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INTRODUCTION TO INTEGRABLE MANY-BODY SYSTEMS I

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This is the first volume of a three-volume introductory course about integrable (exactly solvable) systems of interacting bodies. The aim of the course is to derive and analyze, on an elementary mathematical and physical level, the Bethe ansatz solutions, ground-state properties and the thermodynamics of integrable many-body systems in many domains of physics: Nonrelativistic one-dimensional continuum Fermi and Bose gases; One-dimensional quantum models of condensed matter physics like the Heisenberg, Hubbard and Kondo models; Relativistic models of the (1+1)-dimensional Quantum Field Theory like the Luttinger model, the sine-Gordon model and its fermionic analog the Thirring model; Two-dimensional classical models, especially the symmetric Coulomb gas. In the first part of this volume, we deal with nonrelativistic one-dimensional continuum Fermi and Bose quantum gases of spinless (identical) particles with specific types of pairwise interactions like the short-range δ-function and hard-core interactions, and the long-range $1/x^2$ interaction. The second part is devoted to the description of the Quantum Inverse Scattering Method, as the universal method for generating and solving integrable models, and to the analysis of the related Yang-Baxter equation, as the consistency condition for the factorization of the multi-particle scattering. With the aid of this method, we present the complete solution of spin-$\frac{1}{2}$ fermions with δ-function interactions.

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1 Introduction

In classical mechanics, a dynamical system of interacting bodies with $2N$-dimensional phase space is said to be integrable if there exist N conserved functions (charges) which Poisson bracket vanishes. A system in the Quantum Field Theory (QFT) is called integrable if there exists an infinite set of mutually commuting conserved charges. The existence of the conserved charges allows us to solve the physical system exactly and in this way to describe the modeled phenomena without any approximative scheme. Although the integrability property is restricted to low dimensions, the exact solution provides in many cases a fundamental information about physical phenomena. In the present times we know precisely how to generate systematically integrable models and how to solve them, explicitly or implicitly in the form of integral equations.

Integrable models are known in many domains of quantum and statistical physics:

- Nonrelativistic one-dimensional continuum Fermi and Bose quantum gases with specific types of pairwise interactions like the short-range δ-function and hard-core interactions, the long-range $1/x^2$ interaction, etc.

- One-dimensional lattice and continuum quantum models of condensed matter physics like the Heisenberg model of interacting quantum spins, the Hubbard model of hopping electrons with one-site interactions between electrons with opposite spins, the Kondo model of the interaction of a conduction band with a localized spin impurity, etc.

- Relativistic models of the QFT in a (1+1)-dimensional spacetime like the Luttinger model, the sine-Gordon model and its fermionic analog the Thirring model, etc.

- Two-dimensional lattice and continuum classical models in thermal equilibrium like the Ising model of interacting ± 1 spins, the six- and eight-vertex models, the Coulomb gas of ± 1 charges interacting logarithmically, etc.

Integrable systems can be either homogeneous, i.e. formulated in a finite domain with periodic boundary conditions or taken as infinite (the thermodynamic limit), or inhomogeneous, e.g. in the presence of a hard-wall boundary impenetrable to particles. We shall restrict ourselves to homogeneous systems.

The complete solution of an integrable one-dimensional quantum-mechanical model proceeds in few steps.

- The first step is to reduce the problem of calculating the spectrum of a Hamiltonian to solving a set of coupled algebraic equations. In this way the original problem of exponential complexity is reduced to the one of polynomial complexity. The coupled equations are known, for historical reasons, as the Bethe ansatz equations and have an adjective which depends on the type of the system under consideration or on the applied method. The adjectives are "coordinate" for spinless particles treated in the direct format, "nested" for particles with internal degrees of freedom like spin, "algebraic" for an inverse scattering formulation, etc.

- The next step is to find the solution of the Bethe equations which corresponds to the ground state (zero-temperature thermodynamics), i.e. the eigenstate of the Hamiltonian with the

lowest energy. In dependence on the Hamiltonian parameters, this problem can be trivial or nontrivial, explicitly solvable only in the thermodynamic limit by using a continualization procedure.

- The third step consists in the construction of low-lying excitations upon the ground state and in finding the asymptotic expression for their energy in the thermodynamic limit.
- The fourth step is the derivation of the thermodynamics (the free energy) for the system at temperature $T > 0$ (the "Thermodynamic Bethe ansatz").
- The last step is the evaluation of the correlation functions of interacting bodies at some distance. Much attention is devoted to this topic at present, but we shall not discuss it.

The solution of the equilibrium statistical mechanics of an integrable classical model formulated on a two-dimensional lattice consists in the diagonalization of a row-to-row transfer matrix whose largest eigenvalue determines the thermodynamic limit of the free energy. From this point of view, the problem resembles technically the one of finding the ground-state energy of a quantum-mechanical model in spatial dimension reduced by one.

In the next paragraphs, we shall summarize briefly milestones in the history of integrable many-body systems.

The most important integrable system was certainly the quantum-mechanical model of magnetism proposed by Heisenberg [1]. The Heisenberg Hamiltonian of N interacting particles with spin $\frac{1}{2}$ on a one-dimensional chain reads

$$H = -\frac{1}{2}\sum_{n=1}^{N}\left(J_x\boldsymbol{\sigma}_n^x\boldsymbol{\sigma}_{n+1}^x + J_y\boldsymbol{\sigma}_n^y\boldsymbol{\sigma}_{n+1}^y + J_z\boldsymbol{\sigma}_n^z\boldsymbol{\sigma}_{n+1}^z\right), \tag{1.1}$$

where $\boldsymbol{\sigma}_n^\alpha$ $(\alpha = x, y, z)$ are the Pauli spin operators on site $n = 1, 2, \ldots, N$ (see Appendix A for definitions), satisfying periodic boundary conditions $\boldsymbol{\sigma}_{N+1}^\alpha = \boldsymbol{\sigma}_1^\alpha$, and $\{J_x, J_y, J_z\}$ are real coupling constants. When $J_x \neq J_y \neq J_z$, this model is called the XYZ model. The special cases $J_x = J_y \neq J_z$ and $J_x = J_y = J_z = J$ correspond to the XXZ and XXX models, respectively. The eigenvectors and the eigenvalues of the completely isotropic XXX Hamiltonian were found in the pioneering work [2] by Bethe in 1931. In the ferromagnetic case $J > 0$, the Bethe ansatz equations provide an exact answer for the (trivial) ground-state properties and low-lying excitations of string type (an n-string is a group of n roots in the complex momentum/rapidity plane distributed symmetrically and equidistantly around the real axis). In the antiferromagnetic case $J < 0$, the nontrivial ground state was constructed by Hultén [3]. He derived from the asymptotic $N \to \infty$ limit of Bethe's equations a linear integral equation for a particle distribution function in the momentum space, the solution of which provides an explicit expression for the ground-state energy per site. More than twenty years later des Cloizeaux and Pearson [4] constructed excitations upon the antiferromagnetic ground state and found the asymptotic expression for their energy. The generalization of Bethe's method to the XXZ model, made by Yang and Yang [5, 6], was straightforward and brought the topic to a higher mathematical level. The solution of the XYZ model, given by Baxter in 1971 [7]- [10], was a breakthrough. Baxter discovered a link between the quantum one-dimensional XXZ and XYZ models and the equilibrium statistical mechanics of two-dimensional classical systems, the so-called six-vertex and

eight-vertex models, respectively. He observed that the eigenstates of the transfer matrix of the six-vertex model are independent of one of the model parameters. Consequently, there exists an infinite family of commuting transfer matrices which originates from the so-called "Yang-Baxter equation" fulfilled by the Boltzmann weights of the six-vertex model. The same observation takes place also in the case of the eight-vertex model, for which Baxter obtained a system of Bethe-like transcendental equations. With the aid of these equations he was able to calculate the ground-state energy of the XYZ model and its critical properties which are non-universal in a weak sense: although the critical indices depend on model's parameters, but their ratios do not. The asymptotic energy of low-lying excitations of the XYZ model was obtained by Johnson, Krinsky and McCoy [11].

The fundamental property of integrable particle systems, possessing an infinite number of conservation laws, is the factorization property of the multi-particle scattering into a sequence of two-particle ones. The two-particle scattering is elastic, i.e. not only the total momentum but also both individual particle momenta are conserved. In this context, the Yang-Baxter equation is the consistency condition for elements of the two-particle scattering matrix which ensures the invariance of the three-particle (and, consequently, multi-particle) scattering with respect to the order in which two-particle scatterings are accomplished. The concept of the transfer matrix and the Yang-Baxter equation as the consistency condition played a central role in a program called the "Quantum Inverse Scattering Method" (QISM), established in late seventies by Faddeev, Sklyanin, Takhtajan and their co-workers [12,13]. The method is based on a relationship between integrable many-body models and integrable evolution equations [14, 15]. An important feature of the method, the algebraic construction of eigenstates of the transfer matrix [16, 17], gave an alternative name for it: the "algebraic Bethe ansatz". The systematic search for the solutions of the Yang-Baxter equation [18] resulted in the appearance of "Quantum Groups" [19, 20].

Another important group of integrable models are nonrelativistic one-dimensional continuum Fermi and Bose (the relationship between the spin and statistics is usually ignored, for mathematical reasons) quantum gases with specific types of pairwise interactions. The crucial model was the one of spinless (identical) bosons with attractive or repulsive δ-function interactions, initiated in 1963 by Lieb and Liniger [21, 22]. While the attractive bosons exhibit a collapse in the thermodynamic limit, the thermodynamic limit of the repulsive boson system is well behaved and the Bethe ansatz equations provide the ground-state (zero-temperature) properties as well as the energy of low-lying excitations. In 1969 Yang and Yang [23] derived from the Bethe equations the thermodynamic properties of repulsive δ-function bosons at finite temperatures; this was the first exact treatment of thermodynamics for an interacting many-body system. The crucial observation was that also the holes, i.e. unoccupied energy levels, contribute to the entropy of the system. Since the spectrum of excitations energies is relatively simple (the momenta are real, so there are only strings of length $n = 1$), the thermodynamics is determined by a coupled pair of integral equations for the distribution functions of the excitation energy and of the equilibrium particle (hole) densities in the momentum space. The other spinless particle systems with integrable interactions, like the hard-core and the long-range $1/x^2$ ones, were treated analogously [24, 25]. The generalization of the Bethe ansatz method to systems of particles with internal degrees of freedom turned out to be complicated because in the scattering the internal states of the particles can be changed. The problem of spin-$\frac{1}{2}$ (two-state) fermions with δ-function interactions was solved in 1967 by Yang [26] and Gaudin [27] by using the "nested Bethe ansatz" and the Yang-Baxter equation as the consistency condition. The excited states of this model form strings of

various lengths $n = 1, 2, \ldots$. The final result for the thermodynamics [28–30] is thus expressible in terms of the solution of an infinite set of coupled nonlinear integral equations, one for each string length n, known as the "Thermodynamic Bethe ansatz". These equations can be analyzed analytically only in special limits, e.g. in the limits of zero and infinite temperatures or interaction strengths. The same structure of the Thermodynamic Bethe ansatz was observed in the case of the Heisenberg model [31, 32]. The strings can be avoided in a simpler method developed by Destri and de Vega [33, 34] which leads to a single nonlinear integral equation.

The technique of the nested Bethe ansatz was applied to other models of the condensed matter physics. The lattice version of the spin-$\frac{1}{2}$ fermion system with δ-function interactions, the Hubbard model, was solved by Lieb and Wu in 1968 [35]. The exact solution showed the absence of a conducting-insulating Mott transition in the one-dimensional version of this model. The Kondo model, reflecting effects of the interaction of a conduction band with a localized spin impurity, was solved by Andrei [36] and Wiegmann [37].

The Bethe ansatz technology was successfully applied also to integrable models of the QFT in a (1+1)-dimensional spacetime, like the sine-Gordon model and its fermionic equivalent the Thirring model [38], to obtain their exact scattering matrices and the mass spectrum [39, 40], the vacuum energy as a function of renormalized parameters of the theory [41], the relation between the coupling constant and the physical mass scale [42], etc.

As concerns the equilibrium statistical mechanics of classical systems, the first milestone occurred in 1944 when Onsager solved the two-dimensional Ising model [43]. His exact solution showed the universality of critical phenomena and the fact that the critical indices in two dimensions are not mean-field like. Further lattice models of special interest were the vertex ones, in which the local state variables are localized on the edges connecting nearest-neighbor sites. Three cases of the six-vertex model – antiferroelectric F [44], ferroelectric KDP [45] and ice [46] – were solved by Lieb. The general case of the six-vertex model was solved by Sutherland [47]. The exact solution of the eight-vertex model by Baxter [7, 9] has already been mentionned in the context of the XYZ Heisenberg model.

The statistical models mentionned above are defined on a regular discrete lattice structure. There exists another family of classical statistical models, the so-called fluids, formulated in the continuum space. Concepts and methods used in the two fields are usually very different and the overlap between the physical communities is small. While there exist many exactly solvable two-dimensional lattice models, nontrivial fluid systems were solvable only in one dimension. A modest contribution of the present author and his co-workers consists in solving exactly the thermodynamics of the first continuum fluid in dimension higher than one: the two-dimensional Coulomb gas of ± 1 pointlike charges interacting via the logarithmic potential [48, 49]. The exact solution of a two-dimensional Coulomb gas with the charge asymmetry $+1, -1/2$ is also available [50].

There exist few monographs about the present subject. Those, which I consider as the most relevant and therefore belong to my library, are presented in the chronological order in what follows. The famous book by Baxter [51] concerns integrable models of the equilibrium statistical mechanics. Gaudin summarizes his experience with the Bethe ansatz and the ground-state analysis in the technically rather difficult book [52]. Mattis's encyclopedia of exactly solved models in one dimension [53] contains over 80 reprinted papers with a short summary for each topic. The book by Korepin and Essler [54] contains reprinted articles in the field of condensed matter physics. The Yang-Baxter equation, the general structure of its solutions and the Quantum

groups are in the center of interest of the book [55]. Takahashi's book [56] is an encyclopedia of the results about the thermodynamics of integrable many-body systems. Sutherland concentrates in his book [57] on nonrelativistic Fermi and Bose gases in one dimension with internal degrees of freedom, a special interest is devoted to models with long-range interactions. The one-dimensional Hubbard model is reviewed in detail in a recent book [58].

A natural question arises: why to write another course about integrable systems? The main motivations are the following:

- The published books are usually oriented to a restricted area of models and methods. The present course encompasses all important kinds of integrable models, including the ones of the (1+1)-dimensional QFT. It is intended for non-specialists who would like to understand methods of other branches of physics and potentially use these methods in their own field.

- The mathematical level of some of the books is very high and requires a preliminary study of specific topics from the literature. The present course is self-contained, made mathematically as simple as possible. Only an elementary knowledge of the quantum mechanics and the equilibrium statistical physics is required. This makes the text accessible to graduate students, which is in agreement with the novel strategy of Acta Physica Slovaca (APS).

- The methods and techniques presented in the published books are usually traditional. We intend to include also modern trends in the Thermodynamic Bethe ansatz which, to my knowledge, are not included in the standard textbooks.

- The course is not intended as an encyclopedia of the obtained results in the field of integrable systems; I apologize myself to everybody who contributed relevantly to the field and his work is not cited. For each particular model, the course contains a detailed derivation of the Bethe ansatz equations, the specification of the ground state, the construction of the thermodynamic Bethe ansatz and a discussion about physical consequences which follow from the exact results.

The character and the aims of the course reflect my own experience in the equilibrium statistical mechanics of lattice models and continuum fluids. In the past, a small group of theoretical physicists at our Institute of Physics created a "lecture club" in the field of integrable models. There was no particular relationship between our work of that time and this topic, which we considered to be of general importance and therefore appealing. I benefited personally from the series of lectures given by all members of our group and afterwards I was able to teach the subject occasionally PhD. students of our Institute and Comenius University in Bratislava. One of the topics of my special interest became Coulomb fluids, classical and quantum, two-dimensional and higher-dimensional. This was just in time of great discoveries in the QFT in a (1+1)-dimensional spacetime. Being able to read the papers and to adopt the results from the sine-Gordon model, we contributed to the equilibrium statistical mechanics by solving exactly the two-dimensional Coulomb gas. Although this was a relatively simple task, a deep understanding of integrable models was a necessary condition. My long-time collaborator Bernard Jancovici from LPT (Université de Paris Sud) in Orsay told me that, of course, he and his coworkers were aware about the relationship between the two-dimensional Coulomb gas and the (1+1)-dimensional sine-Gordon theory, but they were not able to grasp recent results obtained in this integrable field theory, although they participated on the same conferences as well-known

experts in the QFT. To my opinion, a narrow specialization and a separation of communities is a feature of the contemporary physics. When Editor in Chief of APS asked me to write a review about the subject of my interest, I hesitated between writing a specialized article about Coulomb gases and a more extensive course about integrable many-body systems. Since I remember how many articles I had to find and read to understand the subject in his many relevant aspects, I decided to write an extensive course. I hope that, perhaps, this will help somebody to save time and to find new results in his own field.

The material is divided into three volumes as follows:

- The first volume is divided into two parts. In the first part, we deal with nonrelativistic one-dimensional continuum Fermi and Bose quantum gases of identical spinless particles. The second part is devoted to the description of the QISM and to the analysis of the Yang-Baxter equation. We present the complete solution of spin-$\frac{1}{2}$ fermions with δ-function interactions.

- The second volume will concern integrable Heisenberg models, especially with spin-$\frac{1}{2}$, but also the isotropic ones with general spin S. The thermodynamics will be derived by using traditional methods based on the string hypothesis as well as a simpler method [33, 34] which leads to a single nonlinear integral equation.

- In the first part of the third volume devoted to systems of the condensed matter physics, we plan to review the exact solutions of the Hubbard model and of the Kondo effect. The second part concerns the results of the Thermodynamic Bethe ansatz for relativistic (1+1)-dimensional models of the QFT, namely the sine-Gordon model and its fermionic equivalent – the Thirring model. The models will be first treated semiclassically, then their full quantum description will be given. The relationship between these models and the two-dimensional classical Coulomb gas will be explained and the exact thermodynamics of the latter model will be derived.

This volume is organized as follows.

As a first one-dimensional nonrelativistic model of spinless Bose and Fermi gases, we treat in Sect. 2 the one with δ-function interactions which is nontrivial only in the case of bosons. After a detailed derivation of the Bethe ansatz equations, the ground state of the model and its energy are discussed. In the attractive regime, the ground state in the sector of N particles corresponds to an N-string of energy $\propto N^3$ which leads to the thermodynamic collapse. In the repulsive regime, the ground state has all momenta real and the thermodynamic limit $N \to \infty$ is well behaved. This is the opportunity to document Hultén's continualization procedure [3] and to derive an integral equation for the ground-state particle density in momentum space which is analyzed in special limits of the interaction strength.

The form of the Bethe ansatz equations is common for various integrable spinless Bose and Fermi gases, only a phase-shift function depends on the particular type of pair interaction. Also the choice of quantum numbers corresponding to the ground state is the same, which enables us to perform in Sect. 3 a general analysis of low-lying excited states and of the zero-temperature thermodynamics.

