - 2\varepsilon + \gamma_\nu), \quad \beta_w = w(\eta - \gamma_\nu). \tag{4.39}$$

In the rapid-change limit $w \to 0$ ($u \to \infty$) the two-loop contribution to γ_ν is equal to zero [217]. It is not surprising because in the rapid-change model there are no higher-loop corrections to the self-energy operator [110, 111], thus we are coming to the one-loop result of Ref. [74] with the anomalous dimension γ_ν of the form

$$\gamma_\nu = \lim_{w\to 0} \frac{(d-1)g\bar{S}_d}{2d(1+w)} \equiv \frac{(d-1)\bar{g}'}{2d}. \tag{4.40}$$

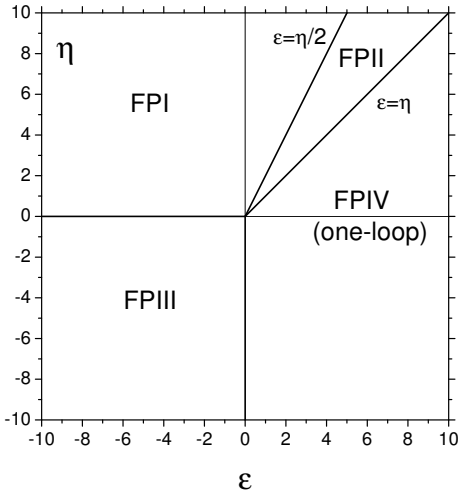


Figure 4.2. Regions of the stability for the fixed points in one-loop approximation. The regions of stability for fixed points FPI, FPII, and FPIII are exact, i.e., are not influenced by loop corrections. The fixed point FPIV is shown in one-loop approximation. Dependence of the FPIV on the value of d in two-loop approximation is shown in Fig. 4.3.

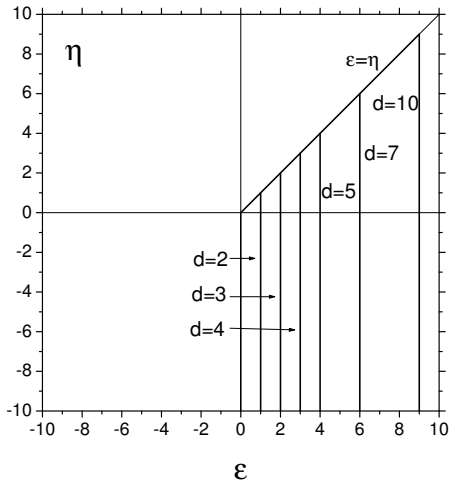


Figure 4.3. Regions of the stability for the fixed point FPIV in two-loop approximation without helicity for different space dimensions d. The IR fixed point is stable in the region given by inequalities: $\varepsilon > 0, \varepsilon > \eta$, and $\varepsilon < d - 1$.

In this regime we have two fixed points denoted as FPI and FPII in Ref. [74]. The first fixed point is trivial, namely

$$\text{FPI}: \quad w^* = g'^* = 0, \tag{4.41}$$

with $\gamma_\nu^* = 0$, and diagonal matrix Ω with eigenvalues (diagonal elements)

$$\Omega_1 = \eta, \quad \Omega_2 = \eta - 2\varepsilon. \tag{4.42}$$

The region of stability is shown in Fig. 4.2. The second point is defined as

$$\text{FPII}: \quad w^* = 0, \;\; \bar{g}'^* = \frac{2d}{d-1}(2\varepsilon - \eta), \tag{4.43}$$

with $\gamma_\nu^* = 2\varepsilon - \eta$. These are exact one-loop expressions as a result of non-existence of the higher-loop corrections. That means that they have no corrections of order $\mathcal{O}(\varepsilon^2)$ and higher (we work with assumption that $\varepsilon \simeq \eta$, therefore it also includes corrections of the type $\mathcal{O}(\eta^2)$ and $\mathcal{O}(\eta\varepsilon)$). The corresponding "stability matrix" is triangular with diagonal elements (eigenvalues):

$$\Omega_1 = 2(\eta - \varepsilon), \quad \Omega_2 = 2\varepsilon - \eta. \tag{4.44}$$

The region of stability of this fixed point is shown in Fig. 4.2.

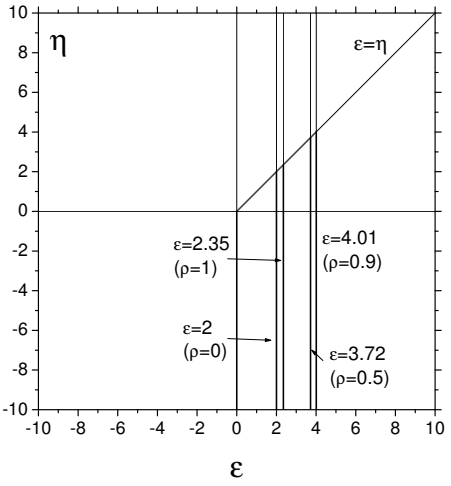


Figure 4.4. Regions of the stability for the fixed point FPIV in two-loop approximation with helicity. The IR fixed point is stable in the region given by inequalities: $\varepsilon > 0, \varepsilon > \eta$, and $\varepsilon < \varepsilon_\rho$.

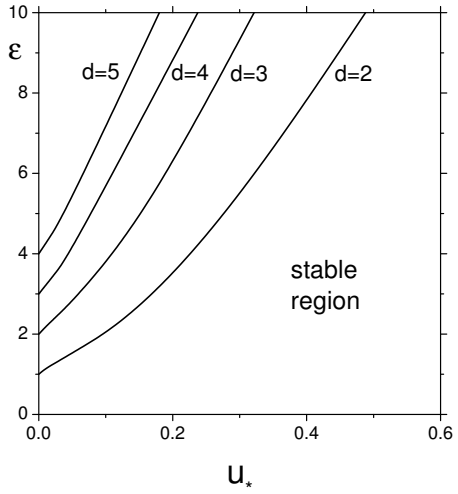


Figure 4.5. Regions of the stability for the fixed point FPV in two-loop approximation without helicity. d dependence of the stability is shown.

The frozen velocity field is mathematically obtained from the model under consideration in the limit $u \to 0$. To study this transition it is appropriate to change the variable g to the new variable $g'' \equiv g/u$. Then the β functions are transform to the following ones:

$$\beta_{g''} = g''(-2\varepsilon + 2\gamma_\nu), \quad \beta_u = u(-\eta + \gamma_\nu). \tag{4.45}$$

The system of β functions (4.45) exhibits two fixed points, denoted as FPIII and FPIV in Ref. [74], related to the corresponding two scaling regimes. One of them is trivial,

$$\text{FPIII}: \quad u^* = g''^* = 0, \tag{4.46}$$

with $\gamma_\nu^* = 0$. The eigenvalues of the corresponding matrix Ω, which is diagonal in this case, are

$$\Omega_1 = -2\varepsilon, \quad \Omega_2 = -\eta. \tag{4.47}$$

Thus, this regime is IR stable only if both parameters ε, and η are negative simultaneously as can be seen in Fig. 4.2. The second, non-trivial, point is

$$\text{FPIV}: \quad u^* = 0, \quad \bar{g}''^* = -\frac{\varepsilon}{2\mathcal{A}''_0} - \frac{\mathcal{B}''_0}{2\mathcal{A}''^2_0}\varepsilon^2, \tag{4.48}$$

where explicit expressions for $\mathcal{A}''_0$ and $\mathcal{B}''_0$ can be found in [217].

What is the influence of two-loop approximation on this IR scaling regime without helicity in general d-dimensional case? We denote the corresponding fixed point as FPIV_0, and its

coordinates are

$$\mathrm{FPIV}_0: \quad u^* = 0, \quad \bar{g}''^* = \frac{2d}{d-1}\left(\varepsilon + \frac{1}{d-1}\varepsilon^2\right), \tag{4.49}$$

with anomalous dimension γ_ν defined as

$$\gamma_\nu^* = \frac{d-1}{2d}\left(\bar{g}''^* - \frac{(\bar{g}''^*)^2}{2d}\right) = \varepsilon, \tag{4.50}$$

which is the exact one-loop result [74]. The eigenvalues of the matrix Ω (taken at the fixed point) are

$$\Omega_1 = 2\left(\varepsilon + \frac{1}{1-d}\varepsilon^2\right), \quad \Omega_2 = \varepsilon - \eta. \tag{4.51}$$

The eigenvalue $\Omega_2 = \partial_u \beta_u|_* = -\eta + \gamma_\nu^*$ is also exact one-loop result. The conditions $\bar{g}''^* > 0, \Omega_1 > 0$, and $\Omega_2 > 0$ for the IR stable fixed point lead to the following restrictions on the values of the parameters ε and η:

$$\varepsilon > 0, \quad \varepsilon > \eta, \quad \varepsilon < d - 1. \tag{4.52}$$

The region of stability is shown in Fig. 4.3. The region of stability of this IR fixed point increases when the dimension of the coordinate space d is increasing.

For the system with helicity the dimension of the space is fixed for $d = 3$. Thus, our starting conditions for stable IR fixed point of this type are obtained from conditions (4.52) with explicit value $d = 3$: $\varepsilon > 0, \varepsilon > \eta, \varepsilon < 2$. But they are valid only if helicity is vanishing and could be changed when non-zero helicity is present. When helicity is present the fixed point FPIV is given as

$$u^* = 0 \quad \bar{g}''^* = 3\varepsilon + \frac{3}{2}\left(1 - \frac{3\pi^2\rho^2}{16}\right)\varepsilon^2, \tag{4.53}$$

Therefore, in helical case, the situation is a little bit more complicated as a result of a competition between non-helical and helical term within two-loop corrections. The matrix Ω is triangular with diagonal elements (taken already at the fixed point)

$$\Omega_1 = 2\varepsilon + \left(-1 + \frac{3\pi^2\rho^2}{16}\right)\varepsilon^2, \quad \Omega_2 = \varepsilon - \eta, \tag{4.54}$$

where explicit dependence of eigenvalue Ω_1 on parameter ρ takes place. The requirement to have positive values for parameter $\bar{g}''^*$, and at the same time for eigenvalues Ω_1, Ω_2 leads to the region of stable fixed point. The results are shown in Fig. 4.4. The picture is rather complicated due to the very existence of the "critical" absolute value of ρ:

$$\rho_c = \frac{4}{\sqrt{3}\pi}, \tag{4.55}$$

which is defined from the condition of vanishing of the two-loop corrections in Eqs. (4.53), and (4.54):

$$\left(-1+\frac{3\pi^2\rho^2}{16}\right)=0. \tag{4.56}$$

As was already discussed above, when the helicity is not present, the system exhibits this type of fixed point (and, of course, the corresponding scaling behavior) in the region restricted by inequalities: $\varepsilon>0, \varepsilon>\eta$, and $\varepsilon<2$. The last condition is changing when helicity is switched on. The important feature here is that the two-loop contributions to $\bar{g}'^{*}$ and Ω_1 have the same structure but opposite sign. This leads to the different sources of conditions in the case when $|\rho|<\rho_c$ and $|\rho|>\rho_c$, respectively. In the situation with $|\rho|<\rho_c$ the positiveness of Ω_1 plays crucial role and one has the following region of stability of IR fixed point FPIV:

$$\varepsilon>0,\ \ \varepsilon>\eta,\ \ \varepsilon<\frac{32}{16-3\pi^2\rho^2}. \tag{4.57}$$

On the other hand, in the case with $|\rho|>\rho_c$, the principal restriction on the IR stable regime is yield by condition $\bar{g}''_*>0$ with final IR stable region defined as

$$\varepsilon>0,\ \ \varepsilon>\eta,\ \ \varepsilon<\frac{32}{-16+3\pi^2\rho^2}. \tag{4.58}$$

Therefore, if we are continuously increasing absolute value of helicity parameter ρ, the region of stability of the fixed point defined by the last inequality in Eq. (4.57) is increasing too. This restriction vanishes completely when $|\rho|$ reaches the "critical" value ρ_c, and the picture becomes the same as in the one-loop approximation [74]. In this rather specific situation the two-loop influence on the region of stability of fixed point is exactly zero: the helical and non-helical two-loop contributions are canceled by each other. Then if the absolute value of parameter ρ increases further, the last condition appears again, namely the third condition in Eq. (4.58), and restriction becomes stronger when $|\rho|$ tends to its maximal value, $|\rho|=1$. In this case of the maximal breaking of mirror symmetry (maximal helicity), $|\rho|=1$, the region of the IR stability of the fixed point is defined by inequalities: $\varepsilon>0, \varepsilon=\eta$, and $\varepsilon<2.351$ (see Fig. 4.4). It is interesting enough that the presence of helicity in the system leads to the enlargement of the stability region.

The most interesting scaling regime is the one with finite value of the fixed point for the variable u. But by short analysis one immediately concludes that the system of equations (see also [74])

$$\beta_g=g(-2\varepsilon-\eta+3\gamma_\nu)=0,\quad \beta_u=u(-\eta+\gamma_\nu)=0 \tag{4.59}$$

can be fulfilled simultaneously for finite values of g,u only in the case when the parameter ε is equal to η: $\varepsilon=\eta$. In this case, the function β_g is proportional to function β_u. As a result we have not one fixed point of this type but a curve of fixed points in the (g,u)- plane with exact one-loop result for $\gamma^*_\nu=\varepsilon=\eta$ (this is already directly given by Eq. (4.59)). We denote the corresponding point as in Ref. [74] as FPV. The possible values of the fixed point for variable u

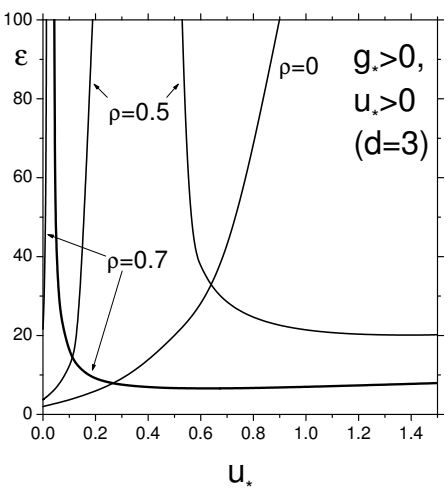


Figure 4.6. Regions of the stability for the fixed point FPV in two-loop approximation with helicity in the situation when $\rho < \rho_c = 4/(3^{1/2}\pi)$.

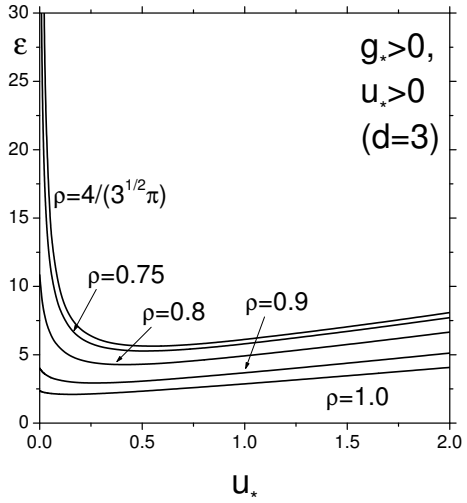


Figure 4.7. Regions of the stability for the fixed point FPV in two-loop approximation with helicity in the situation when $\rho \geq \rho_c = 4/(3^{1/2}\pi)$.

can be restricted (and will be restricted) as we shall discuss below. The matrix Ω defined in Eq. (2.24) has the following eigenvalues

$$\Omega_1 = 0, \;\; \Omega_2 = 3\bar{g}^* \left(\frac{\partial \gamma_\nu}{\partial g}\right)_* + u^* \left(\frac{\partial \gamma_\nu}{\partial u}\right)_* . \tag{4.60}$$

The vanishing of the Ω_1 is an exact result which is related to the degeneracy of the system of Eqs. (4.59) when nonzero solutions in respect to g, and u are assumed, or, equivalently, it reflects the existence of a marginal direction in the (g, u)-plane along the line of the fixed points.

The analysis of the last fixed point can start with the investigation of influence of the two-loop correction on the corresponding scaling regime when helicity is not present in the system ($\rho = 0$). In this situation it is interesting to determine the dependence of scaling regime on dimension d. In Fig. 4.5, the regions of stability for the fixed point FPV without helicity in the $\varepsilon - u$ plane for different space dimension d are shown. It is interesting that in two-loop case nontrivial d-dependence of IR stability appears in contrast to one-loop approximation [74].

Now the investigation of the situation with helicity follows and its influence on the stability of the IR fixed point is analyzed. In this case we work in three-dimension space. The competition between helical and non-helical terms appears again which will lead to a nontrivial restriction for the fixed point values of variable u to have positive fixed values for variable

Numerical analysis [217] shows important role is played by $\rho_c = 4/(\sqrt{3}\pi)$. First, the case $|\rho| < \rho_c$ is studied. The corresponding regions of stable IR fixed points with $g^* > 0$ is shown in Fig. 4.6. In the case when helicity is not present ($\rho = 0$, see the corresponding curve in Fig. 4.6), the only restriction is given by condition that $\Omega_2 > 0$, on the other hand, the condition $g^* > 0$ is fulfilled without restriction on the parameter space. When arbitrary small helicity is

present, i.e., $\rho > 0$, the restriction related to positiveness of g^* arises and is stronger when $|\rho|$ is increasing (the right curve for the concrete value of ρ in Fig. 4.6) and becomes to play the dominant role. At the same time, with increasing of $|\rho|$ the importance of the positiveness of the eigenvalue Ω_2 decreases (the left curve for the concrete value of ρ in Fig. 4.6). For a given $|\rho| < \rho_c$ there exists an interval of values of the variable u^* for which there is no restriction on the value of the parameter ε. For example, for $|\rho| = 0.1$, it is $1.128 < u^* < 13.502$, for $|\rho| = 0.5$, $0.217 < u^* < 0.394$, and for $|\rho| = 0.7$, $0.019 < u^* < 0.029$. Now turn to the case $|\rho| \geq \rho_c$. When $|\rho|$ acquires its "critical" value ρ_c, the IR fixed point is stable for all values of $u^* > 0$ and $\varepsilon > 0$, i.e., the condition $\Omega_2 > 0$ becomes fulfilled without restrictions on parameter space. On the other hand, the condition $g^* > 0$ yields strong enough restriction and it becomes stronger when $|\rho|$ tends to its maximal value $|\rho| = 1$ as it can be seen in Fig. 4.7).

The most important conclusion of two-loop approximation [217] of the model is the fact that the possible restrictions on the regions of stability of IR fixed points are "pressed" to the region with rather large values of ε, namely, $\varepsilon \geq 2$, and do not disturb the regions with relatively small ε. For example, the Kolmogorov point ($\varepsilon = \eta = 4/3$) is not influenced.

If F denotes some multiplicatively renormalized quantity (a parameter, a field or composite operator) then its critical dimension is given by the expression

$$\Delta[F] \equiv \Delta_F = d_F^k + \Delta_\omega d_F^\omega + \gamma_F^* \,, \tag{4.61}$$

see, e.g., Refs. [25, 32, 71] for details. In Eq. (4.61) d_F^k and d_F^ω are the canonical dimensions of F, $\Delta_\omega = 2 - \gamma_\nu^*$ is the critical dimension of frequency, and γ_F^* is the value of the anomalous dimension $\gamma_F \equiv \tilde{\mathcal{D}}_\mu \ln Z_F$ at the corresponding fixed point. Because the anomalous dimension γ_ν is already exact for all fixed points at one-loop level, the critical dimensions of frequency ω and of fields $\Phi \equiv \{\boldsymbol{v}, \theta, \theta'\}$ are also found exactly at one-loop level approximation [74]. In our notation they read

$$\begin{aligned} &\Delta_\omega = 2 - 2\varepsilon + \eta && \text{for FPII,} \\ &\Delta_\omega = 2 - \varepsilon && \text{for FPIV,} \\ &\Delta_\omega = 2 - \varepsilon = 2 - \eta && \text{for FPV,} \end{aligned} \tag{4.62}$$

and

$$\Delta_{\boldsymbol{v}} = 1 - \gamma_\nu^*, \quad \Delta_\theta = -1, \quad \Delta_{\theta'} = d + 1. \tag{4.63}$$

General equal-time two-point quantity $F(r)$ depends on a single distance parameter r which is multiplicatively renormalizable ($F = Z_F F^R$, where Z_F is the corresponding renormalization constant). Then the renormalized function F^R must satisfy the RG equation of the form

$$(\mathcal{D}_{RG} + \gamma_F) F(r) = 0, \tag{4.64}$$

with operator $\mathcal{D}_{RG}$ given explicitly in Eq. (4.32) and standardly $\gamma_F \equiv \tilde{\mathcal{D}}_\mu \ln Z_F$. The difference between functions F and F^R is only in normalization, choice of parameters (bare or renormalized), and related to this choice the form of the perturbation theory (in g_0 or in g). The existence of a nontrivial IR stable fixed point means that in the IR asymptotic region $r/l \gg 1$ and any fixed r/L the function $F(r)$ takes on the self-similar form

$$F(r) \simeq \nu_0^{d_F^\omega} l^{-d_F} (r/l)^{-\Delta_F} f(r/L), \tag{4.65}$$

where the values of the critical dimensions correspond to the given fixed point (see above in this section and Table 4.1). The dependence of the scaling functions on the argument r/L in the region $r/L \ll 1$ can be studied using the well-known OPE technique discussed in Sec. 2.7.

4.1.5 Effective diffusivity

One of the interesting object from the theoretical as well as experimental point of view is so-called effective diffusivity $\bar{\nu}$. In this section let us briefly investigate the effective diffusivity $\bar{\nu}$, which replaces initial molecular diffusivity ν_0 in equation (4.1) due to the interaction of a scalar field θ with random velocity field $\boldsymbol{v}$. Molecular diffusivity ν_0 governs exponential attenuation in time of all fluctuations in the system in the lowest approximation, which is given by the propagator (response function)

$$G(t-t', \boldsymbol{k}) = \langle \theta(t, \boldsymbol{k}) \theta'(t', \boldsymbol{k}) \rangle_0 = \theta(t-t') \exp(-\nu_0 k^2 (t-t')). \tag{4.66}$$

Analogously, the effective diffusivity $\bar{\nu}$ governs exponential attenuation of all fluctuations described by full response function, which is defined by Dyson equation (4.36). Its explicit expression can be obtained by the RG approach. In accordance with general rules of the RG (see, e.g., Ref. [25]) all principal parameters of the model g_0, u_0 and ν_0 are replaced by their effective (running) counterparts, which satisfy Gell-Mann-Low RG equations

$$s\frac{\mathrm{d}\bar{g}}{\mathrm{d}s} = \beta_g(\bar{g}, \bar{u}), \quad s\frac{\mathrm{d}\bar{u}}{\mathrm{d}s} = \beta_u(\bar{g}, \bar{u}), \quad s\frac{\mathrm{d}\bar{\nu}}{\mathrm{d}s} = -\bar{\nu}\gamma_\nu(\bar{g}, \bar{u}), \tag{4.67}$$

with initial conditions $\bar{g}|_{s=1} = g, \bar{u}|_{s=1} = u, \bar{\nu}|_{s=1} = \nu$. Here $s = k/\mu$, β and γ functions are defined in (4.33) - (4.35) and all running parameters clearly depend on variable s. Straightforward integration (at least numerical) of equations (4.67) gives way how to find their fixed points. Instead, one very often solves the set of equations $\beta_g(g^*, u^*) = \beta_u(g^*, u^*) = 0$ which defines all fixed points g^*, u^*. Just this approach was used above when we classified all fixed points. Due to special form of β-functions (4.34), (4.35) we are able to solve equation (4.67) analytically. Using Eqs. (4.67), and (4.34) one immediately rewrites (4.67) in the form

$$s\frac{\mathrm{d}\bar{\nu}}{\bar{\nu}} = \frac{\gamma_\nu}{2\varepsilon + \eta - 3\gamma_\nu}\frac{\mathrm{d}\bar{g}}{\bar{g}} \tag{4.68}$$

which can be easily integrated. Using initial conditions the solution acquires the form

$$\bar{\nu} = \left(\frac{g\nu^3}{\bar{g}s^{2\epsilon+\eta}}\right)^{1/3} = \left(\frac{D_0}{\bar{g}k^{2\epsilon+\eta}}\right)^{1/3}, \tag{4.69}$$

where to obtain the last expression we used the equations $g\mu^{2\epsilon+\eta}\nu^3 = g_0\nu_0^3 = D_0$. We emphasize that above solution is exact, i.e., the exponent $2\epsilon + \eta$ is as well. However, in infrared region $k \ll \Lambda \sim l^{-1}$, $\bar{g} \to g^*$, which can be calculated only pertubatively. In the two-loop approximation $g^* = g^{(1)*}\varepsilon + g^{(2)*}\varepsilon^2$ and after the Taylor expansion of $(g^*)^{1/3}$ in Eq. (4.69) we obtain

$$\bar{\nu} \approx \nu_* \left(\frac{D_0}{g^{(1)*}\varepsilon}\right)^{1/3} k^{-\frac{2\epsilon+\eta}{3}}, \qquad \nu^* \equiv 1 - \frac{g^{(2)*}\varepsilon}{3g^{(1)*}}. \tag{4.70}$$

Remind that for Kolmogorov values $\varepsilon = \eta = 4/3$ the exponent in (4.70) becomes equal to $-4/3$. Let us estimate the contribution of helicity to the effective diffusivity in nontrivial point above denoted as FPV. At this point $\varepsilon = \eta$ ($(2\varepsilon + \eta)/3 = \varepsilon$) and

$$\nu^* = 1 - \frac{\varepsilon}{12(1+u^*)}\left(\frac{2(3+u^*)}{5(1+u^*)^2}\,{}_2F_1\left(1,1;\frac{7}{2};\frac{1}{(1+u^*)^2}\right)\right.$$
$$\left. - \pi\rho^2\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{5}{2};\frac{1}{(1+u^*)^2}\right)\right). \tag{4.71}$$

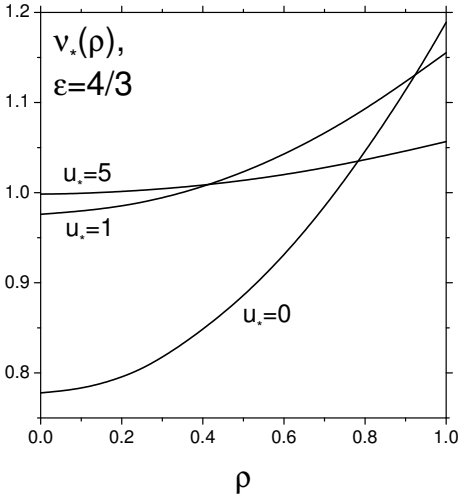


Figure 4.8. The dependence of ν^* on the helicity parameter ρ for definite IR fixed point values u^* of the parameter u.

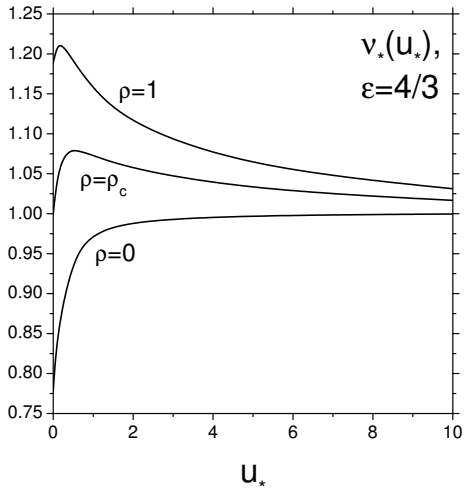


Figure 4.9. The dependence of ν^* on the IR fixed point u^* for the concrete values of the helicity parameter ρ.

In Figs. 4.8, and 4.9 the dependence of the ν^* on the helicity parameter ρ and the IR fixed point u^* of the parameter u is shown. As one can see from these figures when $u^* \to \infty$ (the rapid change model limit) the two-loop corrections to $\nu^* = 1$ are vanishing. Such behavior is related to the fact, which was already stressed in the text, that within the rapid change model there are no two and higher loop corrections at all. On the other hand, the largest two-loop corrections to the ν^* are given in the frozen velocity field limit ($u^* \to 0$). It is interesting that for all finite values of the parameter u^* there exists a value of the helicity parameter ρ for which the two-loop contribution to ν^* are canceled. For example, for the frozen velocity field limit ($u^* = 0$) such situation arises when the helicity parameter ρ is equal to its "critical" value $\rho_c = 4/(\sqrt{3}\pi)$ (this situation can be seen in Fig. 4.9). It is again the result of the competition between the non-helical and helical parts of the the two-loop corrections as is shown in Eq. (4.71). Further important feature of the expression (4.71) is that it is linear in the parameter ε. Thus, when one varies the value of ε the picture is the same as in Figs. 4.8, and 4.9 and only the scale of corrections is changed. In Figs. 4.8, and 4.9 we have shown the situation for the most interesting case when ε

is equal to its "Kolmogorov" value, namely, $\varepsilon = 4/3$.

Finally, time behavior of the retarded response function $G \equiv W_{\theta\theta'}\langle\theta\theta'\rangle_{\text{conn}}$ in the limit $t \to \infty$ is analyzed. In frequencywave vector representation $G(\omega, \boldsymbol{p})$ satisfies the Dyson equation (4.36) when one takes into account the relation $\Gamma_{\theta'\theta}\mathcal{W}_{\theta\theta'} = 1$ that follows from Eq. (2.6). The self-energy operator Σ is expressed via multi-loop Feynman graphs and can be calculated perturbatively. Its divergent part was found [216] up to the two-loop approximation and its finite part with the one-loop precision was calculated. Using the Dyson equation the response function can be written in the timewave vector representation

$$G(t, \boldsymbol{p}) = \int \frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{-i\omega t}G(\omega, \boldsymbol{p}) = \int \frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{e}^{-i\omega t}}{-i\omega + \nu_0 p^2 - \Sigma(\omega, \boldsymbol{p})}. \tag{4.72}$$

In the lowest approximation $\Sigma(\omega, \boldsymbol{p}) = 0$; thus the integral can be easily calculated: $G_0(t, \boldsymbol{p}) = \theta(t)\mathrm{e}^{-i\omega_r t}$. Here $\theta(t)$ denotes the usual step function and ω_r is a residuum in complex plain ω in point $i\nu_0 p^2$. Let us suppose that this situation remains the same for the full response function G; i.e., the leading contribution to its asymptotic behavior for $t \to \infty$ is determined by the residuum $\omega = \omega_r$, which corresponds to the smallest root of the dispersion relation

$$G^{-1}(\omega, \boldsymbol{p}) = i\omega_r + \nu_0 p^2\Sigma(\omega_r, \boldsymbol{p}) = 0. \tag{4.73}$$

It is advantageous to rewrite the last relation in the dimensionless form:

$$1 - z - I(1, z) = 0, \quad z \equiv \frac{i\omega_r}{\nu_0 p^2}, \quad I(1, z, g) \equiv \frac{\Sigma(\omega, \boldsymbol{p})}{\nu_0 p^2}, \tag{4.74}$$

which after renormalization can be rewritten in the fixed point g^* () as follows:

$$1 - z - I = 0, \tag{4.75}$$

where $\bar{\nu}$ is effective diffusivity (4.70) and $I^* \equiv I^*(1, z^*, g^*)$ is the renormalized (finite) part of the dimensionless self-energy operator I at the fixed point g^*. After some algebraic manipulations [216] the decay law $G_0(t, \boldsymbol{p}) \sim \exp(-\nu_0 p^2 t)$ is changed into

$$G(t, \boldsymbol{p}) \sim \exp(-i\omega_r t) = \exp(-z^*\bar{\nu}p^2 t), \quad t \to \infty. \tag{4.76}$$

Due to the existence of two complex conjugate values z^* the response function $G(t, p^2)$ can be written in the asymptotic limit $t \to \infty$ in the following final form:

$$G(t, p^2) \cong \sin(\nu_f p^{2-\varepsilon} t)\exp(-\nu_{\text{eff}} p^{2-\varepsilon} t) \tag{4.77}$$

where

$$\begin{aligned}
\nu_{\text{eff}} &= \left\{1 - \varepsilon\left[\frac{8}{3} + 2\ln\frac{1+u^*}{2} + \frac{3+u^*}{30(1+u^*)^3}\,{}_2F_1\left(1, 1; \frac{7}{2}; \frac{1}{(1+u^*)^2}\right)\right.\right. \\
&\quad \left.\left. - \frac{\pi\rho^2}{12(1+u^*)}\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{5}{2}; \frac{1}{(1+u^*)^2}\right)\right]\right\}\left(\frac{2D_0}{3(1+u^*)\varepsilon}\right)^{1/3}, \\
\nu_f &= \frac{\pi\varepsilon}{2}\left(\frac{2D_0}{3(1+u^*)\varepsilon}\right)^{1/3}.
\end{aligned} \tag{4.78}$$

It is clear that the exponential attenuation is accompanied by the oscillations.

4.1.6 Operator product expansion, critical dimensions of composite operators, and anomalous scaling

The behavior of the scaling function in Eq. (4.65) can be analyzed using OPE technique (Sec. 2.7).

In what follows, we shall concentrate on the equal-time structure functions of the scalar field defined as

$$S_n(r) \equiv \langle [\theta(t,\boldsymbol{x}) - \theta(t,\boldsymbol{x}')]^n \rangle, \quad r = |\boldsymbol{x} - \boldsymbol{x}'|, \tag{4.79}$$

which are also interesting from experimental point of view. The representation (4.65) is valid with the dimensions $d_F^\omega = 0$ and $d_F = \Delta_F = n\Delta_\theta = -n$. In general, not only the operators which are present in the corresponding Taylor expansion are entering into the OPE but also all possible operators that admix to them in renormalization. In present model the leading contribution of the Taylor expansion for the structure functions (4.79) is given by the tensor composite operators constructed solely of the scalar gradients

$$F[n,p] \equiv \partial_{i_1}\theta \cdots \partial_{i_p}\theta(\partial_i\theta\partial_i\theta)^l, \tag{4.80}$$

where $n = p+2l$ is the total number of the fields θ entering into the operator and p is the number of the free vector indices.

4.1.7 Anomalous scaling: two-loop approximation

The influence of the helicity on the anomalous scaling is the most interesting for the degenerate fixed point, namely, the fixed point denoted as FPV in Sec.4.1.4. In this case, the dimensions $\Delta_{F_{np}}$ are represented in the following series in the only independent exponent $\varepsilon = \eta$

$$\Delta_{F_{np}} = \varepsilon\Delta^{(1)}_{F_{np}} + \varepsilon^2\Delta^{(2)}_{F_{np}}\,. \tag{4.81}$$

The one-loop contribution has the form

$$\Delta^{(1)}_{F_{np}} = \frac{2n(n-1) - (n-p)(d+n+p-2)(d+1)}{2(d+2)(d-1)}\,, \tag{4.82}$$

which is independent of the parameter u. Although the fixed point value g_* depends on helicity parameter ρ, the two-loop contribution to the critical dimension $\Delta^{(2)}_{F_{np}}$ is independent of ρ. Thus, the result is the same as that obtained in Ref. [121] (the correct formula was republished in Ref. [218]).

In this section, the influence of helicity on the stability of asymptotic regimes, on the anomalous scaling, and on the effective diffusivity was briefly reviewed in the framework of the passive scalar advected by the turbulent flow with finite correlations in time of the velocity field [216, 217]. Such investigation is important and useful for understanding of efficiency of toy models (like Kraichnan model, and related ones) to study the real turbulent motions by means of modern theoretical methods including renormalization group approach. Thus, it can be consider as the first step in investigation of the influence of helicity in real turbulent environment.

The RG calculations [217] are necessary to two-loop approximation in order to observe effects of helicity. It has been mentioned that the anomalous scaling of the structure functions,

which is typical for the Kraichnan model and its numerous extensions, is not changed by the inclusion of helicity to the incompressible fluid. It is given mathematically by the very interesting fact that although separated two-loop Feynman diagrams of the corresponding composite operators strongly depend on the helicity parameter ρ, their sum - the critical dimension Δ_n is independent of ρ in the asymptotic regime defined by IR stable fixed point.

On the other hand, stability of possible asymptotic regimes, values of the fixed RG points and the turbulent diffusivity strongly depend on amount of helicity. The presence of helicity in the system leads to the restrictions on the possible values of the parameters of the model. The most interesting fact is the existence of a "critical" value ρ_c of the helicity parameter ρ which divides the interval of possible absolute values of ρ into two parts with completely different behavior. It is related to the existence of a competition between non-helical and helical contributions within two-loop approximation. As a result of this competition, within of the so-called frozen limit, the presence of helicity enlarges the region of parameter space with stable scaling regime, and if $|\rho| = \rho_c$ the corresponding two-loop restriction is vanished completely and one is coming to the one-loop results [74]. Similar splitting, although more complicated, into two nontrivial behavior of the fixed point was also obtained in the general case with finite correlations in time of the velocity field.

Another quantity which rather strongly depends on the helicity parameter ρ is effective diffusivity. The value of effective diffusivity can be 50% larger in helical case in comparison with non-helical case.

4.2 Effect of strong anisotropy

Another important question addressed is the effects of large-scale anisotropy on inertial-range statistics of passively advected fields [74, 89, 91, 101, 102, 114, 115, 204, 219–224] and the velocity field itself [225–229]. According to the classical Kolmogorov–Obukhov theory, the anisotropy introduced at large scales by the forcing (boundary conditions, geometry of an obstacle etc.) dies out when the energy is transferred down to the smaller scales owing to the cascade mechanism [70, 79]. A number of works confirms this picture for the *even* correlation functions, thus giving some quantitative support to the aforementioned hypothesis on the restored local isotropy of the inertial-range turbulence for the velocity and passive fields [74, 114, 115, 219–222, 225, 227–229]. More precisely, the exponents describing the inertial-range scaling exhibit universality and hierarchy related to the degree of anisotropy, and the leading contribution to an even function is given by the exponent from the isotropic shell [74, 114, 115, 204, 220–222, 227–229]. Nevertheless, the anisotropy survives in the inertial range and reveals itself in *odd* correlation functions, in disagreement with what was expected on the basis of the cascade ideas. The so-called skewness factor decreases down the scales much slower than expected [86–89, 89, 91, 93, 101, 102, 230], while the higher-order odd dimensionless ratios (hyperskewness etc.) increase, thus signaling of persistent small-scale anisotropy [74, 114, 115, 219, 231]. The effect seems rather universal, being observed for the scalar [74,114] and vector [115] fields, advected by the Gaussian rapid-change velocity, and for the scalar advected by the two-dimensional Navier-Stokes velocity field [219]. Problem of large-scale anisotropy for strongly compressible fluid was recently analyzed in works [232, 233].

Here we demonstrate the anomalous scaling behavior of a passive scalar advected by the

time-decorrelated strongly anisotropic Gaussian velocity field. In contradistinction with the studies of [74, 88, 89, 91, 93, 101, 102, 114], where the velocity was isotropic and the large-scale anisotropy was introduced by the imposed linear mean gradient, the uniaxial anisotropy in considered model persists for all scales, leading to non-universality of the anomalous exponents through their dependence on the anisotropy parameters.

The aim is twofold. First, explicit inertial-range expressions for the structure functions and correlation functions of the scalar gradients are obtained and then the corresponding anomalous exponents to the first order of the ε expansion are computed. The exponents become non-universal through the dependence on the parameters describing the anisotropy of the velocity field. Owing to the anisotropy of the velocity statistics, the composite operators of different ranks mix strongly in renormalization, and the corresponding anomalous exponents are given by the eigenvalues of the matrices which are neither diagonal nor triangular (in contrast with the case of large-scale anisotropy). In the language of the zero-mode technique this means that the $SO(d)$ decompositions of the correlation functions (employed, e.g., in Refs. [221, 222]) do not lead to the diagonalization of the differential operators in the corresponding exact equations.

4.2.1 Definition of the model. Anomalous scaling and "dangerous" composite operators.

The discussion here closely follows exposition in Sec. 4.1. As has been mentioned there the advection of a passive scalar field $\theta(x) \equiv \theta(t, \boldsymbol{x})$ in the rapid-change model is described by the stochastic equation (4.1). The velocity $\boldsymbol{v}(x)$ correlator now instead of Eq. (4.11) reads

$$\langle v_i(x) v_j(x') \rangle = D_0 \, \frac{\delta(t-t')}{(2\pi)^d} \int \mathrm{d}^d \boldsymbol{k} \, T_{ij}(\boldsymbol{k}) \, (k^2 + m^2)^{-d/2-\varepsilon/2} \, \exp[\mathrm{i}\boldsymbol{k} \cdot (\boldsymbol{x} - \boldsymbol{x}')], \quad (4.83)$$

where

$$\frac{D_0}{\nu_0} \equiv g_0 \equiv \Lambda^\varepsilon \quad (4.84)$$

and as was already mentioned in Sec. 4.1.1 the relation $m = 1/L$ holds. In the isotropic case, the tensor quantity $T_{ij}(\boldsymbol{k})$ in (4.83) is taken to be the ordinary transverse projector $T_{ij}(\boldsymbol{k}) = P_{ij}(\boldsymbol{k})$ (See Eq. (3.10)). The velocity statistics is taken to be anisotropic also at small scales. The ordinary transverse projector is replaced by the general transverse structure that possesses the uniaxial anisotropy:

$$T_{ij}(\boldsymbol{k}) = a(\psi) P_{ij}(\boldsymbol{k}) + b(\psi) \tilde{n}_i(\boldsymbol{k}) \tilde{n}_j(\boldsymbol{k}). \quad (4.85)$$

Here the unit vector $\boldsymbol{n}$ determines the distinguished direction ($\boldsymbol{n}^2 = 1$),

$$\tilde{n}_i(\boldsymbol{k}) \equiv P_{ij}(\boldsymbol{k}) n_j, \quad (4.86)$$

and ψ is the angle between the vectors $\boldsymbol{k}$ and $\boldsymbol{n}$, so that $(\boldsymbol{n} \cdot \boldsymbol{k}) = k \cos\psi$ [note that $(\tilde{\boldsymbol{n}} \cdot \boldsymbol{k}) = 0$]. The scalar functions can be decomposed the Gegenbauer polynomials (the d-dimensional generalization of the Legendre polynomials, see Ref. [234]):

$$a(\psi) = \sum_{l=0}^{\infty} a_l P_{2l}(\cos\psi), \quad b(\psi) = \sum_{l=0}^{\infty} b_l P_{2l}(\cos\psi). \quad (4.87)$$

The positivity of the correlator (4.83) leads to the conditions

$$a(\psi) > 0, \quad a(\psi) + b(\psi)\sin^2\psi > 0. \tag{4.88}$$

In practical calculations one works with the special case

$$T_{ij}(\boldsymbol{k}) = \left[1 + \alpha_1 \frac{(\boldsymbol{n}\cdot\boldsymbol{k})^2}{k^2}\right] P_{ij}(\boldsymbol{k}) + \alpha_2 \tilde{n}_i(\boldsymbol{k})\tilde{n}_j(\boldsymbol{k}). \tag{4.89}$$

Then the inequalities (4.88) reduce to $\alpha_{1,2} > -1$. Later it will be shown that this choice represents nicely all the main features of the general model (4.85).

We note that the quantities (4.85), (4.89) possess the symmetry $\boldsymbol{n} \to -\boldsymbol{n}$. The anisotropy makes it possible to introduce mixed correlator $\langle \boldsymbol{v} f\rangle \propto \boldsymbol{n}\delta(t-t')\, C'(r/L)$ with some function $C'(r/L)$ analogous to $C(r/L)$ from Eq. (4.2). This violates the evenness in $\boldsymbol{n}$ and gives rise to non-vanishing odd functions S_{2n+1}. However, this leads to no serious alterations in the RG analysis; this case is discussed in Sec. 4.2.6, and for now we assume $\langle \boldsymbol{v} f\rangle = 0$.

In a number of papers, e.g. [74, 88, 89, 91, 93, 101, 102, 114], the artificial stirring force in Eq. (4.1) was replaced by the term $(\boldsymbol{h}\cdot\boldsymbol{v})$, where $\boldsymbol{h}$ is a constant vector that determines the distinguished direction and therefore introduces large-scale anisotropy. The anisotropy gives rise to non-vanishing odd functions S_{2n+1}. The critical dimensions of all composite operators remain unchanged, but the irreducible tensor operators acquire nonzero mean values and their contributions appear on the right hand side of Eq. (4.30); see [74, 114]. This is easily understood in the language of the zero-mode approach: the noise f and the term $(\boldsymbol{h}\cdot\boldsymbol{v})$ do not affect the differential operators in the equations satisfied by the equal-time correlations functions; the zero modes (homogeneous solutions) coincide in the two cases, but in the latter case the modes with nontrivial angular dependence should be taken into account.

The direct calculation to the order $\mathcal{O}(\varepsilon)$ has shown that the leading exponent associated with a given rank contribution to Eq. (4.1) decreases monotonically with the rank [74, 114]. Hence, the leading term of the inertial-range behavior of an even structure function is determined by the same exponent as was obtained isotropic model [112], while the exponents related to the tensor operators determine only subleading corrections. A similar hierarchy was established in Ref. [220] (see also [115]) for the magnetic field advected passively by the rapid-change velocity in the presence of a constant background field, and in [227–229] within the context of the Navier–Stokes turbulence.

In a number of papers, e.g., [235–239], the RG techniques were applied to the anisotropically driven Navier–Stokes equation, including passive advection and magnetic turbulence, with the expression (4.89) entering into the stirring force correlator. The detailed account can be found in Ref. [32], where some errors of the previous treatments were corrected. However, these studies have up to now been limited to the first stage, i.e., investigation of the existence and stability of the fixed points and calculation of the critical dimensions of basic quantities. Calculation of the anomalous exponents in those models remains an open problem.

4.2.2 Field theoretic formulation and the Dyson–Wyld equations

The stochastic problem is equivalent to the field theoretic model of the set of three fields $\Phi \equiv \{\theta', \theta, \boldsymbol{v}\}$ with action functional

$$S[\Phi] = \frac{1}{2}\theta' D_\theta \theta' + \theta' \left[-\partial_t - (\boldsymbol{v}\cdot\boldsymbol{\nabla}) + \nu_0 \boldsymbol{\nabla}^2 + \frac{1}{2} D_{vij}(0)\partial_i \partial_j \right] \theta - \frac{1}{2}\boldsymbol{v} D_v^{-1} \boldsymbol{v}. \quad (4.90)$$

Here D_θ and D_v are the correlators (4.2) and (4.83), respectively, and

$$D_{vij}(0) = D_0 \int \frac{\mathrm{d}^d \boldsymbol{q}}{(2\pi)^d} \frac{T_{ij}(\boldsymbol{q})}{(q^2+m^2)^{d/2+\varepsilon/2}} \quad (4.91)$$

is the diagonal term (in spatial variables) of the coefficient of the temporal δ function in the velocity pair correlation function (4.83).

The model (4.90) corresponds to a standard Feynman diagrammatic technique with the triple vertex (3.95), propagators (4.21) and

$$\langle \theta\theta \rangle_0 = C(k)\,(\omega^2 + \nu_0^2 k^4)^{-1}, \quad \langle \theta'\theta' \rangle_0 = 0, \quad (4.92)$$

where $C(k)$ is the Fourier transform of the function $C(r/L)$ from Eq. (4.2) and the bare propagator $\langle \boldsymbol{v}\boldsymbol{v} \rangle_0 \equiv \langle \boldsymbol{v}\boldsymbol{v} \rangle$ is given by Eq. (4.83) with the transverse projector from Eqs. (4.85) or (4.89).

The pair correlation functions $\langle \Phi\Phi \rangle$ of the multicomponent field $\Phi \equiv \{\theta', \theta, \boldsymbol{v}\}$ satisfy the standard Dyson equation, which in the component notation reduces to the system of two equations, cf. [79]

$$G^{-1}(\omega, \boldsymbol{k}) = -\mathrm{i}\omega + \nu_0 k^2 - \Sigma_{\theta'\theta}(\omega, \boldsymbol{k}), \quad (4.93)$$

$$D(\omega, \boldsymbol{k}) = |G(\omega, \boldsymbol{k})|^2 \, [C(k) + \Sigma_{\theta'\theta'}(\omega, \boldsymbol{k})], \quad (4.94)$$

where $G(\omega, \boldsymbol{k}) \equiv \langle \theta\theta' \rangle$ and $D(\omega, \boldsymbol{k}) \equiv \langle \theta\theta \rangle$ are the exact response function and pair correlator, respectively, and $\Sigma_{\theta'\theta}$, $\Sigma_{\theta'\theta'}$ are self-energy operators represented by the corresponding 1-irreducible diagrams; all the other functions $\Sigma_{\Phi\Phi}$ in the model (4.90) vanish identically.

The characteristic feature of the models like (4.90) is that all the skeleton multi-loop diagrams entering into the self-energy operators $\Sigma_{\theta'\theta}$, $\Sigma_{\theta'\theta'}$ contain effectively closed circuits of retarded propagators $\langle \theta\theta' \rangle$ (it is crucial here that the propagator $\langle \boldsymbol{v}\boldsymbol{v} \rangle_0$ in Eq. (4.83) is proportional to the δ function in time) and therefore vanish.

Therefore the self-energy operators in (4.93-4.94) are given by the one-loop approximation exactly and have the form

$$\Sigma_{\theta'\theta}(\omega, \boldsymbol{k}) = -\frac{D_0 \, k_i k_j}{2} \int \frac{\mathrm{d}^d \boldsymbol{q}}{(2\pi)^d} \frac{T_{ij}(\boldsymbol{q})}{(q^2+m^2)^{d/2+\varepsilon/2}}. \quad (4.95)$$

$$\Sigma_{\theta'\theta'}(\omega, \boldsymbol{k}) = D_0 \, k_i k_j \int \frac{\mathrm{d}\omega'}{2\pi} \int \frac{\mathrm{d}^d \boldsymbol{q}}{(2\pi)^d} \frac{T_{ij}(\boldsymbol{q})}{(q^2+m^2)^{d/2+\varepsilon/2}} \, D(\omega', \boldsymbol{q}'), \quad (4.96)$$

where $\boldsymbol{q}' \equiv \boldsymbol{k} - \boldsymbol{q}$.

The integration over ω' on the right-hand side of Eq. (4.96) gives the equal-time pair correlator

$$D(\boldsymbol{q}) = (1/2\pi) \int \mathrm{d}\omega'\, D(\omega', \boldsymbol{q}). \tag{4.97}$$

Note that both the self-energy operators are in fact independent of ω.

The integration of Eq. (4.94) over the frequency ω gives a closed equation for the equal-time correlator. Using Eq. (4.95) it can be written in the form

$$2\nu_0 k^2 D(\boldsymbol{k}) = C(k) + D_0\, k_i k_j \int \frac{\mathrm{d}^d \boldsymbol{q}}{(2\pi)^d}\, \frac{T_{ij}(\boldsymbol{q})}{(q^2+m^2)^{d/2+\varepsilon/2}} \Big[D(\boldsymbol{q}') - D(\boldsymbol{k}) \Big]. \tag{4.98}$$

Equation (4.98) can also be rewritten as a partial differential equation for the pair correlator in the coordinate representation, $D(\boldsymbol{r}) \equiv \langle \theta(t,\boldsymbol{x})\theta(t,\boldsymbol{x}+\boldsymbol{r})\rangle$ [we use the same notation D for the coordinate function and its Fourier transform]. Noting that the integral in Eq. (4.98) involves convolutions of the functions $D(\boldsymbol{k})$ and $D_0\, T_{ij}(\boldsymbol{q})/(q^2{+}m^2)^{d/2+\varepsilon/2}$, and replacing the momenta by the corresponding derivatives, $\mathrm{i}k_i \to \partial_i$ and so on, the following equation

$$2\nu_0 \partial^2 D(\boldsymbol{r}) + C(r/L) + D_0\, S_{ij}(\boldsymbol{r})\, \partial_i \partial_j\, D(\boldsymbol{r}) = 0 \tag{4.99}$$

can be obtained, where the "effective eddy diffusivity" is given by

$$S_{ij}(\boldsymbol{r}) \equiv \int \frac{\mathrm{d}^d \boldsymbol{q}}{(2\pi)^d}\, \frac{T_{ij}(\boldsymbol{q})}{(q^2+m^2)^{d/2+\varepsilon/2}} \Big[1 - \exp\left(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}\right) \Big]. \tag{4.100}$$

For $0 < \varepsilon < 2$, equations (4.98)–(4.100) allow for the limit $m \to 0$: the possible IR divergence of the integrals at $\boldsymbol{q} = 0$ is suppressed by the vanishing of the expressions in the square brackets. For the isotropic case (i.e., after the substitution $T_{ij} \to P_{ij}$) Eq. (4.99) coincides (up to the notation) with the well-known equation for the equal-time pair correlator in the model [77, 84, 240, 241].

4.2.3 Renormalization, RG functions, and RG equations

The analysis of the UV divergences is based on the analysis of canonical dimensions introduced in Sec. 2. The dimensions for the model (4.90) are given in Table 4.2.3, including the parameters

F	θ	θ'	$\boldsymbol{v}$	ν, ν_0	m, μ, Λ	g_0	g, α, α_0
d_F^k	0	d	-1	-2	1	ε	0
d_F^ω	$-1/2$	$1/2$	1	1	0	0	0
d_F	-1	$d+1$	1	0	1	ε	0

Table 4.2. Canonical dimensions of the fields and parameters in the model (4.101).

which will be introduced later on.

In the presence of anisotropy, it is necessary to also introduce new counterterm of the form $\theta'(\boldsymbol{n}\cdot\boldsymbol{\nabla})^2\theta$, which is absent in the unrenormalized action functional (4.90). Therefore, the model

(4.90) in its original formulation is not multiplicatively renormalizable, and in order to use the standard RG techniques it is necessary to extend the model by adding the new contribution to the unrenormalized action:

$$\mathcal{S}[\Phi] = \theta' D_\theta \theta'/2 + \theta' \left[-\partial_t - (\boldsymbol{v}\cdot\boldsymbol{\nabla}) + \nu_0 \boldsymbol{\nabla}^2 + \chi_0 \nu_0 (\boldsymbol{n}\cdot\boldsymbol{\nabla})^2\right] \theta - \boldsymbol{v} D_v^{-1} \boldsymbol{v}/2. \quad (4.101)$$

Here χ_0 is a new dimensionless unrenormalized parameter. The stability of the system implies the positivity of the total viscous contribution $\nu_0 k^2 + \chi_0 \nu_0 (\boldsymbol{n}\cdot\boldsymbol{k})^2$, which leads to the inequality $\chi_0 > -1$. Its real ("physical") value is zero, but this fact does not hinder the use of the RG techniques, in which it is first assumed to be arbitrary, and the equality $\chi_0 = 0$ is imposed as the initial condition in solving the equations for invariant variables (see Sec. 4.2.4). The zero value of χ_0 corresponds to certain nonzero value of its renormalized analog, which can be found explicitly.

For the action (4.101), the nontrivial bare propagators in (4.92) are replaced with

$$\langle \theta\theta' \rangle_0 = \langle \theta'\theta \rangle_0^* = \frac{1}{-\mathrm{i}\omega + \nu_0 k^2 + \chi_0 \nu_0 (\boldsymbol{n}\cdot\boldsymbol{k})^2}, \quad (4.102)$$

$$\langle \theta\theta \rangle_0 = \frac{C(k)}{|-\mathrm{i}\omega + \nu_0 k^2 + \chi_0 \nu_0 (\boldsymbol{n}\cdot\boldsymbol{k})^2|^2}\,. \quad (4.103)$$

After the extension, the model has become multiplicatively renormalizable: inclusion of the counterterms is reproduced by the inclusion of two independent renormalization constants $Z_{1,2}$ as coefficients in front of the counterterms. This leads to the renormalized action of the form

$$\mathcal{S}_R[\Phi] = \theta' D_\theta \theta'/2 + \theta' \left[-\partial_t - (\boldsymbol{v}\cdot\boldsymbol{\nabla}) + \nu Z_1 \boldsymbol{\nabla}^2 + \chi \nu Z_2 (\boldsymbol{n}\cdot\boldsymbol{\nabla})^2\right] \theta - \boldsymbol{v} D_v^{-1} \boldsymbol{v}/2, \quad (4.104)$$

or, equivalently, to the multiplicative renormalization of the parameters ν_0, g_0 and χ_0 in the action functional (4.101):

$$\nu_0 = \nu Z_\nu, \quad g_0 = g\mu^\varepsilon Z_g, \quad \chi_0 = \chi Z_\chi. \quad (4.105)$$

The correlator (4.83) in (4.104) is expressed in renormalized variables using Eqs. (4.105). The comparison of Eqs. (4.101), (4.104), and (4.105) leads to the relations

$$Z_1 = Z_\nu, \quad Z_2 = Z_\chi Z_\nu, \quad Z_g = Z_\nu^{-1}. \quad (4.106)$$

The beta functions are given by

$$\beta_g(g,\alpha) \equiv \widetilde{\mathcal{D}}_\mu g = g\,(-\varepsilon - \gamma_g) = g\,(-\varepsilon + \gamma_\nu) = g\,(-\varepsilon + \gamma_1), \quad (4.107)$$

$$\beta_\chi(g,\chi) \equiv \widetilde{\mathcal{D}}_\mu \chi = -\chi\gamma_\chi = \chi(\gamma_1 - \gamma_2). \quad (4.108)$$

The relation between β_g and γ_ν in Eq. (4.107) results from the definitions and the last relation in (4.106).

One-loop calculation [112] yields the following expressions for renormalization constants Z_1 and Z_2

$$Z_1 = 1 - \frac{g\bar{S}_d}{2d(d+2)\varepsilon}\Big[(d-1)(d+2) + \alpha_1(d+1) + \alpha_2\Big], \tag{4.109}$$

$$Z_2 = 1 - \frac{g\bar{S}_d}{2d(d+2)\chi\varepsilon}\Big[-2\alpha_1 + \alpha_2(d^2-2)\Big], \tag{4.110}$$

where the geometrical factor $\bar{S}_d$ stemming from angular integration has been introduced in Eq. (3.62).

For the anomalous dimension $\gamma_1(g)$ one obtains

$$\gamma_1(g) = \frac{-\varepsilon\mathcal{D}_g \ln Z_\nu}{1 - \mathcal{D}_g \ln Z_\nu} = \frac{g\bar{S}_d}{2d(d+2)}\Big[(d-1)(d+2) + \alpha_1(d+1) + \alpha_2\Big], \tag{4.111}$$

and for $\gamma_2(g,\alpha)$ one has

$$\gamma_2(g,\alpha) = \frac{\Big[(-\varepsilon+\gamma_1)\mathcal{D}_g + \gamma_1\mathcal{D}_\chi\Big]\ln Z_2}{1 + \mathcal{D}_\chi \ln Z_2} = \frac{-\varepsilon\mathcal{D}_g \ln Z_2}{1 + \mathcal{D}_\chi \ln Z_2} \tag{4.112}$$

$$= \frac{g\bar{S}_d}{2d(d+2)\chi}\big[-2\alpha_1 + \alpha_2(d^2-2)\big] \tag{4.113}$$

[note that $(\mathcal{D}_g + \mathcal{D}_\chi)\ln Z_2 = 0$]. The cancellation of the poles in ε in Eqs. (4.111) and (4.113) is a consequence of the UV finiteness of the anomalous dimensions γ_F; their independence of ε is a property of the MS scheme. Note also that the expressions (4.110)–(4.113) are exact, i.e., have no corrections of order g^2 and higher; this is a consequence of the fact that the one-loop approximation (4.95) for the self-energy operator is exact.

The coordinates g^*, χ^* of the fixed points are found from the equations

$$\beta_g(g^*,\chi^*) = \beta_\chi(g^*,\chi^*) = 0, \tag{4.114}$$

with the beta functions from Eqs. (4.107)-(4.108). The type of the fixed point is determined by the eigenvalues of the matrix Ω defined in Eq. (2.24). The IR asymptotic behavior is governed by the IR stable fixed points. From the explicit expressions (4.111), (4.113) it then follows that the RG equations of the model have the only IR stable fixed point with the coordinates

$$g^*\bar{S}_d = \frac{2d(d+2)\varepsilon}{(d-1)(d+2) + \alpha_1(d+1) + \alpha_2}, \quad \chi^* = \frac{-2\alpha_1 + \alpha_2(d^2-2)}{(d-1)(d+2) + \alpha_1(d+1) + \alpha_2}. \tag{4.115}$$

For this point, both the eigenvalues of the matrix Ω equal to ε; the values $\gamma_1^* = \gamma_2^* = \gamma_\nu^* = \varepsilon$ are also found exactly from Eqs. (4.107)-(4.108) [here and below, $\gamma_F^* \equiv \gamma_F(g^*,\chi^*)$]. The fixed point (4.115) is degenerate in the sense that its coordinates depend continuously on the anisotropy parameters $\alpha_{1,2}$.*

* Formally, $\alpha_{1,2}$ can be treated as the additional coupling constants. The corresponding beta functions $\beta_{1,2} \equiv \widetilde{\mathcal{D}}_\mu\alpha_{1,2}$ vanish identically owing to the fact that $\alpha_{1,2}$ are not renormalized. Therefore the equations $\beta_{1,2} = 0$ give no additional constraints on the values of the parameters g, χ at the fixed point.

4.2.4 Solution of the RG equations. Invariant variables

The solution of the RG equations is discussed in Sec. 2.4; below we confine ourselves to only the information we need.

Consider the solution of the RG equation on the example of the even different-time structure functions

$$S_{2n}(\boldsymbol{r},\tau) \equiv \langle[\theta(t,\boldsymbol{x}) - \theta(t',\boldsymbol{x}')]^{2n}\rangle, \quad \boldsymbol{r} \equiv \boldsymbol{x} - \boldsymbol{x}', \quad \tau \equiv t - t'. \tag{4.116}$$

It satisfies the RG equation $\mathcal{D}_{RG}S_{2n} = 0$ with the operator $\mathcal{D}_{RG} = \mathcal{D}_\mu + \beta_g\partial_g + \beta_\chi\partial_\chi - \gamma_{nu}\mathcal{D}_\nu$.

In renormalized variables, dimensionality considerations give

$$S_{2n}(\boldsymbol{r},\tau) = \nu^{-n}r^{2n}\tilde{R}_{2n}(\mu r, \tau\nu/r^2, r/L, g, \chi), \tag{4.117}$$

where $\tilde{R}_{2n}$ is a scaling function of completely dimensionless arguments (the dependence on d, ε, $\alpha_{1,2}$ and the angle between the vectors $\boldsymbol{r}$ and $\boldsymbol{n}$ is also implied). From the RG equation the identical representation follows,

$$S_{2n}(\boldsymbol{r},\tau) = (\bar{\nu})^{-n}r^{2n}\tilde{R}_{2n}(1, \tau\bar{\nu}/r^2, r/L, \bar{g}, \bar{\chi}), \tag{4.118}$$

where the invariant variables $\bar{e} = \bar{e}(\mu r, e)$ satisfy Eq. (2.18). The identity $\bar{L} \equiv L$ is a consequence of the fact that L is not renormalized. The relation between the bare and invariant variables has the form

$$\nu_0 = \bar{\nu}Z_\nu(\bar{g}), \quad g_0 = \bar{g}r^{-\varepsilon}Z_g(\bar{g}), \quad \chi_0 = \bar{\chi}Z_\chi(\bar{g},\bar{\chi}). \tag{4.119}$$

Equation (4.119) determines implicitly the invariant variables as functions of the bare parameters; it is valid because both sides of it satisfy the RG equation, and because Eq. (4.119) at $\mu r = 1$ coincides with (4.105) owing to the normalization conditions.

In general, the large μr behavior of the invariant variables is governed by the IR stable fixed point: $\bar{g} \to g^*$, $\bar{\chi} \to \chi^*$ for $\mu r \to \infty$. However, in multi-charge problems one has to take into account that even when the IR point exists, not every phase trajectory (i.e., solution of Eq. (4.119)) reaches it in the limit $\mu r \to \infty$. It may first pass outside the natural region of stability [physical region is given by the inequalities $g > 0$, $\chi > -1$] or go to infinity within this region. Fortunately, in our case the constants Z_F entering into Eq. (4.119) are known exactly from Eqs. (4.110), and it is readily checked that the RG flow indeed reaches the fixed point (4.115) for any initial conditions $g_0 > 0$, $\chi_0 > -1$, including the physical case $\chi_0 = 0$. Furthermore, the large μr behavior of the invariant viscosity $\bar{\nu}$ is also found explicitly from Eq. (4.119) and the last relation in (4.106): $\bar{\nu} = D_0 r^\varepsilon/\bar{g} \to D_0 r^\varepsilon/g^*$ (we recall that $D_0 = g_0\nu_0$). Then for $\mu r \to \infty$ and any fixed mr we obtain

$$S_{2n}(\boldsymbol{r},\tau) = D_0^{-n}r^{n(2-\varepsilon)}g^{*n}\,R_{2n}(\tau D_0 r^{\Delta_t}, r/L), \tag{4.120}$$

where the scaling function

$$R_{2n}(D_0\tau r^{\Delta_t}, r/L) \equiv \tilde{R}_{2n}(1, D_0\tau r^{\Delta_t}, r/L, g_*, \alpha_*), \tag{4.121}$$

has appeared and $\Delta_t \equiv -2 + \gamma_\nu^* = -2 + \varepsilon$ is the critical dimension of time. For the equal-time structure function (4.27), the first argument of R_{2n} in the representation (4.121) is absent:

$$S_{2n}(\boldsymbol{r}) = D_0^{-n} r^{n(2-\varepsilon)} g^{*n} R_{2n}(r/L), \tag{4.122}$$

where the definition of R_{2n} is obvious from (4.121). It is noteworthy that Eqs. (4.120)–(4.122) prove the independence of the structure functions in the IR range (large μr and any r/L) of the viscosity coefficient or, equivalently of the UV scale: the parameters g_0 and ν_0 enter into Eq. (4.120) only in the form of the combination $D_0 = g_0\nu_0$. A similar property was established in Ref. [242] for the stirred Navier–Stokes equation.

In contrast to the previously mentioned models the scaling function $\tilde{R}$ in (4.118) contains two different scales - corresponding to spatial and time differences, respectively. The information about its behavior can be again using OPE method from Sec. 2.7. Therefore, let $F(r,\tau)$ stands for some multiplicatively renormalized quantity. Dimensionality considerations give

$$F_R(r,\tau) = \nu^{d_F^\omega} r^{-d_F} \tilde{R}_F(\mu r, \tau\nu/r^2, r/L, g, \alpha), \tag{4.123}$$

where d_F^ω and d_F are the frequency and total canonical dimensions of F (see Sec. 2.3) and R_F is a function of dimensionless arguments. The analog of Eq. (4.118) has the form

$$F(r,\tau) = Z_F(g,\alpha)F_R = Z_F(\bar{g},\bar{\alpha})\,(\bar{\nu})^{d_F^\omega} r^{-d_F} \tilde{R}_F(1, \tau\bar{\nu}/r^2, r/L, \bar{g}, \bar{\alpha}). \tag{4.124}$$

In the large μr limit, one has $Z_F(\bar{g},\bar{\alpha}) \simeq \mathrm{const}\,(\Lambda r)^{-\gamma_F^*}$; see, e.g., [243]. The UV scale appears in this relation from Eq. (4.84). Then in the IR range ($\Lambda r \sim \mu r$ large, r/L arbitrary) Eq. (4.124) takes on the form

$$F(r,\tau) \simeq \mathrm{const}\,\Lambda^{-\gamma_F^*} D_0^{d_F^\omega} r^{-\Delta[F]} R_F(D_0\tau r^{\Delta_t}, r/L). \tag{4.125}$$

Here

$$\Delta[F] \equiv \Delta_F = d_F^k - \Delta_t d_F^\omega + \gamma_F^*, \quad \Delta_t = -2 + \varepsilon \tag{4.126}$$

is the critical dimension of the function F and the scaling function R_F is related to $\tilde{R}_F$ as in Eq. (4.120). For nontrivial γ_F^*, the function F in the IR range retains the dependence on Λ or, equivalently, on ν_0.

4.2.5 Renormalization and critical dimensions of composite operators

Operators of the form $\theta^N(x)$ with the canonical dimension $d_F = -N$ enter into the structure functions (4.27). From Table 4.2.3 in Sec. 4.2.3 and Eq. (2.49) the relation $d_\Gamma = -N + N_\theta - N_{\boldsymbol{v}} - (d+1)N_{\theta'}$ can be obtained, and from the analysis of the diagrams it follows that the total number of the fields θ entering into the function Γ can never exceed the number of the fields θ in the operator θ^N itself, i.e., $N_\theta \le N$ (cf. item (i) in Sec. 4.2.3). Therefore, the divergence can only exist in the functions with $N_{\boldsymbol{v}} = N_{\theta'} = 0$, and arbitrary value of $N = N_\theta$, for which the formal index vanishes, $d_\Gamma = 0$. However, at least one of N_θ external "tails" of the field θ is attached to a vertex $\theta'(\boldsymbol{v}\cdot\boldsymbol{\nabla})\theta$ (it is impossible to construct nontrivial, superficially divergent diagram of the desired type with all the external tails attached to the vertex F), at least

one derivative ∂ appears as an extra factor in the diagram, and, consequently, the real index of divergence d'_Γ is necessarily negative.

This means that the operator θ^N requires no counterterms at all, i.e., it is in fact UV finite, $\theta^N = Z\,[\theta^N]^R$ with $Z = 1$. It then follows that the critical dimension of $\theta^N(x)$ is simply given by the expression (4.126) with no correction from γ_F^* and is therefore reduced to the sum of the critical dimensions of the factors:

$$\Delta[\theta^N] = N\Delta[\theta] = N(-1 + \varepsilon/2). \tag{4.127}$$

Since the structure functions (4.27) or (4.116) are linear combinations of pair correlators involving the operators θ^N, equation (4.127) shows that they indeed satisfy the RG equation of the form (2.12), discussed in Sec. 4.2.4. We stress that the relation (4.127) was not clear *a priori*; in particular, it is violated if the velocity field becomes non-solenoidal [113].

In the following, an important role will be also played by the tensor composite operators $\partial_{i_1}\theta\cdots\partial_{i_p}\theta\,(\partial_i\theta\partial_i\theta)^n$ constructed solely of the scalar gradients. It is convenient to deal with the scalar operators obtained by contracting the tensors with the appropriate number of the vectors $\boldsymbol{n}$,

$$F[N,p] \equiv [(\boldsymbol{n}\cdot\boldsymbol{\nabla})\theta]^p(\partial_i\theta\partial_i\theta)^n, \quad N \equiv 2n + p. \tag{4.128}$$

Their canonical dimensions depend only on the total number of the fields θ and have the form $d_F = 0$, $d_F^\omega = -N$.

In this case, from Table 4.2.3 and Eq. (2.49) we obtain $d_\Gamma = N_\theta - N_{\boldsymbol{v}} - (d+1)N_{\theta'}$, with the necessary condition $N_\theta \le N$, which follows from the structure of the diagrams. It is also clear from the analysis of the diagrams that the counterterms to these operators can involve the fields θ, θ' only in the form of derivatives, $\partial\theta$, $\partial\theta'$, so that the real index of divergence has the form $d'_\Gamma = d_\Gamma - N_\theta - N_{\theta'} = -N_{\boldsymbol{v}} - (d+2)N_{\theta'}$. It then follows that superficial divergences can exist only in the Green functions with $N_{\boldsymbol{v}} = N_{\theta'} = 0$ and any $N_\theta \le N$, and that the corresponding operator counterterms reduce to the form $F[N',p']$ with $N' \le N$. Therefore, the operators (4.128) can mix only with each other in renormalization, and the corresponding infinite renormalization matrix

$$F[N,p] = \sum_{N',p'} Z_{[N,p]\,[N',p']}\,F^R[N',p'] \tag{4.129}$$

is in fact block-triangular, i.e., $Z_{[N,p]\,[N',p']} = 0$ for $N' > N$. It is then obvious that the critical dimensions associated with the operators $F[N,p]$ are completely determined by the eigenvalues of the finite subblocks with $N' = N$.