The general analysis of the finite-temperature thermodynamics, based on the hole concept of Yang and Yang [23], is the subject of Sect. 4. For both zero-temperature and finite-temperature

thermodynamics, the model system worked out into detail is that of bosons with repulsive δ-function interactions.

Probably the simplest integrable model – the one-dimensional gas of Bose or Fermi particles with hard-core interactions, is studied in Sect. 5.

The one-dimensional system of Bose and Fermi particles with a periodic version of the long-range interaction $1/x^2$ is the subject of Sect. 6. In contrast to systems with short-range interactions, the two-particle wavefunction becomes a superposition of plane waves only at asymptotically large distances between particles; this fact is behind the name "asymptotic Bethe ansatz". The model is obtained naturally as an answer to the following task: find a pair interaction for which the ground-state wavefunction is of a pair-product form. Excited states, ground-state properties and finite-temperature thermodynamics of the model are also derived.

Sect. 7 is the first one devoted to the explanation of principles and applications of the QISM. The Yang-Baxter equation for the elements of the two-particle scattering matrix is derived and special types of its solutions are shown. An elementary introduction to noncommutative geometry and Quantum groups is given. We introduce important operators constructed from the elements of the scattering matrix, like the Lax operators, the transfer and monodromy matrices, and derive for them a hierarchy of Yang-Baxter relationships. A family of commuting transfer matrices is obtained.

In Sect. 8 we show how to diagonalize the family of commuting transfer matrices and how to generate from these transfer matrices integrable one-dimensional lattice quantum systems like the Heisenberg model. This section explains a link between two-dimensional classical and one-dimensional quantum models.

Sect. 9 brings another application of the QISM formalism in the context of the periodic boundary conditions for integrable systems of particles with internal degrees of freedom. In this case, the transfer matrix is inhomogeneous since the building elements of the scattering matrix are site-dependent. We show how to diagonalize these inhomogeneous transfer matrices. The resulting equations are the ones of the nested Bethe ansatz.

The formalism of Sect. 9 is applied to spin-$\frac{1}{2}$ fermions with δ-function interactions. The nested Bethe ansatz equations and the ground state properties, which are well behaved in the thermodynamic limit, are obtained for both repulsive and attractive regimes in Sect. 10. The finite-temperature thermodynamics of the model is derived in Sect. 11.

Appendix A describes an explicit construction of spin operators on a chain. The subject of Appendix B is the description of doubly-periodic elliptic functions which are generalizations of the trigonometric functions in the complex plane.

BOSE AND FERMI GASES

2 Spinless particles with δ-function interactions: Bethe ansatz and the ground state

Elementary particles have internal degrees of freedom called spins (see Appendix A). The particle with spin $S = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ can be in one of $(2S + 1)$ different states, which means that the corresponding spin Hilbert space has dimension $(2S + 1)$. The integer values of the spin $S = 0, 1, 2, \ldots$ correspond to bosons, the wavefunction of which is symmetric with respect to any interchange of two particles. The half-odd integer values $S = \frac{1}{2}, \frac{3}{2}, \ldots$ correspond to fermions, the wavefunction of which is antisymmetric with respect to any interchange of two particles. The relation between the spin and the statistics of identical particles is the physical postulate. The boson and fermion systems are mathematically well defined for an arbitrary value of the spin and we shall ignore this physical postulate.

2.1 Definitions and basic formalism

We shall first consider a general one-dimensional system of N identical spinless (one-state) $S = 0$ particles $j = 1, 2, \ldots, N$ of mass m. The particles are localized on a circle of length L, i.e. their coordinates are constrained to the line

$$0 \le x_j \le L, \qquad j = 1, 2, \ldots, N \tag{2.1}$$

with imposed periodic boundary conditions, and interact pairwisely by a potential $v(x - x') = v(x' - x)$. In the units of $\hbar = 2m = 1$, the quantum Hamiltonian reads

$$\mathcal{H} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{j>k=1}^{N} v(x_j - x_k). \tag{2.2}$$

The energy spectrum of the particle system is determined by the time-independent Schrödinger equation

$$\mathcal{H}\psi(x_1, x_2, \ldots, x_N) = E\psi(x_1, x_2, \ldots, x_N), \tag{2.3}$$

where the wavefunction ψ satisfies periodic boundary conditions

$$\psi(\cdots, x_j, \cdots) = \psi(\cdots, x_j + L, \cdots), \qquad j = 1, 2, \ldots, N. \tag{2.4}$$

The wavefunction ψ is determined up to a constant which is fixed by the normalization to unity of the integral of $|\psi|^2$ over the N-particle coordinate space. In this review, the proper normalization of the wavefunction will be irrelevant and therefore not required.

The spinless particles are supposed to obey boson/fermion statistics expressed by the symmetry/antisymmetry property of the wavefunction under the exchange of an arbitrary pair of particles. Since the system is one-dimensional, the N particles can be ordered from the left to the right in $N!$ ways. Let S_N denotes the symmetric group of all $N!$ permutations of numbers $(12 \ldots N)$. The ordering sectors are labeled by the permutations $Q = (Q1, Q2, \ldots, QN) \in S_N$ according to the prescription

$$Q: \qquad 0 \le x_{Q1} < x_{Q2} < \cdots < x_{QN} \le L. \tag{2.5}$$

For example, in the case of $N = 3$ particles, $Q = (132)$ labels the ordering $x_1 < x_3 < x_2$, $Q = (213)$ labels the ordering $x_2 < x_1 < x_3$, etc. The *fundamental* sector, identified with the identity permutation $I = (12 \dots N)$, corresponds to the particle ordering

$$I : \qquad 0 \le x_1 < x_2 < \cdots < x_N \le L. \tag{2.6}$$

Each sector Q is adjacent to $N - 1$ other sectors at the points $x_{Q1} = x_{Q2}$, $x_{Q2} = x_{Q3}$,..., $x_{Q(N-1)} = x_{QN}$. According to principles of quantum mechanics, the wavefunction must be continuous at these points.

Let us denote the wavefunction corresponding to the ordering sector Q as ψ_Q and assume that the wavefunction of the fundamental sector (2.6),

$$\psi_I \equiv \psi_I(x_1, x_2, \dots, x_N), \tag{2.7}$$

is known. In general, this function has no exchange symmetry properties with respect to particle coordinates $x_1, x_2, \dots, x_N$. In accordance with the standard symmetrization/antisymmetrization procedure, we have

$$\psi_Q(x_1, x_2, \dots, x_N) = (\pm 1)^{\eta_Q} \psi_I(x_{Q1}, x_{Q2}, \dots, x_{QN}), \tag{2.8}$$

where the $+/-$ sign corresponds to bosons/fermions and η_Q is the number of transpositions of nearest-neighbor elements which bring the permutation $(Q1, Q2, \dots, QN)$ to $(12 \dots N)$. We shall often use the notation $(-1)^{\eta_Q} = \mathrm{sign}(Q)$. The general wavefunction ψ can be formally represented as follows

$$\psi(x_1, x_2, \dots, x_N) = \sum_{Q \in S_N} \theta(x_{Q2} - x_{Q1}) \theta(x_{Q3} - x_{Q2}) \cdots \theta(x_{QN} - x_{Q(N-1)}) \psi_Q, \tag{2.9}$$

where $\theta(x)$ is the Heaviside step function

$$\theta(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x > 0. \end{cases} \tag{2.10}$$

For solving the particle system, it is sufficient to find the wavefunction ψ_I associated with the fundamental sector I (2.6). A minor problem is that the periodic boundary conditions (2.4) relate different Q-sectors. Indeed, the shift of $x_1 = 0$ in the I-sector by L is identified with

$$\psi_I(x_1 = 0, x_2, \dots, x_N) = \psi_Q(x_1 = L, x_2, \dots, x_N), \qquad Q = (23 \dots N1). \tag{2.11}$$

Since $x_1 = L$ is the largest of particle coordinates, the wavefunction on the rhs of (2.11) corresponds to the ordering sector $Q = (23 \dots N1)$ with $\eta_Q = N - 1$. Using the formula (2.8), we finally obtain the periodic boundary condition

$$\psi_I(x_1 = 0, x_2, \dots, x_N) = (\pm 1)^{N-1} \psi_I(x_2, \dots, x_N, x_1 = L), \tag{2.12}$$

relating the wavefunctions in the same I-sector.

We first consider the case of the short-range δ-function potential $v(x) = 2c\delta(x)$, where $2c$ is the interaction amplitude. Although this potential is not the simplest one of integrable potentials, it was studied extensively in the past. It serves as an "etalon" for standard methods in integrable models. Many of the results can be proven rigorously. The case $c = 0$ corresponds to free

particles, $c < 0$ to the particle attraction and $c > 0$ to the particle repulsion. Particles can pass through each other, except the impenetrable limit $c = \infty$.

The Hamiltonian of the N-particle system reads

$$\mathcal{H} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{j>k=1}^{N} \delta(x_j - x_k). \tag{2.13}$$

When particle coordinates differ from each other, like it is in any ordering sector Q (2.5), the δ-functions have no effect and the original Schrödinger equation can be replaced by a Helmholtz equation for free particles

$$-\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} \psi_Q = E\psi_Q, \qquad 0 \le x_{Q1} < x_{Q2} < \cdots < x_{QN} \le L. \tag{2.14}$$

The presence of δ-functions in the Hamiltonian is equivalent to specific "boundary" conditions whenever two nearest-neighbor particles touch one another. To see this, let us consider the case of $N = 2$ particles for which the wavefunction (2.9) can be written as

$$\psi(x_1, x_2) = \theta(x_2 - x_1)\psi_I(x_1, x_2) + \theta(x_1 - x_2)\psi_{(21)}(x_1, x_2). \tag{2.15}$$

The continuity of the wavefunction at $x_1 = x_2$ requires that it holds $\lim_{x_2 - x_1 \to 0^+} \psi_I(x_1, x_2) = \lim_{x_2 - x_1 \to 0^-} \psi_{(21)}(x_1, x_2)$. Using the Helmholtz equation (2.14) for ψ_I and $\psi_{(21)}$ it is easy to show that

$$\begin{aligned} \left(-\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right)\psi(x_1, x_2) \quad &= \quad E\psi(x_1, x_2) + \delta(x_1 - x_2)\left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1}\right) \\ &\quad \times \left[\psi_{(21)}(x_1, x_2) - \psi_I(x_1, x_2)\right]. \end{aligned} \tag{2.16}$$

To fulfill the Schrödinger equation with the Hamiltonian (2.13), we obtain the following boundary condition for the discontinuity in derivatives:

$$\left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1}\right)\psi\Big|_{x_2 - x_1 \to 0^+} - \left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1}\right)\psi\Big|_{x_2 - x_1 \to 0^-} = 2c\psi\Big|_{x_2 = x_1}. \tag{2.17}$$

The same formula

$$\left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k}\right)\psi\Big|_{x_j - x_k \to 0^+} - \left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k}\right)\psi\Big|_{x_j - x_k \to 0^-} = 2c\psi\Big|_{x_j = x_k}. \tag{2.18}$$

holds for an N-particle wavefunction, where x_j and x_k denote the coordinates of nearest neighbor particles.

2.2 Fermi gas with δ-function interactions

If the particles are fermions, the problem of the δ-function potential is trivial and reduces to the problem of free fermions.

• $N = 1$: For a single particle, the solution of the Helmholtz equation (2.14) is the plane wave

$$\psi(x) = A\exp(\mathrm{i}kx), \qquad E = k^2. \tag{2.19}$$

With regard to the periodic boundary condition $\psi(x) = \psi(x + L)$, the wave number (quasi-momentum) k is quantized as follows

$$kL = 2\pi I, \qquad I = 0, \pm 1, \pm 2, \ldots . \tag{2.20}$$

The particle system is formulated on a continuous line; the Hilbert space has therefore infinite dimension and the values of the quantum number I are unbounded.

• $N = 2$: In the case of two fermions at positions in the fundamental sector $0 \leq x_1 < x_2 \leq L$, the most general solution of the Helmholtz equation (2.14) is a superposition of plane waves

$$\psi_I(x_1, x_2) = A(12)\mathrm{e}^{\mathrm{i}(k_1x_1+k_2x_2)} - A(21)\mathrm{e}^{\mathrm{i}(k_2x_1+k_1x_2)} \tag{2.21}$$

with the energy

$$E = k_1^2 + k_2^2. \tag{2.22}$$

Here, the coefficients $A(12) \equiv A(k_1, k_2)$, $A(21) \equiv A(k_2, k_1)$ and the minus sign is attached to $A(21)$ for the sake of the simplification of the formalism. In the sector $Q = (21)$, i.e. when $0 \leq x_2 < x_1 \leq L$, the formula (2.8) tells us that

$$\begin{aligned} \psi_{(21)}(x_1, x_2) &= -\psi_I(x_2, x_1) \\ &= -A(12)\mathrm{e}^{\mathrm{i}(k_1x_2+k_2x_1)} + A(21)\mathrm{e}^{\mathrm{i}(k_2x_2+k_1x_1)} \end{aligned} \tag{2.23}$$

The wavefunction $\psi(x_1, x_2)$ must be continuous at $x_1 = x_2$, which implies that

$$A(21) = A(12) \tag{2.24}$$

and, as a consequence of the Pauli exclusion principle, $\psi(x, x) = 0$. Both sides of Eq. (2.17) vanish identically under this condition and the δ-potential has no effect on particles which therefore behave like free fermions. The scattering S-matrix, which is in the present case of spinless (one-state) fermions the scalar, is defined as follows

$$A(k_2, k_1) = S(k_1, k_2)A(k_1, k_2). \tag{2.25}$$

In view of the relation (2.24), we have trivially

$$S(k_1, k_2) = 1. \tag{2.26}$$

The wave numbers k_1 and k_2 are quantized according to the periodic boundary condition (2.12), $\psi_I(0, x_2) = -\psi_I(x_2, L)$, as follows

$$A(12) = A(21)\exp(\mathrm{i}k_1L), \qquad A(21) = A(12)\exp(\mathrm{i}k_2L). \tag{2.27}$$

With regard to (2.24), these conditions are equivalent to

$$\left.\begin{array}{l} k_1L = 2\pi I_1 \\ k_2L = 2\pi I_2 \end{array}\right\} \qquad I_1, I_2 = \text{integer}. \tag{2.28}$$

Because the wavefunction vanishes for $k_1 = k_2 = k$, the wave numbers must be unequal, $k_1 \neq k_2$ or $I_1 \neq I_2$. The integers I_1 and I_2 may be interchanged without affecting the solution and so we can restrict ourselves to $I_1 < I_2$.

• Arbitrary N: In the problem of N fermions, the wavefunction in the fundamental ordering sector I ($0 \le x_1 < x_2 < \ldots < x_N \le L$) has the form

$$\psi_I(x_1, x_2, \ldots, x_N) = \sum_{P \in S_N} \mathrm{sign}(P) A(P) \exp\left(\mathrm{i} \sum_{j=1}^{N} k_{Pj} x_j \right). \tag{2.29}$$

The corresponding energy is

$$E = \sum_{j=1}^{N} k_j^2. \tag{2.30}$$

The coefficients $A(P) \equiv A(k_{P1}, k_{P2}, \ldots, k_{PN})$ fulfill the scattering relation

$$A(\ldots k_v, k_u \ldots) = S(k_u, k_v) A(\ldots, k_u, k_v \ldots), \qquad S(k_u, k_v) = 1. \tag{2.31}$$

A successive application of this relation allows us to express an arbitrary coefficient $A(P)$ in terms of say $A(I)$ and in this way to find, up to the normalization, the wavefunction. In our trivial case $A(P)$ is a constant, independent of the permutation P. Setting $A(P) = 1$ in (2.29), the (unnormalized) wavefunction ψ_I is thus expressible as the familiar Slater determinant

$$\psi_I(x_1, x_2, \ldots, x_N) = \operatorname*{Det}_{1 \le j,l \le N} \exp(\mathrm{i} k_j x_l). \tag{2.32}$$

To satisfy the periodic boundary condition for the wavefunction, the wave numbers are given by

$$k_j = 2\pi I_j \qquad I_j = \text{integer} \qquad (j = 1, 2, \ldots, N). \tag{2.33}$$

To avoid the nullity of the wavefunction, only the sets of integers $I_1 < I_2 < \ldots < I_N$ are allowed.

Combining Eq. (2.8) with the representation (2.29), the wavefunction in the ordering sector $Q \in S_N$ is formally expressible as

$$\psi_Q(x_1, x_2, \ldots, x_N) = \sum_{P \in S_N} [Q, P]_f \exp\left(\mathrm{i} \sum_{j=1}^{N} k_{Pj} x_{Qj} \right), \tag{2.34}$$

where the "fermion" symbol $[Q, P]_f$ means

$$[Q, P]_f = \mathrm{sign}(Q)\mathrm{sign}(P) A(k_{P1}, k_{P2}, \ldots, k_{PN}). \tag{2.35}$$

The A-coefficients in different permutation P sectors are related via the scattering formula (2.31). Since S relates directly the A-coefficients in two P sectors which differ from one another by one nearest neighbor transposition of two numbers, the presence of $\mathrm{sign}(P)$ in the definition (2.35) fixes only the sign of S. It is more convenient to have $S = 1$ than $S = -1$.

2.3 Bosons: Bethe ansatz equations

For the symmetric Bose wavefunction in the I sector, the condition (2.18) takes a simpler form

$$\left(\frac{\partial}{\partial x_{j+1}} - \frac{\partial}{\partial x_j}\right)\psi_I\big|_{x_{j+1}=x_j} = c\psi_I\big|_{x_{j+1}=x_j}, \qquad j = 1, \ldots, N-1. \tag{2.36}$$

Lieb and Liniger [21, 22] showed that the system of δ-function bosons is solvable by the coordinate Bethe ansatz. The usual way how to derive the Bethe ansatz equations is to increase successively the number of particles $N = 2, 3, \ldots$ in order to reveal the structure of the solution for an arbitrary value of N.