In the isotropic case, as well as in the presence of large-scale anisotropy, the elements $Z_{[N,p]\,[N,p']}$ vanish for $p < p'$, and the block $Z_{[N,p]\,[N,p']}$ is triangular along with the corresponding blocks of the matrices U_F and Δ_F from Eqs. (2.47), (4.126). In the isotropic case it can be diagonalized by changing to irreducible operators (scalars, vectors, and traceless tensors), but even for nonzero imposed gradient its eigenvalues are the same as in the isotropic case. Therefore, the inclusion of large-scale anisotropy does not affect critical dimensions of the operators (4.128); see [74, 114]. In the case of small-scale anisotropy, the operators with different values of p mix heavily in renormalization, and the matrix $Z_{[N,p]\,[N,p']}$ is neither diagonal nor triangular here.

The calculation of the renormalization constants $Z_{[N,p]\,[N,p']}$ can be illustrated within the one-loop approximation. Let $\Gamma(x;\theta)$ be the generating functional of the 1-irreducible Green functions with one composite operator $F[N,p]$ from Eq. (4.128) and any number of fields θ. Here x is the argument of the operator and θ is the functional argument, the "classical counterpart" of the random field θ. The general interest is in the N-th term of the expansion of $\Gamma(x;\theta)$ in θ, which is denoted as $\Gamma_N(x;\theta)$; it has the form

$$\Gamma_N(x;\theta) = \frac{1}{N!}\int \mathrm{d}x_1 \cdots \int \mathrm{d}x_N\, \theta(x_1)\cdots\theta(x_N)\,\langle F[N,p](x)\theta(x_1)\cdots\theta(x_N)\rangle_{1-\mathrm{ir}}. \quad (4.130)$$

The matrix of critical dimensions (4.126) is given in the one-loop approximation by the expression

$$\Delta_{[N,p][N,p']} = N\varepsilon/2 + \gamma^*_{[N,p][N,p']}, \quad (4.131)$$

where the asterisk implies the substitution (4.115). The details of calculation of $\gamma_{[N,p][N,p']}$ can be found in [112].

As already said above, the critical dimensions themselves are given by the eigenvalues of the matrix (4.131). One can check that for the isotropic case ($\alpha_{1,2} = 0$), its elements with $p' > p$ vanish, the matrix becomes triangular, and its eigenvalues are simply given by the diagonal elements $\Delta[N,p] \equiv \Delta_{[N,p][N,p]}$. They are found explicitly and have the form

$$\Delta[N,p] = N\varepsilon/2 + \frac{2p(p-1) - (d-1)(N-p)(d+N+p)}{2(d-1)(d+2)}\,\varepsilon + \mathcal{O}(\varepsilon^2). \quad (4.132)$$

It is easily seen from Eq. (4.132) that for fixed N and any $d \geq 2$, the dimension $\Delta[N,p]$ decreases monotonically with p and reaches its minimum for the minimal possible value of $p = p_N$, i.e., $p_N = 0$ if N is even and $p_N = 1$ if N is odd:

$$\Delta[N,p] > \Delta[N,p'] \quad \text{if} \quad p > p'\,. \quad (4.133)$$

Furthermore, this minimal value $\Delta[N,p_N]$ decreases monotonically as N increases for odd and even values of N separately, i.e.,

$$0 \geq \Delta[2n,0] > \Delta[2n+2,0], \quad \Delta[2n+1,1] > \Delta[2n+3,1]. \quad (4.134)$$

A similar hierarchy is demonstrated by the critical dimensions of certain tensor operators in the stirred Navier–Stokes turbulence; see Ref. [244] and Sec. 2.3 of [32]. However, no clear hierarchy is demonstrated by neighboring even and odd dimensions: from the relations

$$\Delta[2n+1,1] - \Delta[2n,0] = \frac{\varepsilon(d+2-4n)}{2(d+2)}, \quad \Delta[2n+2,0] - \Delta[2n+1,1] = \frac{\varepsilon(2-d)}{2(d+2)} \quad (4.135)$$

it follows that the inequality $\Delta[2n+1,1] > \Delta[2n+2,0]$ holds for any $d > 2$, while the relation $\Delta[2n,0] > \Delta[2n+1,1]$ holds only if n is sufficiently large, $n > (d+2)/4$. ⋆

⋆The situation is different in the presence of the linear mean gradient: the first term $N\varepsilon/2$ in Eq. (4.132) is then absent owing to the difference in canonical dimensions, and the complete hierarchy relations hold, $\Delta[2n,0] > \Delta[2n+1,1] > \Delta[2n+2,0]$; see [74, 114].

In what follows, we shall use the notation $\Delta[N,p]$ for the eigenvalue of the matrix (4.131) which coincides with (4.132) for $\alpha_{1,2}=0$. Since the eigenvalues depend continuously on $\alpha_{1,2}$, this notation is unambiguous at least for small values of $\alpha_{1,2}$.

The dimension $\Delta[2,0]$ vanishes identically for any $\alpha_{1,2}$ and to all orders in ε. Like in the isotropic model, this can be demonstrated using the Schwinger equation of the form

$$\int \mathcal{D}\Phi \frac{\delta}{\delta\theta'(x)}\left[\theta(x)\mathrm{e}^{\mathcal{S}_R[\Phi]+A\Phi}\right]=0, \tag{4.136}$$

(in the general sense of the word, Schwinger equations are any relations stating that any functional integral of a total variational derivative is equal to zero; see, e.g., [24, 25]). In (4.136), S_R is the renormalized action (4.104) and Eq. (4.136) can be rewritten in the form

$$\Big\langle \theta' D_\theta \theta - \nabla_t[\theta^2/2] + \nu Z_1 \Delta[\theta^2/2] + \alpha\nu Z_2 (\boldsymbol{n}\cdot\boldsymbol{\nabla})^2[\theta^2/2] - \nu Z_1 F[2,0] = \\ - \alpha\nu Z_2 F[2,2]\Big\rangle_A - A_{\theta'}\frac{\delta W_R(A)}{\delta A_\theta}. \tag{4.137}$$

Here D_θ is the correlator (4.2), $\langle\cdots\rangle_A$ denotes the averaging with the weight $\exp[\mathcal{S}_R[\Phi]+A\Phi]$, $\mathcal{W}_R$ is determined by Eq. (1.80) with the replacement $\mathcal{S}\to\mathcal{S}_R$, and the argument x common to all the quantities in (4.137) is omitted.

The quantity $\langle F\rangle_A$ is the generating functional (defined in Eq. (2.44)) of the correlation functions with one insertion of the operator F and any number of the primary fields Φ, therefore the UV finiteness of the operator F is equivalent to the finiteness of the functional $\langle F\rangle_A$. The quantity in the right hand side of Eq. (4.137) is UV finite (a derivative of the renormalized functional with respect to finite argument), and so is the operator in the left hand side. Operators $F[2,0]$, $F[2,2]$ do not admix in renormalization to $\theta' D_\theta\theta$ (no needed diagrams can be constructed), and to the operators $\nabla_t[\theta^2/2]$ and $\boldsymbol{\nabla}^2[\theta^2/2]$ (they have the form of total derivatives, and $F[N,p]$ do not reduce to this form). On the other hand, all the operators in (4.137) other than $F[N,p]$ do not admix to $F[N,p]$, because the counterterms of the operators (4.128) can involve only operators of the same type; see above. Therefore, the operators $F[N,p]$ entering into Eq. (4.137) are independent of the others, and so they must be UV finite separately: $\nu Z_1 F[2,0]+\alpha\nu Z_2 F[2,2]=$ UV finite. Since the operator in (4.137) is UV finite, it coincides with its finite part,

$$\nu Z_1 F[2,0]+\alpha\nu Z_2 F[2,2]=\nu F^R[2,0]+\alpha\nu F^R[2,2], \tag{4.138}$$

which along with the relation (4.129) gives

$$Z_1 Z_{[2,0][2,0]}+\alpha Z_2 Z_{[2,2][2,0]}=1, \quad Z_1 Z_{[2,0][2,2]}+\alpha Z_2 Z_{[2,2][2,2]}=\alpha, \tag{4.139}$$

and therefore for the anomalous dimensions in the MS scheme one obtains

$$\gamma_1+\gamma_{[2,0][2,0]}+\alpha\gamma_{[2,2][2,0]}=0, \quad \gamma_{[2,0][2,2]}+\alpha\gamma_2+\alpha\gamma_{[2,2][2,2]}=0. \tag{4.140}$$

Bearing in mind that $\gamma_1^*=\gamma_2^*=\varepsilon$ (see Sec. 4.2.3), the conclusion can be made that among the four elements of the matrix γ_F^* only two, which we take to be $\gamma^*_{[2,2][2,0]}$ and $\gamma^*_{[2,2][2,2]}$, are independent. Then the matrix of critical dimensions (4.132) takes on the form

$$\Delta_{[2,p][2,p']}=\varepsilon+\begin{pmatrix} -\varepsilon-\alpha_*\gamma^*_{[2,2][2,0]} & -\alpha_*\varepsilon-\alpha_*\gamma^*_{[2,2][2,2]} \\ \gamma^*_{[2,2][2,0]} & \gamma^*_{[2,2][2,2]}\end{pmatrix}. \tag{4.141}$$

It is then easily checked that the eigenvalue of the matrix (4.141), which is identified with $\Delta[2,0]$, does not involve unknown anomalous dimensions and vanishes identically, $\Delta[2,0] \equiv 0$, while the second one is represented as

$$\Delta[2,2] = \varepsilon - \alpha_* \gamma^*_{[2,2][2,0]} + \gamma^*_{[2,2][2,2]}. \tag{4.142}$$

Using the explicit $\mathcal{O}(\varepsilon)$ expressions [112] one obtains to the order $\mathcal{O}(\varepsilon)$:

$$\Delta[2,2]/\varepsilon = 2 + \Big\{-(d-2)d(d+2)(d+4)F_0^* - (d+2)(d+4)(2+(d-2)\alpha_1 + d\alpha_2)F_1^* + +3(d+4)(d-2\alpha_1+2d\alpha_2)F_2^* + 15d(\alpha_1-\alpha_2)F_3^*\Big\}\Big/\Big\{(d-1) \times (d+4)[(d-1)(d+2)+(d+1)\alpha_1+\alpha_2]\Big\}, \tag{4.143}$$

where $F_n^* \equiv F(1, 1/2+n; d/2+n; -\alpha_*)$ with α_* from Eq. (4.115).

In Fig. 4.10, we present the levels of the dimension (4.143) on the (α_1, α_2)-plane for $d = 3$. We note that the dependence on $\alpha_{1,2}$ is quite smooth, and that $\Delta[2,2]$ remains positive on the whole of the (α_1, α_2)-plane, i.e., the first of the hierarchy relations (4.133)-(4.135) remains valid also in the presence of anisotropy. A similar behavior takes place also for $d = 2$.

For $N > 2$, the eigenvalues can be found analytically only within the expansion in $\alpha_{1,2}$. The explicit expressions can be found in [112]. They illustrate two facts which seem to hold for all N:

(i) The leading anisotropy correction is of order $\mathcal{O}(\alpha_{1,2})$ for $p \neq 0$ and $\mathcal{O}(\alpha_{1,2}^2)$ for $p = 0$, so that the dimensions $\gamma^*[N,0]$ are anisotropy independent in the *linear* approximation, and

(ii) This leading contribution depends on $\alpha_{1,2}$ only through the combination $\alpha_3 \equiv 2\alpha_1 + d\alpha_2$.

This conjecture is confirmed by the following expressions for $N = 6, 8$ and $p = 0$:

$$\gamma^*[6,0]/\varepsilon = \frac{-2(d+6)}{(d+2)} - \frac{12(d-2)^2(d+1)(d^2+14d+48)\alpha_3^2}{(d-1)^2 d(d+2)^4(d+4)^2}, \tag{4.144}$$

$$\gamma^*[8,0]/\varepsilon = \frac{-4(d+8)}{(d+2)} - \frac{24(d-2)^2(d+1)(d^2+18d+80)\alpha_3^2}{(d-1)^2 d(d+2)^4(d+4)^2}. \tag{4.145}$$

The eigenvalues beyond the small $\alpha_{1,2}$ expansion have been obtained numerically [112]. Some of them are presented in Figs. 4.11-4.14, namely, the dimensions $\Delta[n,p]$ for $n = 3, 4, 5, 6$ *vs* α_1 for $\alpha_2 = 0$, *vs* $\alpha_1 = \alpha_2$, and *vs* α_2 for $\alpha_1 = 0$. The main conclusion that can be drawn from these diagrams is that the hierarchy (4.133-4.135) demonstrated by the dimensions for the isotropic case ($\alpha_{1,2} = 0$) holds valid for all the values of the anisotropy parameters.

4.2.6 Operator product expansion and anomalous scaling

From the operator product expansion (Sec. 2.7) we find the following expression for the scaling function $R(r/L)$ in the representation (4.122) for the correlator $\langle F_1(x) F_2(x')\rangle$:

$$R(r/L) = \sum_F A_F \left(\frac{r}{L}\right)^{\Delta_F}, \quad r \ll L, \tag{4.146}$$

with the coefficients A_F regular in $(r/L)^2$.

Now let us turn to the equal-time structure functions S_N from (4.27). From it is assumed that the mixed correlator $\langle \boldsymbol{v} f \rangle$ differs from zero (see Sec. 4.2.1); this does not affect the critical dimensions, but gives rise to non-vanishing odd structure functions. In general, the operators entering into the OPE are those which appear in the corresponding Taylor expansions, and also all possible operators that admix to them in renormalization [24,25]. The leading term of the Taylor expansion for the function S_N is obviously given by the operator $F[N,N]$ from Eq. (4.128); the renormalization gives rise to all the operators $F[N',p]$ with $N' \leq N$ and all possible values of p. The operators with $N' > N$ (whose contributions would be more important) do not appear in Eq. (4.146), because they do not enter into the Taylor expansion for S_N and do not admix in renormalization to the terms of the Taylor expansion; see Sec. 4.2.5. Therefore, combining the RG representation (4.120) with the OPE representation (4.146) gives the desired asymptotic expression for the structure function in the inertial range:

$$S_N(\boldsymbol{r}) = D_0^{-N/2} r^{N(1-\varepsilon/2)} \sum_{N' \leq N} \sum_{p} \Big\{ C_{N',p} \, (r/L)^{\Delta[N',p]} + \cdots \Big\} . \tag{4.147}$$

The second summation runs over all values of p, allowed for a given N'; $C_{N',p}$ are numerical coefficients dependent on ε, d, $\alpha_{1,2}$ and the angle ϑ between $\boldsymbol{r}$ and $\boldsymbol{n}$. The dots stand for the contributions of the operators other than $F[N,p]$, for example, $\partial^2\theta\partial^2\theta$; they give rise to the terms of order $(r/L)^{2+\mathcal{O}(\varepsilon)}$ and higher and will be neglected in what follows.

Some remarks are now in order.

(i) If the mixed correlator $\langle \boldsymbol{v} f \rangle$ is absent, the odd structure functions vanish, while the contributions to even functions are given only by the operators with even values of N'. In the isotropic case ($\alpha_{1,2} = 0$) only the contributions with $p = 0$ survive; see [110, 112]. In the presence of the anisotropy, $\alpha_{1,2} \neq 0$, the operators with $p \neq 0$ acquire nonzero mean values, and their dimensions $\Delta[N',p]$ also appear on the right hand side of Eq. (4.147).

(ii) The leading term of the small r/L behavior is obviously given by the contribution with the minimal possible value of $\Delta[N',p]$. Now we recall the hierarchy relations (4.133), (4.134), which hold for $\alpha_{1,2} = 0$ and therefore remain valid at least for $\alpha_{1,2} \ll 1$. This means that, if the anisotropy is weak enough, the leading term in Eq. (4.147) is given by the dimension $\Delta[N,0]$ for any S_N. For all the special cases studied in Sec. 4.2.5, this hierarchy persists also for finite values of the anisotropy parameters, and the contribution with $\Delta[N,0]$ remains the leading one for such N and $\alpha_{1,2}$.

(iii) Of course, it is not impossible that the inequalities (4.133), (4.134) break down for some values of n, d and $\alpha_{1,2}$, and the leading contribution to Eq. (4.147) is determined by a dimension with $N' \neq N$ and/or $p > 0$.

Furthermore, it is not impossible that the matrix (4.131) for some $\alpha_{1,2}$ had a pair of complex conjugate eigenvalues, Δ and Δ^*. Then the small r/L behavior of the scaling function $\xi(r/L)$ entering into Eq. (4.147) would involve oscillating terms of the form

$$(r/L)^{\mathrm{Re}\,\Delta} \Big\{ C_1 \cos\big[\mathrm{Im}\,\Delta \, (r/L)\big] + C_2 \sin\big[\mathrm{Im}\,\Delta \, (r/L)\big] \Big\},$$

with some constants C_i.

Another exotic situation emerges if the matrix (4.131) cannot be diagonalized and is only reduced to the Jordan form. In this case, the corresponding contribution to the scaling function would involve a logarithmic correction to the powerlike behavior, $(r/L)^{\Delta}\left[C_1 \ln(r/L) + C_2\right]$, where Δ is the eigenvalue related to the Jordan cell. However, these interesting hypothetical possibilities are not actually realized for the special cases studied above in Sec. 4.2.5.

(iv) The inclusion of the mixed correlator $\langle \boldsymbol{v} f\rangle \propto \boldsymbol{n}\delta(t-t')\, C'(r/L)$ violates the evenness in $\boldsymbol{n}$ and gives rise to non-vanishing odd functions S_{2n+1} and to the contributions with odd N' to the expansion (4.147) for even functions. If the hierarchy relations (4.133), (4.134) hold, the leading term for the even functions will still be given by the contribution with $\Delta[N,0]$. If the relations (4.135) hold, the leading term for the odd function S_{2n+1} will be given by the dimension $\Delta[2n,0]$ for $n < (d+2)/4$ and by $\Delta[2n+1,1]$ for $n > (d+2)/4$. Note that for the model with an imposed gradient, the leading terms for S_{2n+1} are given by the dimensions $\Delta[2n+1,1]$ for all n; see [74, 114].

Representations similar to Eqs. (4.122), (4.147) can easily be written down for any equal-time pair correlator, provided its canonical and critical dimensions are known. In particular, for the operators $F[N,p]$ in the IR region ($\Lambda r \to \infty$, r/L fixed) one obtains

$$\langle F[N_1,p_1]F[N_2,p_2]\rangle = \nu_0^{-(N_1+N_2)/2} \sum_{N,p} \sum_{N',p'} (\Lambda r)^{-\Delta_{[N,p]}-\Delta_{[N',p']}} R_{N,p;N',p'}(r/L), \quad (4.148)$$

where the summation indices N, N' satisfy the inequalities $N \le N_1$, $N' \le N_2$, and the indices p, p' take on all possible values allowed for given N, N'. The small r/L behavior of the scaling functions $R_{N,p;N',p'}(r/L)$ has the form

$$\xi_{N,p;N',p'}(mr) = \sum_{N'',p''} C_{N'',p''}\,(r/L)^{\Delta[N'',p'']}, \quad (4.149)$$

with the restriction $N'' \le N + N'$ and corresponding values of p''; $C_{N'',p''}$ are some numerical coefficients.

So far, we have discussed the special case of the velocity correlator given by Eqs. (4.83) and (4.89). From the explicit calculations [112] follows that only even polynomials in the expansion (4.87) can give contributions to the renormalization constants, and consequently, to the coordinates of the fixed point and the anomalous dimensions. For this reason, the odd polynomials were omitted in Eq. (4.87) from the very beginning. Moreover, it is clear from Eq. (4.95) that only the coefficients a_l with $l = 0, 1$ and b_l with $l = 0, 1, 2$ contribute to the constants $Z_{1,2}$ in Eq. (4.104) and therefore to the basic RG functions (4.107) and (4.108) to the coordinates of the fixed point in Eq. (4.115). Therefore, the fixed point in the general model (4.85) is parametrized completely by these five coefficients; the higher coefficients enter only via the positivity conditions (4.88).

Furthermore, for $\chi = 0$, only coefficients a_l with $l \le 2$ and b_l with $l \le 3$ can contribute to the integrals H_n and, consequently, to the one-loop critical dimensions (4.131). Therefore, the calculation of the latter essentially simplifies for the special case $a_0 = 1$, $a_1 = 0$ and $b_l = 0$ for $l \le 2$ in Eq. (4.85). Then the coordinates of the fixed point (4.115) are the same as in the isotropic model, in particular, $\alpha_* = 0$, and the anomalous exponents will depend on the only two parameters a_2 and b_3. We have performed a few sample calculations for this situation; the results are presented in Figs. 6–9 for $\Delta[n,p]$ with $n = 3, 4, 5, 6$ *vs* a_2 for $b_3 = 0$, *vs* $a_2 = b_3$, and

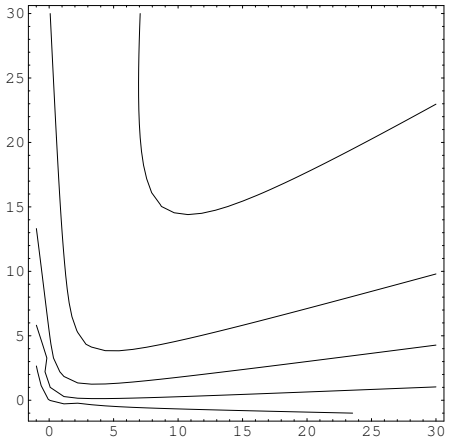


Figure 4.10. Levels of the dimension $\Delta[2,2]$ for $d = 3$ on the plane α_1–α_2. Value changes from 1.15 (left-bottom) to 1.4 (right-top) with step 0.05.

vs b_3 for $a_2 = 0$. In all cases studied, the general picture has appeared similar to that outlined above for the case (4.89). In particular, the hierarchy of the critical dimensions, expressed by the inequalities (4.133)-(4.135), persists also for this case. We may conclude that the special case (4.89) case represents nicely all the main features of the general model (4.85).

The exponents are determined by the critical dimensions of composite operators (4.128) built of the scalar gradients. In contrast with the isotropic velocity field, these operators in the model under consideration mix in renormalization such that the matrices of their critical dimensions are neither diagonal nor triangular. These matrices are calculated explicitly to the order $\mathcal{O}(\varepsilon)$, but their eigenvalues (anomalous exponents) can be found explicitly only as series in $\alpha_{1,2}$ [Eqs. (4.144), (4.145)] or numerically [Figs. 4.10–4.18].

In the limit of vanishing anisotropy, the exponents can be associated with definite tensor composite operators built of the scalar gradients, and exhibit a kind of hierarchy related to the degree of anisotropy: the less is the rank, the less is the dimension and, consequently, the more important is the contribution to the inertial-range behavior [see Eqs. (4.133)-(4.135)].

The leading terms of the even (odd) structure functions are given by the scalar (vector) operators. For the finite anisotropy, the exponents cannot be associated with individual operators (which are essentially "mixed" in renormalization), but, surprising enough, the aforementioned hierarchy survives for all the cases studied, as is shown in Figs. 4.11–4.18.

The short comment about the second-order structure function $S_2(\boldsymbol{r})$ is appropriate. It can be studied using the RG and zero-mode techniques [112]; like in the isotropic case [77,84,240,241], its leading term has the form $S_2 \propto r^{2-\varepsilon}$, but the amplitude now depends on $\alpha_{1,2}$ and the angle

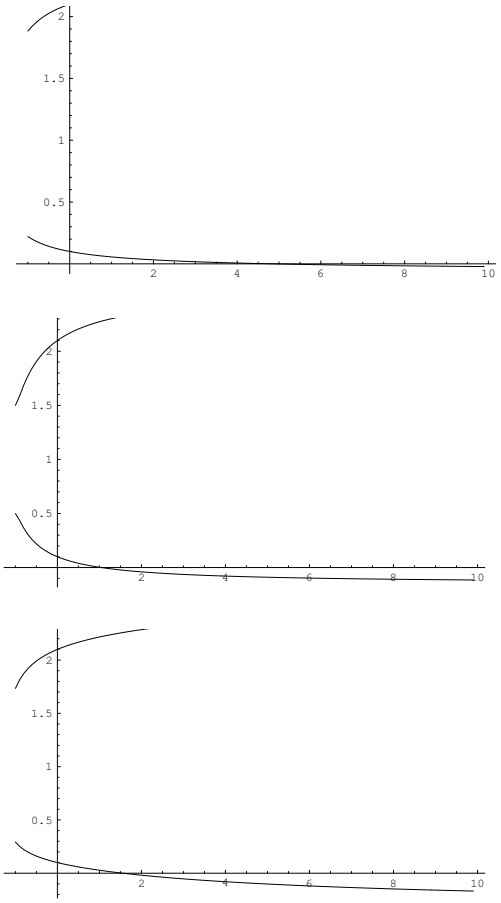


Figure 4.11. Behavior of the critical dimension $\Delta[3,p]$ for $d = 3$ with $p = 1, 3$ (from below to above) *vs* α_1 for $\alpha_2 = 0$—*top*, *vs* $\alpha \equiv \alpha_1 = \alpha_2$—*center*, *vs* α_2 for $\alpha_1 = 0$—*bottom*.

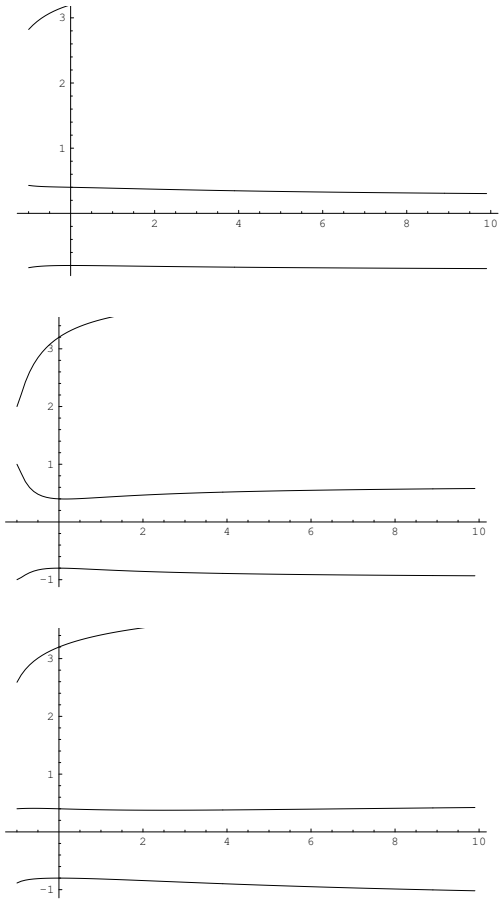


Figure 4.12. Behavior of the critical dimension $\Delta[4,p]$ for $d = 3$ with $p = 0, 2, 4$ (from below to above) *vs* α_1 for $\alpha_2 = 0$—*top*, *vs* $\alpha \equiv \alpha_1 = \alpha_2$—*center*, *vs* α_2 for $\alpha_1 = 0$—*bottom*.

between the vectors $\boldsymbol{r}$ and $\boldsymbol{n}$ from Eq. (4.89). The first anisotropic correction has the form $(r/L)^{\Delta[2,2]}$ with the exponent $\Delta[2,2] = O(\varepsilon)$ from Eq. (4.143).

It is well known that, for the isotropic velocity field, the anisotropy introduced at large scales by the external forcing or imposed mean gradient, persists in the inertial range and reveals itself in *odd* correlation functions: the skewness factor $S_3/S_2^{3/2}$ decreases for $r/L \to 0$ but slowly (see Refs. [86–89, 89, 91, 93, 101, 102, 230]), while the higher-order ratios $S_{2n+1}/S_2^{n+1/2}$ increase (see, e.g., [74, 114, 115, 219]).

In the case at hand, the inertial-range behavior of the skewness is given by $S_3/S_2^{3/2} \propto (r/L)^{\Delta[3,1]}$. For $\alpha_{1,2} \to 0$, the exponent $\Delta[3,1]$ is given by Eq. (4.132) with $n = 3$ and $p = 1$; it is positive and coincides with the result of Ref. [89, 91]. The levels of the dimension

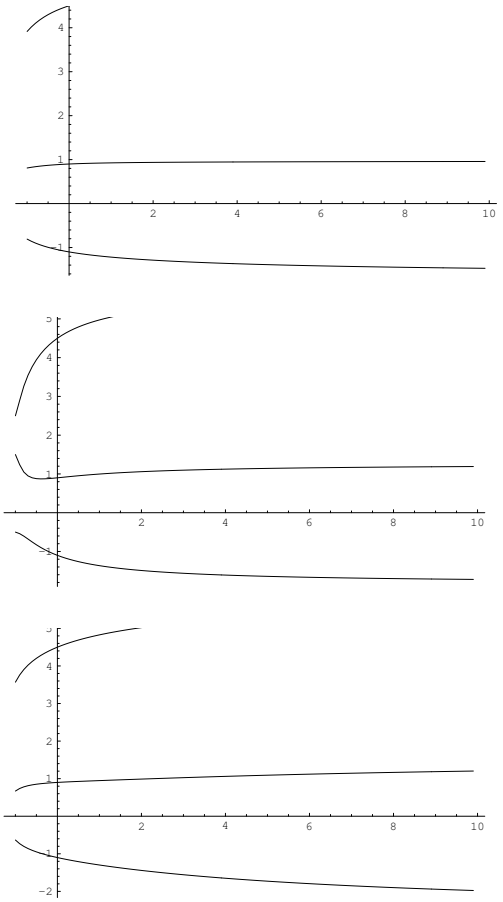


Figure 4.13. Behavior of the critical dimension $\Delta[5,p]$ for $d=3$ with $p=1,3,5$ (from below to above) *vs* α_1 for $\alpha_2=0$—*top*, *vs* $\alpha\equiv\alpha_1=\alpha_2$—*center*, *vs* α_2 for $\alpha_1=0$—*bottom*.

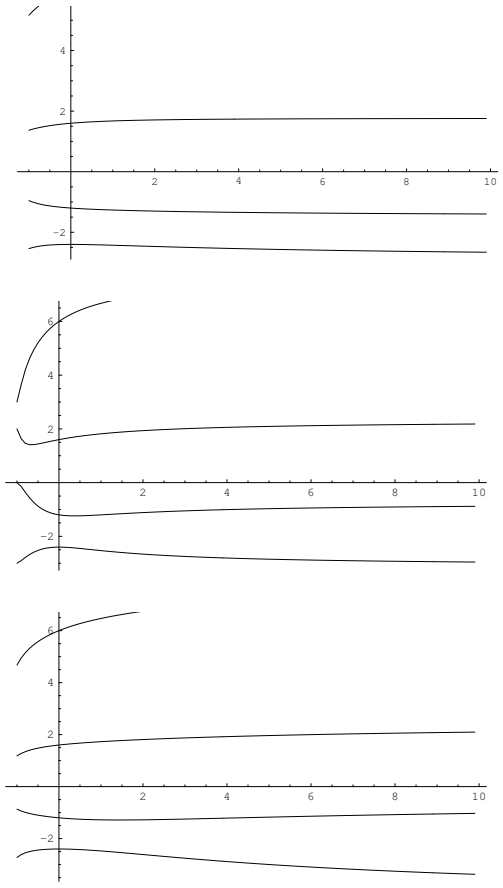


Figure 4.14. Behavior of the critical dimension $\Delta[6,p]$ for $d=3$ with $p=0,2,4,6$ (from below to above) *vs* α_1 for $\alpha_2=0$—*top*, *vs* $\alpha\equiv\alpha_1=\alpha_2$—*center*, *vs* α_2 for $\alpha_1=0$—*bottom*.

$\Delta[3,1]$ on the (α_1,α_2)-plane are shown in Fig. 4.19. One can see that, if the anisotropy becomes strong enough, $\Delta[3,1]$ becomes negative and the skewness factor *increases* going down towards to the depth of the inertial range; the higher-order odd ratios increase already when the anisotropy is weak.

4.3 Passively advected magnetic field in the presence of strong anisotropy

It is of general interest to compare passive advection of scalar and vector quantity. Though there are some common features, also important differences are observed. Here, the spatial struc-

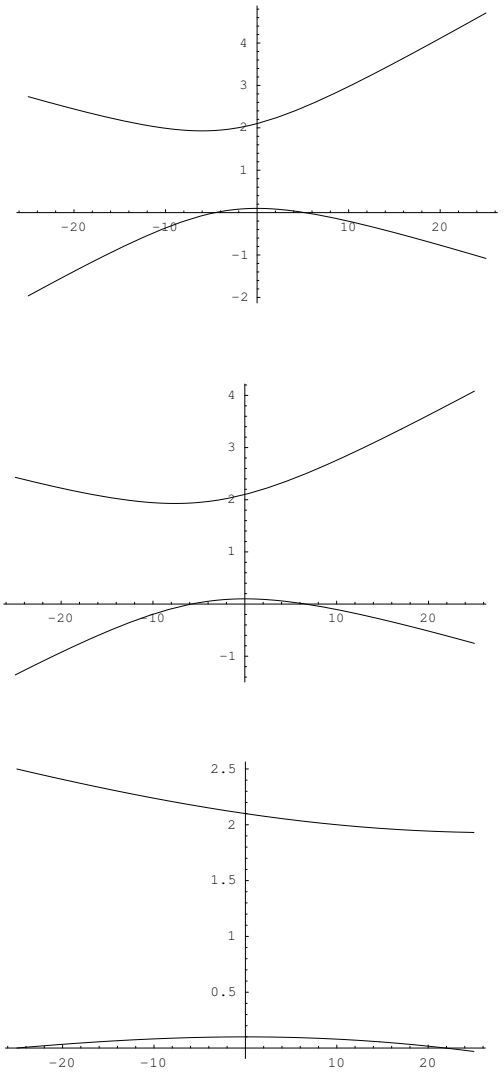


Figure 4.15. Behavior of the critical dimension $\Delta[3,p]$ for $d=3$ with $p=1,3$ (from below to above) *vs* a_2 for $b_3=0$—*top*, *vs* $a_2=b_3$—*center*, *vs* b_3 for $a_2=0$—*bottom*.

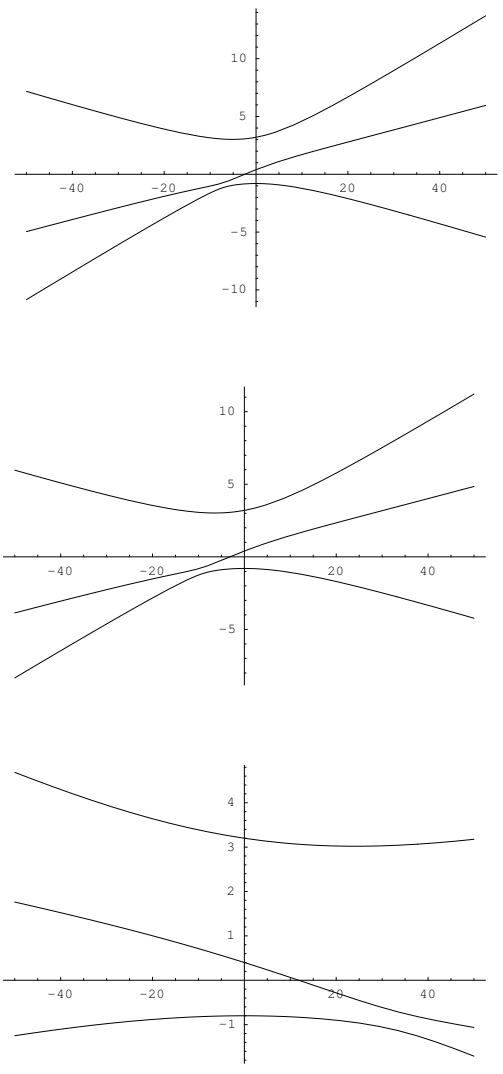


Figure 4.16. Behavior of the critical dimension $\Delta[4,p]$ for $d=3$ with $p=0,2,4$ (from below to above) *vs* a_2 for $b_3=0$—*top*, *vs* $a_2=b_3$—*center*, *vs* b_3 for $a_2=0$—*bottom*.

ture of correlations of fluctuations of the magnetic (vector) field $\boldsymbol{b}$ in a given turbulent fluid in the framework of the kinematic MHD Kazantsev-Kraichnan model (KMHD) is studied. These fluctuations are generated stochastically by a Gaussian random emf and a white in time and anisotropic self-similar in space Gaussian drift. The main goal is the calculation of the anomalous exponents as functions of the anisotropy parameters of the drift.

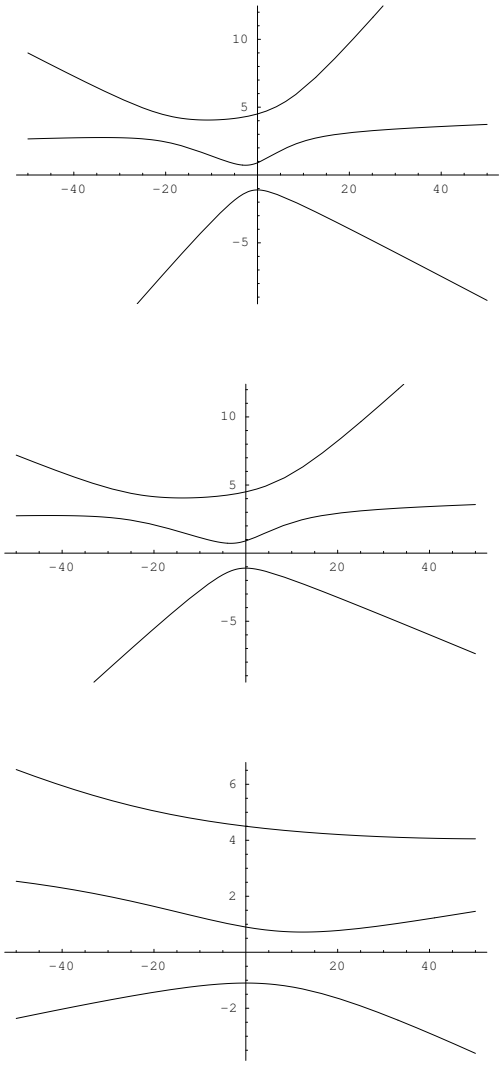


Figure 4.17. Behavior of the critical dimension $\Delta[5,p]$ for $d=3$ with $p=1,3,5$ (from below to above) *vs* a_2 for $b_3=0$—*top*, *vs* $a_2=b_3$—*center*, *vs* b_3 for $a_2=0$—*bottom*.

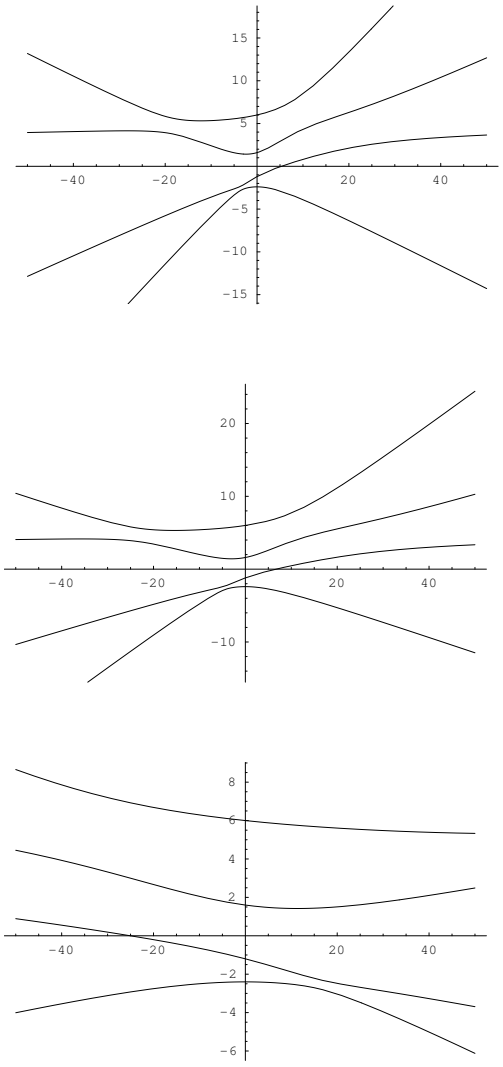


Figure 4.18. Behavior of the critical dimension $\Delta[6,p]$ for $d=3$ with $p=0,2,4$ (from below to above) *vs* a_2 for $b_3=0$—*top*, *vs* $a_2=b_3$—*center*, *vs* b_3 for $a_2=0$—*bottom*.

4.3.1 Kinematic MHD Kazantsev-Kraichnan model

Consider passive advection of a solenoidal magnetic field $\boldsymbol{b}\equiv\boldsymbol{b}(t,\boldsymbol{x})$ in the framework of the KMHD model described by the stochastic equation

$$\partial_t\boldsymbol{b}=\nu_0\boldsymbol{\nabla}^2\boldsymbol{b}-(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{b}+(\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{v}+\boldsymbol{f}, \tag{4.150}$$

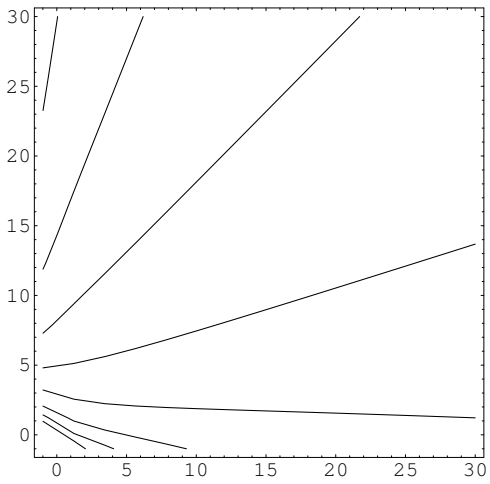


Figure 4.19. Levels of the dimension $\Delta[3,1]$ for $d = 3$ on the plane α_1–α_2. Value changes from -0.3 (top) to 0.1 (bottom) with step 0.05.

where ν_0 is the coefficient of the magnetic diffusivity, and $\boldsymbol{v} \equiv \boldsymbol{v}(t,\boldsymbol{x})$ is a random solenoidal velocity field. Thus, both $\boldsymbol{v}$ and $\boldsymbol{b}$ are divergence-free vector fields: $\boldsymbol{\nabla}\cdot\boldsymbol{v} = \boldsymbol{\nabla}\cdot\boldsymbol{b} = 0$. A transverse Gaussian emf flux density $\boldsymbol{f} \equiv \boldsymbol{f}(\boldsymbol{x},t)$ with zero mean and the correlation function

$$D^f_{ij} \equiv \langle f_i(t,\boldsymbol{x}) f_j(t',\boldsymbol{x}')\rangle = \delta(t-t')C_{ij}(\boldsymbol{r}/L), \quad \boldsymbol{r} = \boldsymbol{x}-\boldsymbol{x}' \tag{4.151}$$

is the source of the fluctuations of the magnetic field $\boldsymbol{b}$. The parameter L represents an integral scale related to the stirring, and C_{ij} is a function finite in the limit $L \to \infty$. In the present treatment its precise form is irrelevant, and with no loss of generality, we take $C_{ij}(0) = 1$ in what follows. The random velocity field $\boldsymbol{v}$ obeys Gaussian statistics with zero mean and the correlation function

$$D^v_{ij}(t,\boldsymbol{x}) \equiv \langle v_i(t,\boldsymbol{x}) v_j(0,\boldsymbol{0})\rangle = \frac{D_0\delta(t)}{(2\pi)^d}\int \mathrm{d}^d\boldsymbol{k}\frac{\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\,T_{ij}(\boldsymbol{k})}{(k^2+r_l^{-2})^{d/2+\varepsilon/2}}, \tag{4.152}$$

where r_l is another integral scale. In general, the scale r_l may be different from the integral scale L, below we, however, take $r_l \simeq L$. $D_0 > 0$ is an amplitude factor related to the coupling constant g_0 of the model by the relation (4.84). In the isotropic case, the second-rank tensor $T_{ij}(\boldsymbol{k})$ in Eq. (4.152) has the simple form of the ordinary transverse projector: $T_{ij}(\boldsymbol{k}) = P_{ij}(\boldsymbol{k})$. The latter was defined in Eq. (3.10).

Although the structure functions $S_N(r)$ of the magnetic field defined in analogy with (4.27) as

$$S_N(r) \equiv \langle [b_r(t,\boldsymbol{x}) - b_r(t,\boldsymbol{x}')]^N\rangle, \qquad r \equiv |\boldsymbol{x}-\boldsymbol{x}'|, \qquad b_r = \frac{\boldsymbol{b}\cdot\boldsymbol{r}}{r}. \tag{4.153}$$

are important tools in the analysis of MHD turbulence in the inertial range [defined by the inequalities $l \ll r \ll L$, where $l \simeq \Lambda^{-1}$ is an internal (viscous) scale], here the analysis of simpler

quantities (the equal-time two-point correlation functions of the composite operators) is invoked $b_r^{N-m}(t,\boldsymbol{x})$ and $b_r^m(t,\boldsymbol{x})$)

$$B_{N-m,m}(r) \equiv \langle b_r^{N-m}(t,\boldsymbol{x}) b_r^m(t,\boldsymbol{x}')\rangle \qquad r \equiv |\boldsymbol{x}-\boldsymbol{x}'|. \tag{4.154}$$

for two reasons: first, the field-theoretic approach yields the scaling behavior of these quantities in the first place, while the scaling behavior of the structure functions (4.153) emerges from their representation as linear combinations of the two-point correlation functions (4.154). Second, contrary to the problems of turbulent velocity of incompressible fluid and passive scalar advected by such fluid, the basic stochastic equation (4.150) is not invariant under the shift $\boldsymbol{b} \to \boldsymbol{b} + \boldsymbol{c}$, where $\boldsymbol{c}$ is a constant vector. Thus, there is no compelling need to aim at the analysis of more complex quantities, the structure functions, instead of their building blocks, the correlation functions (4.154).

Dimensional analysis yields

$$B_{N-m,m}(r) = \nu_0^{-N/2} r^N \tilde{R}_{N,m}(r/l, r/L), \tag{4.155}$$

where $R_{N,m}$ are functions of dimensionless parameters. When the random source field $\boldsymbol{f}$ and the velocity field $\boldsymbol{v}$ are uncorrelated, the correlation functions of odd order $B_{2n+1-m,m}$ vanish, however. The standard perturbation expansion (series in g_0) is ill suited for calculation of correlation functions (4.155) in the limit $r/l \to \infty$ and $r/L \to 0$, due to the singular behavior of the coefficients of the expansion. Therefore, to find the correct IR behavior it is necessary to sum the whole series. Such a summation can be carried out within the field-theoretic RG and OPE given in Sec. 2.

First, the UV renormalization of correlation functions (4.154) is carried out. As a consequence of this the asymptotic behavior of these functions for $r/l \gg 1$ and arbitrary but fixed r/L is given by IR stable fixed point(s) of the corresponding RG equations and for correlation functions (4.154) the following asymptotic form is obtained

$$B_{N-m,m}(r) \sim \nu_0^{-N/2} r_d^N \left(\frac{r}{l}\right)^{-\Delta[B_{N-m,m}]} R_{N,m}(r/L) \qquad \frac{r}{l} \gg 1\,, \tag{4.156}$$

where the critical dimensions $\Delta[B_{N-m,m}]$ are expressed in terms of the "anomalous dimensions" γ_ν^* and γ_N^* of the viscosity ν and the composite operators b_r^N, respectively, as:

$$\Delta[B_{N-m,m}] = -N\left(1 - \frac{\gamma_\nu^*}{2}\right) + \gamma_{N-m}^* + \gamma_m^*\,. \tag{4.157}$$

The scaling functions $R_{N,m}(r/r_l)$ in relations (4.156) remain unknown. The critical dimensions $\Delta[B_{N-m,m}]$ are calculated as (asymptotic) series in ε with the use of renormalized perturbation theory.

Second, the small r/L behavior of the functions $R_{N,m}(r/L)$ has to be estimated. This may be done using the OPE, which leads to the following asymptotic form in the limit $r/L \to 0$

$$R_{N,m}(r/L) = \sum_F C_F(r/L) \left(\frac{r}{L}\right)^{\Delta_F}$$

where $C_F(r/L)$ are coefficients regular in r/L. The summation is implied over all possible renormalized scale-invariant composite operators F, and Δ_F are their critical dimensions.

In the limit $r/L \to 0$ correlation function (4.151) of the random source field is uniform in space, which – as usual in stochastic models describing passive transport [74, 110, 112–114] – brings about composite operators with negative critical dimensions (dangerous composite operators) at the outset in the asymptotic analysis. This takes place because the limiting behavior of the correlation function determines the canonical scaling dimension of the magnetic field which in this case becomes equal to -1. Origin of the dangerous operators is thus different from that of the stochastic Navier-Stokes problem, where canonical field dimensions are positive and composite operators become dangerous (i.e. acquire negative scaling dimension) only for large enough values of the RG expansion parameter [32, 71].

The velocity fluctuation contribution to the scaling dimension in the passive transport problems is independent of the statistical properties of the source field. It is important to bear this in mind, because below it will be shown that calculation of the fluctuation corrections in the present problem is very similar to that in the case of passively advected scalar if the magnetic field $\boldsymbol{b}$ is traded for the vector field $\boldsymbol{\nabla}\theta$ – the gradient of the scalar. This occurs because – as a consequence of the invariance of the transport equation for the scalar θ with respect to the shift $\theta \to \theta + c$ with any constant c – in the scalar problem fluctuation corrections to scaling behavior are determined by composite operators constructed not from the scalar field itself but its derivatives $\nabla\theta$. Physically, however, these problems are different, because the usual random source for the scalar with a correlation function $C(r/L) \to 1$ in the limit $r/L \ll 1$ corresponds to random source for the vector field $\boldsymbol{\nabla}\theta$ with correlations concentrated at small separations (large wave numbers) instead of the asymptotically flat correlation function in the coordinate space (corresponding to strong correlation at small wave numbers in the wave-vector space).

Contributions of these dangerous operators with negative scaling dimensions to the OPE imply singular behavior of the scaling functions in the limit $r/L \to 0$. The leading term is given by the operator with the most negative critical dimension Δ_F. The leading contributions to correlation functions of even order $B_{N-m,m}$ ($N = 2n$) are given by scalar operators $F_N = (\boldsymbol{b} \cdot \boldsymbol{b})^{N/2}$ with their critical dimensions $\Delta_N = -N(1-\gamma_\nu^*/2)+\gamma_N^*$, which eventually determine the nontrivial asymptotic behavior of the correlation functions $B_{N-m,m}$ of the form (the correlation function $B_{N,0} = B_{0,N}$ is a constant)

$$B_{N-m,m}(r) \sim \nu_0^{-N/2} L^N \left(\frac{l}{L}\right)^{N\gamma_\nu^*/2} \left(\frac{r}{l}\right)^{-\gamma_{N-m}^*-\gamma_m^*} \left(\frac{r}{L}\right)^{\gamma_N^*} \sim r^{\gamma_N^*-\gamma_{N-m}^*-\gamma_m^*}. \quad (4.158)$$

In the isotropic case, the anomalous dimensions γ_N^* in the one-loop approximation are related [113] to anomalous dimensions of composite operators of a simpler model of passively advected scalar field [110], viz. γ_N^* are given by

$$\gamma_N^* = -\frac{N(N+d)\varepsilon}{2(d+2)} + \mathcal{O}(\varepsilon^2), \quad N \geq 2, \qquad \gamma_1^* = 0. \quad (4.159)$$

From this relation it follows that the scaling exponent in expression (4.158) $\gamma_N^*-\gamma_{N-m}^*-\gamma_m^* < 0$ at this order. Below it will be shown that these relations are stable against small-scale anisotropy.

In the anisotropic case we will assume that the statistics of the velocity field is anisotropic at all scales and replace the ordinary transverse projection operator in Eq. (4.152) with the operator (4.89), which has been introduced in Sec. 4.2.1.

The strong small-scale anisotropy (4.85) affects the inertial-range asymptotic behavior of the correlation functions (4.158) in two respects: the anomalous dimensions γ_N^* become dependent on the anisotropy parameters α_1 and α_2 and powerlike corrections with new anomalous dimensions appear. Combining the results of multiplicative renormalization and OPE in the manner sketched above for the isotropic case, we arrive at the following expression for the inertial-range asymptotics of the correlation functions of the passively advected vector field $B_{N-m,m}$:

$$B_{N-m,m}(r) \sim \nu_0^{-N/2} L^N \left(\frac{l}{L}\right)^{\gamma_N^* + N\varepsilon/2} \left(\frac{r}{l}\right)^{\gamma_N^* - \underline{\gamma}_{N-m}^* - \underline{\gamma}_m^*}, \qquad \underline{\gamma}_1^* = 0\,, \qquad m \geq 1\,, \tag{4.160}$$

with negative exponents $\gamma_N^* + N\varepsilon/2 < 0$ and $\gamma_N^* - \underline{\gamma}_{N-m}^* - \underline{\gamma}_m^* < 0$. The anomalous dimensions γ_N^*, $\underline{\gamma}_{N-m}^*$, and $\underline{\gamma}_m^*$ in (4.160) will be defined in relation (4.187) and results of their numerical calculation discussed in detail in Sec. 4.3.3. The scaling behavior in expression (4.160) is similar to that of the correlation functions of the passive scalar advected by a *compressible* vector field [113].

4.3.2 Field-theoretic formulation, renormalization, and RG analysis

The stochastic problem (4.150)–(4.152) in the Stratonovich interpretation is equivalent to the field-theoretic model of the set of the three fields $\Phi = \{\boldsymbol{b}', \boldsymbol{b}, \boldsymbol{v}\}$ with the action functional

$$\mathcal{S}[\Phi] \equiv \frac{1}{2}\,\boldsymbol{b}' D_b \boldsymbol{b}' + \boldsymbol{b}' \cdot [-\partial_t - (\boldsymbol{v}\cdot\boldsymbol{\nabla}) + \nu_0 \boldsymbol{\nabla}^2 + \Sigma_{b'b}]\boldsymbol{b} + \boldsymbol{b}'(\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{v} - \frac{1}{2}\boldsymbol{v} D_v^{-1} \boldsymbol{v}, \tag{4.161}$$

where $\boldsymbol{b}'$ is an auxiliary field (all required integrations over space-time coordinates and summations over the vector indices are implied). To keep notation simple, we have denoted the white-noise contraction term in dynamic action (4.161) in terms of the self energy operator (cf. the dynamic action of the passive scalar case (4.90) and the relation to the self-energy operator there (4.95). We recall that the self energy operator is given by a single one-loop graph corresponding to the white-noise contraction term in the limit of white noise in the SDE). The first five terms in Eq. (4.161) represent the De Dominicis-Janssen action corresponding to the stochastic problem at fixed $\boldsymbol{v}$ (see, e.g., Refs. [28]), the sixth term comes from the Stratonovich interpretation of the SDE, whereas the last term represents the Gaussian averaging over $\boldsymbol{v}$. The kernel functions D_b and D_v are the correlation functions (4.151) and (4.152), respectively.

In this field-theoretic language, the correlation functions (4.154) are defined as

$$B_{N-m,m}(r) \equiv \int \mathcal{D}\Phi\, b_r^{N-m}(t,\boldsymbol{x}) b_r^m(t,\boldsymbol{x}') \mathrm{e}^{\mathcal{S}[\Phi]} \tag{4.162}$$

with the action $\mathcal{S}[\Phi]$ defined above.

From the action (4.161) the propagators for the fields $\boldsymbol{b}'$ and $\boldsymbol{b}$ are directly obtained and in the wave-vector-frequency representation they read

$$\langle b_i b_j' \rangle_0 = \langle b_j' b_i \rangle_0^* = \frac{P_{ij}(\boldsymbol{k})}{-i\omega + \nu_0 k^2}, \quad \langle b_i b_j \rangle_0 = \frac{C_{ij}(\boldsymbol{k})}{\omega^2 + \nu_0^2 k^4}, \quad \langle b_i' b_j' \rangle_0 \quad = 0, \tag{4.163}$$

where $C_{ij}(\boldsymbol{k})$ is the Fourier transform of the function $C_{ij}(\boldsymbol{r}/L)$ from Eq. (4.151). The bare propagator of the velocity field $\langle \boldsymbol{v}\boldsymbol{v}\rangle_0 \equiv \langle \boldsymbol{v}\boldsymbol{v}\rangle$ is defined by Eq. (4.152) with the transverse projector given by Eq. (4.89). The interaction in the model is given by the nonlinear terms $-b_i'(\boldsymbol{v}\cdot\boldsymbol{\nabla})b_i + b_i'(\boldsymbol{b}\cdot\boldsymbol{\nabla})v_i \equiv b_i' V_{ijl} v_j b_l$ with the vertex factor which in the wave-number-frequency representation has the following form

$$V_{ijl} = i(\delta_{ij}k_l - \delta_{il}k_j),$$

where $\boldsymbol{k}$ is momentum flowing through the corresponding prime field. With the use of the standard power counting [24, 25] correlation functions with superficial UV divergences may be identified. In the present model superficial divergences exist only in the 1PI Green function $\Gamma_{\boldsymbol{b}'\boldsymbol{b}}$. In the isotropic case this Green function gives rise only to the renormalization of the term $\nu_0 \boldsymbol{b}'\Delta\boldsymbol{b}$ of action (4.161) and the corresponding UV divergences may be fully absorbed in the proper redefinition of the existing parameters g_0, ν_0.

When anisotropy is introduced, however, the situation becomes more complicated, because the 1PI Green function $\Gamma_{\boldsymbol{b}'\boldsymbol{b}}$ produces divergences corresponding to the structure $\boldsymbol{b}'(\boldsymbol{n}\cdot\boldsymbol{\nabla})^2\boldsymbol{b}$ in the action of the model [due to peculiarities of the rapid-change models [110] the term $(\boldsymbol{b}'\cdot\boldsymbol{n})\boldsymbol{\nabla}^2(\boldsymbol{b}\cdot\boldsymbol{n})$ possible on dimensional and symmetry grounds does not appear]. The term $\boldsymbol{b}'(\boldsymbol{n}\cdot\boldsymbol{\nabla})^2\boldsymbol{b}$ is not present in the original unrenormalized action (4.161), but has to be added to the renormalized action, therefore the model is not multiplicatively renormalizable. In such a case it is customary to extend the original action (4.161) by including all terms needed for the renormalization of the correlation functions and thus adding new parameters. As a result the extended model is described [245] by a new action of the form:

$$\mathcal{S}[\Phi] \equiv \frac{1}{2}\boldsymbol{b}' D_b \boldsymbol{b}' + \boldsymbol{b}'[-\partial_t - (\boldsymbol{v}\cdot\boldsymbol{\nabla}) + \nu_0\boldsymbol{\nabla}^2 + \chi_0\nu_0(\boldsymbol{n}\cdot\boldsymbol{\nabla})^2 + \Sigma_{b'b}]\boldsymbol{b} + \boldsymbol{b}'(\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{v} - \frac{1}{2}\boldsymbol{v}D_v^{-1}\boldsymbol{v}, \quad (4.164)$$

where a new unrenormalized parameter χ_0 has been introduced in the same sense as in the action (4.101) and the self-energy operator is calculated in terms of the extended model.

Of course, the bare propagators (4.163) of the isotropic model are modified and for the extended action (4.164) assume the form

$$\langle b_i b_j'\rangle_0 = \langle b_j' b_i\rangle_0^* = \frac{P_{ij}(\boldsymbol{k})}{-i\omega + \nu_0 k^2 + \chi_0\nu_0(\boldsymbol{n}\cdot\boldsymbol{k})^2}\,, \quad (4.165)$$

$$\langle b_i b_j\rangle_0 = \frac{C_{ij}(\boldsymbol{k})}{|-i\omega + \nu_0 k^2 + \chi_0\nu_0(\boldsymbol{n}\cdot\boldsymbol{k})^2|^2}\,, \quad (4.166)$$

$$\langle b_i' b_j'\rangle_0 = 0\,.$$

After this modification all terms needed to remove the divergences are present in action (4.164), therefore the model becomes multiplicatively renormalizable allowing for the standard RG analysis. The corresponding renormalized action may be written down immediately:

$$\mathcal{S}_R[\Phi] \equiv \frac{1}{2}\boldsymbol{b}' D_b \boldsymbol{b}' + \boldsymbol{b}'[-\partial_t - (\boldsymbol{v}\cdot\boldsymbol{\nabla}) + \nu Z_1 \boldsymbol{\nabla}^2 + \chi\nu Z_2 (\boldsymbol{n}\cdot\boldsymbol{\nabla})^2 + \overline{\Sigma}_{b'b}]\boldsymbol{b} + \boldsymbol{b}'(\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{v} - \frac{1}{2}\boldsymbol{v} D_v^{-1}\boldsymbol{v}\,. \quad (4.167)$$

Here, Z_1 and Z_2 are the renormalization constants in which the UV divergent parts of the 1PI response function $\Gamma_{\boldsymbol{b}'\boldsymbol{b}}$ are absorbed. Thus, from the renormalized self-energy operator $\overline{\Sigma}_{b'b}$ in (4.167) the divergent parts corresponding to the chosen renormalization scheme have been subtracted. The renormalized action (4.167) leads to the multiplicative renormalization of the parameters ν_0, g_0 and χ_0 given by the relations (4.105).

Identification of the unrenormalized action (4.164) with the renormalized one (4.167) leads to the following relations between the renormalization constants:

$$Z_1 = Z_\nu, \quad Z_2 = Z_\chi Z_\nu, \quad Z_g = Z_\nu^{-1}\,. \quad (4.168)$$

As has been discussed after Eq. (4.94) the rapid-change-like models (4.164) have a distinguished property that in all multi-loop diagrams of the self-energy operator $\Sigma_{b'b}$ closed circuits of the retarded bare propagators $\langle \boldsymbol{b}\boldsymbol{b}'\rangle_0$ are produced, because the propagator $\langle \boldsymbol{v}\boldsymbol{v}\rangle_0$ is proportional to the δ function in time. As a result, also the one-loop self energy operator $\Sigma_{b'b}$ with the graphical notation of Fig. 4.20 is exact.

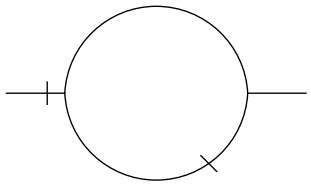


Figure 4.20. The (exact) graphical expression for the self-energy operator $\Sigma_{b'b}$ of the response function of the passive vector field. The plain line denotes the bare propagator (4.166), and the line with slash (denoting the end corresponding to the arguments of the field $\boldsymbol{b}'$) corresponds to the bare propagator (4.165).

The divergent part of the graph in Fig. 4.20 is

$$\Sigma_{b'b}(p) = -\frac{g\nu\bar{S}_d}{2d(d+2)\varepsilon}\left\{\Big[(d-1)(d+2) + \alpha_1(d+1) + \alpha_2\Big]p^2 - (2\alpha_1 - (d^2-2)\alpha_2) \times (\boldsymbol{n}\cdot\boldsymbol{p})^2\right\}. \quad (4.169)$$

The expression (4.169) leads to a straightforward determination of the renormalization constants Z_1 and Z_2:

$$Z_1 = 1 - \frac{gC_d}{2d(d+2)\varepsilon}\left[(d-1)(d+2) + \alpha_1(d+1) + \alpha_2\right],$$
$$Z_2 = 1 - \frac{gC_d}{2d(d+2)\chi\varepsilon}\left[-2\alpha_1 + (d^2-2)\alpha_2\right]. \quad (4.170)$$

The RG differential equations for the renormalized correlation functions of the fields read

$$(\mathcal{D}_\mu + \beta_g\partial_g + \beta_\chi\partial_\chi - \gamma_\nu\mathcal{D}_\nu)\langle \boldsymbol{b}(t,\boldsymbol{x})\boldsymbol{b}(t,\boldsymbol{x}')\rangle_R = 0 \tag{4.171}$$

with the definition standard definition from Sec. 2.3. The β functions can be written as follows

$$\beta_g \equiv \widetilde{\mathcal{D}}_\mu g = g(-\varepsilon + \gamma_1)\,, \quad \beta_\chi \equiv \widetilde{\mathcal{D}}_\mu \chi = \chi(\gamma_1 - \gamma_2)\,. \tag{4.172}$$

The anomalous dimensions γ_1 and γ_2 can be computed using definition (2.34) and Eqs. (4.170)

$$\gamma_1 = \frac{gC_d}{2d(d+2)}\left[(d-1)(d+2) + \alpha_1(d+1) + \alpha_2\right]\,, \tag{4.173}$$

$$\gamma_2 = 1 - \frac{gC_d}{2d(d+2)\chi}\left[-2\alpha_1 + (d^2-2)\alpha_2\right]\,. \tag{4.174}$$

It should be emphasized that both the renormalization constants (4.170) and the corresponding anomalous dimensions (4.173) and (4.174) in the present model are exact, i.e., they have no corrections of order g^2 or higher.

The fixed points (g^*, χ^*) of the RG equations are defined by the system of two equations

$$\beta_g(g^*,\chi^*) = 0\,, \qquad \beta_\chi(g^*,\chi^*) = 0\,. \tag{4.175}$$

It should be noted that as a consequence of relations (4.168), (4.172) and (4.175) at any fixed point with $g^* \neq 0$ the anomalous dimension of viscosity assumes the exact value $\gamma_\nu^* = \varepsilon$.

The IR stability of a fixed point is determined by the eigenvalues of the matrix

$$\Omega = \begin{pmatrix} \frac{\partial\beta_g}{\partial g} & \frac{\partial\beta_g}{\partial\chi} \\ \frac{\partial\beta_\chi}{\partial g} & \frac{\partial\beta_\chi}{\partial\chi} \end{pmatrix}_*$$

which for the IR stable fixed point have to have positive real parts. Calculation shows that the RG equations have only one non-trivial IR stable fixed point defined by expressions

$$g_* = \frac{2d(d+2)\varepsilon}{C_d\left[(d-1)(d+2) + \alpha_1(d+1) + \alpha_2\right]}\,, \tag{4.176}$$

$$\chi_* = \frac{-2\alpha_1 + (d^2-2)\alpha_2}{(d-1)(d+2) + \alpha_1(d+1) + \alpha_2}. \tag{4.177}$$

Both eigenvalues of the stability matrix Ω are equal to ε at this fixed point, therefore, the IR fixed point (4.176), (4.177) is stable for $\varepsilon > 0$ and all values of the anisotropy parameters α_1 and α_2.

Rather unexpectedly, the β functions and, consequently, the fixed points of the present model of passively advected vector field are exactly the same as in the model of passively advected scalar field [112]. In Sec. 4.3.3 it will be shown that this similarity is extended to the anomalous scaling dimensions of the composite operators in the OPE representation of the correlation functions as well.

The fixed point (4.176), (4.177) governs the behavior of solutions of Eqs. (4.171) and the like, and at large scales far from viscous length $r \gg l$ at any fixed ratio r/L yields the scaling form

$$\langle \boldsymbol{b}(t,\boldsymbol{x})\boldsymbol{b}(t,\boldsymbol{x}-\boldsymbol{r})\rangle = D_0^{-1} r^{2-\varepsilon} R_2(r/L)\,, \tag{4.178}$$

for the *unrenormalized* correlation function (we remind that due to the absence of field renormalization renormalized and unrenormalized correlation functions are equal but expressed in terms of different variables). It should be noted, however, that the scaling function $R_2(r/L)$ in Eq. (4.178) is not determined by the RG Eqs. (4.171).

The correlation functions (4.154) contain UV divergences additional to those included in the renormalization constants (4.170). These additional divergences due to composite operators (products of fields and their derivatives with coinciding space and time arguments) can be dealt with in a manner similar to that applied to the divergences in the usual correlation functions [25].

4.3.3 Renormalization and critical dimensions of composite operators

Two-point correlation functions (4.162) are averages of products of composite operators at two separate space points. These composite operators are integer powers of the field b_r and contain additional UV divergences, which also may be removed by a suitable renormalization procedure discussed in Sec. 2.7.

The two-point correlation functions are, however, quantities with insertions of two composite operators. This is the crucial difference with the previous models (and also Sec. 2.7). Nevertheless, field theoretic RG technique is capable of dealing with this problem. It would seem that we would have to consider renormalization of products of two composite operators as well, the aim being then to render UV finite all 1PI correlation functions with two insertions of composite operators. Superficially divergent correlation functions with operator insertions are identified by power counting similar to that of the basic renormalization. In the present model such a power counting shows that insertion of products of composite operators of the structure $\boldsymbol{b}^m(t,\boldsymbol{x})\boldsymbol{b}^n(t,\boldsymbol{x}')$ does not bring about any new superficial divergences and it is thus sufficient to renormalize the composite operators themselves only in order to make the two-point correlation functions UV finite. Therefore, from the RG analysis of composite operators it follows – by virtue of relations (2.45) and (2.47) – that the two-point correlation function $B_{N-m,m}$ given in Eq. (4.160) may be expressed as a functional average of a quadratic form

$$B_{N-m,m}(r) = \sum_{\alpha,\beta} B_{\alpha\beta} \left\langle \bar{F}_\alpha^R\left(t,\boldsymbol{x}+\frac{1}{2}\boldsymbol{r}\right) \bar{F}_\beta^R\left(t,\boldsymbol{x}-\frac{1}{2}\boldsymbol{r}\right)\right\rangle_R \tag{4.179}$$

with coefficients $B_{\alpha\beta}$ independent of spatial coordinates and basis operators have been introduced in Eq. (2.47). Each term in expression (4.179) obeys the following asymptotic form in the limit $l \ll r, r \lesssim L$

$$\left\langle \bar{F}_\alpha^R\left(t,\boldsymbol{x}+\frac{1}{2}\boldsymbol{r}\right) \bar{F}_\beta^R\left(t,\boldsymbol{x}-\frac{1}{2}\boldsymbol{r}\right)\right\rangle_R \sim D_0^{d_\alpha^\omega + d_\beta^\omega} r^{-\Delta_\alpha-\Delta_\beta} r_d^{\gamma_\alpha^*+\gamma_\beta^*} \Xi_{\alpha\beta}\left(\frac{r}{L}\right) \tag{4.180}$$

with the scaling functions $\Xi_{\alpha\beta}$ still to be determined.

The physically interesting range of scales, however, is the inertial range, specified by the inequalities $l \ll r \ll L$. The limit $r \ll L$ may be explored with the use of the OPE [24, 25] as was already discussed in Sec. 2.7. The basic statement of the OPE theory is summarized in the relation (2.50), which we know write as follows

$$F_\alpha^R\left(t, \boldsymbol{x} + \frac{1}{2}\boldsymbol{r}\right) F_\beta^R\left(t, \boldsymbol{x} - \frac{1}{2}\boldsymbol{r}\right) = \sum_\gamma C_{\alpha\beta\gamma}(\boldsymbol{r}) F_\gamma^R(t, \boldsymbol{x}), \tag{4.181}$$

where the functions $C_{\alpha\beta\gamma}$ are the Wilson coefficients regular in $1/L$, and F_γ^R are renormalized local composite operators which appear in the formal Taylor expansion with respect to $\boldsymbol{r}$ together with all operators that mix with them in renormalization. If these operators have additional vector indices, they are contracted with the corresponding indices of the coefficients $C_{\alpha\beta\gamma}$.

Without loss of generality we may take the expansion on the right-hand side of Eq. (4.181) in terms of the basis operators with definite critical dimensions $\Delta_{\mathcal{F}}$. The renormalized correlation function $\langle F_\alpha^R F_\beta^R \rangle_R$ is obtained by averaging Eq. (4.181) with the weight generated by the renormalized action, the quantities $\langle \mathcal{F} \rangle_R$ appear now only on the right-hand side. Their asymptotic behavior for $r/L \to 0$ is found from the corresponding RG equations and is of the form $\langle \mathcal{F} \rangle \propto L^{-\Delta_{\mathcal{F}}}$. Comparison of the expression for a given function $\langle F_\alpha^R F_\beta^R \rangle_R$ in terms of the IR scaling representation of correlation functions of the basis operators (4.180) on one hand and the OPE representation brought about by relation (4.181) on the other in the limit $L \to \infty$ allows to find the asymptotic form of the scaling functions $\Xi_{\alpha\beta}(r/L)$ in relation (4.180).