• $N = 2$: As in the fermion case, the wavefunction is assumed to be a superposition of plane waves in both particle ordering sectors $I = (12)$ and (21):

$$\psi_I(x_1, x_2) = A(12)\mathrm{e}^{\mathrm{i}(k_1x_1+k_2x_2)} - A(21)\mathrm{e}^{\mathrm{i}(k_2x_1+k_1x_2)}, \tag{2.37}$$

$$\psi_{(21)}(x_1, x_2) = +\psi_I(x_2, x_1)$$
$$= A(12)\mathrm{e}^{\mathrm{i}(k_1x_2+k_2x_1)} - A(21)\mathrm{e}^{\mathrm{i}(k_2x_2+k_1x_1)}. \tag{2.38}$$

This function is continuous at $x_1 = x_2$ for an arbitrary choice of the coefficients $A(12)$ and $A(21)$. The boundary condition (2.36) taken at $x_1 = x_2$ implies the following relation between the amplitudes $A(12)$ and $A(21)$,

$$\frac{A(21)}{A(12)} = \frac{k_2 - k_1 + \mathrm{i}c}{k_1 - k_2 + \mathrm{i}c} = \exp(-\mathrm{i}\theta_{12}). \tag{2.39}$$

The phase shift $\theta_{12} \equiv \theta(k_1 - k_2)$ is readily shown to be given by

$$\theta(k) = -2\arctan\left(\frac{k}{c}\right), \qquad -\pi < \theta < \pi. \tag{2.40}$$

The phase function is antisymmetric with respect to the exchange of indices,

$$\theta_{12} = -\theta_{21}. \tag{2.41}$$

The (unnormalized) A-coefficients are expressible simply as

$$A(12) = \exp\left(\frac{\mathrm{i}}{2}\theta_{12}\right), \qquad A(21) = \exp\left(\frac{\mathrm{i}}{2}\theta_{21}\right). \tag{2.42}$$

Another possible representation of the coefficients reads

$$A(P) = k_{P1} - k_{P2} + \mathrm{i}c. \tag{2.43}$$

The solution for the two-body wavefunction (2.21) with the A-coefficients defined by (2.42) has a simple interpretation. For $k_1 > k_2$, the wavefunction is the sum of an incoming wave $\exp[\mathrm{i}(k_1x_1 + k_2x_2)]$ and an outgoing wave $-\exp[\mathrm{i}(k_2x_1 + k_1x_2) - \mathrm{i}\theta(k_1 - k_2)]$, with a phase shift $\theta(k_1 - k_2)$ between them. The outgoing wave is the result of the "point scattering" of the particles due to the δ-function potential. In the two-body scattering, the total momentum $K = k_1 + k_2$ and the energy $E = k_1^2 + k_2^2$ are conserved. Consequently, the scattering of the particles must be purely elastic: the particles only exchange their momenta, so that the outgoing

momenta are $k_1' = k_2$ and $k_2' = k_1$. The scattering matrix, defined as in the fermion case by scalar relation (2.25), reads

$$S(k_1, k_2) \equiv S(k_1 - k_2) = \mathrm{e}^{-\mathrm{i}\theta(k_1 - k_2)} = \frac{k_2 - k_1 + \mathrm{i}c}{k_1 - k_2 + \mathrm{i}c}. \tag{2.44}$$

The wave numbers k_1 and k_2 are quantized according to the periodic boundary condition $\psi_I(0, x_2) = \psi_I(x_2, L)$ as follows

$$A(12) = -A(21)\exp(\mathrm{i}k_1 L), \qquad A(21) = -A(12)\exp(\mathrm{i}k_2 L). \tag{2.45}$$

With regard to (2.39), these conditions are equivalent to

$$\left.\begin{array}{l} k_1 L = 2\pi I_1 + \theta_{12} \\ k_2 L = 2\pi I_2 + \theta_{21} \end{array}\right\} \qquad I_1, I_2 = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots. \tag{2.46}$$

Because half-odd integers I_1 and I_2 may be interchanged without affecting a solution, we can restrict ourselves to $I_1 \leq I_2$. The wave numbers must be unequal, $k_1 \neq k_2$. In the opposite case of $k_1 = k_2 = k$, it follows from Eq. (2.39) that $A(21) = A(12)$ and the wavefunction $\psi_I(x_1, x_2) = [A(12) - A(21)]\exp[\mathrm{i}k(x_1 + x_2)]$ vanishes. Since $\theta(0) = 0$, an immediate consequence of $k_1 \neq k_2$ is the inequality $I_1 \neq I_2$, i.e. $I_1 < I_2$. Note that this exclusion rule for the quantum numbers resembles that for fermions.

• $N = 3$: The solution of the three-particle problem in the fundamental sector $0 \leq x_1 < x_2 < x_3 \leq L$ can be represented as a superposition of plane waves with quasi-momenta permutated among the particle coordinates,

$$\begin{aligned} \psi_I(x_1, x_2, x_3) \;=\; & A(123)\mathrm{e}^{\mathrm{i}(k_1 x_1 + k_2 x_2 + k_3 x_3)} + A(132)\mathrm{e}^{\mathrm{i}(k_1 x_1 + k_3 x_2 + k_2 x_3)} \\ & + A(213)\mathrm{e}^{\mathrm{i}(k_2 x_1 + k_1 x_2 + k_3 x_3)} + A(231)\mathrm{e}^{\mathrm{i}(k_2 x_1 + k_3 x_2 + k_1 x_3)} \\ & + A(312)\mathrm{e}^{\mathrm{i}(k_3 x_1 + k_1 x_2 + k_2 x_3)} + A(321)\mathrm{e}^{\mathrm{i}(k_3 x_1 + k_2 x_2 + k_1 x_3)}. \end{aligned} \tag{2.47}$$

The corresponding energy is given by

$$E = k_1^2 + k_2^2 + k_3^2. \tag{2.48}$$

The condition (2.36) taken at $x_1 = x_2$ implies the following relation among the amplitudes,

$$\frac{A(213)}{A(123)} = \mathrm{e}^{-\mathrm{i}\theta_{12}}, \quad \frac{A(312)}{A(132)} = \mathrm{e}^{-\mathrm{i}\theta_{13}}, \quad \frac{A(321)}{A(231)} = \mathrm{e}^{-\mathrm{i}\theta_{23}}, \tag{2.49}$$

where $\theta_{\alpha\beta} \equiv \theta(k_\alpha - k_\beta)$ with $\alpha, \beta = 1, 2, 3$ is the obvious generalization of the previously defined two-particle phase shift (2.39). The condition (2.36) taken at $x_2 = x_3$ implies the relations

$$\frac{A(132)}{A(123)} = \mathrm{e}^{-\mathrm{i}\theta_{23}}, \quad \frac{A(231)}{A(213)} = \mathrm{e}^{-\mathrm{i}\theta_{13}}, \quad \frac{A(321)}{A(312)} = \mathrm{e}^{-\mathrm{i}\theta_{12}}. \tag{2.50}$$

Using the symmetry relations $\theta_{\alpha\beta} = -\theta_{\beta\alpha}$, the solution of 6 homogeneous relations in Eqs. (2.49) and (2.50) for 6 unknown coefficients can be written in the form

$$\begin{aligned} A(123) &= \exp\left[\frac{\mathrm{i}}{2}(\theta_{12} + \theta_{13} + \theta_{23})\right], \\ A(213) &= \exp\left[\frac{\mathrm{i}}{2}(\theta_{21} + \theta_{23} + \theta_{13})\right], \\ A(321) &= \exp\left[\frac{\mathrm{i}}{2}(\theta_{32} + \theta_{31} + \theta_{21})\right], \end{aligned} \tag{2.51}$$

etc. The formal structure of A-coefficients is evident. Another possible representation of the coefficients is

$$A(P) = (k_{P1} - k_{P2} + \mathrm{i}c)(k_{P1} - k_{P3} + \mathrm{i}c)(k_{P2} - k_{P3} + \mathrm{i}c). \tag{2.52}$$

The result for the three-body wavefunction (2.47) with the A-coefficients (2.51) has two important features.

• Firstly, assuming that $k_1 > k_2 > k_3$, the conservation of the total momentum $K = k_1 + k_2 + k_3$ and energy $E = k_1^2 + k_2^2 + k_3^2$ does not exclude a diffractive three-body (outgoing) scattering term

$$\int\limits_{\substack{k_1' < k_2' < k_3' \\ K,E \text{ fixed}}} \int\limits \int\limits \mathrm{d}k_1' \mathrm{d}k_2' \mathrm{d}k_3' S(k_1', k_2', k_3') \exp\left[\mathrm{i}(k_1' x_1 + k_2' x_2 + k_3' x_3)\right] \tag{2.53}$$

in a general three-body representation of the wavefunction. This diffractive term is absent in the elastic Bethe ansatz (2.47). Multiparticle scatterings are *nondiffractive* in integrable models, i.e. not only the total momentum and energy, but also a sum of any power of momenta, $\sum_j k_j^m$ $(m = 1, 2, \ldots)$, is conserved.

• The second important aspect is that, according to Eqs. (2.49) and (2.50), the interchange (scattering) of two particles is independent of the third particle. This means that a multiparticle scattering factorizes onto a product of two-body scatterings and the result does not depend on the particular sequence of two-body collisions. If we start from the incoming amplitude $A(123)$, we can reach the outgoing one $A(321)$ in two different paths:

$$A(123) \to \left\{ \begin{array}{l} A(213) \to A(231) \\ A(132) \to A(312) \end{array} \right\} \to A(321). \tag{2.54}$$

It is a simple task to check by using relations (2.49) and (2.50) that both paths give the same result

$$\frac{A(321)}{A(123)} = \exp\left[-\mathrm{i}(\theta_{12} + \theta_{13} + \theta_{23})\right] = S(k_1, k_2) S(k_1, k_3) S(k_2, k_3). \tag{2.55}$$

This equivalence of two different paths is the precursor of the Yang-Baxter equation for the scattering matrix.

The periodic boundary condition $\psi_I(0, x_2, x_3) = \psi_I(x_2, x_3, L)$ implies that

$$A(123) = A(231)\mathrm{e}^{\mathrm{i}k_1 L}, \quad A(213) = A(132)\mathrm{e}^{\mathrm{i}k_2 L}, \quad A(312) = A(123)\mathrm{e}^{\mathrm{i}k_3 L}. \tag{2.56}$$

The wave numbers k_1, k_2 and k_3, no two of which are identical in order to avoid the nullity of the wavefunction, are thus quantized as follows

$$\left. \begin{array}{l} k_1 L = 2\pi I_1 + \theta_{12} + \theta_{13} \\ k_2 L = 2\pi I_2 + \theta_{21} + \theta_{23} \\ k_3 L = 2\pi I_3 + \theta_{31} + \theta_{32} \end{array} \right\} \qquad I_1 < I_2 < I_3 = 0, \pm 1, \pm 2, \ldots. \tag{2.57}$$

We see that the three-particle solution is constructed by using the two-particle phase function $\theta_{\alpha\beta}$. This property is maintained also for higher $N = 4, 5, \ldots$ particle numbers.

• Arbitrary N: In the problem of N I-ordered particles $0 \leq x_1 < x_2 < \ldots < x_N \leq L$, the Bethe ansatz has the form

$$\psi_I(x_1, x_2, \ldots, x_N) = \sum_{P \in S_N} \mathrm{sign}(P) A(P) \exp\left(\mathrm{i} \sum_{j=1}^{N} k_{Pj} x_j \right). \tag{2.58}$$

This formal solution fulfills the Helmholtz equation (2.14), the corresponding energy is given by

$$E = \sum_{j=1}^{N} k_j^2. \tag{2.59}$$

Introducing the total momentum operator by

$$\mathcal{K} = \sum_{j=1}^{N} -\mathrm{i} \frac{\partial}{\partial x_j}, \tag{2.60}$$

the wavefunction (2.58) is also an eigenstate of this operator,

$$\mathcal{K}\psi = K\psi, \qquad K = \sum_{j=1}^{N} k_j. \tag{2.61}$$

We shall now show the conditions for the coefficients $A(P)$ under which the Bethe ansatz form of the wavefunction (2.58) satisfies the set of $N-1$ conditions (2.36), say the one relating the nearest-neighbor particles j and $j+1$. The $N!$ terms in the wavefunction can be paired, since each permutation P has as a pair permutation $P_{j,j+1}$ which is generated from P by the transposition of the nearest-neighbors components Pj and $P(j+1)$, i.e. if $P = (P1, \ldots, Pj, P(j+1), \ldots, PN)$ then $P_{j,j+1} = (P1, \ldots, P(j+1), Pj, \ldots, PN)$. The pair of terms occurs in the form

$$\begin{aligned} \psi_I \quad = \quad & \cdots \mathrm{sign}(P) A(P) \exp(\cdots + k_{Pj} x_j + k_{P(j+1)} x_{j+1} + \cdots) \\ & -\mathrm{sign}(P) A(P_{j,j+1}) \exp(\cdots + k_{P(j+1)} x_j + k_{Pj} x_{j+1} + \cdots) + \cdots. \end{aligned} \tag{2.62}$$

For each pair of terms to individually satisfy the condition (2.36) at $x_j = x_{j+1}$, we must have

$$\frac{A(P_{j,j+1})}{A(P)} = \frac{k_{P(j+1)} - k_{Pj} + \mathrm{i}c}{k_{Pj} - k_{P(j+1)} + \mathrm{i}c} = \exp(-\mathrm{i}\theta_{Pj,P(j+1)}). \tag{2.63}$$

This formula represents a generalization of the scattering relation (2.25) to an arbitrary number of particles:

$$A(\ldots k_v, k_u \ldots) = S(k_u, k_v) A(\ldots k_u, k_v \ldots), \qquad S(k_u, k_v) = \mathrm{e}^{-\mathrm{i}\theta(k_u - k_v)}. \tag{2.64}$$

Eq. (2.63) leads to the general result valid for any permutation P:

$$A(P) = \exp\left(\frac{\mathrm{i}}{2} \sum_{\substack{j,l=1 \\ (j<l)}}^{N} \theta_{Pj,Pl} \right). \tag{2.65}$$

There exists an alternative representation of the coefficients

$$A(P) = \prod_{j<l} (k_{Pj} - k_{Pl} + \mathrm{i}c). \tag{2.66}$$

The periodic boundary condition (2.12) is equivalent to the set of conditions

$$A(P) = (-1)^{N-1} A(PC) \exp(\mathrm{i}k_{P1}L) \qquad \text{for arbitrary } P, \tag{2.67}$$

where PC is the cyclic transposition of P, i.e. when $P = (P1, P2, \ldots, PN)$ then $PC = (P2, \ldots, PN, P1)$. With respect to the result for the A-amplitudes (2.65), we obtain

$$\exp(\mathrm{i}k_{P1}L) = (-1)^{N-1} \exp\left(\mathrm{i}\sum_{j=1}^{N} \theta_{P1,Pj}\right) \quad \text{for arbitrary } P, \tag{2.68}$$

where we have used that $\theta_{jj} = 0$. We conclude that the wave numbers k_1, k_2, ..., k_N are quantized according to the set of N coupled Bethe equations

$$\begin{aligned} k_j L &= 2\pi I_j + \sum_{\substack{l=1 \\ (l \neq j)}}^{N} \theta(k_j - k_l) \\ &= 2\pi I_j - \sum_{l=1}^{N} 2 \arctan\left(\frac{k_j - k_l}{c}\right), \qquad j = 1, 2, \ldots, N, \end{aligned} \tag{2.69}$$

where

$$I_j = \begin{cases} 0, \pm 1, \pm 2, \ldots, & \text{if } N\text{=odd}, \\ \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots, & \text{if } N\text{=even}. \end{cases} \tag{2.70}$$

Only solutions with distinct wave numbers are allowed in order to avoid the nullity of the wavefunction. With regard to the equality $\theta(k) + \theta(-k) = 0$, the total particle momentum is given by

$$K = \frac{2\pi}{L} \sum_{j=1}^{N} I_j. \tag{2.71}$$

The set of Bethe ansatz equations (2.69) has the same form for many other integrable systems, only the two-body phase shift $\theta(k)$ depends on the type of the particle interaction in the particular model. The only variables which have to be determined are the quasi-momenta $k_1, k_2, \ldots, k_N$ which number equals to the number of particles.

The wavefunction in the ordering sector $Q \in S_N$ is formally expressible as

$$\psi_Q(x_1, x_2, \ldots, x_N) = \sum_{P \in S_N} [Q, P]_b \exp\left(\mathrm{i}\sum_{j=1}^{N} k_{Pj} x_{Qj}\right), \tag{2.72}$$

where the "boson" symbol $[Q, P]_b$ means

$$[Q, P]_b = \mathrm{sign}(P) A(k_{P1}, k_{P2}, \ldots, k_{PN}). \tag{2.73}$$

The A-coefficients with different permutation P sectors are related via the scattering formula (2.64).

2.4 Bound states for attractive bosons

The general analysis of the set of nonlinear Bethe equations (2.69) is very complicated for a finite circle length L, but simplifies substantially in the limit $L \to \infty$. We shall study this limit in order to show fundamental differences in the energy spectra of the attractive $c < 0$ and repulsive $c > 0$ regimes.

In the case of $N = 1$ particle, it is clear from Eq. (2.20) that the wave number k covers continuously the whole real axis $(-\infty, \infty)$. The energy $E = k^2$ is bounded from below by 0.

For $N = 2$ particles, the Bethe equations can be expressed as follows

$$\exp(\mathrm{i}k_1 L) = \frac{k_1 - k_2 + \mathrm{i}c}{k_1 - k_2 - \mathrm{i}c}, \qquad \exp(\mathrm{i}k_2 L) = \frac{k_1 - k_2 - \mathrm{i}c}{k_1 - k_2 + \mathrm{i}c}. \tag{2.74}$$

Let us first look for the real solutions for momenta. Denoting $(k_1 - k_2 + \mathrm{i}c)/(k_1 - k_2 - \mathrm{i}c) = \exp(\mathrm{i}\varphi)$ with $\varphi \in R$, one has

$$\exp(\mathrm{i}k_1 L) = \exp(\mathrm{i}\varphi), \qquad \exp(\mathrm{i}k_2 L) = \exp(-\mathrm{i}\varphi). \tag{2.75}$$

In the limit $L \to \infty$, k_1 and k_2 again cover continuously the whole real axis and one has the state of two independent particles with the total energy $E = k_1^2 + k_2^2 \geq 0$. The system of two equations (2.74) may also exhibit complex solutions of type

$$k_1 = u_1 + \mathrm{i}v_1, \qquad k_2 = u_2 + \mathrm{i}v_2, \tag{2.76}$$

where all u's and v's are real numbers. Comparing the modulus of both sides of the first equation in (2.74), we get

$$\exp(-2v_1 L) = \frac{(u_1 - u_2)^2 + (v_1 - v_2 + c)^2}{(u_1 - u_2)^2 + (v_1 - v_2 - c)^2}. \tag{2.77}$$

Let us assume that $v_1 > 0$. Then, as $L \to \infty$, the lhs of Eq. (2.77) goes to 0 and, consequently,

$$u_1 = u_2 \equiv u, \qquad v_1 - v_2 + c = 0. \tag{2.78}$$

The multiplication of the two equation in (2.74) implies that $\exp[\mathrm{i}(k_1 + k_2)L] = 1$. In the limit $L \to \infty$, the possible values of u cover the whole real axis and $v_1 + v_2 = 0$, so that $v_1 = -c/2$. Since $v_1 > 0$, the resulting complex solution

$$k_1 = u + \mathrm{i}\frac{|c|}{2}, \qquad k_2 = u - \mathrm{i}\frac{|c|}{2}, \qquad u \in (-\infty, \infty) \tag{2.79}$$

exists only in the attractive case $c < 0$. The total momentum is $K = k_1 + k_2 = 2u$ and the energy

$$E = k_1^2 + k_2^2 = \frac{K^2}{2} - \frac{c^2}{2}. \tag{2.80}$$

This is a bound state of two particles because the mass is doubled and the binding energy is $-c^2/2$. The ground state with the minimum energy $-c^2/2$ is given by $K = 0$, the corresponding (symmetrized) wavefunction reads

$$\psi(x_1, x_2) = \exp\left(\frac{c}{2}|x_1 - x_2|\right). \tag{2.81}$$

The complex roots of the Bethe ansatz equations for an infinite line are usually symmetrically and equidistantly distributed around the real axis. They are called strings; the simplest 1-string is the point on the real axis $k_1 = u$, the 2-string is the complex solution of type (2.79), etc.

In the three-body problem, the wave numbers k_1, k_2 and k_3 can also be real numbers (for both $c > 0$ and $c < 0$) or complex numbers with a nonzero imaginary part (exclusively for $c < 0$). In the later case, they can form either a mixture of one 2-string and one 1-string

$$k_1 = u + \mathrm{i}\frac{|c|}{2}, \qquad k_2 = u - \mathrm{i}\frac{|c|}{2}, \qquad k_3 = u' \tag{2.82}$$

(u and u' cover the whole real axis) or just one string of length 3

$$k_1 = u + \mathrm{i}|c|, \qquad k_2 = u, \qquad k_3 = u - \mathrm{i}|c|, \tag{2.83}$$

which is a bound state of three particles with the total momentum $K = 3u$ and the energy $E = K^2/3 - 2c^2$.

For N attractive bosons [59], there exists one N-string

$$k_j = u + \mathrm{i}\frac{|c|}{2}(N + 1 - 2j), \qquad j = 1, \ldots, N \tag{2.84}$$

with the total momentum $K = Nu$ and the energy

$$E = \frac{K^2}{N} - N(N^2 - 1)\frac{c^2}{12}. \tag{2.85}$$

The minimum of the energy at $K = 0$ is the ground state energy in the sector of N particles, the corresponding symmetrized wavefunction reads

$$\psi = \exp\left(\frac{c}{2}\sum_{j<l}|x_j - x_l|\right). \tag{2.86}$$

The ground state energy is of order N^3 which corresponds to a kind of "attraction collapse" of bosons with no thermodynamic limit. The ground state would be well behaved if we make the negative c proportional to $1/N$ and then let $N \to \infty$. The situation is different in the repulsive case $c > 0$, where the ground state energy is of order N.

2.5 Uniqueness of the Bethe ansatz solution for repulsive bosons

In what follows, we shall reconsider the case of an arbitrary length L and restrict ourselves to the repulsive regime $c > 0$.

We claim that for any set of quantum numbers $\{I_1, I_2, \ldots, I_N\}$ the system of Bethe equations (2.69) has a unique real solution for the momenta $\{k_1, k_2, \ldots, k_N\}$. To prove this statement, we introduce the function

$$B(k_1, k_2, \ldots, k_N) = \frac{L}{2}\sum_{j=1}^{N} k_j^2 - 2\pi\sum_{j=1}^{N} I_j k_j + \sum_{j>l=1}^{N} \Theta(k_j - k_l), \tag{2.87}$$

where

$$\Theta(k) = \int_0^k \mathrm{d}k'\, \theta(k') = 2k \arctan\left(\frac{k}{c}\right) - c \ln\left[1 + \left(\frac{k}{c}\right)^2\right]. \tag{2.88}$$

The extremal point of the B-function, given by

$$\frac{\partial B(k_1, k_2, \ldots, k_N)}{\partial k_j} = 0, \qquad j = 1, 2, \ldots, N, \tag{2.89}$$

is the solution of the Bethe ansatz equations (2.69). Let us now consider the $N \times N$ matrix of second derivatives of the B-function:

$$B_{jl} \equiv \frac{\partial^2 B}{\partial k_j \partial k_l} = \delta_{jl}\left[L + \sum_m \frac{2c}{c^2 + (k_j - k_m)^2}\right] - \frac{2c}{c^2 + (k_j - k_l)^2}. \tag{2.90}$$

Since for an arbitrary real vector $(u_1, u_2, \ldots, u_N)$ it holds

$$\sum_{jl} u_j B_{jl} u_l = L \sum_j u_j^2 + \sum_{j>l} \frac{2c}{c^2 + (k_j - k_l)^2}(u_j - u_l)^2 \geq 0, \tag{2.91}$$

the matrix B_{jl} is positive definite and the B-function is a concave function in the N-dimensional k-space. Moreover, for large values of momenta the B-function behaves like $L\sum_j k_j^2/2$ and so it has one and only one extremum, namely a minimum. The position of this minimum in the k-space depends, for a fixed set of quantum numbers $\{I_j\}$, on the value of c.

A convenient reference value of c is the limit $c \to \infty$ ($c^{-1} = 0$), when the phase shift $\theta(k) = 0$. According to (2.69), the extremal (minimum) point occurs at

$$\lim_{c\to\infty} k_j = \frac{2\pi}{L} I_j, \qquad j = 1, 2, \ldots, N. \tag{2.92}$$

From Eq. (2.65) we have $A(P) = 1$, so that the wavefunction (2.58) in the fundamental ordering sector I reads

$$\psi_I(x_1, x_2, \ldots, x_N) = \operatorname*{Det}_{1\leq j,l\leq N} \exp(\mathrm{i}k_j x_l). \tag{2.93}$$

The determinant on the rhs of this equation is the familiar Slater determinant of free fermions with momenta (2.92). The wavefunction is nonzero only when no two k's are identical; the quantum numbers $\{I_j\}$ and the corresponding momenta $\{k_j\}$ can be ordered as follows

$$I_1 < I_2 < \cdots < I_N, \qquad \frac{2\pi}{L} I_1 = k_1 < k_2 < \cdots < k_N = \frac{2\pi}{L} I_N. \tag{2.94}$$

The interacting bosons have in the limit $c \to \infty$ paradoxically the energy spectrum of free fermions; we shall refer to such bosons as the (pointlike) hard-core bosons. The symmetrized wavefunctions of the hard-core bosons *is not* equal to that of the corresponding free fermions, but

$$\Psi_{\text{hard-core bosons}} = |\Psi_{\text{free fermions}}|. \tag{2.95}$$

These wavefunctions constitute a complete set of eigenfunctions for the boson Hamiltonian.

Since the matrix (2.90) is positive definite, its determinant is nonzero. The minimum point thus moves continuously in k-space as c is changing from ∞ to a positive finite value. The subtraction of two Bethe equations (2.69) with different indices $j \neq j'$ implies

$$\begin{aligned} |(k_j - k_{j'})L - 2\pi(I_j - I_{j'})| &= 2\sum_{l=1}^{N} \left| \arctan\left(\frac{k_j - k_l}{c}\right) - \arctan\left(\frac{k_{j'} - k_l}{c}\right) \right| \\ &\leq \frac{2N}{c}|k_j - k_{j'}|, \end{aligned} \tag{2.96}$$

where we used the inequality $|\arctan x - \arctan y| \leq |x - y|$. Since $|I_j - I_{j'}| \geq 1$, from Eq. (2.96) we find that

$$|k_j - k_{j'}| \geq \frac{2\pi}{L}\left(1 + \frac{2n}{c}\right)^{-1} \tag{2.97}$$

with $n = N/L$ being the particle density. The differences $|k_j - k_{j'}|$ therefore cannot turn to 0 for an arbitrary $c > 0$, which means that the ordering of momenta (2.94) is preserved. It is also clear from the structure of the Bethe equations (2.69) that the lower and upper bounds for momenta in (2.94) do not change for finite $c > 0$, so that

$$\frac{2\pi}{L}I_1 < k_1 < k_2 < \cdots < k_N < \frac{2\pi}{L}I_N, \qquad k_{j+1} - k_j \geq \frac{2\pi}{L}\left(1 + \frac{2n}{c}\right)^{-1}. \tag{2.98}$$

The possible sets of quantum numbers $\{I_j\}$ are the same as in the limit $c \to \infty$ and so the eigenfunctions constitute a complete set of the boson Hamiltonian.

2.6 Ground-state energy

In the limit $c \to \infty$, the ground state of N repulsive bosons with the lowest energy can be easily determined. With regard to the permissible values of the momenta (2.92) no two of which are equal, the minimum of the energy $E = \sum_{j=1}^{N} k_j^2$ is attained by an ordered sequence of quantum I-numbers with a unity elementary step, distributed symmetrically around 0:

$$-I_{\max}, -I_{\max} + 1, \cdots, I_{\max} - 1, I_{\max}.$$

$I_{\max}$ is determined by the obvious equality for the number of momenta $N = 2I_{\max} + 1$. Thus, the sequence of quantum numbers corresponding to the ground state is

$$\{I_1, I_2, \ldots, I_N\} = \left\{-\frac{N-1}{2}, -\frac{N-1}{2} + 1, \cdots, \frac{N-1}{2}\right\}. \tag{2.99}$$

The quantum numbers $\{I_j\}$ are integers for odd N and half-odd integers for even N, in agreement with the requirement (2.70). The total momentum is $K_0 = 0$ and the ground-state energy reads

$$E_0 = \left(\frac{2\pi}{L}\right)^2 \sum_{j=1}^{N} I_j^2 = \frac{\pi^2}{3L^2}(N-1)N(N+1). \tag{2.100}$$

The Slater determinant (2.93) for the ground state (2.99) is

$$\psi_I(x_1, x_2, \ldots, x_N) = \underset{1\le j,k\le N}{\mathrm{Det}} \exp\left[\mathrm{i}\pi(2j-1-N)x_k/L\right]. \tag{2.101}$$

With $z_k = \exp(\mathrm{i}2\pi x_k/L)$, we rewrite the wavefunction as

$$\psi_I = \exp\left[\mathrm{i}\pi\frac{(1-N)}{L}\sum_{k=1}^{N} x_k\right] \underset{1\le j,k\le N}{\mathrm{Det}} z_k^{j-1}. \tag{2.102}$$

The determinant in this expression is a van der Monde determinant,

$$\underset{1\le j,k\le N}{\mathrm{Det}} z_k^{j-1} = \prod_{j>k=1}^{N} (z_j - z_k). \tag{2.103}$$

The antisymmetrized/symmetrized ground-state wavefunctions of free fermions/(pointlike) hard-core bosons are thus expressible as

$$\Psi_0 = \prod_{j>k=1}^{N} \psi(x_j - x_k), \qquad \psi(x) = \begin{cases} \sin(\pi x/L) & \text{for free fermions,} \\ \sin(\pi|x|/L) & \text{for hard-core bosons.} \end{cases} \tag{2.104}$$

The product form of the ground-state wavefunctions is appealing. Note that the function $\psi(x)$ has no nodes in the interval $0 < x < L$, as it must be in the ground state.

Based on the continuity arguments presented in the previous subsection 2.5, the same set of quantum I-numbers (2.99) determines the ground state for an arbitrary interaction amplitude $c > 0$. Let us rewrite the Bethe equations (2.69) as follows

$$k_j = 2\pi\frac{I_j}{L} + \frac{1}{L}\sum_{l=1}^{N}\theta(k_j - k_l), \qquad j = 1, 2, \ldots, N \tag{2.105}$$

and consider the thermodynamic limit $L, N \to \infty$, keeping the particle density $n = N/L$ finite. Following Hultén's continualization procedure [3], we introduce the state density

$$f_j = \frac{I_j}{L}, \qquad f \in \left\langle -\frac{n}{2}, \frac{n}{2} \right\rangle. \tag{2.106}$$

For large L,

$$L\,\mathrm{d}f = \text{number of } I\text{'s in the interval } (f, f+\mathrm{d}f). \tag{2.107}$$

We assume from the structure of the Bethe equations (2.105) that the difference $k_{j+1} - k_j > 0$ is of order $1/L$ and define

$$k_{j+1} - k_j = \frac{1}{L\rho(k_j)}, \qquad \rho(k) \ge 0. \tag{2.108}$$

The meaning of the distribution $\rho(k)$ is that, for large L,

$$L\,\rho(k)\mathrm{d}k = \text{number of } k\text{'s in the interval } (k, k+\mathrm{d}k). \tag{2.109}$$

$\rho(k)$ can be understood as the ground-state particle density in k-space. Since it holds

$$f_{j+1} - f_j = \frac{I_{j+1} - I_j}{L} = \frac{1}{L} = \rho(k_j)\,(k_{j+1} - k_j)\,, \tag{2.110}$$

we have $\mathrm{d}f = \rho(k)\mathrm{d}k$. Equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}k} f(k) = \rho(k), \qquad f(k) = \int_0^k \mathrm{d}k' \rho(k'). \tag{2.111}$$

With respect to the inequalities (2.98), we expect the k's to be distributed symmetrically with the density $\rho(k) = \rho(-k)$ between some as-yet unspecified limits $-q$ and q. Thus, we can replace any summation over k by an integral over k according to

$$\sum_k \cdots \to L \int_{-q}^{q} \mathrm{d}k\, \rho(k) \cdots. \tag{2.112}$$

Since there are exactly N momenta, it holds $N = L \int_{-q}^{q} \mathrm{d}k \rho(k)$, or

$$n \equiv \frac{N}{L} = \int_{-q}^{q} \mathrm{d}k\, \rho(k). \tag{2.113}$$

It is tempting to refer to q as the Fermi momentum. However, the accepted definition of the Fermi momentum, even for an interacting fermion system, is $k_F = \pi n$. The momentum limit q of the integrals is equal to πn only in the free fermion (hard-core boson) limit $c \to \infty$. For a fixed finite $c > 0$, the dependence of q on the particle density n is determined by the condition (2.113). It will be shown that q is related to the chemical potential of bosons. As concerns the ground-state momentum density, it vanishes

$$\frac{K_0}{L} = \int_{-q}^{q} \mathrm{d}k\, k\rho(k) = 0. \tag{2.114}$$

The ground-state energy per unit length is expressible as

$$e_0 \equiv \frac{E_0}{L} = \int_{-q}^{q} \mathrm{d}k\, k^2 \rho(k). \tag{2.115}$$

To obtain an equation for the density ρ of the k's in the ground state, we first continualize the Bethe equations (2.105),

$$\begin{aligned} k &= 2\pi f(k) + \frac{1}{L} \sum_{k'} \theta(k - k') \\ &= 2\pi f(k) + \int_{-q}^{q} \mathrm{d}k'\, \theta(k - k')\rho(k'). \end{aligned} \tag{2.116}$$

The subsequent differentiation of these equations with respect to k leads to

$$\frac{1}{2\pi} = \rho(k) + \int_{-q}^{q} \mathrm{d}k' \frac{\theta'(k - k')}{2\pi} \rho(k'), \qquad |k| \le q, \tag{2.117}$$

where

$$\theta'(k) \equiv \frac{\partial\theta(k)}{\partial k} = -\frac{2c}{k^2+c^2}. \tag{2.118}$$

We make the substitutions $k = qp$ and $k' = qp'$ leading to the dimensionless variables p and p', redefine $\rho(qp) \equiv \tilde{\rho}(p)$ and finally put $\lambda = c/q$, to obtain

$$\tilde{\rho}(p) = \frac{1}{2\pi} + \frac{1}{\pi}\int_{-1}^{1} \mathrm{d}p' \, \frac{\lambda}{(p-p')^2+\lambda^2}\tilde{\rho}(p'), \qquad |p| \le 1. \tag{2.119}$$

The energy density (2.115) and the particle density (2.113) are now expressible as

$$e_0 = q^3 E(\lambda), \qquad E(\lambda) \equiv \int_{-1}^{1} \mathrm{d}p \, p^2 \tilde{\rho}(p), \tag{2.120}$$

$$n = qF(\lambda), \qquad F(\lambda) \equiv \int_{-1}^{1} \mathrm{d}p \, \tilde{\rho}(p). \tag{2.121}$$

We can eliminate from the formalism the momentum limit q by considering the combinations

$$\frac{e_0}{n^3} = \frac{E(\lambda)}{F^3(\lambda)}, \qquad \frac{c}{n} = \frac{\lambda}{F(\lambda)}. \tag{2.122}$$

Thus,

$$e_0 = n^3 u\left(\frac{c}{n}\right) \tag{2.123}$$

with a certain function of interest $u(x)$.

Eq. (2.119) is an inhomogeneous Fredholm equation of the second kind which exhibits exactly one solution $\tilde{\rho}(p)$ [21,22]. Although it cannot be solved explicitly for an arbitrary $\lambda > 0$, systematic perturbation expansions can be constructed around two limits.

• In the limit $\lambda \to \infty$ ($c \to \infty$, hard-core bosons), Eq. (2.119) tells us that

$$\tilde{\rho}(p) \underset{\lambda\to\infty}{\sim} \frac{1}{2\pi}, \qquad u(x) \underset{x\to\infty}{\sim} \frac{\pi^2}{3}. \tag{2.124}$$

Note that the asymptotic value of u is in agreement with the previous result (2.100). Substituting the $\lambda \to \infty$ value of $\tilde{\rho}(p)$ into the rhs of Eq. (2.119), we obtain its leading $1/\lambda$ correction, and so on. The result of the iteration procedure for the function of interest $u(x)$ is

$$u(x) = \frac{\pi^2}{3}\left[1 - \frac{4}{x} + \frac{12}{x^2} + O\left(\frac{1}{x^3}\right)\right]. \tag{2.125}$$

This series expansion is convergent for $x > 2$.

• The limit $\lambda \to 0$ ($c \to 0$, free bosons) is tricky. As $\lambda \to 0$, the kernel in Eq. (2.119) becomes a well-known representation of the δ-function,

$$\lim_{\lambda\to 0} \frac{1}{\pi}\frac{\lambda}{(p-p')^2+\lambda^2} = \delta(p-p'). \tag{2.126}$$

The considered equation then takes the form $\tilde{\rho}(p) = 1/(2\pi) + \tilde{\rho}(p)$, which is the indication of the divergence of $\tilde{\rho}(p)$ as $\lambda \to 0$.