The two-point correlation functions are products of integer powers of the field b_r of the form $b_r^{N-m}(t, \boldsymbol{x}) b_r^m(t, \boldsymbol{x}')$. Thus, at the leading order in $\boldsymbol{r}$ their OPE contains operators of the closed set generated by the operator $b_r^N(t, \boldsymbol{x})$. Power counting and analysis of the structure of graphs shows that this set of composite operators contains only operators consisting of exactly N components of the vector field $\boldsymbol{b}$, viz. the tensor operators constructed solely of the fields $\boldsymbol{b}$ without derivatives: $b_{i_1} ... b_{i_p} (b_i b_i)^l$ with $p + 2l = N$. It is convenient to deal with the scalar operators obtained by contracting the tensor with the appropriate number of the anisotropy vectors $\boldsymbol{n}$:

$$F[N, p](t, \boldsymbol{x}) \equiv [\boldsymbol{n} \cdot \boldsymbol{b}(t, \boldsymbol{x})]^p [\boldsymbol{b}^2(t, \boldsymbol{x})]^l \tag{4.182}$$

with $N \equiv 2l + p$. Analysis of graphs shows that composite operators (4.182) with different N do not mix in renormalization, and therefore the corresponding renormalization matrix $Z_{[N,p][N',p']}$ is in fact block-diagonal, i.e., $Z_{[N,p][N',p']} = 0$ for $N' \neq N$ and

$$F[N, p] = \sum_{l=0}^{[N/2]} Z_{[N,p][N,N-2l]} F^R[N, N - 2l]\,,$$

where $[N/2]$ stands for the integer part of the rational number $N/2$ for odd N (although the odd-order correlation functions $B_{2n+1-m,m}$ vanish, renormalization of the even-order correlation functions $B_{2n-m,m}$ involves odd-order composite operators). Each block with fixed N gives rise to a $(N + 1) \times (N + 1)$ matrix of critical dimensions whose eigenvalues at the IR stable fixed point are the critical dimensions $\Delta[N, p]$ of the set of operators $F[N, p]$.

Taking into account that renormalization of the composite operators b_r^{N-m} and b_r^m in the correlation function $B_{N-m,m}$ involves operators of the sets $F[N - m, p]$ and $F[m, q]$, respectively, whereas the leading contribution to the OPE involves the set $F[N, s]$, the basis-operator

decomposition of the correlation function may be written as

$$B_{N-m,m}(r) \sim \frac{r_d^N}{\nu_0^{N/2}} \sum_{l_1=0}^{[(N-m)/2]} \sum_{l_2=0}^{[m/2]} \sum_{l_3=0}^{N/2} A^{Nm}_{l_1 l_2 l_3}(r/L) \left(\frac{r}{L}\right)^{\Delta[N,N-2l_3]} \times \left(\frac{r}{l}\right)^{-\Delta[N-m,N-m-2l_1]-\Delta[m,m-2l_2]} \tag{4.183}$$

where the coefficients $A^{Nm}_{l_1 l_2 l_3}(r/L)$ are regular in $(r/L)^2$.

The decomposition (4.183) reveals the inertial-range scaling form of the correlation functions. The leading singular contribution in the limit $L \to \infty, l \to 0$ is given by the basis operator $\mathcal{F}[N,s]$ with the minimal critical dimension $\Delta[N,s]$ and operators $\mathcal{F}[N-m,p]$ and $\mathcal{F}[m,q]$ with the minimal sum of critical dimensions $\Delta[N-m,p]+\Delta[m,q]$. We also remind that the critical dimensions of the basis operators have the structure $\Delta[N,p] = -N\left(1-\frac{\varepsilon}{2}\right) + \gamma^*_{[N,p]}$, where $\gamma^*_{[N,p]}$ is the anomalous dimension. Therefore, in expression (4.183) other contributions than the anomalous dimensions cancel in the power of separation distance r. As a result, the correlation functions have asymptotic powerlike behavior as $r/L \to 0$ with the minimal anomalous dimension in the basis set generated by the composite operator $b_r^N(t,\boldsymbol{x})$ and the sum of minimal anomalous dimensions in the basis set brought about by operators $b_r^{N-m}(t,\boldsymbol{x})$ and $b_r^m(t,\boldsymbol{x})$. Calculation shows that these anomalous dimensions grow with the number of field components in the anisotropy direction and thus the minimal anomalous dimension $\underline{\gamma}^*_N$ in any set with fixed N is $\gamma^*_{[N,0]} = \gamma^*_N$ for N even and $\gamma^*_{[N,1]}$ for N odd. Therefore, the leading asymptotic term of the correlation functions in the inertial range is of the form (we remind that N is an even integer here)

$$B_{N-m,m}(r) \sim \nu_0^{-N/2} L^N \left(\frac{l}{L}\right)^{\gamma^*_N + N\varepsilon/2} \left(\frac{r}{l}\right)^{\gamma^*_N - \underline{\gamma}^*_{N-m} - \underline{\gamma}^*_m}, \qquad \underline{\gamma}^*_1 = 0\,, \qquad m \geq 1\,. \tag{4.184}$$

Numerical results at one-loop order yield negative exponents $\gamma^*_N + N\varepsilon/2 < 0$ and $\gamma^*_N - \underline{\gamma}^*_{N-m} - \underline{\gamma}^*_m < 0$, see Figs. 4.21 – 4.26. Note that two-loop approximation has been analyzed in the works [246–250] for models without anisotropy but helicity and compressibility taken into account.

A few words about the structure functions (4.153) are in order. These structure functions may be expressed as linear combinations of two-point correlation functions, in which the constant term corresponding to $B_{N,0}$ is always present in even-order structure functions. The present asymptotic analysis does not allow for direct comparison of this constant and the powerlike asymptotics predicted in relation (4.184) for the "genuine" two-point correlation functions in the inertial range. If a powerlike behavior is to be detected, however, this constant must be small at least compared to the leading powerlike term. One-loop results show that the leading powerlike term in the inertial range corresponds to the two-point correlation function $B_{N-1,1}$ with the asymptotic behavior ($\gamma^*_1 = 0$)

$$B_{N-1,1}(r) \sim \nu_0^{-N/2} L^N \left(\frac{l}{L}\right)^{\gamma^*_N + N\varepsilon/2} \left(\frac{r}{l}\right)^{\gamma^*_N - \underline{\gamma}^*_{N-1}}. \tag{4.185}$$

A detailed account of calculation of the matrix of the renormalization constants $Z_{[N,p][N,p']}$ (which may be readily extended to investigation of all related problems) has been given in Ref. [112] for the advection of a passive scalar and in Ref. [245] for passive vector admixture, therefore we will not describe all details of the determination of renormalization constants in the present vector model, rather we will discuss its specific features.

It turned out that not only the β functions in the vector and scalar models coincide, but the one-loop renormalization matrices as well. This nontrivial fact stems from the similarities of the mathematical structure of both models. In the model of scalar advection [112] the composite operators $\partial_{i_1}\theta ... \partial_{i_p}\theta(\partial_i\theta\partial_i\theta)^l$ constructed solely of the scalar gradients of the scalar admixture θ are needed for calculation of the asymptotic behavior of the correlation functions, whereas in vector case [245] the main contribution is given by composite operators constructed solely of the fields $\boldsymbol{b}$ without derivatives. As direct inspection of the relevant diagrams shows, the tensor structures arising upon functional averaging in both cases are in fact identical, which yields the same renormalization matrix $Z_{[N,p][N,p']}$ in both models.

The only nonzero elements of the matrix $Z_{[N,p][N,p']}$ are

$$Z_{[N,p][N,p-2]} = \frac{g\bar{S}_d}{d^2-1}\frac{1}{4\varepsilon}Q_1\,, \qquad Z_{[N,p][N,p]} = 1 + \frac{g\bar{S}_d}{d^2-1}\frac{1}{4\varepsilon}Q_2\,,$$
$$Z_{[N,p][N,p+2]} = \frac{g\bar{S}_d}{d^2-1}\frac{1}{4\varepsilon}Q_3\,, \qquad Z_{[N,p][N,p+4]} = \frac{g\bar{S}_d}{d^2-1}\frac{1}{4\varepsilon}Q_4\,,$$

with the coefficients Q_i given in [245] and $\bar{S}_d$ was given in Eq. (3.62). The nontrivial elements of the matrix of anomalous dimensions $\gamma_{[N,p][N,p']}$ are

$$\gamma_{[N,p][N,p-2]} = -\frac{g\bar{S}_d}{4(d^2-1)}Q_1\,, \qquad \gamma_{[N,p][N,p]} = -\frac{g\bar{S}_d}{4(d^2-1)}Q_2\,,$$
$$\gamma_{[N,p][N,p+2]} = -\frac{g\bar{S}_d}{4(d^2-1)}Q_3\,, \qquad \gamma_{[N,p][N,p+4]} = -\frac{g\bar{S}_d}{4(d^2-1)}Q_4\,, \tag{4.186}$$

and the matrix of critical dimensions (2.46) is thus

$$\Delta_{[N,p][N,p']} = -N\left(1-\frac{\varepsilon}{2}\right)\delta_{pp'} + \gamma^*_{[N,p][N,p']}\,, \tag{4.187}$$

where the asterisk stands for the value at the fixed point (4.176), (4.177). This represents the critical dimensions of the composite operators (4.182) at the first order in ε. It should to be stressed that in contrast to the value of the fixed point (4.176), (4.177), which has no higher order corrections, the expressions for anomalous dimensions (4.186) have non-vanishing corrections of order g^2 and higher.

The critical dimensions $\Delta[N,p] = -N\,(1-\varepsilon/2) + \gamma^*_{[N,p]}$ are given by the eigenvalues of the matrix (4.187).

Since the result for the anomalous dimensions is the same as in Ref. [112] for the admixture of a passive scalar, all conclusions about the hierarchical behavior of the critical dimensions of the composite operators are also valid in the analysis of the vector model. Nevertheless, the inertial-range asymptotic behavior of the correlation functions in these two problems is completely different, because, first, in the scalar problem single-point products of the scalar are not

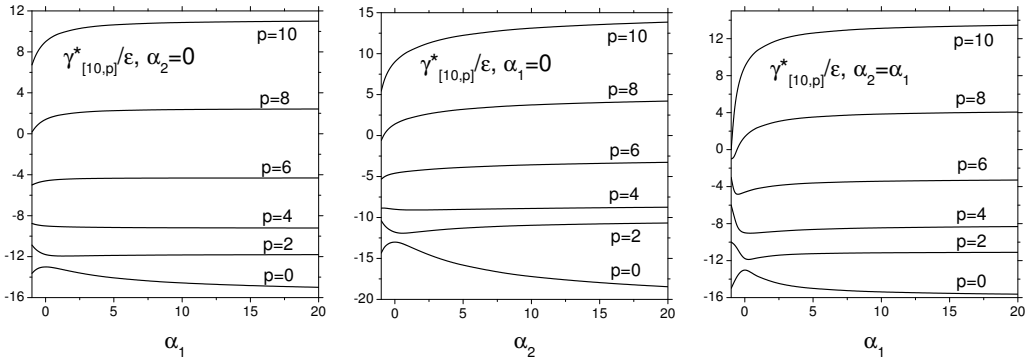


Figure 4.21. Behavior of the anomalous dimension $\gamma^*_{[10,p]}/\varepsilon$ in space dimension $d = 3$ and for representative values of p as functions of anisotropy parameters α_1 and α_2.

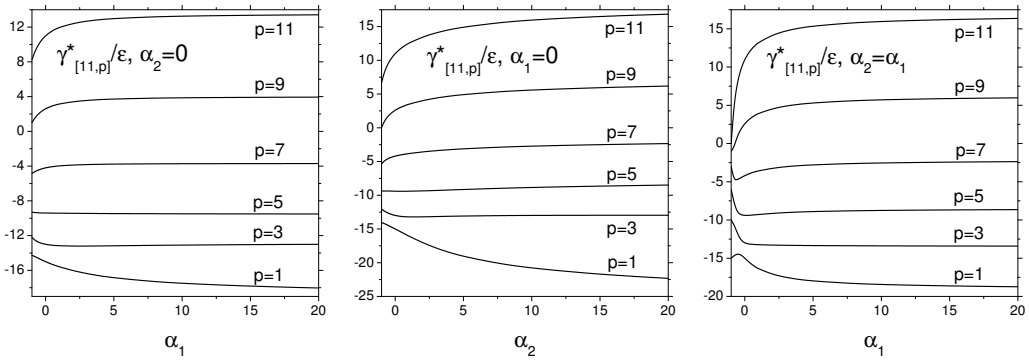


Figure 4.22. Behavior of the anomalous dimension $\gamma^*_{[11,p]}/\varepsilon$ in space dimension $d = 3$ and for representative values of p as functions of anisotropy parameters α_1 and α_2.

renormalized, while in the vector problem they are, and, second, the leading contribution to the OPE is given by the products of derivatives of the scalar, whereas in the vector problem products of the field components themselves yield the leading contribution.

In Ref. [112] the behavior of the critical dimensions $\Delta[N, p]$ for $N = 2, 3, 4, 5$, and 6 was numerically studied. The main conclusion is that the dimensions Δ_N remain negative in anisotropic case and decrease monotonically as N increases for odd and even values of N separately.

Here the main attention is paid to the investigation of the composite operators (4.182) for relatively large values of N, namely we will analyze cases with $N = 10, 11, 20, 21, 30, 31, 40, 41, 50$, and 51. The aim has been to find out whether hierarchies which hold for small values of N remain valid for significantly larger values of N, and the answer turned out to be in the affirmative.

In Figs. 4.21-4.26 behavior of the eigenvalues of the matrix of anomalous dimensions $\gamma^*_{[N,p]}$ for relatively large values of the N are shown. It can be seen that only real eigenvalues exist in all cases, and also their hierarchical behavior discussed in Ref. [112] is conserved. At first sight the curves for $p = 0$ and $p = 2$ in the even case and the curves for $p = 1$ and $p = 3$ in the odd

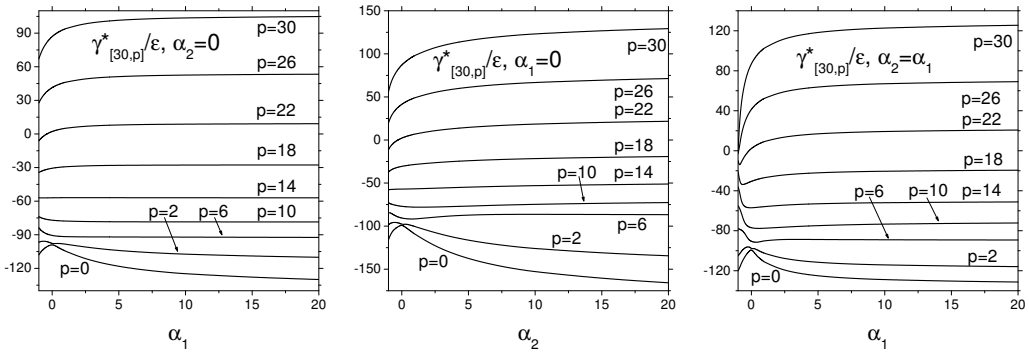


Figure 4.23. Behavior of the anomalous dimension $\gamma^*_{[30,p]}/\varepsilon$ in space dimension $d = 3$ and for representative values of p as functions of anisotropy parameters α_1 and α_2.

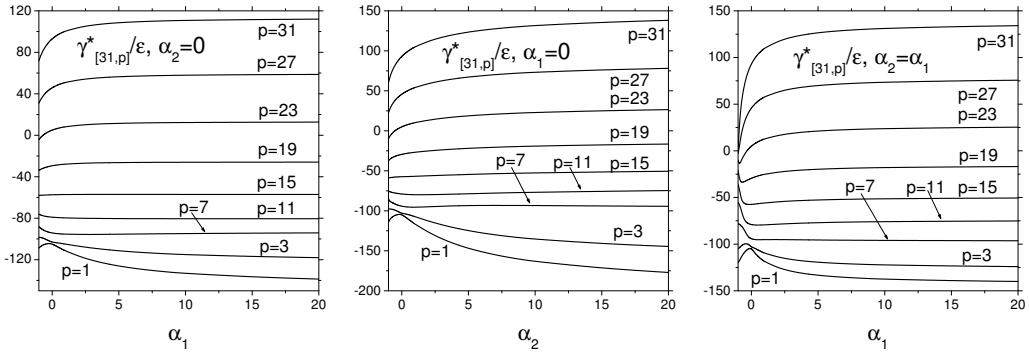


Figure 4.24. Behavior of the anomalous dimension $\gamma^*_{[31,p]}/\varepsilon$ in space dimension $d = 3$ and for representative values of p as functions of anisotropy parameters α_1 and α_2.

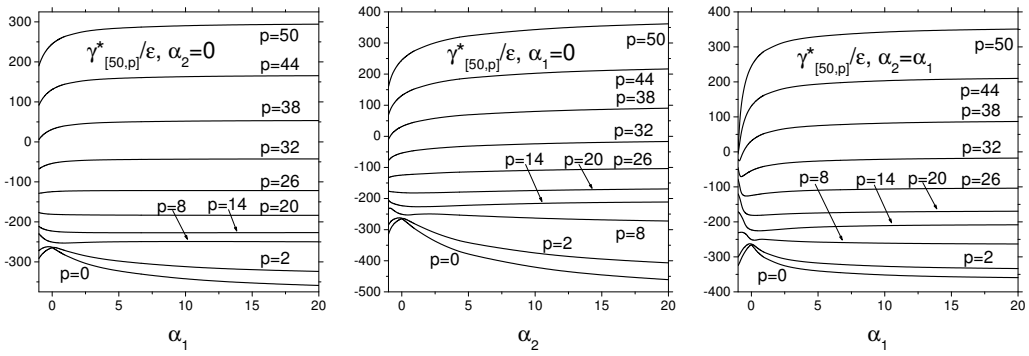


Figure 4.25. Behavior of the anomalous dimension $\gamma^*_{[50,p]}/\varepsilon$ in space dimension $d = 3$ and for representative values of p as functions of anisotropy parameters α_1 and α_2.

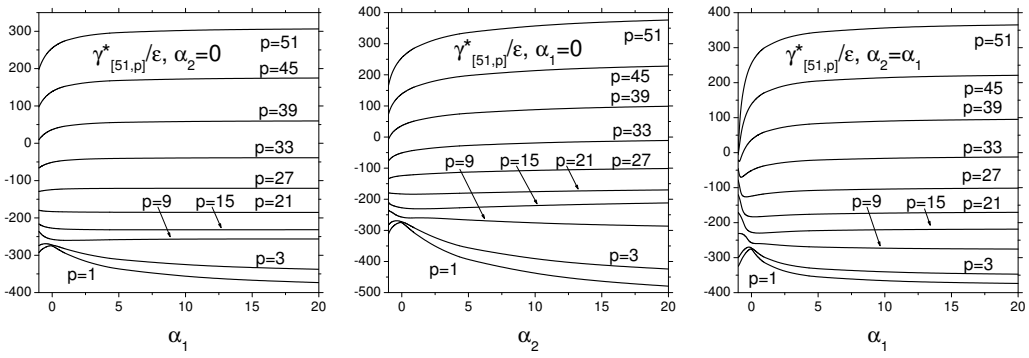


Figure 4.26. Behavior of the anomalous dimension $\gamma^*_{[51,p]}/\varepsilon$ in space dimension $d = 3$ and for representative values of p as functions of anisotropy parameters α_1 and α_2.

case in Figs. 4.23-4.26 appear to be crossing at the point $\alpha_1 = \alpha_2 = 0$ but in fact the curves are only visually running very near together at that point which is a mathematical consequence of the formulas for critical dimensions in the infinitesimal limit $\alpha_1 \to 0$ and $\alpha_2 \to 0$.

In Figs. 4.27, 4.28 the eigenvalues are presented as the functions of two variables α_1 and α_2 for $N = 30, 31, 50, 51$ for the first the most singular modes $p = 0$ for even N and $p = 1$ for odd ones. It can clearly be seen that as values of the parameters α_1 and α_2 increase the anomalous dimensions become more negative approaching some saturated values, therefore the anisotropy amplifies the anomalous scaling.

From the mathematical point of view the present model is found to be similar to the model of a passive scalar quantity advected by a Gaussian strongly anisotropic velocity field [112] in that the fluctuation contributions to the critical dimensions Δ_F of the OPE representation of the structure functions coincide in both cases and thus the hierarchical dependence on the degree of anisotropy is also the same. Here, numerical calculation of the critical dimensions Δ_F in the one-loop approximation has been extended to dimensions related to correlation functions of order $N = 51$ to explore possible departures from powerlike asymptotic behavior. However, contrary to the scalar case, in the inertial range the leading terms of the structure functions of the magnetic field themselves are shown to be coordinate independent with powerlike corrections whose exponents are generated by the calculated critical dimensions.

It is shown that in the inertial range the leading-order powerlike asymptotic behavior of the correlation functions is determined by the critical dimensions brought about already in the isotropic case, which, however, acquire rather strong dependence on the parameters of anisotropy. Powerlike corrections also appear with critical dimensions generated entirely by anisotropic velocity fluctuations. We have calculated numerically the anomalous correction exponents up to order $N = 51$ to explore possible oscillatory modulation or logarithmic corrections to the leading powerlike asymptotics, but have found no sign of this kind of behavior: all calculated corrections have had purely powerlike behavior. Our results show that the exponents of the powerlike corrections tend to decrease with increasing relative impact of the anisotropy.

From the renormalization-group point of view the present model of passively advected vector field in the presence of strong anisotropy is technically similar to that of passively advected scalar

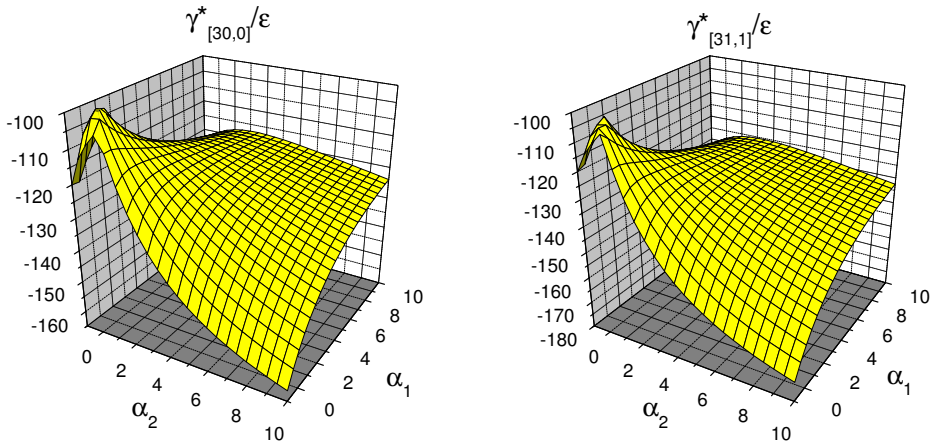


Figure 4.27. Behavior of the anomalous dimensions $\gamma^*_{[30,0]}/\varepsilon$ and $\gamma^*_{[31,1]}/\varepsilon$ in space dimension $d = 3$ as functions of anisotropy parameters α_1 and α_2.

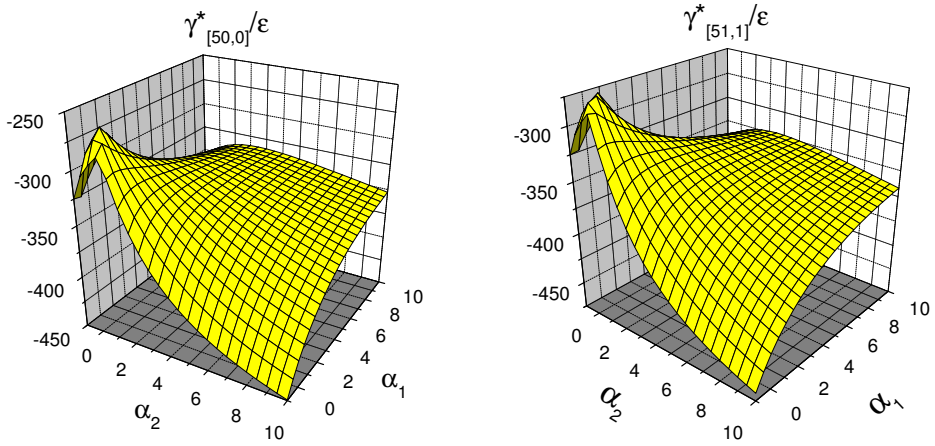


Figure 4.28. Behavior of the anomalous dimensions $\gamma^*_{[50,0]}/\varepsilon$ and $\gamma^*_{[51,1]}/\varepsilon$ in space dimension $d = 3$ as functions of anisotropy parameters α_1 and α_2.

field [112]. In particular, the β functions and the one-loop contributions to renormalization matrices of relevant composite operators are the same. Since in the published analysis of the scalar problem [112] there were some misprints, we have also presented corrected complete results of the calculation of the renormalization matrices.

However, physically the two models differ significantly: instead of the anomalous powerlike growth of the correlation and structure functions of the scalar problem in the inertial range, in the present vector case a powerlike falloff is predicted for the correlation and structure functions. Moreover, in the scalar problem velocity fluctuation contributions to the inertial-range scaling exponents are (small in ε) corrections to the exponents determined by canonical dimensions of the

fields, whereas in the vector problem the inertial-range scaling exponents are solely determined by the fluctuation contributions.

4.4 Developed turbulence with weak anisotropy

Usually, the RG approach is applied to the isotropic models of developed turbulence. However, the field theoretical approach can be used (with some modifications) in the theory of anisotropic developed turbulence. A crucial question immediately arises, whether the principal properties of the isotropic case and anisotropic one are the same at least at the qualitative level. If they are, then it is possible to consider the isotropic case as a first step in the investigation of real systems. On this way of transition from the isotropic developed turbulence into the anisotropic one we have to learn whether the scaling regime does remain stable under this transition. That means, whether the stable fixed points of the RG equations remain stable under the influence of anisotropy.

During the last thirty years a few papers have appeared in which the above question has been considered. In some cases it has been found out that stability really takes place (see,e.g. [235,238]). On the other hand, existence of systems without such a stability has been proved too. As has been shown in Ref. [237], in the anisotropic magnetohydrodynamic developed turbulence a stable regime generally does not exist. In [238, 239], d-dimensional models with $d > 2$ were investigated for two cases: weak anisotropy [238] and strong one [239], and it has been shown that the stability of the isotropic fixed point is lost for dimensions $d < d_c = 2.68$. It has also been shown that stability of the fixed point even for dimension $d = 3$ takes place only for sufficiently weak anisotropy. The only problem in these investigations is that it is impossible to use them in the case $d = 2$ because new UV divergences appear in the Green functions when one considers $d = 2$, and they were not taken into account in [238, 239].

In paper [146], a correct treatment of the two-dimensional isotropic turbulence has been given. The correctness in the renormalization procedure has been reached by introduction into the model a new local term (with a new coupling constant) which allows one to remove additional UV-divergences. From this point of view, the results obtained earlier for anisotropical developed turbulence presented in [157] and based on the paper [144] (the results of the last paper are in conflict with [146]) cannot be considered as correct because they are inconsistent with the basic requirement of the UV-renormalization, namely with the requirement of the localness of the counterterms [24, 65].

The authors of the paper [251] have used the double-expansion procedure introduced in Sec. 3.2 for investigation of developed turbulence with weak anisotropy for $d = 2$. In such a perturbative approach the deviation of the spatial dimension from $d = 2$, $\Delta = (d - 2)/2$, and the deviation of the exponent of the powerlike correlation function of random forcing from their critical values, ε, play the role of the expansion parameters.

The main result of the paper [251] was the conclusion that the two-dimensional fixed point is not stable under weak anisotropy. It means that two-dimensional turbulence is very sensitive to the anisotropy and no stable scaling regimes exist in this case. In the case $d = 3$, for both the isotropic turbulence and anisotropic one, as it has been mentioned above, existence of the stable fixed point, which governs the Kolmogorov asymptotic regime, has been established by means of the RG approach by using the analytical regularization procedure [235, 238, 239]. One can make analytical continuation from $d = 2$ to the three-dimensional turbulence (in the same

sense as in the theory of critical phenomena) and verify whether the stability of the fixed point (or, equivalently, stability of the Kolmogorov scaling regime) is restored. From the analysis made in Ref. [251] it follows that it is impossible to restore the stable regime by transition from dimension $d = 2$ to $d = 3$. We suppose that the main reason for the above described discrepancy is related to the straightforward application of the standard MS scheme. In the standard MS scheme one works with the purely divergent part of the Green functions only, and in concrete calculations its dependence on the space dimension d resulting from the tensor nature of these Green functions is neglected (see Sect.3). In the case of isotropic models, the stability of the fixed points is independent of dimension d. However, in anisotropic models the stability of fixed points depends on the dimension d, and consideration of the tensor structure of Feynman graphs in the analysis of their divergences becomes important.

Here we describe modified MS scheme [155] in which we keep the $d-$dependence of UV-divergences of graphs. There it was confirmed that after such a modification the d-dependence is correctly taken into account and can be used in investigation of whether it is possible to restore the stability of the anisotropic developed turbulence for some dimension d_c when going from a two-dimensional system to a three-dimensional one. In the limit of infinitesimally weak anisotropy for the physically most reasonable value of $\varepsilon = 2$, the value of the borderline dimension is $d_c = 2.44$. Below the borderline dimension, the stable regime of the fixed point of the isotropic developed turbulence is lost by influence of weak anisotropy.

4.4.1 Description of the model

As has already been discussed above, the aim is now to study fully developed turbulence with assumed weak anisotropy. It means that the parameters that describe deviations from the fully isotropic case are sufficiently small and allow one to forget about corrections of higher degrees (than linear) which are made by them.

In the statistical theory of anisotropically developed turbulence the turbulent flow can be described by a random velocity field $\boldsymbol{v}(t, \boldsymbol{x})$ and its evolution is given by the stochastically forced Navier-Stokes equation

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} - \nu_0 \boldsymbol{\nabla}^2 \boldsymbol{v} - \boldsymbol{f}^A = \boldsymbol{f}, \tag{4.188}$$

which is slightly modified with respect to the isotropic case (3.9). The incompressibility of the fluid is again assumed. The term $\boldsymbol{f}^A$ is related to anisotropy and will be specified later. The large-scale random force per unit mass $\boldsymbol{f}$ is assumed to have Gaussian statistics defined by the averages

$$\langle f_i \rangle = 0, \ \langle f_i(\boldsymbol{x}_1, t) f_j(\boldsymbol{x}_2, t) \rangle = D_{ij}(\boldsymbol{x}_1 - \boldsymbol{x}_2, t_1 - t_2). \tag{4.189}$$

The two-point correlation matrix

$$D_{ij}(\vec{x}, t) = \delta(t) \int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} \tilde{D}_{ij}(\boldsymbol{k}) \exp(i\boldsymbol{k} \cdot \boldsymbol{x}) \tag{4.190}$$

is convenient to be parameterized in the following way [235, 237]:

$$\tilde{D}_{ij}(\boldsymbol{k}) = g_0 \nu_0^3 k^{4-d-2\varepsilon} \left\{ \left[1 + \alpha_{10} \frac{(\boldsymbol{n} \cdot \boldsymbol{k})^2}{k^2} \right] P_{ij}(\boldsymbol{k}) + \alpha_{20} \tilde{n}_i(\boldsymbol{k}) \tilde{n}_j(\boldsymbol{k}) \right\}, \tag{4.191}$$

where $\tilde{\boldsymbol{n}}_i$ has been introduced in Eq. (4.86). In what it is assumed that parameters α_{10} and α_{20} are small enough and generate only small deviations from the isotropy case.

The action of the fields $\boldsymbol{v}$ and $\boldsymbol{v}'$ is given in the compact form

$$\mathcal{S} = \frac{1}{2} v_i' D_{ij} v_j' + \boldsymbol{v}' \cdot \left[-\partial_t \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} + \nu_0 \boldsymbol{\nabla}^2 \boldsymbol{v} + \boldsymbol{f}^A \right]. \tag{4.192}$$

Now we can return back to give an explicit form of the anisotropic dissipative term $\boldsymbol{f}^A$. When $d > 2$ the UV-divergences are only present in the one-particle-irreducible Green function $\langle \boldsymbol{v}'\boldsymbol{v} \rangle$. To remove them, one needs to introduce into the action in addition to the counterterm $\boldsymbol{v}'\boldsymbol{\nabla}^2\boldsymbol{v}$ (the only counterterm needed in the isotropic model) the following ones $\boldsymbol{v}'(\boldsymbol{n}\cdot\boldsymbol{\nabla})^2\boldsymbol{v}$, $(\boldsymbol{n}\cdot\boldsymbol{v}')\boldsymbol{\nabla}^2(\boldsymbol{n}\cdot\boldsymbol{v})$ and $(\boldsymbol{n}\cdot\boldsymbol{v}')(\boldsymbol{n}\cdot\boldsymbol{\nabla})^2(\boldsymbol{n}\cdot\boldsymbol{v})$. These additional terms are needed to remove divergences related to anisotropic structures. In this case ($d > 2$), one can use the above action (4.192) with (4.191) to solve the anisotropic turbulent problem. Therefore, in order to arrive at the multiplicative renormalizable model, it is necessary to take the term $\boldsymbol{f}^A$ in the form

$$\boldsymbol{f}^A = \nu_0 \left[\chi_{10} (\boldsymbol{n}\cdot\boldsymbol{\nabla})^2 \boldsymbol{v} + \chi_{20} \boldsymbol{n} \nabla^2 (\boldsymbol{n}\cdot\boldsymbol{v}) + \chi_{30} \boldsymbol{n} (\boldsymbol{n}\cdot\boldsymbol{\nabla})^2 (\boldsymbol{n}\cdot\boldsymbol{v}) \right]. \tag{4.193}$$

Bare parameters χ_{10}, χ_{20} and χ_{30} characterize the weight of the individual structures in (4.193).

As was discussed in Sec. 3.2 more complicated situation arises in the specific case $d = 2$ where new divergences appear. They are related to the 1-irreducible Green function $\langle \boldsymbol{v}'\boldsymbol{v}' \rangle$ which is finite when $d > 2$. Here one comes to a problem how to remove these divergences because the term in our action, which contains a structure of this type is nonlocal, namely $\boldsymbol{v}' k^{4-d-2\varepsilon} \boldsymbol{v}'$. The only correct way of solving the above problem is to introduce into the action a new local term of the form $\boldsymbol{v}'\nabla^2\boldsymbol{v}'$ (isotropic case) [146]. In the anisotropic case, we have to introduce additional counterterms $\boldsymbol{v}'(\boldsymbol{n}\cdot\boldsymbol{\nabla})^2\boldsymbol{v}'$, $(\boldsymbol{n}\cdot\boldsymbol{v}')\boldsymbol{\nabla}^2(\boldsymbol{n}\cdot\boldsymbol{v}')$ and $(\boldsymbol{n}\cdot\boldsymbol{v}')(\boldsymbol{n}\cdot\boldsymbol{\nabla})^2(\boldsymbol{n}\cdot\boldsymbol{v}')$.

In this case, the kernel (4.191) corresponding to the correlation matrix $D_{ij}(\boldsymbol{x}_1 - \boldsymbol{x}_2, t_2 - t_1)$ in action (4.192) is replaced by the expression

$$\begin{aligned} \tilde{D}_{ij}(\boldsymbol{k}) = {} & g_{10} \nu_0^3 k^{2-2\Delta-2\varepsilon} \left\{ \left[1 + \alpha_{10} \frac{(\boldsymbol{n}\cdot\boldsymbol{k})^2}{k^2} \right] P_{ij}(\vec{k}) + \alpha_{20} \tilde{\boldsymbol{n}}_i(\boldsymbol{k}) \tilde{\boldsymbol{n}}_j(\boldsymbol{k}) \right\} \\ & + g_{20} \nu_0^3 k^2 \left\{ \left[1 + \alpha_{30} \frac{(\boldsymbol{n}\cdot\boldsymbol{k})^2}{k^2} \right] P_{ij}(\boldsymbol{k}) + \left[\alpha_{40} + \alpha_{50} \frac{(\boldsymbol{n}\cdot\boldsymbol{k})^2}{k^2} \right] \tilde{\boldsymbol{n}}_i(\boldsymbol{k}) \tilde{\boldsymbol{n}}_j(\boldsymbol{k}) \right\}. \end{aligned} \tag{4.194}$$

Here g_{20}, α_{30}, α_{40} and α_{50} are new parameters of the model, and the parameter g_0 in Eq. (4.191) is now renamed as g_{10}. One can see that in such a formulation the counterterm $\boldsymbol{v}'\boldsymbol{\nabla}^2\boldsymbol{v}'$ and all anisotropic terms can be taken into account by renormalization of the coupling constant g_{20}, and the parameters α_{30}, α_{40} and α_{50}.