To treat the $c \to 0$ limit correctly, we apply the equality

$$\arctan x + \arctan(1/x) = \begin{cases} \pi/2 & \text{for } x > 0, \\ -\pi/2 & \text{for } x < 0, \end{cases} \tag{2.127}$$

to all θ-functions in the discrete Bethe ansatz equations (2.69) with the momenta ordering $k_1 < k_2 < \cdots < k_N$. We obtain

$$k_j L = 2\pi n_j + \sum_{\substack{l=1 \\ (l \neq j)}}^{N} 2 \arctan\left(\frac{c}{k_j - k_l}\right), \qquad j = 1, 2, \ldots, N \tag{2.128}$$

with

$$n_j = I_j + \frac{1}{2}(N+1) - j. \tag{2.129}$$

For the ground state defined by the quantum numbers (2.99), $n_j = 0$ for all j. In the vicinity of $c = 0$, we look for a solution of the type

$$k_j = \sqrt{\frac{2c}{L}} x_j + O(c). \tag{2.130}$$

From Eq. (2.128), all the x_j have to be distinct and must satisfy

$$x_j = \sum_{\substack{l=1 \\ (l \neq j)}} \frac{1}{x_j - x_l}. \tag{2.131}$$

The x_j are the zeros of the Hermite polynomial of degree N. This can be shown by inserting the representation $H_N(x) = C_N \prod_{j=1}^{N} (x - x_j)$ into the differential equation satisfied by this Hermite polynomial,

$$H_N''(x) - 2x H_N'(x) + 2N H_N(x) = 0. \tag{2.132}$$

The density of zeros of $H_N(x)$ is given in the asymptotic limit $N \to \infty$ by the Wigner semicircle law,

$$L\rho_H(x) = \frac{1}{\pi}\sqrt{2N - x^2}. \tag{2.133}$$

Thus,

$$\rho(k) = \rho_H(x)\frac{\mathrm{d}x}{\mathrm{d}k} = \frac{1}{2\pi c}\sqrt{4cn - k^2}. \tag{2.134}$$

Taking into account that the momentum limit $q = 2\sqrt{cn}$ and $\lambda = c/q$, we finally arrive at

$$\tilde{\rho}(p) \underset{\lambda \to 0}{\sim} \frac{1}{2\pi\lambda}\sqrt{1 - p^2}, \qquad u(x) \underset{x \to 0}{\sim} x. \tag{2.135}$$

The Fredholm integral equation (2.119) is known in the electrostatic potential theory as the Love equation for the circular disk condenser, consisting of two circular metallic disks separated by distance λ [60]. This analogy enables one to calculate next terms of the expansion of $u(x)$ around $x = 0$ [61]:

$$u(x) = x - \frac{4}{3\pi}x^{3/2} + \left(\frac{1}{6} - \frac{1}{\pi^2}\right)x^2 + O(x^{5/2}). \tag{2.136}$$

3 Bethe ansatz: zero-temperature thermodynamics and excited states

In the previous sections, we have derived the (fundamental) set of Bethe equations (2.69) for bosons with δ-function interaction, formulated on the circle of length L. For many other integrable systems, the Bethe ansatz equations have the analogous form

$$k_j L = 2\pi I_j + \sum_{\substack{l=1 \\ (l\neq j)}}^{N} \theta(k_j - k_l), \qquad j = 1, \ldots, N, \tag{3.1}$$

where the phase-shift function $\theta(k)$ depends on the particular type of particle interaction and $\{I_j\}_{j=1}^N$ is a set of distinct integers or half-odd integers. We shall often use the abbreviated form of the Bethe equations

$$kL = 2\pi I(k) + \sum_{k'} \theta(k - k'), \tag{3.2}$$

where the term $k' = k$ is omitted from the summation; this restriction is superfluous for the usual case with $\theta(0) = 0$.

All studied models possess a special point, which corresponds to either free fermions or pointlike hard-core bosons. At this point, the ground state is characterized by the quantum numbers $I(k)$ symmetrically distributed around 0 and all k's are real. Based on the continuity arguments, the ground state is characterized by the same set of quantum numbers $I(k)$ also in a region outside of the special point. In the thermodynamic limit, the probability distribution of real k's, $\rho(k)$, satisfies the integral Eq. (2.117). To simplify the notation, this equation will be written in a compact form

$$\frac{1}{2\pi} = \rho + G\rho = (I + G)\rho. \tag{3.3}$$

Here, the functions of k, like the constant $1/(2\pi)$ and $\rho(k)$, are understood as column vectors, I is the identity operator and G is an integral operator with a real, symmetric kernel $G(k, k') = \theta'(k-k')/(2\pi)$, over the interval $-q \leq k' \leq q$. To avoid confusion, we shall denote the constant function 1 as the vector η, i.e. $1/(2\pi) \equiv \eta/(2\pi)$. Let us introduce the resolvent operator J such that $I + J$ is the inverse operator of $I + G$,

$$(I + J)(I + G) = (I + G)(I + J) = I. \tag{3.4}$$

The operator J is an integral operator with real, symmetric kernel $J(k, k')$ which is not a difference kernel like $G(k, k')$. From Eq. (3.3), we express formally the distribution ρ as

$$\rho = (I + J)\frac{\eta}{2\pi}. \tag{3.5}$$

Within the proposed notation, the relations for the ground-state number density (2.113), momentum density (2.114) and energy density (2.115) can be written compactly as

$$\frac{N}{L} = \eta^+\rho, \qquad \frac{K_0}{L} = k^+\rho = 0, \qquad \frac{E_0}{L} = (k^2)^+\rho, \tag{3.6}$$

where the upperscript $^+$ denotes a row vector.

3.1 Response of the ground state

Let us study the response of the system in the ground state to an external perturbation $\phi(k)$, say a phase shift coming from the scattering on an impurity, which induces a phase factor $\exp[-\mathrm{i}\phi(k)]$ to each particle. This perturbation causes the real ground-state momenta k to shift by a small amount $\Delta(k)$ of the order $1/L$, $k \to k + \Delta(k)$. The Bethe equations (3.2) for the shifted k's become

$$[k + \Delta(k)]\, L = 2\pi I(k) + \sum_{k'} \theta[k + \Delta(k) - k' - \Delta(k')] + \phi[k + \Delta(k)]. \tag{3.7}$$

Expanding this equation in the small quantity $\Delta(k)$ and subtracting Eq. (3.2), we obtain

$$\Delta(k)L = \sum_{k'} \theta'(k - k')\,[\Delta(k) - \Delta(k')] + \phi(k). \tag{3.8}$$

The replacement of the summation by an integral results in

$$\Delta(k)L = \Delta(k)L \int_{-q}^{q} \mathrm{d}k'\; \theta'(k - k')\rho(k') - \int_{-q}^{q} \mathrm{d}k'\; \theta'(k - k')\rho(k')\Delta(k')L + \phi(k). \tag{3.9}$$

With regard to Eq. (2.117) for $\rho(k)$, this equation reduces to

$$2\pi\rho(k)\Delta(k)L + \int_{-q}^{q} \mathrm{d}k'\; \theta'(k - k')\rho(k')\Delta(k')L = \phi(k). \tag{3.10}$$

Introducing the new function $\omega(k) = \rho(k)\Delta(k)L$, the response of the system to the perturbation ϕ is described by

$$(I + G)\omega = \frac{\phi}{2\pi}. \tag{3.11}$$

The formal solution for ω is

$$\omega = (I + J)\frac{\phi}{2\pi}. \tag{3.12}$$

Let us explore the effect of the perturbation ϕ on the extensive thermodynamic quantities N, K and E. We did not change the number of particles in the system, so that

$$\Delta N = 0. \tag{3.13}$$

The change of the total momentum K is given by the sum of k's shifts,

$$\Delta K = \sum_{k} \Delta(k) \to L \int_{-q}^{q} \mathrm{d}k\; \rho(k)\Delta(k) = \int_{-q}^{q} \mathrm{d}k\; \omega(k) = \eta^{+}\omega. \tag{3.14}$$

Using the formal solution (3.12) for ω, this expression can be rewritten as

$$\Delta K = \eta^{+}(I + J)\frac{\phi}{2\pi} = \phi^{+}(I + J)\frac{\eta}{2\pi} = \phi^{+}\rho. \tag{3.15}$$

Here, we have used the symmetricity of the kernel $I(k,k')+J(k,k')$. As concerns the total energy, since

$$E=\sum_k [k+\Delta(k)]^2 \sim \sum_k k^2+\sum_k 2k\Delta(k), \tag{3.16}$$

its change is expressed by

$$\begin{aligned}\Delta E &= \sum_k 2k\Delta(k) \to L\int_{-q}^{q} \mathrm{d}k\; 2k\Delta(k)\rho(k) = \int_{-q}^{q}\mathrm{d}k\; 2k\omega(k)\\ &= k^+(2\omega)=k^+(I+J)\frac{\phi}{\pi}=\phi^+(I+J)\frac{k}{\pi}.\end{aligned} \tag{3.17}$$

3.2 Zero-temperature thermodynamics

To study the zero-temperature thermodynamics, it is useful to define a new function $\epsilon(k)$ with $|k|\le q$ as the solution of the integral equation

$$(I+G)\epsilon = k^2-\mu. \tag{3.18}$$

$\epsilon(k)$ is an even function of k. The constant μ is chosen such that $\epsilon(k)$ vanishes at the limits $\pm q$,

$$\epsilon(-q)=\epsilon(q)=0. \tag{3.19}$$

The formal solution for $\epsilon(k)$ reads

$$\epsilon=(I+J)k^2-(I+J)\mu=(I+J)k^2-2\pi\mu\rho. \tag{3.20}$$

It is easy to show from Eq. (3.18), by integration by parts and using $\epsilon(\pm q)=0$, that $\epsilon'(k)$, being an odd function of k, satisfies the relation

$$(I+G)\epsilon'=2k. \tag{3.21}$$

The formula (3.17) for ΔE can be thus rewritten as

$$\Delta E=\phi^+\frac{\epsilon'}{2\pi}=-\epsilon^+\frac{\phi'}{2\pi}. \tag{3.22}$$

In order to reveal the physical meaning of the introduced parameter μ, we return to the basic formula for the ground-state energy in Eq. (3.6) and rewrite it in terms of the new function $\epsilon(k)$,

$$\begin{aligned}\frac{E_0}{L} &= (k^2)^+\rho=(k^2)^+(I+J)\frac{\eta}{2\pi}=\eta^+(I+J)\frac{k^2}{2\pi}\\ &= \eta^+\frac{\epsilon+2\pi\mu\rho}{2\pi}=\eta^+\frac{\epsilon}{2\pi}+\mu\frac{N}{L}.\end{aligned} \tag{3.23}$$

Our N-particle system of length L is in the ground state or, equivalently, at zero temperature with zero entropy. The Gibbs relation reads

$$E_0=-PL+\mu N, \tag{3.24}$$

where P is the pressure and μ is the chemical potential. Comparing this relation with Eq. (3.23), the pressure is given by

$$P = -\eta^+ \frac{\epsilon}{2\pi} \equiv -\frac{1}{2\pi} \int_{-q}^{q} \mathrm{d}k\, \epsilon(k) \tag{3.25}$$

and the parameter μ introduced in the definition (3.18) of $\epsilon(k)$ is nothing but the chemical potential of particles, i.e. energy necessary to add one particle to the system.

The consistency of the formalism can be checked by increasing the size of the system $L \to L + \Delta L$, which manifests itself in the Bethe equations (3.2) as a perturbation $\phi(k) = -k\Delta L$. Using Eq. (3.22), the change of the system length implies the following change of the energy

$$\Delta E = \epsilon^+ \frac{\eta}{2\pi} \Delta L = \eta^+ \frac{\epsilon}{2\pi} \Delta L. \tag{3.26}$$

As follows from the Gibbs relation (3.24), $\Delta E = -P\Delta L + \mu \Delta N$ ($\Delta N = 0$ in our case). Thus, the pressure P is identified with $-\eta^+\epsilon/(2\pi)$, which is in agreement with the previous result (3.25). The chemical potential μ is given by

$$\mu = \frac{\partial E_0}{\partial N}\Big|_{\text{fixed } L} = \frac{\partial e_0}{\partial n}. \tag{3.27}$$

The general thermodynamic relation $L\mathrm{d}P(T,\mu) = S\mathrm{d}T + N\mathrm{d}\mu$ gives at $T = 0$ (entropy $S = 0$)

$$\frac{\partial P}{\partial \mu} = n. \tag{3.28}$$

This relation follows also from the present formalism. Differentiating the integral Eq. (3.18) with respect to μ and taking into account that $\epsilon(\pm q) = 0$, we obtain

$$(I + G)\frac{\partial \epsilon}{\partial \mu} = -1, \qquad \frac{\partial \epsilon}{\partial \mu} = -2\pi\rho. \tag{3.29}$$

Consequently, we verify from Eq. (3.25) that

$$\frac{\partial P}{\partial \mu} = -\frac{1}{2\pi}\left[\epsilon(q) + \epsilon(-q)\right]\frac{\partial q}{\partial \mu} - \frac{1}{2\pi}\int_{-q}^{q} \mathrm{d}k \frac{\partial \epsilon(k)}{\partial \mu} = \int_{-q}^{q} \mathrm{d}k\, \rho(k) = n. \tag{3.30}$$

3.3 Low-lying excitations

We turn now to the excitation spectrum. To understand its structure, we first consider the trivial point of a noninteracting Fermi gas or a (pointlike) hard-core Bose gas. The uncoupled momenta k's are given by Eq. (2.92), the quantum numbers $I_1, I_2, \ldots, I_N$ of the ground state are distributed according to (2.99). The "Fermi momentum" $q = \pi n$. The spectrum is evidently the Fermi one: an elementary excitation consists in taking a particle from a "hole" state with momentum k_h ($|k_h| < q$) to a "particle" state with momentum k_p above the Fermi level, $|k_p| > q$. The energy change $\Delta E = E - E_0$ and momentum change $\Delta K = K$ (recall that $K_0 = 0$) of this state with respect to the ground state are

$$\Delta E(k_h, k_p) = k_p^2 - k_h^2, \qquad K(k_h, k_p) = k_p - k_h. \tag{3.31}$$

This spectrum describes adequately excitations in Fermi gases, but it is different from what we would expect for Bose gases. The difficulty is that every excitation is described in terms of two parameters instead of the one as is anticipated for Boson systems; there is no unique $\Delta E(K)$ dispersion curve. Our main task in describing the Fermi spectrum in boson terms is to find a small number of elementary energy levels from which the others can be constructed by addition.

In order to make the Fermi spectrum appear boson-like, we define two types of elementary one-parameter excitations.

• Type I "particle excitations": Take a particle from q to $k_p > q$, or alternatively from $-q$ to $k_p < -q$. This state has an energy and momentum given by

$$\Delta E(k_p) = k_p^2 - q^2, \qquad K(k_p) = \begin{cases} k_p - q & \text{for } k_p > q, \\ k_p + q & \text{for } k_p < -q. \end{cases} \tag{3.32}$$

The dispersion relation is obtained by eliminating k_p between these equations,

$$\Delta E_p(K) = K^2 + 2\pi n|K|, \qquad -\infty < K < \infty. \tag{3.33}$$

Note that although we refer to this type of excitations as to the particle ones, no particle was added into the system ($\Delta N = 0$).

• Type II "hole excitations": Take a particle from $0 < k_h < q$ to $q + 2\pi/L$, or alternatively from $-q < k_h < 0$ to $-q - 2\pi/L$. This state has an energy and momentum given by

$$\Delta E(k_h) = q^2 - k_h^2, \qquad K(k_h) = \begin{cases} q - k_h & \text{for } 0 < k_h < q, \\ -q - k_h & \text{for } -q < k_h < 0. \end{cases} \tag{3.34}$$

The dispersion relation thus reads

$$\Delta E_h(K) = 2\pi n|K| - K^2, \qquad -\pi n < K < \pi n. \tag{3.35}$$

Note that type II excitations are defined only for momenta with the absolute value less than πn.

With the proposed description, we have achieved a boson-like spectrum at the expense of introducing two, particle and hole, dispersion curves. Any type I excitation can be repeated as many times as desired, with the proviso that we take the last available k below q. The same holds for a type II excitation. Both type I and II excitations may occur simultaneously; the fermion-type excitation (3.31) may be generated by a successive application of type I and II excitations say with momenta $k_p - q$ and $q - k_h$, respectively. When we make a finite number of excitations, the excitation energies and momenta will be additive to order $1/N$. Namely, if we make n type I excitations with momenta $K_1, \ldots, K_n$ and m type II excitations with momenta $K'_1, \ldots, K'_m$, the energy shift will be

$$\Delta E = \sum_{j=1}^{n} \Delta E_p(K_j) + \sum_{j=1}^{m} \Delta E_h(K'_j), \tag{3.36}$$

and the total momentum will be

$$K = \sum_{j=1}^{n} K_j + \sum_{j=1}^{m} K'_j. \tag{3.37}$$

When model's parameters lie outside of the trivial free-fermion or hard-core boson point, the picture of excitations is qualitatively the same. The only technical difficulty is that, since all k's are now coupled via the phase-shift functions, creating a particle or hole excitation will shift of all momenta. These shifts can be computed with the aid of the response technique presented in subsection 3.1. There is a minor complication. The functions of primary importance $\rho(k)$ and $\epsilon(k)$ are defined by the integral equations (3.3) and (3.18), respectively, only in the momentum domain $|k| \leq q$. However, the momenta of particle excitations $|k_p| \geq q$, so we use the analytic continuations of these integral equations to define $\rho(k)$ and $\epsilon(k)$ for all real values of k, including $|k| > q$:

$$\rho(k) \equiv \frac{1}{2\pi} - \int_{-q}^{q} \mathrm{d}k' \frac{\theta'(k-k')}{2\pi} \rho(k'), \qquad -\infty < k < \infty; \tag{3.38}$$

$$\epsilon(k) \equiv k^2 - \mu - \int_{-q}^{q} \mathrm{d}k' \frac{\theta'(k-k')}{2\pi} \epsilon(k'), \qquad -\infty < k < \infty. \tag{3.39}$$

Creating a type I particle excitation, say at $k_p > q$, produces the perturbation

$$\phi(k) = \theta(k-k_p) - \theta(k-q) \tag{3.40}$$

in the Bethe equations. According the the response Eq. (3.22), the energy change is given by

$$\Delta E(k_p) = k_p^2 - q^2 - \epsilon^+ \frac{\phi'}{2\pi} = \epsilon(k_p), \tag{3.41}$$

where we used the extended definition (3.39) of $\epsilon(k)$ and the fact that $\epsilon(q) = 0$. This relation gives us a clear physical interpretation of the quantity $\epsilon(k)$ as the excitation energy for an elementary excitation. The momentum is determined by using Eq. (3.15),

$$K(k_p) = k_p - q + \phi^+ \rho = 2\pi \left[f(k_p) - f(q) \right]. \tag{3.42}$$

Here, $f(k)$ is an analytic continuation of the state density, given by Eq. (2.116), to $|k| > q$:

$$f(k) \equiv \frac{k}{2\pi} - \int_{-q}^{q} \mathrm{d}k' \frac{\theta(k-k')}{2\pi} \rho(k'), \qquad |k| > q. \tag{3.43}$$

Comparing this formula with Eq. (3.38) we see that the equality $f'(k) = \rho(k)$ takes place also for $|k| > q$. The dispersion relation $\Delta E_p(K)$ is thus obtained either parametrically or explicitly by eliminating k_p between the expressions for the energy and momentum. When $k_p < q$ we obtain the same results, except for $K \to -K$ which reflects the symmetry $\Delta E(|K|)$.

Similarly, creating a type II hole excitation at k_h, $0 < k_h < q$, the perturbation in the Bethe equations becomes

$$\phi(k) = -\theta(k-k_h) + \theta(k-q). \tag{3.44}$$

The change of the energy and momentum are now given by

$$\Delta E(k_h) = -k_h^2 + q^2 - \epsilon^+ \frac{\phi'}{2\pi} = -\epsilon(k_h), \tag{3.45}$$

$$K(k_h) = -k_h + q + \phi^+ \rho = 2\pi \left[f(q) - f(k_h) \right]. \tag{3.46}$$

These relations define the dispersion curve $\Delta E_h(|K|)$.

We notice that the energy changes $\Delta E(k_p)$ and $\Delta E(k_h)$, being associated with excited states, must be positive. This is equivalent to saying that

$$\begin{array}{ll} \epsilon(k) < 0 & \text{for } |k| < q, \\ \epsilon(k) > 0 & \text{for } |k| > q. \end{array} \tag{3.47}$$

Assuming that $\epsilon(k)$ is a continuous function of k, these inequalities are consistent with the relation (3.19).