It has to be stressed that the last term of the $\vec{f}^A$ in Eq. (4.193) which is characterized by the parameter χ_{30} and the term of the correlation matrix (4.194) related to the parameter α_{50} are of the order $\mathcal{O}(n^4)$ in contrast to the others which are either $\mathcal{O}(n^0)$ (the isotropic terms) or $\mathcal{O}(n^2)$. Because we work in the limit of weak anisotropy this fact has its consequence in the vanishing of their values at the fixed point. On the other hand, the eigenvalues of the stability matrix which correspond to the parameters χ_{30} and α_{50} are of the same order, $\mathcal{O}(\varepsilon)$, as the eigenvalues which

correspond to the other parameters and they play important role in determination of stability of the regime.

The action (4.192) with the kernel $\tilde{D}_{ij}(\boldsymbol{k})$ (4.194) is given in the form convenient for realization of the quantum field perturbation analysis with the standard Feynman diagram technique. From the quadratic part of the action one obtains the matrix of bare propagators. Their wave-number - frequency representation reads

$$\begin{aligned}
\Delta_{ij}^{vv}(\omega_k,\boldsymbol{k}) &= -\frac{K_3}{K_1K_2}P_{ij} + \frac{1}{K_1(K_2+\tilde{K}(1-\xi_k^2))} \\
&\quad\times\left[\frac{\tilde{K}K_3}{K_2} + \frac{\tilde{K}(K_3+K_4(1-\xi_k^2))}{(K_1+\tilde{K}(1-\xi_k^2))} - K_4\right]\tilde{\boldsymbol{n}}_i(\boldsymbol{k})\tilde{\boldsymbol{n}}_j(\boldsymbol{k}) \\
\Delta_{ij}^{vv'}(\omega_k,\boldsymbol{k}) &= \frac{1}{K_2}P_{ij} - \frac{\tilde{K}}{K_2(K_2+\tilde{K}(1-\xi_k^2))}\tilde{\boldsymbol{n}}_i(\boldsymbol{k})\tilde{\boldsymbol{n}}_j(\boldsymbol{k})\,,
\end{aligned} \tag{4.195}$$

where for brevity following notation has been used

$$\begin{aligned}
\xi_k &= \frac{\boldsymbol{n}\cdot\boldsymbol{k}}{k}, \\
K_1 &= i\omega_k + \nu_0 k^2 + \nu_0\chi_{10}(\boldsymbol{n}\cdot\boldsymbol{k})^2\,, \\
K_2 &= -i\omega_k + \nu_0 k^2 + \nu_0\chi_{10}(\boldsymbol{n}\cdot\boldsymbol{k})^2\,, \\
K_3 &= -g_{10}\nu_0^3 k^{2-2\Delta-2\varepsilon}(1+\alpha_{10}\xi_k^2) - g_{20}\nu_0^3k^2(1+\alpha_{30}\xi_k^2)\,, \\
K_4 &= -g_{10}\nu_0^3 k^{2-2\Delta-2\varepsilon}\alpha_{20} - g_{20}\nu_0^3k^2(\alpha_{40}+\alpha_{50}\xi_k^2)\,, \\
\tilde{K} &= \nu_0\chi_{20}k^2 + \nu_0\chi_{30}(\boldsymbol{n}\cdot\boldsymbol{k})^2\,.
\end{aligned} \tag{4.196}$$

The propagators are written in the form suitable also for strong anisotropy when the parameters α_{i0} are not small. In the case of weak anisotropy, it is possible to make the expansion and work only with linear terms with respect to all parameters which characterize anisotropy.

4.4.2 RG-analysis and stability of the fixed point

Using the standard analysis of quantum field theory (see e.g. [24, 25, 65, 71]), one can find out that the UV divergences of one-particle-irreducible Green functions $\langle \boldsymbol{v}'\boldsymbol{v}\rangle_{\text{1-ir}}$ and $\langle \boldsymbol{v}'\boldsymbol{v}'\rangle_{\text{1-ir}}$ are quadratic in the wave vector. The last one takes place only in the case when dimension of the space is two. All terms needed for removing the divergences are included in the action (4.192) with (4.193) and kernel (4.194). This leads to the fact that the model is multiplicatively renormalizable. Thus, one can immediately write down the renormalized action in wave-number - frequency representation with $\boldsymbol{\nabla}\to i\boldsymbol{k}, \partial_t\to -i\omega_k$ (all needed integrations and summations are assumed)

$$\begin{aligned}
\mathcal{S}^R[\boldsymbol{v},\boldsymbol{v}'] = \frac{1}{2}v_i'\Big\{ & g_1\nu^3\mu^{2\varepsilon}k^{2-2\Delta-2\varepsilon}\Big(1+\alpha_1\xi_k^2\Big)P_{ij} + \alpha_2\tilde{\boldsymbol{n}}_i(\boldsymbol{k})\tilde{\boldsymbol{n}}_j(\boldsymbol{k}) \\
& + g_2\nu^3\mu^{-2\Delta}k^2\Big[\Big(Z_5+Z_6\alpha_3\xi_k^2\Big)P_{ij} + \Big(Z_7\alpha_4+Z_8\alpha_5\xi_k^2\Big)\tilde{\boldsymbol{n}}_i(\boldsymbol{k})\tilde{\boldsymbol{n}}_j(\boldsymbol{k})\Big]\Big\}v_j'
\end{aligned}$$

$$+ v_i^{\cdot}\Bigg\{(i\omega_k - Z_1\nu k^2)P_{ij} - \nu k^2\Bigg[Z_2\chi_1\xi_k^2 P_{ij} + \left(Z_3\chi_2 + Z_4\chi_3\xi_k^2\right) \times \tilde{\boldsymbol{n}}_i(\boldsymbol{k})\tilde{\boldsymbol{n}}_j(\boldsymbol{k})\Bigg]\Bigg\}v_j + \frac{1}{2}v_i^{\cdot}v_j v_l V_{ijl}. \tag{4.197}$$

Quantities g_i, χ_i, α_3, α_4, α_5 and ν are the renormalized counterparts of bare ones and Z_i are renormalization constants which are expressed via the UV divergent parts of the functions $\langle \boldsymbol{v}'\boldsymbol{v}\rangle_{\text{1-ir}}$ and $\langle \boldsymbol{v}'\boldsymbol{v}'\rangle_{\text{1-ir}}$. Their general form in one loop approximation is

$$Z_i = 1 - F_i\,\text{Poles}_i^{\Delta,\varepsilon}\,. \tag{4.198}$$

In the standard MS scheme the amplitudes F_i are only some functions of g_i, χ_i, α_3, α_4, α_5 and are independent of d and ε. The terms $\text{Poles}_i^{\Delta,\varepsilon}$ are given by linear combinations of the poles $\frac{1}{2\varepsilon}$, $\frac{1}{2\Delta}$ and $\frac{1}{4\varepsilon+2\Delta}$ (for $\Delta \to 0, \varepsilon \to 0$). The amplitudes $F_i = F_i^{(1)}F_i^{(2)}$ are a product of two multipliers $F_i^{(1)}, F_i^{(2)}$. One of them, say, $F_i^{(1)}$ is a multiplier originating from the divergent part of the Feynman diagrams, and the second one $F_i^{(2)}$ is connected only with the tensor nature of the diagrams. For illustration purposes let us consider the following simple example (UV-divergent diagram)

$$I(\boldsymbol{k},\boldsymbol{n}) \equiv n_i n_j k_l k_m \int \mathrm{d}^d\boldsymbol{q}\frac{1}{(q^2+m^2)^{1+2\Delta}}\left(\frac{q_i q_j q_l q_m}{q^4} - \frac{\delta_{ij}q_l q_m + \delta_{il}q_j q_m + \delta_{jl}q_i q_m}{3q^2}\right), \tag{4.199}$$

where summations over repeated indices are implied. It can be simplified in the following way:

$$I(\boldsymbol{k},\boldsymbol{n}) \equiv n_i n_j k_l k_m S_{ijlm}\int_0^\infty \mathrm{d}q^2\frac{q^{2\Delta}}{2(q^2+m^2)^{1+2\Delta}}, \tag{4.200}$$

where

$$S_{ijlm} = \frac{S_d}{d(d+2)}\left(\delta_{ij}\delta_{lm}+\delta_{il}\delta_{jm}+\delta_{im}\delta_{jl} - \frac{(d+2)}{3}(\delta_{ij}\delta_{lm}+\delta_{il}\delta_{jm}+\delta_{im}\delta_{jl})\right), \tag{4.201}$$

$$\int_0^\infty \mathrm{d}q^2\frac{q^{2\Delta}}{2(q^2+m^2)^{1+2\Delta}} = \frac{\Gamma(\Delta+1)\Gamma(\Delta)}{2m^{2\Delta}\Gamma(2\Delta+1)}. \tag{4.202}$$

The purely UV divergent part manifests itself as the pole in Δ; therefore, one finds

$$\text{UV div. part of } \; I = \frac{1}{2\delta}(F_1^{(2)}k^2 + F_2^{(2)}(\boldsymbol{n}\cdot\boldsymbol{k})^2),$$

where $F_1^{(2)} = F_2^{(2)}/2 = (1-d)S_d/3d(d+2)$ ($F_1^{(1)} = F_2^{(1)} = 1$).

In the standard MS scheme one puts $d = 2$ in $F_1^{(2)}, F_2^{(2)}$; therefore the d-dependence of these multipliers is ignored. For the theories with vector fields and, consequently, with tensor diagrams, where the sign of values of fixed points and/or their stability depend on the dimension d, the

procedure, which eliminates the dependence of multipliers of the type $F_1^{(2)}, F_2^{(2)}$ on d, is not completely correct because one is not able to control the stability of the fixed point when drives to $d = 3$. In the analysis of Feynman diagrams slightly modified MS scheme was proposed [155] in such a way that we keep the d-dependence of F in (4.198). The following calculations of RG functions (β functions and anomalous dimensions) allow one to arrive at the results which are in qualitative agreement with the results obtained in the framework of the simple analytical regularization scheme [239], i.e., the fixed point was obtained [155] which is not stable for $d = 2$, but whose stability is restored for a borderline dimension $2 < d_c < 3$.

The transition from the action (4.192) to the renormalized one (4.197) is given by the introduction of the following renormalization constants Z:

$$\nu_0 = \nu Z_\nu\,,\ g_{10} = g_1\mu^{2\varepsilon}Z_{g_1}\,,\ g_{20} = g_2\mu^{-2\Delta}Z_{g_2}\,,\ \chi_{i0} = \chi_i Z_{\chi_i}\,,\ \alpha_{(i+2)0} = \alpha_{i+2}Z_{\alpha_{i+2}}, \tag{4.203}$$

where $i = 1, 2, 3$. By comparison of the corresponding terms in the action (4.197) with definitions of the renormalization constants Z for the parameters (4.203), one can immediately write down relations between them

$$Z_\nu = Z_1, \quad Z_{g_1} \ = Z_1^{-3}, \quad Z_{g_2} = Z_5 Z_1^{-3}, \quad Z_{\chi_i} \ = Z_{1+i}Z_1^{-1}, \quad Z_{\alpha_{i+2}} = Z_{i+5}Z_5^{-1}, \tag{4.204}$$

where again $i = 1, 2, 3$.

In the one-loop approximation, divergent one-irreducible Green functions are $\langle \boldsymbol{v}'\boldsymbol{v}\rangle_{\text{1-ir}}$ and $\langle \boldsymbol{v}'\boldsymbol{v}'\rangle_{\text{1-ir}}$ Corresponding divergent parts of these diagrams $\Gamma^{v'v'}$, $\Gamma^{v'v}$ have the structure

$$\begin{aligned}
\Gamma_{v'v'} &= \frac{1}{2}\nu^3 A\Bigg[\frac{g_1^2}{4\varepsilon + 2\Delta}\left(a_1\Delta_{ij}k^2 + a_2\delta_{ij}(\boldsymbol{n}\cdot\boldsymbol{k})^2 + a_3 n_i n_j k^2 + a_4 n_i n_j (\boldsymbol{n}\cdot\boldsymbol{k})^2\right)\\
&\quad + \frac{g_1 g_2}{2\varepsilon}\left(b_1\delta_{ij}k^2 + b_2\delta_{ij}(\boldsymbol{n}\cdot\boldsymbol{k})^2 + b_3 n_i n_j k^2 + b_4 n_i n_j (\boldsymbol{n}\cdot\boldsymbol{k})^2\right)\\
&\quad + \frac{g_2^2}{-2\Delta}\left(c_1\delta_{ij}k^2 + c_2\delta_{ij}(\boldsymbol{n}\cdot\boldsymbol{k})^2 + c_3 n_i n_j k^2 + c_4 n_i n_j (\boldsymbol{n}\cdot\boldsymbol{k})^2\right)\Bigg],\\
\Gamma_{v'v} &= -\nu A\Bigg[\frac{g_1}{2\varepsilon}\left(d_1\delta_{ij}k^2 + d_2\delta_{ij}(\boldsymbol{n}\cdot\boldsymbol{k})^2 + d_3 n_i n_j k^2 + d_4 n_i n_j (\boldsymbol{n}\cdot\boldsymbol{k})^2\right)\\
&\quad + \frac{g_2}{-2\Delta}\left(e_1\delta_{ij}k^2 + e_2\delta_{ij}(\boldsymbol{n}\cdot\boldsymbol{k})^2 + e_3 n_i n_j k^2 + e_4 n_i n_j (\boldsymbol{n}\cdot\boldsymbol{k})^2\right)\Bigg],
\end{aligned} \tag{4.205}$$

where parameter A and functions a_i, b_i, c_i, d_i and e_i are not not needed for a discussion of basic idea. Their explicit expressions can be found in [155]. The counterterms are built up from these divergent parts which lead to the following equations for renormalization constants:

$$\begin{aligned}
Z_1 &= 1 - A\left(\frac{g_1}{2\varepsilon}d_1 - \frac{g_2}{2\Delta}e_1\right),\\
Z_{1+i} &= 1 - \frac{A}{\chi_i}\left(\frac{g_1}{2\varepsilon}d_{1+i} - \frac{g_2}{2\Delta}e_{1+i}\right),
\end{aligned}$$

$$
\begin{aligned}
Z_5 &= 1 + \frac{A}{2}\left(\frac{g_1^2}{g_2}\frac{a_1}{4\varepsilon+2\Delta} + \frac{g_1}{2\varepsilon}b_1 - \frac{g_2}{2\Delta}c_1\right), \\
Z_{5+i} &= 1 + \frac{A}{2\alpha_{i+2}}\left(\frac{g_1^2}{g_2}\frac{a_{i+1}}{4\varepsilon+2\Delta} + \frac{g_1}{2\varepsilon}b_{i+1} - \frac{g_2}{2\Delta}c_{i+1}\right), \\
& i = 1,2,3\,.
\end{aligned} \tag{4.206}
$$

By substitution of the anomalous dimensions γ_i (3.4) into the expressions for the β-functions one obtains

$$
\begin{aligned}
\beta_{g_1} &= g_1\Big[-2\varepsilon + 3A(g_1d_1 + g_2e_1)\Big], \\
\beta_{g_2} &= g_2\left[2\Delta + 3A(g_1d_1 + g_2e_1) + \frac{A}{2}\left(\frac{g_1^2}{g_2}a_1 + g_1b_1 + g_2c_1\right)\right], \\
\beta_{\chi_i} &= -A\left[(g_1d_{i+1} + g_2e_{i+1}) - \chi_i(g_1d_1 + g_2e_1)\right], \\
\beta_{\alpha_{i+2}} &= -\frac{A}{2}\left[-\left(\frac{g_1^2}{g_2}a_{i+1} + g_1b_{i+1} + g_2c_{i+1}\right) + \alpha_{i+2}\left(\frac{g_1^2}{g_2}a_1 + g_1b_1 + g_2c_1\right)\right], \\
& i = 1,2,3\,.
\end{aligned} \tag{4.207}
$$

Now we have all necessary tools at hand to investigate the fixed points and their stability. In *the isotropic case* all parameters which are connected with the anisotropy are equal to zero, and one can immediately find the Kolmogorov fixed point, namely:

$$
\begin{aligned}
g_{1*} &= \frac{1}{A}\,\frac{8(2+d)\varepsilon(2\varepsilon - 3d(\Delta+\varepsilon) + d^2(3\Delta+2\varepsilon))}{9(-1+d)^3 d(1+d)(\Delta+\varepsilon)}, \\
g_{2*} &= \frac{1}{A}\,\frac{8(-4-2d+2d^2+d^3)\varepsilon^2}{9(-1+d)^3 d(1+d)(\Delta+\varepsilon)},
\end{aligned} \tag{4.208}
$$

where corresponding Ω matrix has the following eigenvalues:

$$
\begin{aligned}
\lambda_{1,2} = \frac{1}{6d(d-1)}\Bigg\{ & 6d\Delta(d-1) + 4\varepsilon(2-3d+2d^2) \\
& \pm \Bigg[(6d\Delta(1-d) - 4\varepsilon(2-3d+2d^2))^2 \\
& - 12d(d-1)\varepsilon(12d\Delta(d-1) + 4\varepsilon(2-3d+2d^2))\Bigg]^{\frac{1}{2}}\Bigg\}.
\end{aligned} \tag{4.209}
$$

From detailed analysis of these eigenvalues it follows [155] that in the interesting region of parameters, namely $\varepsilon > 0$ and $\Delta \geq 0$ (it corresponds to $d \geq 2$) the above computed fixed point is stable. In the limit $d = 2$, this fixed point is in agreement with that given in [146, 251].

When one considers *the weak anisotropy case* the situation becomes more complicated because of necessity to use all system of β-functions if one wants to analyze the stability of the

fixed point. It is also possible to find analytical expressions for the fixed point in this more complicated case because in the weak anisotropy limit it is enough to calculate linear corrections of α_1 and α_2 to all the quantities.

To investigate the stability of the fixed point it is necessary to apply it in the matrix (2.24). Analysis of this matrix shows us that it can be written in the block-diagonal form: $(6\times 6)(2\times 2)$. The (2×2) part is given by the β-functions of the parameters α_5 and χ_3 and, namely, this block is responsible for the existence of the borderline dimension d_c because one of its eigenvalues, say $\lambda_1(\varepsilon, d, \alpha_1, \alpha_2)$, has a solution $d_c \in \langle 2, 3\rangle$ of the equation $\lambda_1(\varepsilon, d_c, \alpha_1, \alpha_2) = 0$ for the defined values of $\varepsilon, \alpha_1, \alpha_2$. Te details of determination of fixed points' structure and the corresponding eigenvalues of the stability matrix responsible for instability can be found in [155].

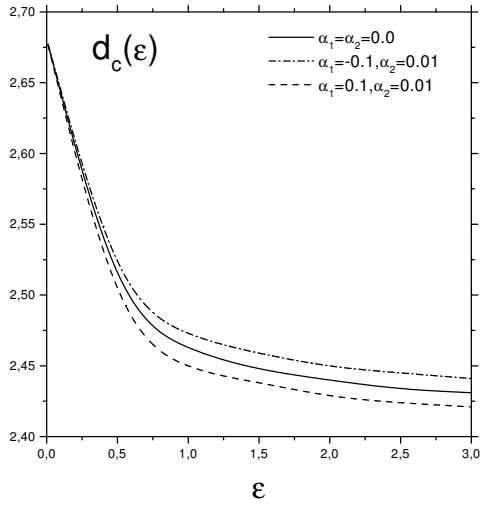


Figure 4.29. Dependence of the borderline dimension d_c on the parameter ϵ for concrete values of α_1 and α_2.

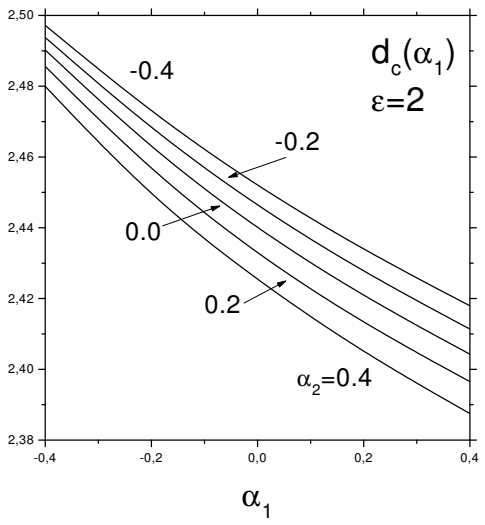


Figure 4.30. Dependence of the borderline dimension d_c on the parameters α_1 and α_2 for physical value $\epsilon = 2$.

For the energy pumping regime ($\varepsilon = 2$) and $\alpha_1 = \alpha_2 = 0$ the critical dimension $d_c = 2.44$ is found. This is the case when one supposes only the fact of anisotropy. Using nonzero values of α_1 and α_2 one can also estimate the influence of these parameters on the borderline dimension d_c. It is interesting to calculate the dependence of d_c on the parameter ε too. In Fig. 4.29, this dependence and the dependence on small values of α_1 and α_2 are presented. As one can see from this figure d_c increases when $\varepsilon \to 0$ and also the parameters α_1 and α_2 give small corrections to d_c. In Fig. 4.30, the dependence of d_c on α_1 and α_2 for $\varepsilon = 2$ is presented.

5 Reaction-diffusion problems

Reaction-diffusion models introduced in Sec. 1.2.2 play an important role in non-equilibrium physics. In contrast to the models of critical dynamics they are often described by non-hermitian operators. This leads to intrigued and interesting phenomena [33, 34]. This will serve us as an illustration of possible use of technique introduced in Secs. 1.2 and 1.4. For an analysis of resulting field-theoretic actions again theoretical approaches from Sec. 2 will be used. The main aim here is to analyze scaling behavior of two important models:

1. annihilation process $A + A \to \varnothing$.
2. directed bond percolation process.

Velocity fluctuations are hardly avoidable in any of experiments. For example, a vast majority of chemical reactions occurs at finite temperature, which is inevitably encompassed with the presence of a thermal noise. Furthermore, disease spreading and chemical reactions could be affected by the turbulent advection to a great extent. Fluid dynamics is in general described by the Navier-Stokes equations [31]. A general solution of these equations remains an open question [70,79]. For both of these problems very important role is played by the properties of the environment in which these processes take place. It can be expected that impurities and defects, which are not taken into account in the original formulation, could cause a change in the universal properties of the model. This is believed to be one of the reasons why there are not so many direct experimental realizations [252,253] of the percolation process itself. A study of deviations from the ideal situation could proceed in different routes and this still constitutes a topic of an ongoing debate [33]. A substantial effort has been made in studying a long-range interaction using Lévy-flight jumps [254–256], effects of immunization [35,257], mutations [258], feedback of the environment on the percolating density [259], or in the presence of spatially quenched disorder [260]. In general, the novel behavior is observed with a possibility that critical behavior is lost or profoundly changed. For example, the latter is observed in annihilation process [261]. There was shown that thermal fluctuations makes otherwise IR stable regime unstable. On the other hand, the presence of a quenched disorder in percolation process causes a shift of the critical fixed point to the unphysical region. This leads to such interesting phenomena as an activated dynamical scaling or Griffiths singularities [262–265].

5.1 Effect of velocity fluctuations on $A + A \to \varnothing$

The irreversible annihilation reaction $A + A \to \varnothing$ is a fundamental model of non-equilibrium physics. The reacting particles perform chaotic motion due to diffusion or some external advection field such as atmospheric eddy and may react after the mutual collision with constant microscopic probability K_0 per unit time. Implicitly it is assumed that resulting molecule is inert, i.e. chemically inactive, and has no influence on the movement of the reacting A particles. Many reactions of this type are observed in diverse chemical, biological or physical systems [266,267]. The usual approach to this kind of problems is based on the use of the kinetic rate equation. It leads to a self-consistent description analogous to the mean-field approximation in the theory of critical phenomena. The basic assumption of the rate-equation approach is that the particle density is spatially homogeneous. This homogeneity can be thought as a consequence of either an infinite mobility of the reactants or of a very small probability that a chemical reaction actually

occurs when reacting entities meet each other. On the other hand, if the particle mobility becomes sufficiently small, or equivalently, if the microscopic reaction probability becomes large enough there is a possible transition to a new regime where is is more probable that the given particle reacts with local neighbors than with distant particles. This behavior is known as the diffusion-controlled regime [58, 268]. For the annihilation reaction limited assumption of the density homogeneity leads to the following equation for the mean particle number

$$\partial_t n(t) = -K_0 n^2(t). \tag{5.1}$$

This equation predicts a long-time asymptotic decay as $n(t) \sim t^{-1}$ and the decay exponent does not depend on the space dimension. This is a common situation observed in the mean field theory. However, it turns out [269, 270] that in lower space dimensions $d \leq 2$ the assumption of spatially uniform density, or equivalently of negligible density fluctuations of reacting particles, is not appropriate. The upper critical dimension for this reaction $d_c = 2$ [269], above which mean field approximation is valid. Re-entrancy of the diffusing particles [271] in low space dimensions leads to effective decrease of the decay exponent.

A typical reaction occurs in liquid or gaseous environment. Thermal fluctuations of this underlying environment cause additional advection of the reacting particles. Therefore, it is interesting to study the influence of the advection field on the annihilation process.

Most of the renormalization-group analyses of the effect of random drift on the annihilation reaction $A + A \to \varnothing$ in the framework of the Doi approach have been carried out for the case of a quenched random drift field. Potential random drift with long-range [272, 273] and short-range correlations [274] have been studied as well as "turbulent" flow (i.e. quenched solenoidal random field) with potential disorder [275, 276]. For a more realistic description of a turbulent flow time-dependent velocity field would be more appropriate. In Ref. [276] dynamic disorder with a given Gaussian distribution has been considered, whereas the most ambitious approach on the basis of a velocity field generated by the stochastic Navier-Stokes equation has been introduced here by two of the present authors [261]. From the point of the Navier-Stokes equation the situation near the critical dimension $d_c = 2$ of the pure reaction model is even more intriguing due to the properties of the Navier-Stokes equation (see Sec. 3.3). The aim of this part is to examine the IR behavior of the annihilation process under the influence of advecting velocity fluctuations and to determine its stability in the second order of the perturbation theory. Using mapping procedure based on the Doi formalism (see Sec. 1.4) an effective field-theoretic model for the annihilation process is constructed. The RG method is applied on the model in the field-theoretic formulation, which is the most efficient in calculations beyond the one-loop order, and within the two-parameter expansion the renormalization constants and fixed points of the renormalization group are determined in the two-loop approximation. The non-linear integro-differential equation, which includes first non-trivial corrections to the (5.1), is obtained for the mean particle number and it is shown how the information about IR asymptotics can be extracted from it in the case of a homogeneous system. This equation allows to investigate heterogeneous systems as well with the account of the effect density and advecting velocity fluctuations. The solution of the equation in the heterogeneous case requires numerical calculations and was considered

in [277].

5.1.1 Field-theoretic model of the annihilation reaction

The goal is to analyze anomalous kinetics of the irreversible single-species annihilation reaction

$$A + A \xrightarrow{K_0} \varnothing, \tag{5.2}$$

with the unrenormalized (mean field) rate constant K_0. This can be considered as a special case of the Verhulst model 1.112. The corresponding field-theoretic action 1.159 with velocity fluctuations taken into account reads

$$\begin{aligned}\mathcal{S}_1 = -\int_0^\infty \mathrm{d}t \int \mathrm{d}^d\boldsymbol{x}\,\{&\psi'\partial_t\psi + \psi'\boldsymbol{\nabla}\cdot(\boldsymbol{v}\psi) - D_0\psi'\boldsymbol{\nabla}^2\psi \\ &+ \lambda_0 D_0[2\psi' + (\psi')^2]\psi^2 + n_0\int \mathrm{d}^d\boldsymbol{x}\,\psi'(\boldsymbol{x},0)\}.\end{aligned} \tag{5.3}$$

In the usual fashion the diffusion constant D_0 has been extracted from the rate constant $K_0 = \lambda_0 D_0$.

The most realistic description of the velocity field $\boldsymbol{v}(x)$ is based on the use of the stochastic Navier-Stokes equation (3.9). Averaging over the random velocity field $\boldsymbol{v}$ is done with the "weight" functional $\mathcal{W} = \mathrm{e}^{\mathcal{S}_{\mathrm{NS}}}$, where S_{NS} is the effective action for the advecting velocity field (3.18).

It is easily seen that the studied model contains three different types of propagators $\Delta^{vv'}$, Δ^{vv} and $\Delta^{\psi\psi'}$. The former two were given in (3.93) and the latter one reads

$$\Delta^{\psi\psi'}(\omega_k, \boldsymbol{k}) = \frac{1}{-i\omega_k + D_0 k^2}. \tag{5.4}$$

The two reaction vertices derived from the functional (5.3) are depicted on Fig. 5.1 and physi-

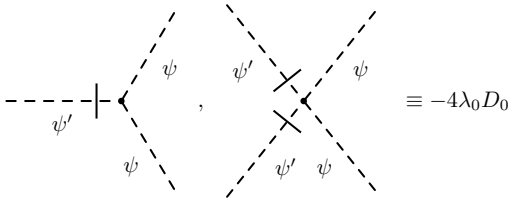


Figure 5.1. Interaction vertices responsible for density fluctuations and their corresponding vertex factor

cally describe the density fluctuations of the reactant particles. It should be stressed that in this model no backward influence of the reactants on the velocity field is assumed. Therefore, the model given by actions (5.3) and (3.9) may be characterized as a model for the advection of the passive chemically active admixture.

In the one-loop approximation velocity fluctuations have no effect on the renormalization of the interaction vertex [261], whereas in the two-loop approximation situation becomes more complicated and all graphs in Fig. 5.2 must be considered [278].

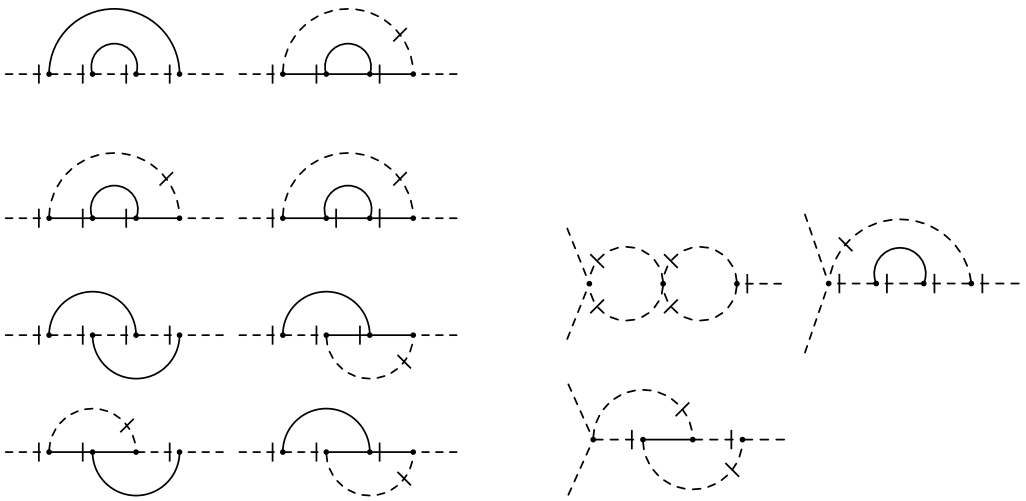


Figure 5.2. Two-loop graphs needed for UV renormalization of $\Gamma_{\psi'\psi}$(left) and $\Gamma_{\psi'\psi\psi}$(left) 1PI functions.

5.1.2 UV renormalization of the model

In what follows we will employ the modified minimal subtraction scheme. What we mean here, is the ray scheme [153] introduced in Sec. 3.2, in which the two regularizing parameters ε, Δ (ε has been introduced in (3.12) and Δ in (3.22)) are taken proportional to each other: $\Delta = \xi\varepsilon$, where the coefficient ξ is arbitrary but fixed. In this case, only one independent small parameter, say, ε remains and the notion of minimal subtraction becomes meaningful. In the modified scheme, as usual, certain geometric factors are not expanded in ε (for details see [153]).

In order to apply the dimensional regularization for the evaluation of renormalization constants, an analysis of possible superficial divergences has to be performed. For the power counting in the actions (5.3) and (3.18) the scheme from Sec. 2.3 is employed, in which to each quantity Q two canonical dimensions are assigned, one with respect to the wave number d_Q^k and the other to the frequency d_Q^ω. The normalization for these dimensions is given by Eq. (2.14). The canonical dimensions for fields and parameters of the model are derived from the condition for action to be a scale-invariant quantity, i.e. to have a zero canonical dimension.

The quadratic part of the action (5.3) determines only the canonical dimension of the quadratic product $\psi^\dagger\psi$. In order to keep both terms in the nonlinear part of the action

$$\lambda_0 D_0 \int \mathrm{d}t \int \mathrm{d}^d\boldsymbol{x}\,[2\psi^\dagger + (\psi^\dagger)^2]\psi^2, \tag{5.5}$$

the field $\psi^\dagger$ must be dimensionless. If the field $\psi^\dagger$ has a positive canonical dimension, which is the case for $d > 2$, then the quartic term should be discarded as irrelevant by the power counting. The action with the cubic term only, however, does not generate any loop integrals corresponding to the density fluctuations and thus is uninteresting for the analysis of fluctuation effects in the space dimension $d = 2$. One-loop calculation has been performed in [261] and the two-loop approximation can be found in the work [278], where all computational details can

Q	ψ	ψ'	ν_0, D_0	λ_0
d_Q^ω	0	0	1	0
d_Q^k	d	0	-2	-2Δ
d_Q	d	0	0	-2Δ

Table 5.1. Canonical dimensions for the parameters and the fields of the model with action (5.3).

found. Similar approach was used using Kraichnan model for generating velocity fluctuations [279]. There the RG analysis was somewhat simpler due to the absence of interaction between velocity fluctuations.

Using the normalization choice (2.14), we are able to obtain the canonical dimensions for all the fields and parameters in the d-dimensional space. The results are summarized in Table 5.1. The dimensions of quantities related to velocity can be found in Tab. 3.1. Here, $d_Q = d_Q^k + 2d_Q^\omega$ is the total canonical dimension and it is determined from the condition that the parabolic differential operator of the diffusion and Navier-Stokes equation scales uniformly under the simultaneous momentum and frequency dilatation $k \to \mu k, \omega \to \mu^2 \omega$.

The total canonical dimension of an arbitrary one-particle irreducible Green 1PI is given by the relation (2.16). Superficial UV divergences may be present only in those Γ functions for which d_Γ is a non-negative integer. Using the dimensions of the fields from Table 5.1 we see that the superficial degree of divergence for a 1PI function Γ is given by the expression

$$d_\Gamma = 4 - N_v - N_{\tilde{v}} - 2N_\psi. \tag{5.6}$$

However, the real degree of divergence δ_Γ is smaller, because of the structure of the interaction vertex (3.95), which allows for factoring out the operator $\boldsymbol{\nabla}$ to each external line v'. Thus the real divergence exponent δ_Γ may be expressed as

$$\delta_\Gamma \equiv d_\Gamma - N_{v'} = 4 - N_v - 2N_{v'} - 2N_\psi \tag{5.7}$$

Although the canonical dimension for the field ψ' is zero, there is no proliferation of superficial divergent graphs with arbitrary number of external ψ' legs. This is due to the condition $n_{\psi'} \leq n_\psi$, which may be established by a straightforward analysis of the Feynman graphs [269]. As has already been shown [121] the divergences in 1PI Green functions containing at least one velocity field $\boldsymbol{v}$ may be removed by a single counterterm of the form $\psi' \boldsymbol{\nabla}^2 \psi$.