The group velocity of the excitations is given by

$$v(k) = \frac{\mathrm{d}(\Delta E)}{\mathrm{d}K} = \frac{\Delta E'}{K'} = \frac{\epsilon'(k)}{2\pi\rho(k)}. \tag{3.48}$$

This expression is valid for both the particle excitations with $|k| \geq q$ and the hole excitations with $|k| \leq q$. As the parameters k_p or k_h approach the momentum q, $\Delta E(k_p)$ or $\Delta E(k_h)$ go continuously to zero, i.e. the energy spectrum is gapless. Since the only long-wavelength, low-energy excitations are localized around q, $v(q)$ is expected to be identical to the velocity of sound v_s,

$$v_s = v(q) = \frac{\epsilon'(q)}{2\pi\rho(q)}. \tag{3.49}$$

3.4 Consistency of zero-temperature thermodynamics

Within the framework of the standard thermodynamics, in the grand canonical ensemble, the velocity of sound v_s is related to the bulk thermodynamic functions by

$$\frac{1}{mv_s^2} = \frac{1}{n}\frac{\partial^2 P}{\partial \mu^2}. \tag{3.50}$$

In this part we shall show that the particle density n and $\partial^2 P/\partial\mu^2$ are expressible in terms of the two quantities $\rho(q)$ and $\epsilon'(q)$, evaluated at the momentum limit q. Consequently, the velocity of sound calculated from Eq. (3.50) will turn out to be identical to the one given by relation (3.49).

We start with the expression for the density n. Integrating by parts, we have

$$n = \eta^+\rho = 2q\rho(q) - k^+\rho'. \tag{3.51}$$

To obtain the equation for ρ', we differentiate Eq. (3.3) for ρ with respect to k, with the result

$$\rho'(k) + \int_{-q}^{q} \mathrm{d}k' \frac{\partial G(k-k')}{\partial k}\rho(k') = 0. \tag{3.52}$$

Transforming the derivative $\partial/\partial k \to -\partial/\partial k'$ and integrating by parts, we find that

$$(I+G)\rho'(k) = [G(k-q) - G(k+q)]\rho(q). \tag{3.53}$$

Multiplying both sides of this equation by the inverse operator of $I+G$, $I+J$, and using that $(I+J)G = -J$, we obtain

$$\rho'(k) = -[J(k,q) - J(k,-q)]\rho(q). \tag{3.54}$$

With regard to the representation (3.51), this relation permits us to express the particle density as

$$n = 2q\rho(q) + \rho(q)[Jk(q) - Jk(-q)]. \tag{3.55}$$

Using Eq. (3.21), we can write

$$Jk = J(I+G)\frac{\epsilon'}{2} = -G\frac{\epsilon'}{2} = \frac{\epsilon'}{2} - k. \tag{3.56}$$

Consequently,

$$n = 2q\rho(q) + \rho(q)\left\{\frac{\epsilon'(q)}{2} - q - \left[\frac{\epsilon'(-q)}{2} + q\right]\right\} = \rho(q)\epsilon'(q). \tag{3.57}$$

The second derivative of the pressure with respect to the chemical potential can be calculated as follows

$$\frac{\partial^2 P}{\partial \mu^2} = \frac{\partial n}{\partial \mu} = \frac{\partial}{\partial \mu}\int_{-q}^{q} \mathrm{d}k\ \rho(k) = 2\rho(q)\frac{\partial q}{\partial \mu} + \int_{-q}^{q} \mathrm{d}k \frac{\partial \rho(k)}{\partial \mu}. \tag{3.58}$$

To obtain the equation for $\partial\rho(k)/\partial\mu$, we differentiate Eq. (3.3) for ρ with respect to μ, with the result

$$\frac{\partial \rho(k)}{\partial \mu} + \rho(q)\frac{\partial q}{\partial \mu}[G(k-q) + G(k+q)] + \int_{-q}^{q} \mathrm{d}k'\ G(k-k')\frac{\partial \rho(k')}{\partial \mu} = 0. \tag{3.59}$$

This equation can be written formally as

$$(I+G)\frac{\partial \rho(k)}{\partial \mu} = -\rho(q)\frac{\partial q}{\partial \mu}[G(k-q) + G(k+q)]. \tag{3.60}$$

The application of the operator $I + J$ to both sides of this equation leads to

$$\frac{\partial \rho(k)}{\partial \mu} = \rho(q)\frac{\partial q}{\partial \mu}[J(k,q) + J(k,-q)]. \tag{3.61}$$

Inserting this representation of $\partial\rho(k)/\partial\mu$ into Eq. (3.58) and using the relation (3.5), we arrive at

$$\frac{\partial^2 P}{\partial \mu^2} = 4\pi\rho^2(q)\frac{\partial q}{\partial \mu}. \tag{3.62}$$

The chemical potential μ and the limit of momentum q are linked through the constraint $\epsilon(q) = 0$. In this constraint, q occurs explicitly, since ϵ is evaluated at $k = q$, and implicitly via μ. The differentiation of the constraint with respect to q thus leads to

$$0 = \epsilon'(q) + \frac{\partial \epsilon(q)}{\partial \mu}\frac{\partial \mu}{\partial q} = \epsilon'(q) - 2\pi\rho(q)\frac{\partial \mu}{\partial q}, \tag{3.63}$$

where we used the formula (3.29). Consequently,

$$\frac{\partial \mu}{\partial q} = \frac{\epsilon'(q)}{2\pi\rho(q)}. \tag{3.64}$$

Substituting this relation into (3.62), we obtain

$$\frac{\partial^2 P}{\partial \mu^2} = \frac{8\pi\rho^3(q)}{\epsilon'(q)}. \tag{3.65}$$

Finally, using (3.57) and (3.65) in the thermodynamic relation (3.50) considered in units of $2m = 1$, we recover the previous result (3.49) for the sound velocity. This confirms the thermodynamic consistency of the zero-temperature formalism of the Bethe ansatz.

3.5 Zero-temperature thermodynamics for repulsive δ-function bosons

We now document the formalism of the zero-temperature thermodynamics on the studied model of bosons with the repulsive δ-function interaction. Due to the scaling form (2.123) of the ground-state energy density e_0, the chemical potential can be calculated from

$$\mu = \frac{\partial e_0}{\partial n} = 3n^2 u\left(\frac{c}{n}\right) - cnu'\left(\frac{c}{n}\right). \tag{3.66}$$

As before, we consider two limits which admit explicit results.

• In the limit of hard-core bosons $c/n \to \infty$, the large-x expansion (2.125) of the function $u(x)$ implies

$$\mu = (\pi n)^2\left[1 - \frac{16}{3}\left(\frac{n}{c}\right) + 20\left(\frac{n}{c}\right)^2 + \cdots\right]. \tag{3.67}$$

In the leading order, we have

$$\mu = (\pi n)^2 = q^2, \qquad e_0 = \frac{\pi^2}{3}n^3. \tag{3.68}$$

Since $\theta'(k) = -2c/(k^2 + c^2)$ goes to 0 in the considered limit, for an arbitrary real k the integral Eqs. (3.38) and (3.39) exhibit trivial solutions

$$\rho(k) = \frac{1}{2\pi}, \qquad \epsilon(k) = k^2 - \mu. \tag{3.69}$$

Note that $\epsilon(k)$ fulfills the requirement $\epsilon(\pm q) = 0$.

The pressure can be calculated from Eq. (3.24) as follows

$$P = \mu n - e_0 = \frac{2}{3}\pi^2 n^3. \tag{3.70}$$

The same result is obtained from the pressure representation (3.25) with $\epsilon(k)$ substituted from Eq. (3.69). The group velocity of excitations (3.48) and the velocity of sound (3.49) are obtained in the form

$$v(k) = 2k, \qquad v_s = 2q = 2\pi n, \tag{3.71}$$

respectively.

• In the free-boson limit $c/n \to 0$, the small-x expansion (2.136) of $u(x)$, when considered in Eq. (3.66), implies

$$\mu = 2cn\left[1 - \frac{1}{\pi}\sqrt{\frac{c}{n}} + \frac{1}{2}\left(\frac{1}{6} - \frac{1}{\pi^2}\right)\frac{c}{n} + \cdots\right]. \tag{3.72}$$

In the leading order, we have

$$\mu = 2cn = \frac{1}{2}q^2, \qquad e_0 = cn^2. \tag{3.73}$$

To derive $\epsilon(k)$, we use the relation (3.29) to obtain

$$\frac{\partial\epsilon}{\partial\mu} = -2\pi\rho = -\frac{1}{c}\sqrt{q^2 - k^2} = -\frac{1}{c}\sqrt{2\mu - k^2}. \tag{3.74}$$

Consequently,

$$\epsilon(k) = -\frac{1}{3c}(2\mu - k^2)^{3/2}, \qquad |k| \le q. \tag{3.75}$$

The group velocity of excitations and the velocity of sound are obtained in the form

$$v(k) = k \qquad (|k| \le q), \qquad v_s = q = 2\sqrt{cn}, \tag{3.76}$$

respectively. The pressure is given by

$$P = \mu n - e_0 = cn^2. \tag{3.77}$$

The same result is attained by inserting the representation (3.75) of $\epsilon(k)$ into Eq. (3.25).

4 Bethe ansatz: finite-temperature thermodynamics

In this section, we shall study the finite-temperature thermodynamics of integrable systems with spectra described by the fundamental set of Bethe equations (3.1).

4.1 The concept of holes

At zero temperature $T = 0$, the particle system is in the ground state characterized by the consecutive sequence of quantum I-numbers given in Eq. (2.99). These quantum numbers

$$\left\{-\frac{N-1}{2}, -\frac{N-3}{2}, \cdots, \frac{N-1}{2}\right\} \tag{4.1}$$

are called the "particle" ones. However, there exist other admissible quantum numbers like

$$\left\{-\frac{N+1}{2}, -\frac{N+3}{2}, \cdots, \frac{N+1}{2}, \frac{N+3}{2}\cdots\right\} \tag{4.2}$$

which are not in the zero-temperature set; they are called the "hole" quantum numbers.

At a finite temperature $T > 0$, due to thermal fluctuations, some of the particle quantum numbers (4.1) become the hole ones and, vice versa, some of the hole quantum numbers (4.2) become the particle ones. Let us denote by $\{I_j\}_{j=1}^N$ a given set of the particle quantum numbers and by $\{\tilde{I}\}$ the infinite set of the remaining hole quantum numbers. In terms of the function $f(k)$ defined by

$$2\pi L f(k) \equiv Lk - \sum_{l=1}^{N} \theta(k - k_l), \tag{4.3}$$

each of the particle momenta k_j is given by

$$Lf(k_j) = I_j \qquad j = 1, 2, \ldots, N. \tag{4.4}$$

On the other hand, a momentum $\tilde{k}$, given by

$$Lf(\tilde{k}) = \tilde{I}, \tag{4.5}$$

is the hole momentum corresponding to the quantum number $\tilde{I}$. Note that the function f, when restricted to the ground state, is equivalent to the f-function introduced in subsection 2.6.

In the thermodynamic limit $L, N \to \infty$ with the fixed particle density $n = N/L$, the particles and holes are distributed as functions of k with densities $\rho(k)$ and $\tilde{\rho}(k)$, respectively:

$$\begin{aligned} &L\rho(k)\mathrm{d}k = \text{number of particles in the interval } (k, k + \mathrm{d}k), \\ &L\tilde{\rho}(k)\mathrm{d}k = \text{number of holes in the interval } (k, k + \mathrm{d}k). \end{aligned} \tag{4.6}$$

The particles and holes cover all admissible values of quantum numbers and so there exists a constraint between the particle and hole densities. Namely, since

$$\begin{aligned} L\left[\rho(k) + \tilde{\rho}(k)\right]\mathrm{d}k &= \text{number of particles and holes in the interval } (k, k + \mathrm{d}k) \\ &= Lf(k + \mathrm{d}k) - Lf(k) = L\mathrm{d}f(k), \end{aligned} \tag{4.7}$$

it holds

$$\rho(k) + \tilde{\rho}(k) = f'(k). \tag{4.8}$$

In the thermodynamic limit, Eq. (4.3) becomes the integral equation for f,

$$2\pi f(k) = k - \int_{-\infty}^{\infty} \mathrm{d}k'\, \theta(k-k')\rho(k'). \tag{4.9}$$

The differentiation of this equation with respect to k leads to

$$\rho(k) + \tilde{\rho}(k) = \frac{1}{2\pi} - \int_{-\infty}^{\infty} \mathrm{d}k'\, \frac{\theta'(k-k')}{2\pi}\rho(k') = \frac{1}{2\pi} - G\rho, \tag{4.10}$$

where G is the same integral operator as in the zero-temperature case, except that the integration limits are extended to $\pm\infty$.

The particle density and the energy per unit length are determined solely by the particle distribution,

$$\frac{N}{L} \equiv n = \int_{-\infty}^{\infty} \mathrm{d}k\, \rho(k), \qquad \frac{E}{L} \equiv e = \int_{-\infty}^{\infty} \mathrm{d}k\, k^2 \rho(k). \tag{4.11}$$

4.2 Thermodynamic equilibrium

We consider the particle system to be in thermodynamic equilibrium, at temperature T, or the inverse temperature $\beta = 1/(k_{\mathrm{B}}T)$ with k_{B} being the Boltzmann constant. In the grand canonical ensemble with the chemical potential μ, the grand canonical partition function is defined by

$$\Xi = \sum_{N=0}^{\infty} \mathrm{e}^{\beta\mu N} \sum_{\substack{\text{eigenstates} \\ \text{of } N \text{ particles}}} \mathrm{e}^{-\beta E} = \sum_{\rho,\tilde{\rho}} \delta[\chi(\rho,\tilde{\rho})] W[\rho,\tilde{\rho}] \mathrm{e}^{\beta\mu N[\rho] - \beta E[\rho]}. \tag{4.12}$$

Here, the delta function enforces the constraint between the particle and hole distributions (4.10), represented formally as

$$\chi(\rho,\tilde{\rho}) = 0, \tag{4.13}$$

and $W[\rho,\tilde{\rho}]$ denotes the total number of microscopic states corresponding to the given macroscopic particle and hole densities in k space. The fact that W has a nontrivial value is associated with the existence of the omitted quantum numbers $\tilde{I}$ which causes a degeneracy: many wavefunctions of approximately the same energy are described by the same ρ and $\tilde{\rho}$. In particular, for given ρ and $\tilde{\rho}$, the total number of admissible k's in the interval $(k, k+\mathrm{d}k)$ is $L[\rho(k)+\tilde{\rho}(k)]\mathrm{d}k$, from which $L\rho(k)\mathrm{d}k$ are particle k's and $L\tilde{\rho}(k)\mathrm{d}k$ are hole k's. The number of possible choices of states in $\mathrm{d}k$ is thus equal to

$$\frac{\{L[\rho(k)+\tilde{\rho}(k)]\mathrm{d}k\}!}{[L\rho(k)\mathrm{d}k]![L\tilde{\rho}(k)\mathrm{d}k]!}.$$

It is natural to introduce the entropy

$$S[\rho,\tilde{\rho}] = \ln W[\rho,\tilde{\rho}]. \tag{4.14}$$

The contribution to the entropy from $\mathrm{d}k$ then reads

$$\begin{aligned} \mathrm{d}S &= \ln\left\{\frac{[L(\rho+\tilde{\rho})\mathrm{d}k]!}{(L\rho\mathrm{d}k)!(L\tilde{\rho}\mathrm{d}k)!}\right\} \\ &\sim L\left[(\rho+\tilde{\rho})\ln(\rho+\tilde{\rho})-\rho\ln\rho-\tilde{\rho}\ln\tilde{\rho}\right]\mathrm{d}k, \end{aligned} \tag{4.15}$$

where we applied Stirling's asymptotic formula $\ln(n!)\sim_{n\to\infty} n(\ln n-1)$ and omit in the notation the k-dependence of ρ and $\tilde{\rho}$. The total entropy per unit length is given by

$$\frac{S}{L}\equiv s=\int_{-\infty}^{\infty}\mathrm{d}k\ \left[(\rho+\tilde{\rho})\ln(\rho+\tilde{\rho})-\rho\ln\rho-\tilde{\rho}\ln\tilde{\rho}\right]. \tag{4.16}$$

For given ρ and $\tilde{\rho}$, the Gibbs free energy per unit length $g=G/L$ is defined by

$$\begin{aligned} g[\rho,\tilde{\rho}] &= \frac{1}{L}\left(-\frac{1}{\beta}S+E-\mu N\right)=-\frac{1}{\beta}s+e-\mu n \\ &= \int_{-\infty}^{\infty}\mathrm{d}k\left\{-\frac{1}{\beta}\left[(\rho+\tilde{\rho})\ln(\rho+\tilde{\rho})-\rho\ln\rho-\tilde{\rho}\ln\tilde{\rho}\right]+(k^2-\mu)\rho\right\}. \end{aligned} \tag{4.17}$$

Using the saddle-point approximation for the thermodynamic $L\to\infty$ limit of the grand canonical partition function (4.12), we have

$$\Xi=\sum_{\rho,\tilde{\rho}}\delta[\chi(\rho,\tilde{\rho})]\mathrm{e}^{-\beta g[\rho,\tilde{\rho}]L}\underset{L,N\to\infty}{\sim}\exp\left(-\beta g[\rho_{eq},\tilde{\rho}_{eq}]L\right), \tag{4.18}$$

where the equilibrium particle and hole distributions are determined by the extremal condition of the minimal Gibbs free energy,

$$\delta g\Big|_{\rho=\rho_{eq},\tilde{\rho}=\tilde{\rho}_{eq}}=0. \tag{4.19}$$

This condition is subject to the constraint (4.10).

With regard to Eq. (4.17), the equilibrium condition (4.19) is equivalent to

$$\delta\tilde{\rho}^{+}\ln\left(\frac{\rho_{eq}+\tilde{\rho}_{\mathrm{eq}}}{\tilde{\rho}_{\mathrm{eq}}}\right)+\delta\rho^{+}\ln\left(\frac{\rho_{\mathrm{eq}}+\tilde{\rho}_{\mathrm{eq}}}{\rho_{\mathrm{eq}}}\right)+\delta\rho^{+}\beta(\mu-k^2)=0. \tag{4.20}$$

From the constraint (4.10) we have

$$\delta\tilde{\rho}=-(I+G)\delta\rho. \tag{4.21}$$

The substitution of this relation into Eq. (4.20) leads to

$$\ln\left(\frac{\tilde{\rho}_{\mathrm{eq}}}{\rho_{\mathrm{eq}}}\right)-G\ln\left(1+\frac{\rho_{\mathrm{eq}}}{\tilde{\rho}_{\mathrm{eq}}}\right)+\beta(\mu-k^2)=0. \tag{4.22}$$

Finally, introducing the notation

$$\frac{\tilde{\rho}_{\mathrm{eq}}(k)}{\rho_{\mathrm{eq}}(k)}\equiv\exp[\beta\epsilon(k)], \tag{4.23}$$

the function $\epsilon(k)$ (whose dependence on μ and β is not explicitly written) is the solution of the nonlinear integral equation

$$\begin{aligned} k^2-\mu &= \epsilon-\frac{1}{\beta}G\ln\left(1+\mathrm{e}^{-\beta\epsilon}\right) \\ &\equiv \epsilon(k)-\frac{1}{\beta}\int_{-\infty}^{\infty}\mathrm{d}k'\,\frac{\theta'(k-k')}{2\pi}\ln\left[1+\mathrm{e}^{-\beta\epsilon(k')}\right]. \end{aligned} \tag{4.24}$$

In dependence on the form of the shift function $\theta(k)$, the function $\epsilon(k)$ can be determined either analytically or perturbatively, e.g. by iteration starting from $\epsilon^{(1)}(k)=k^2-\mu$. $\epsilon(k)$ has physical meaning as the excitation energy for an elementary excitation at a given temperature. From Eq. (4.10) we obtain the relation

$$\rho_{\mathrm{eq}}(k)\left[1+\mathrm{e}^{\beta\epsilon(k)}\right]=\frac{1}{2\pi}-\int_{-\infty}^{\infty}\mathrm{d}k'\,\frac{\theta'(k-k')}{2\pi}\rho_{\mathrm{eq}}(k'), \tag{4.25}$$

which determines $\rho_{\mathrm{eq}}(k)$.