Brief analysis shows that the UV divergences are expected only for the 1PI Green functions listed in Table 5.2. This theoretical analysis leads to the following renormalization of the parameters g_0, D_0 and u_0:

$$\begin{aligned} g_1 &= g_{10}\mu^{-2\epsilon} Z_1^3, & g_2 &= g_{20}\mu^{2\Delta} Z_1^3 Z_3^{-1}, & \nu &= \nu_0 Z_1^{-1} \\ u &= u_0 Z_1 Z_2^{-1}, & \lambda &= \lambda_0 \mu^{2\Delta} Z_2 Z_4^{-1}, & D &= D_0 Z_2^{-1}, \end{aligned} \tag{5.8}$$

where the inverse Prandtl number has been introduced in the same way as was done for advection of passive scalar in action 3.90. From Table 5.1 it follows that u is purely dimensionless quantity

Γ_{1-ir}	$\langle\psi'\psi\rangle$	$\langle\psi'\psi v\rangle$	$\langle v'v\rangle$	$\langle v'vv\rangle$	$\langle v'v'\rangle$	$\langle\psi'\psi^2\rangle$
d_Γ	2	1	2	1	2	0
δ_Γ	2	1	1	0	0	0

Table 5.2. Canonical dimensions for the 1PI divergent Green functions of the model

($d_u^k = d_u^\omega = d_u = 0$). In terms of the introduced renormalized parameters the total renormalized action for the annihilation reaction in a fluctuating velocity field is

$$\mathcal{S}_{1R} = \int \mathrm{d}^d\boldsymbol{r} \int_0^\infty \mathrm{d}t \Big\{ \psi'\partial_t\psi + \psi'\boldsymbol{\nabla}\cdot(\boldsymbol{v}\psi) - u\nu Z_2 \boldsymbol{\nabla}^2\psi + \lambda u\nu\mu^{-2\Delta} Z_4 [2\psi^\dagger + (\psi')^2]\psi^2 \Big\} + n_0 \int \mathrm{d}^d\boldsymbol{r}\psi'(\boldsymbol{r},0). \tag{5.9}$$

The renormalization constants $Z_i, i = 1,2,3,4$ are to be calculated perturbatively through the calculation of the UV divergent parts of the 1PI functions $\Gamma_{v'v}$, $\Gamma_{v'v'}$, $\Gamma_{\psi'\psi}$, $\Gamma_{\psi'\psi^2}$ and $\Gamma_{(\psi')^2\psi^2}$. Interaction terms corresponding to these functions have to be added to the original action $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$ with the aim to ensure UV finiteness of all Green functions generated by the renormalized action $\mathcal{S}_R$. At this stage the main goal is to calculate the renormalization constants $Z_i, i = 1,2,3,4$. The explicit expressions for them could be found in [278] and are not needed here.

5.1.3 IR stable fixed points and scaling regimes

RG analysis reveals an existence of four IR stable fixed points and one IR unstable fixed point. In this section they are presented together with their regions of stability. For brevity, in the following the abbreviation $\overline{g} \equiv g\overline{S_d}$ for the parameters $\{g_{10}, g_{20}, \lambda_0\}$ or their renormalized counterparts is assumed.

(*i*) The trivial (Gaussian) fixed point

$$\overline{g_1}^* = \overline{g_2}^* = \overline{\lambda}^* = 0\,, \tag{5.10}$$

with no restrictions on the inverse Prandtl number u. The Gaussian fixed point is stable, when

$$\varepsilon < 0\,, \qquad \Delta > 0\,. \tag{5.11}$$

and physically corresponds to the case, when the mean-field solution is valid and fluctuation effects negligible.

(*ii*) The short-range (thermal) fixed point

$$\overline{g_1}^* = 0, \qquad \overline{g_2}^* = -16\Delta + 8(1+2R)\Delta^2,$$
$$u^* = \frac{\sqrt{17}-1}{2} - 1.12146\Delta, \qquad \overline{\lambda}^* = -\Delta + \frac{\Delta^2}{2}(\xi - 2.64375), \tag{5.12}$$

where $R = -0.168$ is a numerical constant. At this point local correlations of the random force dominate over the long-range correlations. This fixed point has the following basin of attraction

$$\Delta - \frac{2R-1}{2}\Delta^2 < 0, \qquad 2\varepsilon + 3\Delta - \frac{3\Delta^2}{2} < 0,$$

$$\Delta + \frac{1}{2}\Delta^2 < 0, \qquad \Delta + 0.4529\Delta\varepsilon < 0 \tag{5.13}$$

and corresponds to anomalous decay faster than that due to density fluctuations only, but slower than the mean-field decay.

(*iii*) The kinetic fixed point (corresponds to regime (3.104)) with finite rate coefficient:

$$\begin{aligned} &\overline{g_1}^* = \frac{32}{9}\frac{\varepsilon\,(2\varepsilon+3\Delta)}{\varepsilon+\Delta} + g_{12}^*(\xi)\varepsilon^2, \quad \overline{g_2}^* = \frac{32}{9}\frac{\varepsilon^2}{\Delta+\varepsilon} + g_{22}^*(\xi)\varepsilon^2, \\ &u^* = \frac{\sqrt{17}-1}{2} + u_1^*(\xi)\varepsilon, \qquad \overline{\lambda}^* = -\frac{2}{3}(\varepsilon+3\Delta) + \frac{1}{9\pi}(3\Delta+\varepsilon)(Q\varepsilon-\Delta). \end{aligned} \tag{5.14}$$

Here $Q = 1.64375$. The fixed point (5.14) is stable, when inequalities

$$\mathrm{Re}\;\Omega_\pm > 0\,, \qquad \varepsilon > 0\,, \qquad -\frac{2}{3}\varepsilon < \Delta < -\frac{1}{3}\varepsilon, \tag{5.15}$$

are fulfilled, where

$$\begin{aligned} \Omega_\pm = \Delta + \frac{4}{3}\varepsilon \pm \frac{\sqrt{9\Delta^2 - 12\varepsilon\Delta - 8\varepsilon^2}}{3} + \frac{2\varepsilon}{9}\Bigg(&\frac{4\varepsilon(\varepsilon+3\Delta)R - 6\varepsilon^2 - 12\varepsilon\Delta - 9\Delta^2}{\sqrt{9\Delta^2 - 12\Delta - 8\varepsilon^2}} \\ &- (3+2R)\varepsilon - 3\Delta\Bigg). \end{aligned} \tag{5.16}$$

The decay rate controlled by this fixed point of the average number density is faster than the decay rate induced by dominant local force correlations, but still slower than the mean-field decay rate.

(*iv*) The kinetic fixed point with vanishing rate coefficient:

$$\begin{aligned} &\overline{g_1}^* = \frac{32}{9}\frac{\varepsilon\,(2\varepsilon+3\Delta)}{\varepsilon+\Delta} + g_{12}^*(\xi)\varepsilon^2, \qquad \overline{g_2}^* = \frac{32}{9}\frac{\varepsilon^2}{\Delta+\varepsilon} + g_{22}^*(\xi)\varepsilon^2, \\ &u^* = \frac{\sqrt{17}-1}{2} + u_1^*(\xi)\varepsilon, \qquad \overline{\lambda}^* = 0\,. \end{aligned} \tag{5.17}$$

The expressions for $g_{12}^*(\xi), g_{22}^*(\xi)$ and $u_1^*(\xi)$ are rather cumbersome. They can be found in [278], therefore we do not repeat them here. This fixed point is stable, when the long-range correlations of the random force are dominant

$$\mathrm{Re}\Omega_\pm > 0, \quad \varepsilon > 0, \quad \Delta > -\frac{1}{3}\varepsilon\,, \tag{5.18}$$

and corresponds to reaction kinetics with the normal (mean-field like) decay rate.

(*v*) Driftless fixed point given by

$$\overline{g_1}^* = \overline{g_2}^* = 0, \quad u^* \text{ not fixed}, \quad \overline{\lambda}^* = -2\Delta, \tag{5.19}$$

with the following eigenvalues of stability matrix (2.24)

$$\Omega_1 = -2\varepsilon, \quad \Omega_2 = -\Omega_4 = 2\Delta, \quad \Omega_3 = 0. \tag{5.20}$$

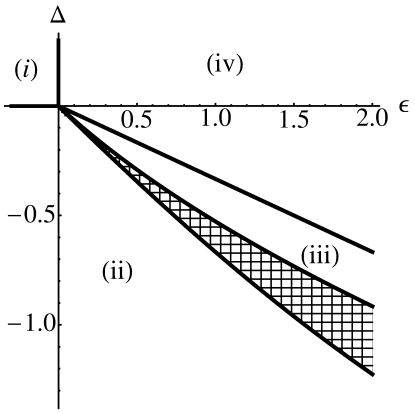


Figure 5.3. Regions of stability for the IR stable fixed points $(i) - (iv)$ in (ε, Δ)-plane.

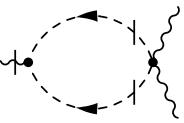
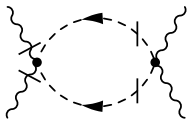
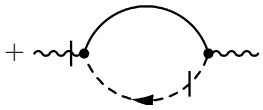
An analysis of the structure of the fixed points and the basins of attraction leads to the following physical picture of the effect of the random stirring on the reaction kinetics. Anomalous behavior always emerges below two dimensions, when the *local* correlations are dominant in the spectrum of the random forcing [the short-range fixed point (*ii*)]. However, the random stirring gives rise to an effective reaction rate faster than the density-fluctuation induced reaction rate even in this case. The anomaly is present (but with still faster decay, see the next Section) also, when the long-range part of the forcing spectrum is effective, but the powerlike falloff of the correlations is fast [this regime is governed by the kinetic fixed point (*iii*)]. Note that this is different from the case in which the solenoidal random velocity field is time-independent, in which case there is no fixed point with $\lambda^* \neq 0$ [275]. At slower spatial falloff of correlations, however, the anomalous reaction kinetics is replaced by a mean-field-like behavior [this corresponds to the kinetic fixed point (*iv*)]. In particular, in dimensions $d > 1$ this is the situation for the value $\varepsilon = 2$ which corresponds to the Kolmogorov spectrum of the velocity field in fully developed turbulence. Thus, long-range correlated forcing gives rise to a random velocity field, which tends to suppress the effect of density fluctuations on the reaction kinetics below two dimensions.

For better illustration, regions of stability for fixed points $(i) - (iv)$ are depicted in Fig.5.3. Wee see that in contrast to the one-loop approximation [261], overlap (dashed region) between regions of stability of fixed points (ii) and (iii) is observed. It is a common situation in the perturbative RG approach that higher order terms lead to either gap or overlap between neighboring stability regions. The physical realization of the large-scale behavior then depends on the initial state of the system.

5.1.4 Long-time asymptotics of number density

Because the renormalization and calculation of the fixed points of the RG are carried out at two-loop level,it is possible to find the first two terms of the ε, Δ expansion of the average number density, which corresponds to solving the stationarity equations at the one-loop level. The simplest way to find the average number density is to calculate it from the stationarity condition of

the functional Legendre transform [25] (which is often called the effective action) of the generating functional obtained by replacing the unrenormalized action by the renormalized one in the weight functional. This is a convenient way to avoid any summing procedures used [269] to take into account the higher-order terms in the initial number density n_0. The standard aim in reaction-diffusion problems is the solution for the number density. Therefore the expectation values of the fields $\boldsymbol{v}$ and $\tilde{\boldsymbol{v}}$ can be set to zero at the outset (but retain, of course, the propagator and the correlation function). Therefore, at the second-order approximation the effective renormalized action for this model is

$\Gamma_R = \mathcal{S}_1 + \frac{1}{4}$  $+\frac{1}{8}$  $+$  $+\dots,$ (5.21)

where $\mathcal{S}_1$ is the action (5.3) (within a convention $\mathcal{S}_{\rm NS} = 0$ in the effective action) and graphs are shown together with their symmetry coefficients. The slashed wavy line corresponds to the field $\psi^\dagger$ and the single wavy line to the field ψ. The stationarity equations for the variational functional

$$\frac{\delta\Gamma_R}{\delta\psi'} = \frac{\delta\Gamma_R}{\delta\psi} = 0 \tag{5.22}$$

give rise to the equations

$$\begin{aligned}
\partial_t\psi = {} & u\nu Z_2\nabla^2\psi - 2\lambda u\nu\mu^{-2\Delta}Z_4\left(1+\psi'\right)\psi^2 + 4u^2\nu^2\lambda^2\mu^{-4\Delta} \\
& \times \int\limits_0^\infty \mathrm{d}t' \int \mathrm{d}^d\boldsymbol{y}\,(\Delta^{\psi\psi'})^2(t-t',\boldsymbol{x}-\boldsymbol{y})\psi^2(t',\boldsymbol{y}) + 4u^2\nu^2\lambda^2\mu^{-4\Delta}\psi'(t,\boldsymbol{x}) \\
& \times \int\limits_0^\infty \mathrm{d}t' \int \mathrm{d}^d\boldsymbol{y}\,(\Delta^{\psi\psi'})^2(t-t',\boldsymbol{x}-\boldsymbol{y})\psi^2(t',\boldsymbol{y}) \\
& + \frac{\partial}{\partial x_i}\int\limits_0^\infty \mathrm{d}t' \int \mathrm{d}^d\boldsymbol{y}\,\Delta_{ij}^{vv}(t-t',\boldsymbol{x}-\boldsymbol{y})\frac{\partial}{\partial x_j}\Delta^{\psi\psi'}(t-t',\boldsymbol{x}-\boldsymbol{y})\psi(t',\boldsymbol{y}) + \dots,
\end{aligned} \tag{5.23}$$

and

$$\begin{aligned}
-\partial_t\psi' = {} & u\nu Z_2\nabla^2\psi' - 2\lambda u\nu\mu^{-2\Delta}Z_4\left[2\psi' + (\psi')^2\right]\psi + 8u^2\nu^2\lambda^2\mu^{-4\Delta} \\
& \times \int\limits_0^\infty \mathrm{d}t' \int \mathrm{d}^d\boldsymbol{y}(\Delta^{\psi\psi'})^2(t'-t,\boldsymbol{y}-\boldsymbol{x})\psi'(t',\boldsymbol{y})\psi(t,\mathbf{x}) + 4u^2\nu^2\lambda^2\mu^{-4\Delta} \\
& \times \int\limits_0^\infty \mathrm{d}t' \int \mathrm{d}^d\boldsymbol{y}(\Delta^{\psi\psi'})^2(t'-t,\boldsymbol{y}-\boldsymbol{x})\left[\psi'(t',\boldsymbol{y})\right]^2\psi(t,\boldsymbol{x})
\end{aligned}$$

$$+\int_0^\infty \mathrm{d}t' \int \mathrm{d}^d\boldsymbol{y}\, \Delta_{ji}^{vv}(t'-t,\boldsymbol{y}-\boldsymbol{x})\frac{\partial}{\partial x_i}\Delta^{\psi\psi'}(t'-t,\boldsymbol{y}-\boldsymbol{x})\frac{\partial}{\partial y_j}\psi'(t',\boldsymbol{y})+\dots \tag{5.24}$$

In (5.23) and (5.24), in the integral terms it is sufficient to put all renormalization constants equal to unity. Substituting the solution $\psi' = 0$ of (5.24) into (5.23) one arrives at the fluctuation-amended rate equation in the form

$$\begin{aligned}\partial_t\psi =& u\nu Z_2\nabla^2\psi - 2\lambda u\nu\mu^{-2\Delta}Z_4\psi^2 + 4u^2\nu^2\mu^{-4\Delta}\lambda^2\int_0^\infty \mathrm{d}t'\int \mathrm{d}^d\boldsymbol{y}(\Delta^{\psi\psi'})^2(t-t',\boldsymbol{x}-\boldsymbol{y})\\ &\times\psi^2(t',\boldsymbol{y}) + \frac{\partial}{\partial x_i}\int_0^\infty \mathrm{d}t'\int \mathrm{d}^d\boldsymbol{y}\,\Delta_{ij}^{vv}(t-t',\boldsymbol{x}-\boldsymbol{y})\frac{\partial}{\partial x_j}\Delta^{\psi\psi'}(t-t',\boldsymbol{x}-\boldsymbol{y})\psi(t',\boldsymbol{y})\\ &+\dots,\end{aligned} \tag{5.25}$$

This is a nonlinear partial integro-differential equation, whose explicit solution is not known. It is readily seen that for a homogeneous solution the term resulting from the third graph in (5.21) vanishes and hence the influence of the velocity field on the homogeneous annihilation process would be only through the renormalization of the coefficients λ and D. However, in case of a nonuniform density field ψ the effect of velocity fluctuations is explicit in (5.25). Such a solution can be most probably found only numerically.

To arrive at an analytic solution, we restrict ourselves to the homogeneous density $n(t) = \langle\psi(t)\rangle$. In this case the last term in (5.25) vanishes together with the Laplace operator term and the remaining coordinate integral may be calculated explicitly . The propagator is the diffusion kernel of the renormalized model (the system is considered in the general space dimension d)

$$\Delta^{\psi\psi'}(t-t',\boldsymbol{x}) = \frac{\theta(t-t')}{[4\pi u\nu(t-t')]^{d/2}}\exp\left[-\frac{x^2}{4u\nu(t-t')}\right]. \tag{5.26}$$

As noted above, for calculation of the one-loop contribution it is sufficient to put the renormalization constant $Z_2 = 1$ in the propagator $\Delta^{\psi\psi'}$. Therefore, evaluation of the Gaussian coordinate integral in (5.25) yields

$$\int \mathrm{d}^d\boldsymbol{y}\,(\Delta^{\psi\psi'})^2(t-t',\boldsymbol{x}-\boldsymbol{y}) = \frac{\theta(t-t')}{[8\pi u\nu(t-t')]^{d/2}} \tag{5.27}$$

and the ordinary integro-differential equation

$$\frac{\mathrm{d}n(t)}{\mathrm{d}t} = -2\lambda u\nu\mu^{-2\Delta}Z_4 n^2(t) + 4\lambda^2u^2\nu^2\mu^{-4\Delta}\int_0^t \mathrm{d}t'\,\frac{n^2(t')}{[8\pi u\nu(t-t')]^{d/2}}. \tag{5.28}$$

is obtained. Naively one can assume that concentrating on homogeneous solution $n(t)$ the information about spatial fluctuations has been lost. However, the integral term in (5.28) corresponds

these fluctuations and cause rather heavy effect even on the homogeneous solution. In particular, the integral in (5.28) diverges at the upper limit in space dimensions $d \geq 2$. This is a consequence of the UV divergences in the model above the critical dimension $d_c = 2$ and near the critical dimension is remedied by the UV renormalization of the model. To see this, subtract and add the term $n^2(t)$ in the integrand to obtain

$$\frac{\mathrm{d}n(t)}{\mathrm{d}t} = -2\lambda u\nu\mu^{-2\Delta} Z_4 n^2(t) + 4\lambda^2 u^2\nu^2\mu^{-4\Delta} n^2(t) \int_0^t \frac{\mathrm{d}t'}{[8\pi u\nu(t-t')]^{d/2}} + 4\lambda^2 u^2\nu^2\mu^{-4\Delta} \int_0^t \mathrm{d}t' \, \frac{n^2(t') - n^2(t)}{[8\pi u\nu(t-t')]^{d/2}} \,. \tag{5.29}$$

The last integral here is now convergent at least near two dimensions, provided the solution $n(t)$ is a continuous function. This is definitely the case for the iterative solution constructed below. The divergence in the first integral in (5.29) may be explicitly calculated below two dimensions and is canceled – in the leading order in the parameter Δ – by the one-loop term of the renormalization constant Z_4 [278]. Expanding the right-hand side of (5.29) in the parameter Δ to the next-to-leading order the equation

$$\frac{\mathrm{d}n(t)}{\mathrm{d}t} = -2\lambda u\nu\mu^{-2\Delta} n^2(t) + 2\lambda u\nu\mu^{-2\Delta} n^2(t) \left\{ \frac{\lambda}{4\pi} \left[\gamma + \ln\left(2u\nu\mu^2 t\right)\right] \right\} + \frac{\lambda^2 u\nu\mu^{-2\Delta}}{2\pi} \int_0^t \mathrm{d}t' \, \frac{n^2(t') - n^2(t)}{t - t'} \tag{5.30}$$

can be derived without divergences near two dimensions. Here, the factor $\mu^{-2\Delta}$ has been retained intact in order not to spoil the consistency of scaling dimensions in different terms of the equation. In (5.30), $\gamma = 0.57721$ is Euler's constant and the coupling constant λ and the parameter Δ have been considered to be small parameters of the same order taking into account the magnitudes of the parameters in the basins of attraction of the fixed points of the RG. The leading-order approximation for $n(t)$ is given by the first term on the right-hand side of (5.30) and it is readily seen that after substitution of this expression the integral term in (5.30) is of the order of λ^3 and thus negligible in the present next-to-leading-order calculation. In this approximation, Eq. (5.30) yields

$$n(t) = \frac{n_0}{1 + 2\lambda u\nu t \left\{1 + \frac{\lambda}{4\pi}\left[1 - \gamma - \ln\left(2u\nu\mu^2 t\right)\right]\right\} \mu^{-2\Delta} n_0} \,, \tag{5.31}$$

where n_0 is the initial number density.

Green functions $\mathcal{W}_R$ differ from the unrenormalized $\mathcal{W} = \langle \Phi \ldots \Phi \rangle$ [25] only by the choice of parameters and thus one may write

$$\mathcal{W}_R(g, \nu, \mu, \ldots) = \mathcal{W}(g_0, \nu_0, \ldots), \tag{5.32}$$

where $g_0 = \{g_{10}, g_{20}, u_0, \lambda_0\}$ is the full set of the bare parameters and dots denotes all variables unaffected by the renormalization procedure. The independence of renormalization mass parameter μ is expressed by the equation $\mu\partial_\mu \mathcal{W}_R = 0$. Using this equation the RG equation for the mean particle number $n(t)$ is readily obtained:

$$\left(\mu\frac{\partial}{\partial\mu} + \sum_g \beta_g \frac{\partial}{\partial g} - \gamma_1 \nu \frac{\partial}{\partial \nu}\right) n(t, \mu, \nu, n_0, g) = 0, \tag{5.33}$$

where in the second term the sum runs over all charges g_1, g_2, u and λ of the model. From macroscopic point of view the most interesting is long-time behavior of the system ($t \to \infty$), therefore the scale setting parameter μ should be traded for the time variable. Canonical scale invariance (Sec. 2.2 and [32]) yields relations

$$\left(\mu\frac{\partial}{\partial\mu} - 2\nu\frac{\partial}{\partial\nu} + dn_0\frac{\partial}{\partial n_0} - d\right) n(t, \mu, \nu, n_0, g) = 0, \tag{5.34}$$

$$\left(-t\frac{\partial}{\partial t} + \nu\frac{\partial}{\partial\nu}\right) n(t, \mu, \nu, n_0, g) = 0, \tag{5.35}$$

where the first equation expresses scale invariance with respect to wave number and the second equation with respect to time. Eliminating partial derivatives with respect to parameter μ and viscosity ν we obtain the Callan-Symanzik equation for the mean particle number:

$$\left[(2-\gamma_1)t\frac{\partial}{\partial t} + \sum_g \beta_g \frac{\partial}{\partial g} - dn_0\frac{\partial}{\partial n_0} + d\right] n\,(t, \mu, \nu, n_0, g) = 0 \tag{5.36}$$

To separate information given by the RG, consider the dimensionless normalized mean particle number

$$\frac{n}{n_0} = \Phi\left(\nu\mu^2 t, \lambda u\,\frac{n_0}{\mu^d}, g\right). \tag{5.37}$$

For the asymptotic analysis, it is convenient to express the particle density in the combination used here. Solution of (5.36) by the method of characteristics yields

$$\Phi\left(\nu\mu^2 t, \lambda u\,\frac{n_0}{\mu^d}, g\right) = \Phi\left(\nu\mu^2\tau, \overline{\lambda}\overline{u}\,\frac{\overline{n}_0}{\mu^d}, \overline{g}\right) \tag{5.38}$$

where τ is the convenient time scale. In Eq. (5.38), $\overline{g}$ and $\overline{n}_0$ are the first integrals (discussed in Sec. 2.4) of the system of differential equations

$$t\frac{\mathrm{d}}{\mathrm{d}t}\overline{g} = -\frac{\beta_g(\overline{g})}{2-\gamma_1(\overline{g})}, \quad t\frac{\mathrm{d}}{\mathrm{d}t}\overline{n}_0 = d\frac{\overline{n}_0}{2-\gamma_1(\overline{g})}. \tag{5.39}$$

Here $\overline{g} = \{\overline{g_1}, \overline{g_2}, \overline{u}, \overline{\lambda}\}$ with initial conditions $\overline{g}|_{t=\tau} = g$ and $\overline{n}_0|_{t=\tau} = n_0$. In particular,

$$\overline{\lambda}\overline{u}\,\overline{n}_0 = \lambda u\, n_0\,\left(\frac{t}{\tau}\right)\exp\left[\int_\tau^t \frac{\gamma_4 \mathrm{d}s}{(2-\gamma_1)s}\right]. \tag{5.40}$$

The asymptotic expression of the integral on the right-hand side of (5.40) in the vicinity of the IR-stable fixed point g^* is of the form

$$\int_\tau^t \frac{\gamma_4 \mathrm{d}s}{(2-\gamma_1)s} \underset{t\to\infty}{\sim} \frac{\gamma_4^*}{2-\gamma_1^*} \ln\left(\frac{t}{\tau}\right) + \frac{2}{2-\gamma_1^*} \int_\tau^\infty \frac{(\gamma_4-\gamma_4^*)\mathrm{d}s}{(2-\gamma_1)s} = \frac{\gamma_4^*}{2-\gamma_1^*} \ln\left(\frac{t}{\tau}\right) + c_n(\tau)\,, \tag{5.41}$$

corrections to which vanish in the limit $t \to \infty$. Quantities with asterisk always refer to corresponding fixed point value. In (5.41) and henceforth, the notation $\gamma_1^* = \gamma_1(g^*)$ has been used. From the point of view of the long-time asymptotic behavior the next-to-leading term in (5.41) is an inessential constant. In the vicinity of the fixed point

$$\overline{\lambda}\overline{u}\,\frac{\overline{n}_0}{\mu^d} \sim \lambda u \frac{n_0}{\mu^d} \left(\frac{t}{\tau}\right)^{1+\frac{\gamma_4^*}{2-\gamma_1^*}} C_n \equiv \lambda u \frac{n_0}{\mu^d} \left(\frac{t}{\tau}\right)^{\alpha} C_n \equiv \overline{y}\, C_n\,, \tag{5.42}$$

where a shorthand notation $\overline{y}$ has been introduced for the long-time scaling of the normalized number density as well as the dimensional normalization constant

$$C_n = \mathrm{e}^{d\tilde{c}_n(\tau)}\,.$$

and the decay exponent

$$\alpha = 1 + \frac{\gamma_4^*}{2-\gamma_1^*} \tag{5.43}$$

The asymptotic behavior of the normalized particle density is described by the scaling function $f(x,y)$:

$$\Phi\left(\nu\mu^2 t, \lambda u \frac{n_0}{\mu^d}, g\right) \sim \Phi\left(\nu\mu^2\tau, C_n\overline{y}, g^*\right) \equiv f\left(\nu\mu^2\tau, C_n\overline{y}\right)\,. \tag{5.44}$$

The scaling function $f(x,y)$ describing the asymptotic behavior of the normalized number density $\Phi = n/n_0$ is a function of two dimensionless argument only, whereas the generic Φ has six dimensionless arguments (all four coupling constants on top of the scaling arguments of $f(x,y)$). We recall that the generic solution of the Callan-Symanzik equation (5.36) does not give the explicit functional form of the function $n = n_0\Phi$, which may to determined from the solution (5.31) of the stationarity equation of the variational problem for the effective potential. The free parameters in the variables of the scaling function $f(x,y)$ correspond to the choice of units of these variables, whereas the objective information is contained in the form of the scaling function [25, 32]. Here, it is convenient to use the explicit solution (5.31) to obtain the ε, Δ expansion for the inverse $h(x,y) = 1/f(x,y)$ of the scaling function. From solution (5.31) the generic expression

$$h(x,y) = \frac{1}{f(x,y)} = 1 + 2xy\left\{1 + \frac{\lambda^*}{4\pi}\left[1 - \gamma - \ln\left(2u^* x\right)\right]\right\} \tag{5.45}$$

is obtained, the substitution in which of the various fixed-point values λ^* (at the leading order $\lambda^* \approx 2\pi\overline{\lambda}^*$) and u^* in the leading approximation yields the corresponding ε, Δ expansions.

Below, we list the scaling functions $h(x,y)$ and the dynamic exponents α at the stable fixed points in the next-to-leading-order approximation.

(*i*) At the trivial (Gaussian) fixed point (5.10) the mean-field behavior takes place with

$$h(x,y) = 1 + 2xy\,, \qquad \alpha = 1\,. \tag{5.46}$$

(*ii*) The thermal (short-range) fixed point (5.12) leads to scaling function and decay exponent

$$h(x,y) = 1 + 2xy\left\{1 - \frac{\Delta}{2}\left[1 - \gamma - \ln\left(\sqrt{17} - 1\right)\,x\right]\right\}, \quad \alpha = 1 + \frac{\Delta}{2} + \frac{\Delta^2}{2}\,. \tag{5.47}$$

Here, the last coefficient is actually a result of numerical calculation, which in the standard accuracy of Mathematica [280] is equal to 0.5. The authors of [278] have not been able to sort out this result analytically, but think that most probably the coefficient of the Δ^2 term in the decay exponent α in (5.47) really is $\frac{1}{2}$.

(*iii*) The kinetic fixed point with an anomalous reaction rate (5.14) corresponds to

$$h(x,y) = 1 + 2xy\left\{1 - \frac{\varepsilon + 3\Delta}{3}\left[1 - \gamma - \ln\left(\sqrt{17} - 1\right)\,x\right]\right\}, \quad \alpha = 1 + \frac{3\Delta + \varepsilon}{3 - \varepsilon}\,, \tag{5.48}$$

with an exact value of the decay exponent.

(*iv*) At the kinetic fixed point with mean-field-like reaction rate (5.17) we obtain

$$h(x,y) = 1 + 2xy, \quad \alpha = 1\,. \tag{5.49}$$

In the actual asymptotic expression corresponding to (5.44) the argument $y \to \widetilde{C}_n\overline{y}$ is different from that of the Gaussian fixed point.

To complete the picture, we recapitulate – with a little bit more detail – the asymptotic behavior of the number density in the physical space dimension $d = 2$ predicted within the present approach [261] (it turns out that for these conclusions the one-loop calculation is sufficient). On the ray $\varepsilon \le 0$, $\Delta = 0$ logarithmic corrections to the mean-field decay take place. The integral determining the asymptotic behavior of the variable (5.40) yields in this case

$$\int_\tau^t \frac{\gamma_4 \mathrm{d}s}{(2-\gamma_1)s} \underset{t\to\infty}{\sim} -\frac{1}{2}\,\ln\ln\left(\frac{t}{\tau}\right) + c_n(\tau)\,, \tag{5.50}$$

with corrections vanishing in the limit $t \to \infty$. Therefore, in the vicinity of the fixed point

$$\overline{\lambda}\overline{u}\,\frac{\overline{n}_0}{\mu^d} \sim \lambda u\,\frac{n_0}{\mu^d}\left(\frac{t}{\tau}\right)\ln^{-1/2}\left(\frac{t}{\tau}\right) C_n \equiv \overline{y}\,C_n\,. \tag{5.51}$$

The scaling function h is of the simple form

$$h(x,y) = 1 + 2xy$$

and gives rise to asymptotic decay slower than in the mean-field case by a logarithmic factor:

$$n \sim \frac{\ln^{1/2}(t/\tau)}{2\nu\lambda u C_n t}\,.$$

It is worth noting that this logarithmic slowing down is weaker than that brought about the density fluctuations only [281] and this change is produced even by the ubiquitous thermal fluctuations of the fluid, when the reaction is taking place in gaseous or liquid media.

On the open ray $\varepsilon > 0$, $\Delta = 0$ the kinetic fixed point with mean-field-like reaction rate (5.17) is stable and the asymptotic behavior is given by (5.49) regardless of the value of the falloff exponent of the random forcing in the Navier-Stokes equation. In particular, only the amplitude factor in the asymptotic decay rate in two dimensions is affected by the developed turbulent flow with Kolmogorov scaling, which corresponds to the value $\varepsilon = 2$. This is in accordance with the results obtained in the case of quenched solenoidal flow with long-range correlations [275, 276] as well as with the usual picture of having the maximal reaction rate in a well-mixed system.

To conclude, the effect of density and velocity fluctuations on the reaction kinetics of the single-species decay $A + A \to \emptyset$ universality class has been analyzed within the framework of field-theoretic renormalization group and calculated the scaling function and the decay exponent of the mean particle density for the four asymptotic patterns predicted by the RG.

5.2 Effect of velocity fluctuations on directed bond percolation

Another important model in non-equilibrium physics is directed percolation (DP) bond process [33, 282]. Its aim is to describe phase transition between absorbing and active phase. In contrast to the annihilation process (5.2) also spreading process $A \to 2A$, and death process $A \to \varnothing$ are allowed [35]. As pointed out by Janssen and Grassberger [283, 284], necessary conditions for corresponding universality class are: i) a unique absorbing state, ii) short-ranged interactions, iii) a positive order parameter and iv) no extra symmetry or additional slow variables. Among a few models described within this framework we name population dynamics, reaction-diffusion problems [285], percolation processes [35], hadron interactions [286], etc.

We focus on the directed bond percolation process in the presence of advective velocity fluctuations. However, to provide more insight we restrict ourselves to a more decent problem. Namely, we assume that the velocity field is given by the Gaussian velocity ensemble with prescribed statistical properties [74, 84]. Although this assumption appears as oversimplified, compared to the realistic flows at the first sight, it nevertheless captures essential physical information about advection processes [32, 73, 84].

Recently, there has been increased interest in different advection problems in compressible turbulent flows [287–290]. These studies show that compressibility plays a decisive role for population dynamics or chaotic mixing of colloids. The aim here is to analyze an influence of compressibility [113, 114] on the critical properties of the directed bond percolation process [33]. To this end, the advective field is described by the Kraichnan model with finite correlation time, in which not only a solenoidal (incompressible) but also a potential (compressible) part of the velocity statistics is involved. Similarly as the annihilation process from the previous part the model under consideration corresponds to the passive advection of the reacting scalar.

The details of the calculations can be found in the work [291]. Here, only main steps are reviewed.

5.2.1 The model

A continuum description of DP in terms of a density $\psi = \psi(t, \boldsymbol{x})$ of infected individuals typically arises from a coarse-graining procedure in which a large number of fast microscopic degrees of freedom are averaged out. A loss of the physical information is supplemented by a Gaussian noise in a resulting Langevin equation. Obviously, a correct mathematical description has to be in conformity regarding the absorbing state condition: $\psi = 0$ is always a stationary state and no microscopic fluctuation can change that. The coarse grained stochastic equation then reads [35]

$$\partial_t \psi = D_0(\boldsymbol{\nabla}^2 - \tau_0)\psi - \frac{g_0 D_0}{2}\psi^2 + \xi, \tag{5.52}$$

where ξ denotes the noise term, D_0 is the diffusion constant, g_0 is the coupling constant and τ_0 measures a deviation from the threshold value for injected probability. It can be thought as an analog to the temperature (mass) variable in the standard φ^4-theory [24,35]. Due to dimensional reasons, we have extracted the dimensional part from the interaction term.

It can be shown [35] that the Langevin equation (5.52) captures the gross properties of the percolation process and contains essential physical information about the large-scale behavior of the non-equilibrium phase transition between the active ($\psi > 0$) and the absorbing state ($\psi = 0$). The Gaussian noise term ξ with zero mean has to satisfy the absorbing state condition. Its correlation function can be chosen in the following form

$$\langle \xi(t_1, \boldsymbol{x}_1)\xi(t_2, \boldsymbol{x}_2)\rangle = g_0 D_0 \psi(t_1, \boldsymbol{x}_1)\delta(t_1 - t_2)\delta^{(d)}(\boldsymbol{x}_1 - \boldsymbol{x}_2) \tag{5.53}$$

up to irrelevant contributions [44]. It should be noted that due to the dependence of the noise correlations on the density, in the stochastic problem (5.52), (5.53) the noise is multiplicative. It is customary to use the Itô interpretation in this case [35] and we stick to this convention.