At zero temperature, the definition (4.24) for $\epsilon(k)$ reduces to the previous one (3.18). This can be seen from the inequalities (3.47). Eq. (4.23) then implies the expected result

$$\begin{array}{ll} \rho_{\mathrm{eq}}(k)=0 & \text{for } |k|>q, \\ \tilde{\rho}_{\mathrm{eq}}(k)=0 & \text{for } |k|<q, \end{array} \qquad \text{at } T=0. \tag{4.26}$$

From the elementary thermodynamics, the pressure P is given by

$$P=-\frac{G}{L}=-g. \tag{4.27}$$

To express the pressure in terms of ϵ, we return to Eq. (4.17) and rewrite it as

$$\begin{aligned} P &= \rho^+\left\{\mu-k^2+\frac{1}{\beta}\left[\left(1+\frac{\tilde{\rho}}{\rho}\right)\ln\left(1+\frac{\tilde{\rho}}{\rho}\right)-\frac{\tilde{\rho}}{\rho}\ln\frac{\tilde{\rho}}{\rho}\right]\right\} \\ &= \rho^+\left[\mu-k^2-\epsilon\mathrm{e}^{\beta\epsilon}+\frac{1}{\beta}\left(1+\mathrm{e}^{\beta\epsilon}\right)\ln\left(1+\mathrm{e}^{\beta\epsilon}\right)\right] \\ &= \rho^+\left[\mu-k^2+\epsilon+\frac{1}{\beta}\left(1+\mathrm{e}^{\beta\epsilon}\right)\ln\left(1+\mathrm{e}^{-\beta\epsilon}\right)\right]. \end{aligned} \tag{4.28}$$

Using the constraint (4.10), reexpressed in the form

$$\rho^+\left(1+\mathrm{e}^{\beta\epsilon}\right)=\frac{\eta}{2\pi}-G\rho, \tag{4.29}$$

we can simplify the last line of Eq. (4.28) as follows

$$P=\rho^+\left[\mu-k^2+\epsilon-\frac{1}{\beta}G\ln\left(1+\mathrm{e}^{-\beta\epsilon}\right)\right]+\frac{1}{2\pi\beta}\eta^+\ln\left(1+\mathrm{e}^{-\beta\epsilon}\right). \tag{4.30}$$

The expression in the brackets vanishes due to Eq. (4.24) defining ϵ, so that

$$P(\mu,\beta)=\frac{1}{2\pi\beta}\eta^+\ln\left(1+\mathrm{e}^{-\beta\epsilon}\right)\equiv\frac{1}{2\pi\beta}\int_{-\infty}^{\infty}\mathrm{d}k\;\ln\left[1+\mathrm{e}^{-\beta\epsilon(k)}\right]. \tag{4.31}$$

In the limit of zero temperature ($\beta \to \infty$), this relation reduces to the previous one (3.25).

The chemical potential μ is usually replaced by the fugacity z defined by

$$z = \exp(\beta\mu). \tag{4.32}$$

Considering P to be the function of (z, β), the particle density is expressible as

$$n = \frac{\partial P}{\partial \mu} = z\frac{\partial}{\partial z}\beta P(z,\beta). \tag{4.33}$$

Having the particle density – fugacity relation $n = n(z, \beta)$, this relation can be (in principle) converted, $z = z(n, \beta)$, and the pressure is expressible as the function of (n, β). The expansion of $P(n, \beta)$ in powers of n is known as the virial expansion.

4.3 Thermodynamics of repulsive δ-function bosons

• In the limit $c \to \infty$, the integrals in (4.24) and (4.25) do not contribute. Therefore,

$$\epsilon = k^2 - \mu \tag{4.34}$$

and

$$2\pi\rho_{\rm eq} = \frac{z\exp(-\beta k^2)}{1 + z\exp(-\beta k^2)}, \tag{4.35}$$

$$2\pi\tilde{\rho}_{\rm eq} = \frac{1}{1 + z\exp(-\beta k^2)}. \tag{4.36}$$

According to Eq. (4.31), the pressure $P(c \to \infty)$ is equal to

$$\begin{aligned} P_{\rm F} &= \frac{1}{2\pi\beta}\int_{-\infty}^{\infty} {\rm d}k\ \ln\left[1 + z\exp(-\beta k^2)\right] \\ &= \frac{1}{2\sqrt{\pi}}(k_{\rm B}T)^{3/2}\sum_{j=1}^{\infty}(-1)^{j+1}\frac{z^j}{j^{3/2}}. \end{aligned} \tag{4.37}$$

These equations correspond to the ones of a free Fermi gas, as it should be.

• In the limit $c \to 0$, we have $\theta'(k) \to -2\pi\delta(k)$. Eq. (4.24) thus gives

$$\epsilon = k^2 - \mu - \frac{1}{\beta}\ln\left[1 + \exp(-\beta\epsilon)\right], \tag{4.38}$$

or

$$\exp(-\beta\epsilon) = \frac{1}{z^{-1}\exp(\beta k^2) - 1}. \tag{4.39}$$

From Eqs. (4.23) and (4.25), we have

$$2\pi\rho_{eq} = \frac{1}{z^{-1}\exp(\beta k^2) - 1}, \tag{4.40}$$

$$2\pi\tilde{\rho}_{eq} = 1. \tag{4.41}$$

The pressure $P(c \to 0)$ is equal to

$$\begin{aligned} P_{\rm B} &= -\frac{1}{2\pi\beta}\int_{-\infty}^{\infty} {\rm d}k\ \ln\left[1 - z\exp(-\beta k^2)\right] \\ &= \frac{1}{2\sqrt{\pi}}(k_{\rm B}T)^{3/2}\sum_{j=1}^{\infty}\frac{z^j}{j^{3/2}}. \end{aligned} \tag{4.42}$$

These equations correspond to the ones of a free Bose gas.

Let us now make a general analysis of the pressure $P(c)$ for an arbitrary coupling constant $c \in (0, \infty)$ [62]. In terms of the fugacity z, the pressure of a 1D particle system of length L is defined by

$$\beta PL = \ln\left(\sum_{N=0}^{\infty} z^N \sum_{\text{states } s} {\rm e}^{-\beta E_s(N)}\right). \tag{4.43}$$

In the case of repulsive δ-function bosons it is clear that the energy E_s of state s increases with increasing c and so the coefficient of z^N decreases with increasing c. For a fixed β and z, we thus have $P(c=0) > P(c\ \text{finite}) > P(c \to \infty)$. Since $P(c=0) = P_{\rm B}$ and $P(c\to\infty) = P_{\rm F}$, the following inequalities take place

$$P_{\rm F} < P(c) < P_{\rm B} \qquad \text{at fixed } \beta \text{ and } z. \tag{4.44}$$

To derive the fugacity and virial expansions of $P(c)$, we write down a formal expansion of $\exp[-\beta\epsilon(k)]$ in powers of z:

$$\exp\left[-\beta\epsilon(k)\right] = \sum_{n=1}^{\infty} a_n(k,\beta) z^n. \tag{4.45}$$

Substituting this expansion into Eq. (4.24), we obtain

$$\begin{aligned} a_1 &= {\rm e}^{-\beta k^2}, \\ a_2 &= -{\rm e}^{-\beta k^2} G a_1, \\ a_3 &= {\rm e}^{-\beta k^2}\left[-Ga_2 + \frac{1}{2}Ga_1^2 + \frac{1}{2}(Ga_1)^2\right], \end{aligned} \tag{4.46}$$

etc. Within the representation (4.31), the pressure is given by

$$\begin{aligned} P &= \frac{1}{2\pi\beta}\int_{-\infty}^{\infty}{\rm d}k\left[a_1 z + \left(a_2 - \frac{a_1^2}{2}\right)z^2 + \left(a_3 - a_1a_2 + \frac{a_1^3}{3}\right)z^3 + \cdots\right] \\ &= P_{\rm F} + \frac{1}{2\sqrt{\pi}}(k_{\rm B}T)^{3/2}{\rm e}^{\beta c^2/2}\left[\frac{1}{\sqrt{2}} - \sqrt{\frac{2}{\pi}}\int_0^{\sqrt{\beta c^2/2}}{\rm d}y\ {\rm e}^{-y^2}\right]z^2 + O(z^3). \end{aligned} \tag{4.47}$$

The fugacity expansion of the particle density n can be determined from this equation by using formula (4.33). The leading terms of the virial expansion of the pressure are finally obtained in the form

$$\frac{\beta P}{n} = 1 + \left\{\frac{1}{2\sqrt{2}} + {\rm e}^{\beta c^2/2}\left[\sqrt{\frac{2}{\pi}}\int_0^{\sqrt{\beta c^2/2}}{\rm d}y\ {\rm e}^{-y^2} - \frac{1}{\sqrt{2}}\right]\right\}\lambda n + \cdots, \tag{4.48}$$

where $\lambda = 2\sqrt{\pi\beta}$ is the thermal de Broglie wavelength, expressed in units of $\hbar = 1$ and $2m = 1$.

Some further developments of the thermodynamic formalism for δ-function bosons can be found in Refs. [63, 64].

5 Particles with hard-core interactions

We now review probably the simplest one-dimensional system solvable by using the Bethe ansatz method. The model consists of N identical spinless particles, either bosons or fermions, with a hard core of diameter a around each particle (hard rod). The hard-core potential around a particle

$$v(x) = \begin{cases} \infty & \text{for } |x| < a, \\ 0 & \text{for } |x| > a, \end{cases} \tag{5.1}$$

is impenetrable for other rods. There is no other interaction among the particles.

The particles are localized on a circle of length $L > Na$, with the density $n < 1/a$. The fundamental sector of the particle ordering I is given by

$$I: \qquad 0 \le x_1 < x_2 - a < x_3 - 2a < \cdots < x_N - (N-1)a \le b, \tag{5.2}$$

where $b = \min\{L - (n-1)a, x_1 + L - Na\}$. The particles are free inside the I-region and therefore their wavefunction fulfill the Helmholtz equation (2.14). Since the hard-core potential is the impenetrable one, the wavefunction has to vanish on the boundary of the I-region:

$$\psi_I(\cdots, x_j, x_{j+1} = x_j + a, \cdots) = 0, \qquad j = 1, 2, \ldots, N-1. \tag{5.3}$$

5.1 Bethe ansatz

For $N = 2$ particles, the general solution of the Helmholtz equation is the superposition of plane waves

$$\psi_I(x_1, x_2) = A(12)\mathrm{e}^{\mathrm{i}(k_1 x_1 + k_2 x_2)} - A(21)\mathrm{e}^{\mathrm{i}(k_2 x_1 + k_1 x_2)} \tag{5.4}$$

with the total momentum $K = k_1 + k_2$ and energy $E = k_1^2 + k_2^2$. The boundary condition at $x_2 = x_1 + a$, $\psi_I(x_1, x_1 + a) = 0$, implies that

$$\frac{A(21)}{A(12)} = \exp\left[-\mathrm{i}\theta(k_1 - k_2)\right], \qquad \theta(k) = ak. \tag{5.5}$$

The function $\theta(k)$ is the two-body phase shift for the hard-core potential. Introducing the variables $y_1 = x_1$ and $y_2 = x_2 - a$, the (unnormalized) wavefunction is written as

$$\psi_I(x_1, x_2) = \operatorname*{Det}_{1 \le j,l \le 2} \exp(\mathrm{i}k_j y_l). \tag{5.6}$$

The wavefunction is nonzero only if $k_1 \ne k_2$. The periodic boundary condition $\psi_I(0, x_2) = \psi_I(x_2, L)$ leads to the previously derived Eqs. (2.46) for the wave numbers k_1 and k_2, with the shift function θ of the hard-core potential.

In analogy with the case of the δ-function potential, it is seen from the form of the boundary conditions (5.3) that for multi-body collisions the phase shift does not depend on the order of two-body collisions and the hard-rod system is integrable. If we define the variables $y_j = x_j - (j-1)a$ $(j = 1, 2, \ldots, N)$, the N-body wavefunction for the fundamental particle ordering I is expressible as

$$\psi_I(x_1, x_2, \ldots, x_N) = \operatorname*{Det}_{1 \le j,l \le N} \exp(\mathrm{i}k_j y_l). \tag{5.7}$$

The total momentum and energy are given by $K = \sum_{j=1}^{N} k_j$ and $E = \sum_{j=1}^{N} k_j^2$, respectively. The wave numbers, no two of which are identical in order to avoid the nullity of ψ_I, are quantized according to the present counterpart of the Bethe equations (2.69),

$$k_j L = 2\pi I_j + a \sum_{l=1}^{N} (k_j - k_l), \qquad j = 1, 2, \ldots, N. \tag{5.8}$$

These equations can be rewritten as follows

$$k_j L(1 - an) = 2\pi I_j - aK, \qquad K = \frac{2\pi}{L} \sum_j I_j. \tag{5.9}$$

The Bethe equations apply to both bosons and fermions which therefore have the same energy spectra. The Bose or Fermi statistics is reflected through the symmetrization/antisymmetrization of the wavefunction (5.7) for other orderings of the particles.

Let us consider first the trivial case in which the rod diameter a is zero. For spinless fermions, the Bethe equations (5.8) correspond to free fermions as was intuitively expected. For bosons, contrary to the classical gas and three-dimensional hard-core Bose gas, letting a be zero does not remove the interaction due to the restriction of the configuration space into the subspace I in (5.2). Consequently, the boson spectrum for pointlike hard cores is identical to that of free fermions; this situation is equivalent to that of the hard-core boson $c \to \infty$ limit of the δ-function potential.

The point $a = 0$ serves as the reference point for the whole region of allowed values of hard-core diameters $0 < a < 1/n$. Based on the continuity arguments, the ground state is characterized by the same set of quantum numbers $\{I_j\}_{j=1}^{N}$ as at the point $a = 0$. This means that the quantum numbers are symmetricaly distributed (with a unity step) around 0, see Eq. (2.99), and so $K_0 = 0$. From Eq. (5.9), we then have

$$k_j L(1 - an) = 2\pi I_j, \qquad j = 1, 2, \ldots, N, \tag{5.10}$$

i.e. the particle system behaves like free fermions in a volume reduced by the hard cores.

5.2 Ground state and zero-temperature thermodynamics

In the thermodynamic limit, Eq. (3.38) for the probability distribution $\rho(k)$, taken with $\theta'(k) = a$, reads

$$\rho(k) = \frac{1}{2\pi} - \frac{a}{2\pi} \int_{-q}^{q} \mathrm{d}k' \rho(k'). \tag{5.11}$$

$\rho(k)$ is thus constant for any real k,

$$\rho(k) = \frac{1}{2\pi}(1 - an), \qquad -\infty < k < \infty. \tag{5.12}$$

The "normalization condition" for the particle density

$$n = \int_{-q}^{q} \mathrm{d}k \; \rho(k) = \frac{q}{\pi}(1 - an) \tag{5.13}$$

determines the density dependence of the momentum limit for the ground state,

$$q = \frac{\pi n}{1 - an}. \tag{5.14}$$

The energy density is given by

$$e_0 = \int_{-q}^{q} \mathrm{d}k\, k^2 \rho(k) = \frac{\pi^2}{3} \frac{n^3}{(1 - an)^2}. \tag{5.15}$$

The chemical potential is calculated as follows

$$\mu = \frac{\partial e_0}{\partial n} = \frac{(\pi n)^2}{(1 - an)^3} \left(1 - \frac{1}{3} an\right). \tag{5.16}$$

The distribution $\epsilon(k)$, defined by Eq. (3.39), obeys the integral equation

$$\epsilon(k) = k^2 - \mu - \int_{-q}^{q} \mathrm{d}k'\, \frac{a}{2\pi} \epsilon(k'). \tag{5.17}$$

The solution of this equation

$$\epsilon(k) = k^2 - q^2, \qquad -\infty < k < \infty \tag{5.18}$$

satisfies the necessary condition $\epsilon(\pm q) = 0$.

The group velocity of excitations and the velocity of sound read

$$v(k) = \frac{2k}{1 - an}, \qquad v_s = \frac{2q}{1 - an} = \frac{2\pi n}{(1 - an)^2}, \tag{5.19}$$

respectively.

To derive dispersion relations for the elementary particle and hole excitations, we first evaluate the f-function

$$f(k) = \int_0^k \mathrm{d}k'\, \rho(k') = \frac{1}{2\pi}(1 - an)k. \tag{5.20}$$

For particle excitations with $k_p > q$, we have

$$\Delta E(k_p) = k_p^2 - q^2, \qquad K(k_p) = (1 - an)(k_p - q). \tag{5.21}$$

Eliminating from these relations k_p, we obtain the dispersion relation for particle excitations

$$\Delta E_p(K) = \frac{1}{(1 - an)^2}(K^2 + 2\pi n|K|), \qquad -\infty < K < \infty. \tag{5.22}$$

Similarly, for hole excitations with $0 < k_h < q$, we have

$$\Delta E(k_h) = q^2 - k_h^2, \qquad K(k_h) = (1 - an)(q - k_p) \tag{5.23}$$

and the dispersion relation for hole excitations read

$$\Delta E_h(K) = \frac{1}{(1 - an)^2}(2\pi n|K| - K^2), \qquad -\pi n < K < \pi n. \tag{5.24}$$

The pressure is determined as follows

$$P = \mu n - e_0 = \frac{2\pi^2}{3} \left(\frac{n}{1 - an}\right)^3. \tag{5.25}$$

This result is reproduced by substituting $\epsilon(k)$ (5.18) in the pressure representation (3.25).