A further step consists in incorporating of the velocity fluctuations into the model (5.52). The standard route [31] based on the replacement (1.46) is not sufficient due to the assumed compressibility. As shown in [292], the following replacement is then adequate

$$\partial_t \to \partial_t + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) + a_0(\boldsymbol{\nabla} \cdot \boldsymbol{v}), \tag{5.54}$$

where a_0 is an additional positive parameter, whose significance will be discussed later.

Note that the last term in (5.54) contains a divergence of the velocity field $\boldsymbol{v}$ and thus $\boldsymbol{\nabla}$ operator does not act on what could possibly follow.

Following [114], we consider the velocity field to be a random Gaussian variable with zero mean and a translationally invariant correlator given as follows:

$$\langle v_i(t, \boldsymbol{x})v_j(0, \boldsymbol{0})\rangle = \int \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d}[P_{ij}^k + \alpha Q_{ij}^k]D_v(\omega, \boldsymbol{k})\mathrm{e}^{-i\omega t + \boldsymbol{k}\cdot\boldsymbol{x}}, \tag{5.55}$$

where in contrast to the Sec. 4.1.3 we consider the kernel function $D_v(\omega, \boldsymbol{k})$ in the form

$$D_v(\omega, \boldsymbol{k}) = \frac{g_{10} u_{10} D_0^3 k^{4-d-y-\eta}}{\omega^2 + u_{10}^2 D_0^2 (k^{2-\eta})^2}. \tag{5.56}$$

The reason is that the model (5.52) has an upper critical dimension $d_c = 4$ and in perturbation theory poles in $d-d_c$ are expected. In accordance with the standard notation [24,25] we therefore retain symbol ε for the expression $4 - d$ and y for a scaling exponent of velocity field. Just for completeness we note that parameter ε from Secs. 3-5.1 is related to y as follows $y = 2\varepsilon$. The incompressible version of the considered model, $\alpha = 0$, has been studied in works [292–296]. The velocity field here is a colored noise field and does not require any specification of interpretation. It should be noted, however, the rapid change limit discussed below corresponds to the white-noise problem in the Stratonovich interpretation.

The stochastic problem (5.52-5.56) can be cast into a field-theoretic form and the resulting dynamic functional [291] reads

$$\mathcal{S}[\Phi] = \mathcal{S}_{\text{diff}}[\psi', \psi] + \mathcal{S}_{\text{vel}}[\boldsymbol{v}] + \mathcal{S}_{\text{int}}[\Phi], \tag{5.57}$$

where $\Phi = \{\psi', \psi, \boldsymbol{v}\}$ stands for the complete set of fields and ψ' is the auxiliary response field [28]. The first term represents a free part of the equation (5.52) and is given by the following expression:

$$\mathcal{S}_{\text{diff}}[\psi', \psi] = \int \mathrm{d}t \int \mathrm{d}^d\boldsymbol{x} \Big\{ \psi'[-\partial_t + D_0\boldsymbol{\nabla}^2 - D_0\tau_0]\psi \Big\}. \tag{5.58}$$

Averaging over the velocity fluctuations is performed with the action (4.20) and fhe final interaction part can be written as

$$\mathcal{S}_{\text{int}}[\Phi] = \int \mathrm{d}t \int \mathrm{d}^d\boldsymbol{x} \Big\{ \frac{D_0\lambda_0}{2}[\psi' - \psi]\psi'\psi + \frac{u_{20}}{2D_0}\psi'\psi\boldsymbol{v}^2 - \psi'(\boldsymbol{v}\cdot\boldsymbol{\nabla})\psi - a_0\psi'(\boldsymbol{\nabla}\cdot\boldsymbol{v})\psi \Big\}. \tag{5.59}$$

All but the third term in (5.59) directly stem from the nonlinear terms in (5.52) and (5.54). The third term proportional to $\propto \psi'\psi\boldsymbol{v}^2$ deserves a special consideration. The presence of this term is prohibited in the original Kraichnan model due to the underlying Galilean invariance. However, in present case the general form of the velocity kernel function does not lead to such restriction. Moreover, by direct inspection of the perturbative expansion, one can show that this kind of term is indeed generated under RG transformation. This term was considered for the first time in [296], where the incompressible case is analyzed.

Let us also note that for the linear advection-diffusion equation [31, 114], the choice $a_0 = 1$ corresponds to the conserved quantity ψ (advection of a density field), whereas for the choice $a_0 = 0$ the conserved quantity is ψ' (advection of a tracer field). From the point of view of the renormalization group, the introduction of a_0 is necessary, because it ensures multiplicative renormalizability of the model [292].

As was mentioned in Sec. 1.2.4, basic ingredients of any stochastic theory, correlation and response functions of the concentration field $\psi(t, \boldsymbol{x})$, can be computed as functional averages with respect to the weight functional $\exp\mathcal{S}$ with action (5.57). Further, the field-theoretic formulation summarized in (5.58)-(5.59) has an additional advantage to be amenable to the full machinery of (quantum) field theory reviewed in Sec. 2. Next, the RG perturbative technique is applied that allows to study the model in the vicinity of its upper critical dimension $d_c = 4$. By direct inspection of the Feynman diagrams one can observe that the real expansion parameter is rather λ_0^2 than λ_0. This is a direct consequence of the duality symmetry [35] of the action for the pure

percolation problem with respect to time inversion

$$\psi(t,\boldsymbol{x}) \to -\psi'(-t,\boldsymbol{x}), \quad \psi'(t,\boldsymbol{x}) \to -\psi(-t,\boldsymbol{x}). \tag{5.60}$$

Therefore, it is convenient to consider a new charge g_{20}

$$g_{20} = \lambda_0^2 \tag{5.61}$$

and express the perturbation calculation in terms of this parameter.

5.2.2 Fixed points and scaling regimes

From the technical point of view the model is an example of multicharge problem with five charges $\{g_1, g_2, u_1, u_2, a\}$. The β-function, are now given by the expressions

$$\begin{aligned} &\beta_{g_1} = g_1(-y + 2\gamma_D - 2\gamma_v), && \beta_{g_2} = g_2(-\varepsilon - \gamma_{g_2}), && \beta_a = -a\gamma_a. \\ &\beta_{u_1} = u_1(-\eta + \gamma_D), && \beta_{u_2} = -u_2\gamma_{u_2}. \end{aligned} \tag{5.62}$$

It turns out [291] that for some fixed points the computation of the eigenvalues of the matrix (2.24) is cumbersome and rather unpractical. In those cases it is possible to obtain information about the stability from analyzing RG flow equations (2.20). Using approach from Sec. (2.3) the following relations

$$\Delta_{\tilde{\psi}} = \frac{d}{2} + \gamma_{\psi'^*}, \quad \Delta_{\psi} = \frac{d}{2} + \gamma_{\psi^*}, \quad \Delta_\tau = 2 + \gamma_\tau^*. \tag{5.63}$$

are derived in straightforward manner. Important information about the physical system can be read out from the behavior of correlation functions, which can be expressed in terms of the cumulant Green functions. In the percolation problems one is typically interested [33, 35] in the behavior of the following functions

a) The number $N(t,\tau)$ of active particles generated by a seed at the origin

$$N(t) = \int \mathrm{d}^d\boldsymbol{r}\ \langle\psi(t,\boldsymbol{r})\psi'(0,\mathbf{0})\rangle_{\text{conn}}, \tag{5.64}$$

where the notation from Eq. (2.10) has been employed.

b) The mean square radius $R^2(t)$ of percolating particles, which started from the origin at time $t = 0$

$$R^2(t) = \frac{\int \mathrm{d}^d\boldsymbol{r}\ \boldsymbol{r}^2\langle\psi(t,\boldsymbol{r})\psi'(0,\mathbf{0})\rangle_{\text{conn}}}{2d\int \mathrm{d}^d\boldsymbol{r}\ \langle\psi(t,\boldsymbol{r})\psi'(0,\mathbf{0})\rangle_{\text{conn}}}. \tag{5.65}$$

c) Survival probability $P(t)$ of an active cluster originating from a seed at the origin (see [297] for derivation)

$$P(t) = -\lim_{k\to\infty}\langle\psi'(-t,\mathbf{0})\mathrm{e}^{-k\int \mathrm{d}^d\boldsymbol{r}\ \psi(0,\boldsymbol{r})}\rangle. \tag{5.66}$$

FP$^{\mathrm{I}}$	$g_1'^*$	g_2^*	u_2^*	a'^*
FP$^{\mathrm{I}}_1$	0	0	NF	NF
FP$^{\mathrm{I}}_2$	0	$\frac{2\varepsilon}{3}$	0	0
FP$^{\mathrm{I}}_3$	$\frac{4\xi}{3+\alpha}$	0	0	NF
FP$^{\mathrm{I}}_4$	$-\frac{4\xi}{3+\alpha}$	0	$\frac{1}{2}$	0
FP$^{\mathrm{I}}_5$	$\frac{24\xi-2\varepsilon}{3(5+2\alpha)}$	$\frac{4\varepsilon(3+\alpha)-24\xi}{3(5+2\alpha)}$	0	0
FP$^{\mathrm{I}}_6$	$\frac{2[\varepsilon-4\xi]}{9+2\alpha}$	$\frac{4\varepsilon(3+\alpha)+24\xi}{3(9+2\alpha)}$	$\frac{(3+\alpha)\varepsilon-3\xi(7+2\alpha)}{3(3+\alpha)[\varepsilon-4\xi]}$	0
FP$^{\mathrm{I}}_7$	$-\frac{\xi}{3+\alpha}$	2ξ	1	$-\frac{3(5+2\alpha)}{\alpha}+\frac{2(3+\alpha)\varepsilon}{\alpha\xi}$

Table 5.3. List of all fixed points obtained in the rapid-change limit. The coordinate w^* is equal to 0 for all points.

By straightforward analysis [35] it can be shown that the scaling behavior of these functions is given by the asymptotic relations

$$R^2(t) \sim t^{2/\Delta_\omega}, \quad N(t) \sim t^{-(\gamma_{\psi^*}+\gamma_{\psi'^*})/\Delta_\omega}, \quad P(t) \sim t^{-(d+\gamma_{\psi^*}+\gamma_{\psi'^*})/2\Delta_\omega}. \tag{5.67}$$

Although to some extent it is possible to obtain coordinates of the fixed points, the eigenvalues of the matrix (2.24) in this case pose a more severe technical problem. Hence, in order to gain some physical insight into the structure of the model, the overall analysis is divided into special cases and analyzed separately.

The fixed points for rapid change model are listed in Tab. 5.3, for frozen velocity field in Tab. 5.4 and for an illustration purposes nontrivial point in Tab. 5.5. For convenience a new parameter a' has been introduced via the relation $a' = (1-2a)^2$ and NF stands for Not Fixed, i.e., for the given FP the corresponding value of a charge coordinate could not be unambiguously determined.

5.2.3 Rapid change

First, an analysis of the rapid-change limit is performed. It is convenient [74, 114] to introduce the new variables g_1' and w given by

$$g_1' = \frac{g_1}{u_1}, \quad w = \frac{1}{u_1}. \tag{5.68}$$

The rapid change limit then corresponds to fixed points with a coordinate $w^* = 0$. The beta-functions for the charges (5.68) are easily obtained

$$\beta_{g_1'} = g_1'(\eta - y + \gamma_D - 2\gamma_v), \qquad \beta_w = w(\eta - \gamma_D). \tag{5.69}$$

Analyzing the resulting system of equations seven possible regimes can be found. Due to the cumbersome form of the matrix (2.24), the determination of all the corresponding eigenvalues in

$\mathrm{FP}^{\mathrm{II}}$	g_1^*	g_2^*	u_2^*	a'^*
$\mathrm{FP}_1^{\mathrm{II}}$	0	0	NF	NF
$\mathrm{FP}_2^{\mathrm{II}}$	0	$\frac{2\varepsilon}{3}$	0	0
$\mathrm{FP}_3^{\mathrm{II}}$	$\frac{2y}{9}(3-\alpha)$	0	$\frac{\alpha}{2(\alpha-3)}$	0
$\mathrm{FP}_4^{\mathrm{II}}$	$\frac{2(\varepsilon-y)}{2\alpha-9}$	$\frac{4[3\varepsilon+2y(\alpha-6)]}{2\alpha-9}$	1	$\frac{\varepsilon(12-\alpha)+5y(\alpha-6)}{\alpha(\varepsilon-y)}$
$\mathrm{FP}_5^{\mathrm{II}}$	$-\frac{2[6\varepsilon+5y(\alpha-3)]}{3(9+\alpha)}$	0	$\frac{3[\varepsilon+y(\alpha-1)]}{6\varepsilon+5y(\alpha-3)}$	$\frac{18\varepsilon-(\alpha-6)(\alpha-3)y}{\alpha[6\varepsilon+5(\alpha-3)y]}$
$\mathrm{FP}_6^{\mathrm{II}}$	NF	0	NF	NF
$\mathrm{FP}_7^{\mathrm{II}}$	g_1^*	g_2^*	u_2^*	0
$\mathrm{FP}_8^{\mathrm{II}}$	g_1^*	g_2^*	u_2^*	0

Table 5.4. List of all fixed points obtained in the frozen velocity limit. The value of the charge u_1^* is equal to 0 for all points.

FP	g_1^*	g_2^*	u_1^*	u_2^*	a'^*
$\mathrm{FP}_7^{\mathrm{II}}$	0.532193	9.89135	0	0.37859	0
$\mathrm{FP}_1^{\mathrm{III}}$	0.365039	6.38225	0.24709	0.352422	0
$\mathrm{FP}_2^{\mathrm{III}}$	0.399062	7.29847	0.148951	0.35954	0

Table 5.5. Coordinates of the IR stable fixed points obtained by numerical integration of flow equations (2.20) the model (5.57). The relevant parameters were fixed as follows $\alpha = 110, \varepsilon = 1$ and $y = 2\eta = 8/3$ (Kolmogorov regime).

an explicit form is impossible. In particular, for nontrivial fixed points (with non-zero coordinates of g_1', g_2 and u_2) the resulting expressions are of a quite unpleasant form. Nevertheless, using numerical software [280] it is possible to obtain all the necessary information about the fixed points' structure and in this way the boundaries between the corresponding regimes have been obtained. In the analysis it is advantageous to exploit additional constraints following from the physical interpretation of the charges. For example, g_1' describes the density of kinetic energy of the velocity fluctuations, g_2 is equal to λ^2 and a' will be later on introduced as $(1-2a)^2$. Hence, it is clear that these parameters have to be non-negative real numbers. Fixed points that violate this condition can be immediately discarded as non-physical.

Out of seven possible fixed points, only four are IR stable: $\mathrm{FP}_1^{\mathrm{I}}$, $\mathrm{FP}_2^{\mathrm{I}}$, $\mathrm{FP}_5^{\mathrm{I}}$ and $\mathrm{FP}_6^{\mathrm{I}}$. Thus, only regimes which correspond to those points could be in principle realized in real physical systems. As expected [114], the coordinates of these fixed points and the scaling behavior of the Green functions depend only on the parameter $\xi = y - \eta$. In what follows, we restrict our discussion only to them.

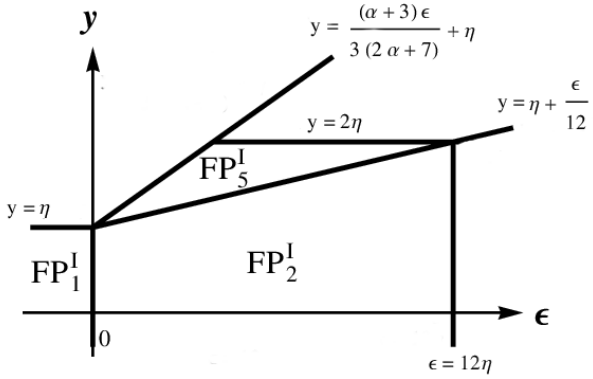


Figure 5.4. A qualitative sketch of the regions of stability for the fixed points in the limit of the rapid-change model. The borders between the regions are depicted with the bold lines.

The $\mathrm{FP}_1^{\mathrm{I}}$ represents the free (Gaussian) FP for which all interactions are irrelevant and ordinary perturbation theory is applicable. As expected, this regime is IR stable in the region

$$y < \eta, \quad \eta > 0, \quad \varepsilon < 0. \tag{5.70}$$

The latter condition ensures that the system is considered above the upper critical dimension $d_c = 4$. For $\mathrm{FP}_2^{\mathrm{I}}$ the correlator of the velocity field is irrelevant and this point describes standard the DP universality class [35] and is IR stable in the region

$$\varepsilon > 0, \quad \varepsilon/12 + \eta > y, \quad \varepsilon < 12\eta. \tag{5.71}$$

The remaining two fixed points constitute nontrivial regimes for which velocity fluctuations as well as percolation interaction become relevant. The $\mathrm{FP}_5^{\mathrm{I}}$ is IR stable in the region given by

$$(\alpha+3)\varepsilon > 3(2\alpha+7)(y-\eta), \quad 12(y-\eta) > \varepsilon, \quad 2\eta > y. \tag{5.72}$$

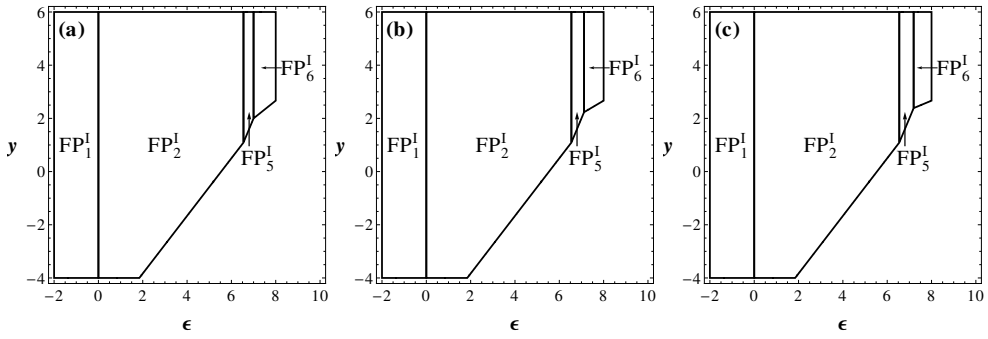


Figure 5.5. Fixed points' structure for the thermal noise situation (5.73). From above to bottom the compressibility parameter α attains consecutively the values: (a) $\alpha = 0$, (b) $\alpha = 5$ and (c) $\alpha = 100$.

The boundaries for $\mathrm{FP}_6^{\mathrm{I}}$ can be only computed by numerical calculations.

Using the information about the phase boundaries, a qualitative picture of the phase diagram can be constructed. In Fig. 5.4 the situation in the plane (ε, y) is depicted. The compressibility affects only the outer boundary of $\mathrm{FP}_5^{\mathrm{I}}$. The larger value of α the larger area of stability. Also the realizability of the regime $\mathrm{FP}_5^{\mathrm{I}}$ crucially depends on the nonzero value of η.

The important subclass of the rapid-change limit constitutes thermal velocity fluctuations, which are characterized by the quadratic dispersion law [61]. In formulation (5.56) this is achieved by considering the following relation:

$$\eta = 6 + y - \varepsilon \tag{5.73}$$

which follows directly from expression (4.15). The situation for increasing values of the parameter α is depicted in Fig. 5.5. We see that for physical space dimensions $d = 3$ $(\varepsilon = 1)$ and $d = 2$ $(\varepsilon = 2)$ the only stable regime is that of pure DP. The nontrivial regimes $\mathrm{FP}_5^{\mathrm{I}}$ and $\mathrm{FP}_6^{\mathrm{I}}$ are realized only in the nonphysical region for large values of ε. This numerical result confirms our previous expectations [292, 293]. It was pointed out [261, 278] that genuine thermal fluctuations could change IR stability of the given universality class. However, this is not realized for the percolation process.

5.2.4 Regime of frozen velocity field

The regime of the frozen velocity field corresponds to the charge constraint $u_1^* = 0$. As analysis show eight possible fixed points are obtained. However, only three of them ($\mathrm{FP}_1^{\mathrm{II}}$, $\mathrm{FP}_2^{\mathrm{II}}$ and $\mathrm{FP}_7^{\mathrm{II}}$) could be physically realized (IR stable).

The fixed point $\mathrm{FP}_1^{\mathrm{II}}$ describes the free (Gaussian) theory. It is stable in the region

$$y < 0, \quad \varepsilon < 0, \quad \eta < 0. \tag{5.74}$$

For $\mathrm{FP}_2^{\mathrm{II}}$ the velocity field is asymptotically irrelevant and the only relevant interaction is due to the percolation process itself. This regime is stable in the region

$$\varepsilon > 6y, \quad \varepsilon > 0, \quad \varepsilon > 12\eta. \tag{5.75}$$

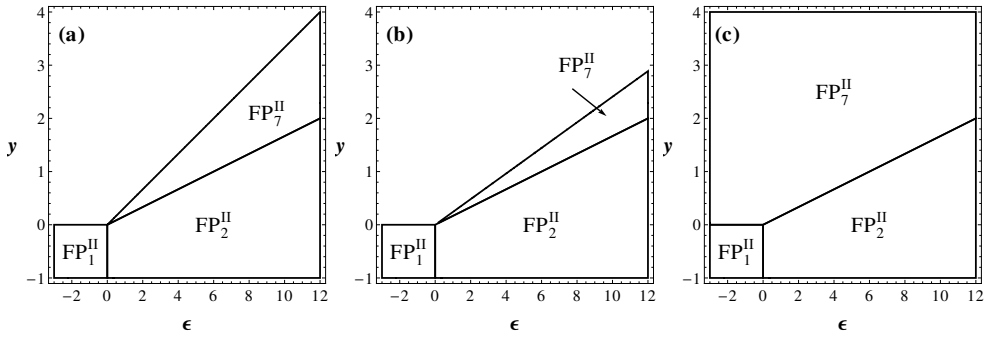


Figure 5.6. Fixed points' structure for frozen velocity case with $\eta = 0$. From above to bottom the compressibility parameter α attains consecutively the values: (a) $\alpha = 0$, —(b) $\alpha = 3.5$ and (c) $\alpha = 8$.

On the other hand, $\mathrm{FP}_7^{\mathrm{II}}$ represents a truly nontrivial regime for which both velocity and percolation are relevant. Since for the points $\mathrm{FP}_1^{\mathrm{II}}$ and $\mathrm{FP}_2^{\mathrm{II}}$ the velocity field could be effectively neglected, the trivial observation is that these boundaries do not depend on the value of the parameter α. The stability region of $\mathrm{FP}_7^{\mathrm{II}}$ can be computed only numerically.

In order to illustrate the influence of compressibility on the stability in the nontrivial regime $\mathrm{FP}_7^{\mathrm{II}}$, let us take a look at situation for $\eta = 0$. For other values of η the situation remains qualitatively the same. The situation for increasing values of α is depicted in Fig. 5.6. For $\alpha = 0$ there is a region of stability for $\mathrm{FP}_7^{\mathrm{II}}$, which shrinks for the immediate value $\alpha = 3.5$ to a smaller area. Numerical analysis [291] shows that this shrinking continues well down to the value $\alpha = 6$. A further increase of α leads to a substantially larger region of stability for the given FP. Already for $\alpha = 8$ this region covers all the rest of the (y, ε) plane. The compressibility thus changes profoundly a simple picture expected from an incompressible case. Altogether the advection process becomes more efficient due to the combined effects of compressibility and the nonlinear terms.

5.2.5 Turbulent advection

In the last part the focus is on a special case of the turbulent advection. Main aim is to determine whether Kolmogorov regime [70], which corresponds to the choice $y = 2\eta = 8/3$, could lead to a new nontrivial regime for the percolation process. In this section, the parameter η is always considered to attain its Kolmogorov value, $4/3$. For a better visualization we present two-dimensional regions of stability in the plane (ε, y) for different values of the parameter α.

First, let us reanalyze the situation for the rapid-change model. The result is depicted in Fig. 5.7. It is clearly visible that for this case a realistic turbulent scenario ($\varepsilon = 1$ or $\varepsilon = 2$) falls out of the possible stable regions. This result is expected because the rapid-change model with vanishing time-correlations could not properly describe well-known turbulent properties [70,79]. We also observe that compressibility mainly affects the boundaries between the regions $\mathrm{FP}_5^{\mathrm{I}}$ and $\mathrm{FP}_6^{\mathrm{I}}$. However, this happens mainly in the nonphysical region.

Next, let us make a similar analysis for the frozen velocity field. The corresponding stability regions are depicted in Fig. 5.8. Here it can be seen that the situation is more complex. The

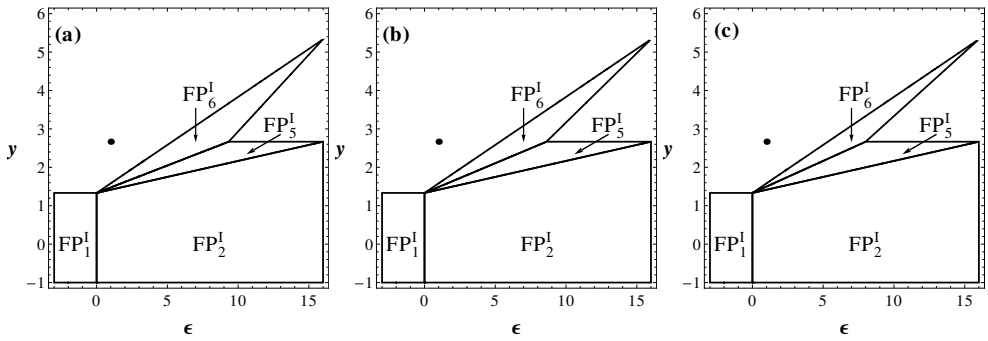


Figure 5.7. Fixed points' structure for rapid change model with $\eta = 4/3$. From above to bottom the compressibility parameter α attains consecutively the values: (a) $\alpha = 0$, (b) $\alpha = 5$ and (c) $\alpha = \infty$. The dot denotes the coordinates of the three-dimensional Kolmogorov regime.

regime $\mathrm{FP}_2^{\mathrm{II}}$ is situated in the non-physical region and could not be realized. For small values of the parameter α the Kolmogorov regime (depicted by a point) does not belong to the frozen velocity limit. However, from a special value $\alpha = 6$ up to $\alpha \to \infty$ the Kolmogorov regime belongs to the frozen velocity limit. Note that the bottom line for the region of stability of $\mathrm{FP}_7^{\mathrm{II}}$ is exactly given by $y = 4/3$. We observe that compressibility affects mainly the boundary of the nontrivial region. We conclude that the presence of compressibility has a stabilizing effect on the regimes where nonlinearities are relevant.

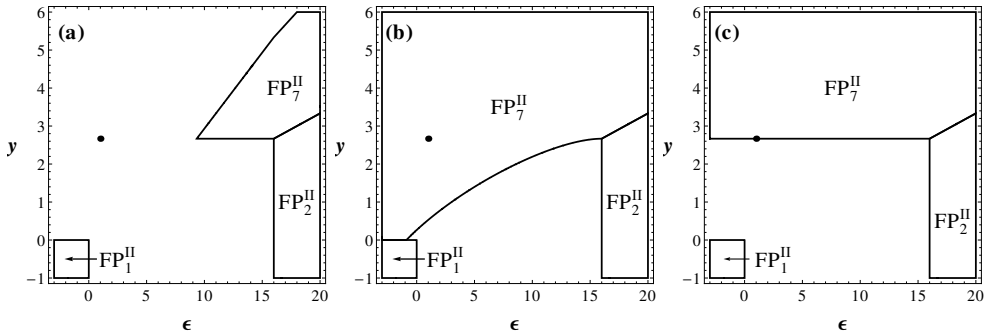


Figure 5.8. Fixed points' structure for the frozen velocity case with $\eta = 4/3$. From above to bottom the compressibility parameter α attains consecutively the values: (a) $\alpha = 0$, (b) $\alpha = 8$ and (c) $\alpha = \infty$. The dot denotes the coordinates of the three-dimensional Kolmogorov regime.

Finally, let us take look at the nontrivial regime, which means that no special requirements were laid upon the parameter u_1. The corresponding differential equations for the RG flow (2.20) have been analyzed numerically [291]. The behavior of the RG flows has been found as follows. There exists a borderline in the plane (ε, α_c) given approximately by the expression

$$\alpha_c = -12.131\varepsilon + 117.165. \tag{5.76}$$

Below α_c, only the frozen velocity regime corresponding to $\mathrm{FP}_7^{\mathrm{II}}$ is stable. Above α_c, three fixed points $\mathrm{FP}_7^{\mathrm{II}}$, $\mathrm{FP}_1^{\mathrm{III}}$ and $\mathrm{FP}_2^{\mathrm{III}}$ are observed. Whereas two of them ($\mathrm{FP}_7^{\mathrm{II}}$ and $\mathrm{FP}_1^{\mathrm{III}}$) are IR stable, the remaining one $\mathrm{FP}_2^{\mathrm{III}}$ is unstable in the IR regime. Again one of the stable FPs corresponds to $\mathrm{FP}_7^{\mathrm{II}}$, but the new FP is a regime with finite correlation time. Since all free parameters $(\varepsilon, \eta, y, \alpha)$ are the same for both points, which of the two points will be realized depends on the initial values of the bare parameters. A similar situation is observed for the stochastic magnetohydrodynamic turbulence [149], where the crucial role is played by a forcing decay-parameter a (See Eq. (3.107).

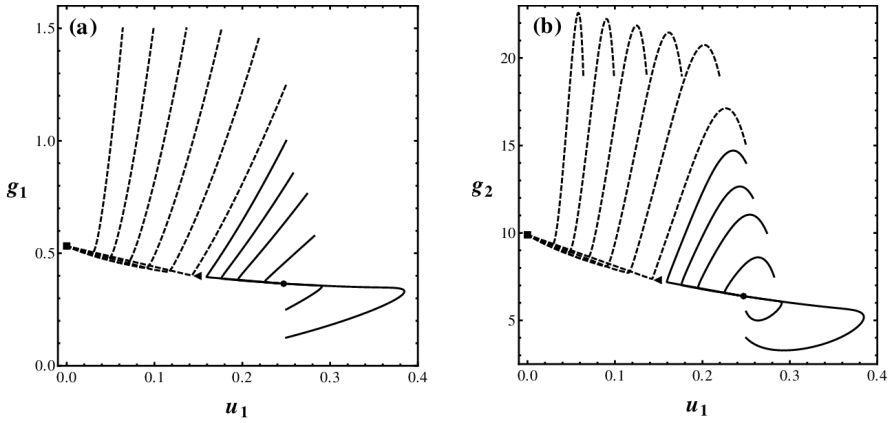


Figure 5.9. Demonstration of the RG trajectories flows' in: (a) the plane (g_1, u_1) and (b) the plane (g_2, u_1) for three dimensional ($\varepsilon = 1$) turbulent advection with $\alpha = 110$. The square ■ denotes frozen velocity regime $\mathrm{FP}_7^{\mathrm{II}}$, triangle ◀ corresponds to the unstable regime $\mathrm{FP}_2^{\mathrm{III}}$ and circle • to the nontrivial regime $\mathrm{FP}_1^{\mathrm{III}}$ for which the time correlations are relevant. Dashed lines corresponds to the chosen flows to the point $\mathrm{FP}_7^{\mathrm{II}}$, whereas the full lines to the flows to the other stable point $\mathrm{FP}_1^{\mathrm{III}}$.

For illustration purposes the projections of the RG flow onto the planes (g_1, u_1) and (g_2, u_1) are depicted in Fig. 5.9. The two stable points are clearly separated by the unstable one.

6 Conclusion

Methods of quantum field theory including functional differential equations, functional integrals, renormalization group and operation product expansion have been successfully applied for a description of second order phase transitions [24, 25, 64]. They cover a broad set of problems, such as critical behavior in ferromagnets, transition to the superfluid phase in ^{4}He, multicriticality and many others. Nowadays, these methods and the related theoretical framework form a cornerstone of the research area of critical phenomena.

Classical dynamic systems, such as developed turbulence, turbulent transport phenomenon, magnetohydrodynamics and reaction-diffusion models, could not be described within the standard equilibrium statistical physics. All these problems are examples of systems far from equilibrium. They are fundamentally different from systems near the thermal equilibrium in two crucial points. First, they do not possess equivalent to the Gibbs thermal state. Second, their underlying dynamics cannot – as a rule – be described by a single hermitian evolution operator.

Nevertheless, these problems and problems in quantum field theory share several common properties. Namely, the fluctuation-dissipation condition is broken in systems far from equilibrium and strong correlations are present over large spatial scales. This often causes diverging correlation length, which allows a description in terms of continuous fields. The fact that classical systems can be considered as euclidean versions of quantum models induced a great stimulus in application of field-theoretic methods in the aforementioned classical problems.

The second half of 70s was a crucial period when quantum field theory and renormalization group were used for the first time in theory of developed isotropic turbulence, stochastic magnetohydrodynamics and for analysis of transport phenomena in random environments. The effort paid off and led, e.g., to a proof of the second Kolmogorov hypothesis about the independence of statistical correlations in velocity fluctuations of viscosity in the inertial interval. This fact is directly related to the scaling with the celebrated universal Kolmogorov exponents. Furthermore, an endeavor has been undertaken with a goal of a controllable calculation of non-universal quantities such as the Kolmogorov constant, Prandtl number and explanation of large-scale generation of magnetic fields due to spontaneous symmetry breaking. The critical exponents of composite operators were computed allowing to analyze behavior of experimentally measurable quantities, e.g., rate of energy dissipation on large scales. It is known that strong fluctuations of this quantity could violate Kolmogorov scaling, especially for higher order structure functions of the velocity field. Generally speaking, the research in this area based on field-theoretical methods was and still is very fruitful and has led to a number of important results summarized in many review articles.

Consequently it became clear that it was important to study not only ideal systems but also more realistic systems with anisotropy, non-homogeneities, broken parity symmetry and compressible fluid. These effects could profoundly change the macroscopic behavior: not only stability of universal regimes but also values of critical exponents.

Due to the fact that most chemical reactions occur in fluids, it was clear that the effect of hydrodynamic fluctuations stemming e.g. from thermal fluctuations must be properly taken into account for reaction-diffusion systems, phase transitions and percolation problems.

During last two decades the anomalous scaling in the developed turbulence (known as intermittency) has been intensively studied. The results have theoretically confirmed the existence of anomalous scaling in open non-equilibrium systems. Study of such systems has also brought

about a further research activity in quantum-field methods. Mainly, the paradigmatic ε expansion has been improved and specific algorithms have been created for more effective calculation of composite operators and multi-loop Feynman diagrams.

In this article it has been our aim summarize and elucidate theoretical approaches and results found in last twenty years in classical stochastic problems far from equilibrium. We have tried to discuss a rather broad set of problems to demonstrate the robustness and effectiveness of functional methods and the renormalization-group technique.

The first part of the article, which is of methodological nature, reveals the importance of functional methods suitable for systems with multiplicative noise. Functional methods in studying systems with intrinsic noise and the double expansion scheme have been presented in detail. The latter is an invaluable tool in problems involving hydrodynamic fluctuations.

The second part has been devoted to the study of systems with some violation from ideality. The effect of helicity and anisotropy has been investigated in the Kraichnan model as well as in the Navier-Stokes equation. A general conclusion is that both violations have a substantial effect on the stability of universal regimes and values of critical exponents. Positions of non-universal fixed points as well as their regions of stability change profoundly with the parameters characterizing quantitatively the violation from ideality. They influence the crossover behavior between different universality classes as well.

An analysis of paradigmatic models that describe spreading of a passive scalar admixture and a vector quantity (magnetic field) has revealed similarities but also differences between these two cases. The rôle played by tensor structures of the fields and effects thereof on the phenomenon of intermittency and the values of critical dimensions of composite operators have been inferred. The results clearly show that critical dimensions for given harmonics describing anisotropy are identical. However, the behavior of scaling functions is completely different due to the leading term. In the last part it is shown that in realistic models, in which, e.g., compressibility of the environment is allowed, new universality classes appear. Their existence can be traced back to new interactions, which in ideal systems are usually prohibited by symmetry reasons.

All the mentioned results and observations generate new questions and challenges. In the future, it is desirable to investigate new universal patterns, crossovers between them and to develop effective techniques for higher loop calculations. From a qualitative point of view, a study of more complicated reaction schemes is called for. In an analysis of intermittent behavior, development of new models with velocity fluctuations – as close to real fluids as possible – is needed.

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