5.3 Finite-temperature thermodynamics

The thermodynamics of the hard-core particle system was solved in Refs. [25, 65]. From the integral equation (4.24) defining the distribution $\epsilon(k)$ and from the pressure representation (4.31) it follows that

$$\epsilon(k) = k^2 - \mu + aP, \tag{5.26}$$

where the pressure P is given self-consistently by

$$P = \frac{1}{2\pi\beta} \int_{-\infty}^{\infty} \mathrm{d}k \, \ln\left[1 + \mathrm{e}^{-\beta(k^2-\mu+aP)}\right]. \tag{5.27}$$

The equilibrium distribution $\rho_{\mathrm{eq}}(k)$, defined by Eq. (4.25), reads

$$\rho_{\mathrm{eq}}(k) = \frac{1}{2\pi}(1-an)\frac{1}{\mathrm{e}^{\beta(k^2-\mu+aP)}+1}. \tag{5.28}$$

The particle number density n and the energy density e then follow from Eq. (4.11):

$$n = \frac{1}{2\pi}(1-an)\int_{-\infty}^{\infty} \mathrm{d}k \, \frac{1}{\mathrm{e}^{\beta(k^2-\mu+aP)}+1}, \tag{5.29}$$

$$e = \frac{1}{2\pi}(1-an)\int_{-\infty}^{\infty} \mathrm{d}k \, k^2 \frac{1}{\mathrm{e}^{\beta(k^2-\mu+aP)}+1}, \tag{5.30}$$

Note that the pointlike hard-core case $a = 0$ is indeed equivalent to free fermions.

Performing in (5.27) an integration by parts, we get

$$P = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}k \, k^2 \frac{1}{\mathrm{e}^{\beta(k^2-\mu+aP)}+1} = \frac{2}{1-an}e. \tag{5.31}$$

Such explicit relation between the pressure and the energy density, which holds for all temperatures and particle densities, is known as the Bernoulli equation.

Let us assume that

$$\exp\left[\beta(aP-\mu)\right] \gg 1. \tag{5.32}$$

In this classical regime of the statistical mechanics, the relations (5.27) and (5.29) become

$$P = \frac{1}{2\pi\beta}\int_{-\infty}^{\infty} \mathrm{d}k \, \mathrm{e}^{-\beta(k^2-\mu+aP)}, \tag{5.33}$$

$$n = \frac{1}{2\pi}(1-an)\int_{-\infty}^{\infty} \mathrm{d}k \, \mathrm{e}^{-\beta(k^2-\mu+aP)}. \tag{5.34}$$

Combining these equations, we obtain the classical Tonks equation of state [66]

$$\beta P = \frac{n}{1-an}. \tag{5.35}$$

Performing the gaussian integrals in Eqs. (5.33) and (5.34), we find

$$\beta\mu = \frac{an}{1-an} + \ln\left(\frac{\lambda n}{1-an}\right), \tag{5.36}$$

where λ is the thermal de Broglie wavelength. Having relations (5.35) and (5.36) it is easy to show that the validity of the classical approximation (5.32) is restricted to the region

$$\lambda \frac{n}{1-an} \ll 1. \tag{5.37}$$

In order to perform a general analysis of the hard-core particle system, we introduce the variables

$$\tilde{\mu} = \mu - aP, \qquad \tilde{n} = \frac{n}{1-an}. \tag{5.38}$$

In terms of these new variables, Eq. (5.27) [or the equivalent one (5.31)] and Eq. (5.29) read

$$\begin{aligned} P &= \frac{1}{2\pi\beta}\int_{-\infty}^{\infty} \mathrm{d}k\ \ln\left[1+\mathrm{e}^{-\beta(k^2-\tilde{\mu})}\right], \\ &\equiv \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}k\ k^2 \frac{1}{\mathrm{e}^{\beta(k^2-\tilde{\mu})}+1}, \qquad (5.39) \\ \tilde{n} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\ \frac{1}{\mathrm{e}^{\beta(k^2-\tilde{\mu})}+1}. \qquad (5.40) \end{aligned}$$

The elimination of $\tilde{\mu}$ from these relations provides P as the function of $\tilde{n}$ and β. The elimination procedure depends on whether the system is in the high-temperature or low-temperature regions.

• High-temperature virial expansion: By expanding the integrated functions in Eqs. (5.39) and (5.40) in powers of $\tilde{z} = \exp(\beta\tilde{\mu})$, we have

$$\begin{aligned} \beta P &= \frac{1}{\lambda}\sum_{j=1}^{\infty}(-1)^{j+1}\frac{\tilde{z}^j}{j^{3/2}}, \qquad (5.41) \\ \tilde{n} &= \frac{1}{\lambda}\sum_{j=1}^{\infty}(-1)^{j+1}\frac{\tilde{z}^j}{\sqrt{j}}. \qquad (5.42) \end{aligned}$$

The inversion of the expansion (5.42),

$$\tilde{z} = \lambda\tilde{n} + \frac{1}{\sqrt{2}}(\lambda\tilde{n})^2 + \left(1-\frac{1}{\sqrt{3}}\right)(\lambda\tilde{n})^3 + \cdots, \tag{5.43}$$

leads to

$$\frac{\beta P}{\tilde{n}} = 1 + \frac{1}{2^{3/2}}(\lambda\tilde{n}) + \left(\frac{1}{2}-\frac{2}{3^{3/2}}\right)(\lambda\tilde{n})^2 + \cdots. \tag{5.44}$$

Since the dimensionless parameter is small if $n\sqrt{\beta} \ll 1$, this expansion converges in the high-temperature region.

• Low-temperature virial expansion: At low temperatures, we assume that

$$\tilde{\mu} \equiv \mu - aP > 0. \tag{5.45}$$

In terms of $\epsilon = k^2$ and $\epsilon_0 = \tilde{\mu} > 0$, Eqs. (5.39) and (5.40) can be written as

$$P = \frac{1}{\pi}\int_0^\infty \mathrm{d}\epsilon\, \sqrt{\epsilon}\frac{1}{\mathrm{e}^{\beta(\epsilon-\epsilon_0)}+1}, \tag{5.46}$$

$$\tilde{n} = \frac{1}{2\pi}\int_0^\infty \mathrm{d}\epsilon\, \frac{1}{\sqrt{\epsilon}}\frac{1}{\mathrm{e}^{\beta(\epsilon-\epsilon_0)}+1}. \tag{5.47}$$

By using elementary algebra it can be shown that for an arbitrary function g it holds

$$\int_0^\infty \mathrm{d}\epsilon\, g(\epsilon)\frac{1}{\mathrm{e}^{\beta(\epsilon-\epsilon_0)}+1} = \int_0^{\epsilon_0} \mathrm{d}\epsilon\, g(\epsilon) + \frac{1}{\beta}\int_{-\beta\epsilon_0}^\infty \mathrm{d}\epsilon\, g\left(\epsilon_0+\frac{\epsilon}{\beta}\right)\frac{\mathrm{sign}(\epsilon)}{\mathrm{e}^{|\epsilon|}+1}. \tag{5.48}$$

In the limit $\beta \to \infty$, the lower bound of the second integral can be set to $-\infty$ since the exponentially small corrections are negligibly small in comparison with the powers of temperature. Expanding $g(\epsilon_0+\epsilon/\beta)$ in a Taylor series around the point ϵ_0 and evaluating the integrals over ϵ, we arrive at

$$\begin{aligned}\int_0^\infty \mathrm{d}\epsilon\, g(\epsilon)\frac{1}{\mathrm{e}^{\beta(\epsilon-\epsilon_0)}+1} &= \int_0^{\epsilon_0} \mathrm{d}\epsilon\, g(\epsilon) + \frac{\pi^2}{6}(k_\mathrm{B}T)^2 g'(\epsilon_0) \\ &\quad + \frac{7\pi^4}{360}(k_\mathrm{B}T)^4 g'''(\epsilon_0) + \cdots.\end{aligned} \tag{5.49}$$

Introducing $k_0 = \sqrt{\epsilon_0}$, the series results for Eqs. (5.46) and (5.47) read

$$P = \frac{2}{3\pi}k_0^3 + \frac{\pi}{12}\frac{(k_\mathrm{B}T)^2}{k_0} + \frac{7\pi^3}{960}\frac{(k_\mathrm{B}T)^4}{k_0^5} + \cdots, \tag{5.50}$$

$$\tilde{n} = \frac{k_0}{\pi} - \frac{\pi}{24}\frac{(k_\mathrm{B}T)^2}{k_0^3} - \frac{7\pi^3}{384}\frac{(k_\mathrm{B}T)^4}{k_0^7} + \cdots. \tag{5.51}$$

The elimination of k_0 implies the desired low-temperature virial expansion

$$P = \frac{2}{3\pi}(\pi\tilde{n})^3 + \frac{\pi}{6}\frac{(k_\mathrm{B}T)^2}{\pi\tilde{n}} + \frac{\pi^3}{30}\frac{(k_\mathrm{B}T)^4}{(\pi\tilde{n})^5} + \cdots. \tag{5.52}$$

6 Particles with $1/x^2$ interactions

In this section, we shall deal with 1D many-body systems of identical particles which interact by a long-ranged pair potential having its origin in the inverse-square potential, written in a standard form as follows

$$v(x) = \frac{2\lambda(\lambda-1)}{x^2}. \tag{6.1}$$

Here, the parameter λ is a dimensionless interaction strength. The potential is repulsive for $\lambda > 1$, the limit $\lambda \to \infty$ corresponds to the classical regime.

The many-body system with the inverse-square interaction is well defined on an infinite line. For a finite circle of length L, the inverse-square potential has to be substituted by a periodic one, say by taking a lattice sum

$$v(x) \to \sum_{n=-\infty}^{\infty} v(x+nL) = \sum_{n=-\infty}^{\infty} \frac{2\lambda(\lambda-1)}{(x+nL)^2} = \frac{2\lambda(\lambda-1)\pi^2}{L^2\sin^2(\pi x/L)}. \tag{6.2}$$

This is the inverse-sin-squared, or trigonometric, potential. The system has a well-defined thermodynamic limit which is expected to be the same as the one of the inverse-square model.

Let us substitute in Eq. (6.2) the circle length L by an imaginary parameter, $\pi/L = \mathrm{i}c$. The resulting potential

$$v(x) = \frac{2\lambda(\lambda-1)c^2}{\sinh^2(cx)} \tag{6.3}$$

is the inverse-sinh-squared, or hyperbolic, potential. There exist two length scales in the model: $1/c$ is the range of the interaction and $L/N = 1/n$ is the average nearest-neighbor spacing between particles. In the limit of high particle densities $n \gg c$, the thermodynamic limit of the hyperbolic model is equivalent to that of the inverse-square model. At low densities $n \ll c$, the hyperbolic potential reduces to

$$v(x) \sim 8\lambda(\lambda-1)c^2\exp(-2c|x|). \tag{6.4}$$

Scaling this potential by $\exp(2c/n)$, it acts effectively only between nearest neighbors and decays exponentially with distance. The model is known as the Toda lattice.

The hyperbolic potential can be made periodic by taking the lattice sum,

$$v(x) \to \sum_{n=-\infty}^{\infty} v(x+nL) = \sum_{n=-\infty}^{\infty} \frac{2\lambda(\lambda-1)c^2}{\sinh^2[c(x+nL)]} = \frac{8\lambda(\lambda-1)K^2}{L^2\mathrm{sn}^2(2Kx/L,k)} + \text{const.} \tag{6.5}$$

Here, $\mathrm{sn}(u,k)$ is a Jacobi elliptic function with modulus k and $K = K(k)$ is the complete elliptic integral of the first kind (see Appendix B). This is the inverse-sn-squared, or elliptic, potential.

We shall study in detail the particle systems with the inverse-square and trigonometric potentials, the results for other potentials are summarized in Sutherland's book [57].

6.1 The two-body scattering problem

The δ-function potential is the short-range one, two particles interact only when they touch one another. The many-body wavefunction is always the superposition of plane waves, for every positions of particles.

The inverse-square potential, and the pair potentials based on this potential, are long-ranged: two interacting particles "feel" one another at arbitrary finite distances. The corresponding wavefunction becomes a superposition of plane waves, incoming and outgoing ones with well defined particle momenta and the phase shift, *only* for asymptotically large distances between particles. In this part, we shall solve explicitly the asymptotic two-body problem and calculate the two-body phase shift for the inverse-square potential of interest.

Let us consider two particles, particle 1 with the coordinate x_1 and particle 2 with the coordinate x_2, localized on an infinite line $x_1, x_2 \in (-\infty, \infty)$. The time-independent Schrödinger equation for a two body wavefunction $\psi(x_1, x_2)$ is

$$-\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)\psi + v(x_1-x_2)\psi = E\psi. \tag{6.6}$$

The interaction potential v is symmetric and goes to zero at asymptotically large distances between the particles,

$$v(x) = v(-x), \qquad \lim_{x\to\infty} v(x) = 0. \tag{6.7}$$

Let k_1 and k_2 be the asymptotic momenta of the two particles. We define the total and relative asymptotic momenta as follows

$$K = k_1 + k_2, \qquad k = k_1 - k_2, \tag{6.8}$$

so the energy

$$E = k_1^2 + k_2^2 = \frac{1}{2}(K^2 + k^2). \tag{6.9}$$

In the basis of the center-of-mass and relative coordinates

$$X = \frac{1}{2}(x_1 + x_2), \qquad x = x_1 - x_2, \tag{6.10}$$

the wavefunction of the Schrödinger equation (6.6) can be expressed in a separation-variables form

$$\psi(x_1, x_2) = \mathrm{e}^{\mathrm{i}KX}\psi(x), \tag{6.11}$$

where $\psi(x)$ satisfies a relative-motion equation

$$-\psi''(x) + \frac{1}{2}v(x)\psi(x) = \left(\frac{k}{2}\right)^2\psi(x). \tag{6.12}$$

For the symmetric potential (6.7), this differential equation exhibits a solution of even (boson) parity

$$\psi_+(x) = \psi_+(-x), \qquad \psi'_+(x) = -\psi'_+(-x), \tag{6.13}$$

and a solution of odd (fermion) parity

$$\psi_-(x) = -\psi_-(-x), \qquad \psi'_-(x) = \psi'_-(-x). \tag{6.14}$$

Let $k_1 > k_2$ ($k > 0$). Then the asymptotic wavefunction is the sum of an incoming and an outgoing wave, with a phase shift $\theta_\pm(k)$ between the two:

$$\psi_\pm(x) \sim \begin{cases} \mathrm{e}^{\mathrm{i}kx/2} - \mathrm{e}^{-\mathrm{i}kx/2-\mathrm{i}\theta_\pm(k)}, & x \to -\infty, \\ \pm\left[\mathrm{e}^{-\mathrm{i}kx/2} - \mathrm{e}^{\mathrm{i}kx/2-\mathrm{i}\theta_\pm(k)}\right], & x \to \infty. \end{cases} \tag{6.15}$$

Note that the phase shift depends, in general, on the particle statistics. Reversing the collision we see that the phase shift must always be an odd function of k, $\theta_\pm(-k) = -\theta_\pm(k)$.

Let us calculate the phase shifts for potentials of the present interest.

• The δ-function potential $v(x) = 2c\delta(x)$: This potential has no effect on the antisymmetric wavefunction which vanishes at the origin. In particular, we have

$$\psi_-(x) = 2\mathrm{i}\sin\left(\frac{kx}{2}\right), \qquad \theta_-(k) = 0. \tag{6.16}$$

In the case of the symmetric wavefunction, integrating the Schrödinger equation

$$-\psi''_+(x) + c\delta(x)\psi_+(x) = \left(\frac{k}{2}\right)^2 \psi_+(x) \tag{6.17}$$

across the origin and using the continuity of the wavefunction, we obtain

$$2\psi'_+(0^+) = c\psi_+(0). \tag{6.18}$$

The asymptotic form (6.15) takes place everywhere, except the origin. We thus have

$$\frac{\psi'_+(x)}{\psi_+(x)} = \frac{k}{2}\cot\left(\frac{kx - \theta_+(k)}{2}\right), \qquad x > 0. \tag{6.19}$$

With regard to Eq. (6.18), we reproduce the previous result $\theta_+(k) = -2\arctan(k/c)$.

• The hard-core potential (5.1): This potential is impenetrable to particles and therefore $\theta_- = \theta_+ = \theta$. Also in this case the asymptotic form (6.15) takes place everywhere, except the hard-core region. The Dirichlet boundary conditions $\psi(\pm a) = 0$ lead to the phase shift $\theta(k) = ak$, in agreement with the previous result (5.5).

• The $1/x^2$ potential $v(x) = 2\lambda(\lambda-1)/x^2$: In the repulsive regime $\lambda \geq 1$, this potential is also impenetrable to particles, so that $\theta_- = \theta_+ = \theta$. The Schrödinger equation (6.12) for this potential reads

$$-\psi''(x) + \frac{\lambda(\lambda-1)}{x^2}\psi(x) = \left(\frac{k}{2}\right)^2 \psi(x). \tag{6.20}$$

Making the substitution $\psi(x) = \sqrt{x}f(y)$ with $y = kx/2$, this equation takes the form

$$f'' + \frac{f'}{y} + f - \left(\lambda - \frac{1}{2}\right)^2 \frac{f}{y^2} = 0. \tag{6.21}$$

Its solution is given by the Bessel function $J_{\lambda-1/2}(y)$. Consequently,

$$\psi(x) = \sqrt{x} J_{\lambda-1/2}\left(\frac{kx}{2}\right) \underset{x\to\infty}{\sim} \frac{2}{\sqrt{\pi k}} \cos\left(\frac{kx - \pi\lambda}{2}\right). \tag{6.22}$$

We conclude that the phase shift and its derivative are given by

$$\theta(k) = \pi(\lambda - 1)\mathrm{sign}(k), \qquad \theta'(k) = 2\pi(\lambda - 1)\delta(k). \tag{6.23}$$

6.2 The ground-state wavefunction of a product form

The ground-state wavefunction (2.104) of free fermions or hard-core bosons on the circle of length L has a simple product form of type

$$\Psi_0(x_1, \ldots, x_N) = \prod_{j>k=1}^{N} \psi(x_j - x_k), \tag{6.24}$$

where $\psi(-x) = \pm\psi(x)$, in dependence on the particle statistics. We might hope that there exist specific potentials $v(x)$ that have a ground-state wavefunction of product form, not only in the trivial free-fermion/hard-core boson point, but for all values of potential's parameters. This is the motivation for the following inverse problem: find all periodic potentials $v(x)$ for which the Hamiltonian

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{j>k=1}^{N} v(x_j - x_k) \tag{6.25}$$

has a ground-state wavefunction of the product form (6.24).

Let us consider the fundamental ordering sector I: $x_1 < x_2 < \cdots < x_N$. Since the ground-state wavefunction has no nodes, we can divide the Schrödinger equation by Ψ_0 to obtain

$$\frac{1}{\Psi_0} \sum_{j=1}^{N} \frac{\partial^2 \Psi_0}{\partial x_j^2} = \sum_{j>k=1}^{N} v(x_j - x_k) - E_0. \tag{6.26}$$

Substituting into this equation the product form (6.24), carrying out the differentiations and introducing the logarithmic derivative of $\psi(x)$,

$$\phi(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x} \ln \psi(x) = \frac{\psi'(x)}{\psi(x)}, \qquad \phi(-x) = -\phi(x) \text{ for both fermions and bosons,} \tag{6.27}$$

we get

$$2\sum_{j>k} \frac{\psi''(x_j - x_k)}{\psi(x_j - x_k)} + 2 \sum_{j>k>l} [\phi(x_j - x_k)\phi(x_j - x_l) - \phi(x_j - x_k)\phi(x_k - x_l) + \phi(x_j - x_l)\phi(x_k - x_l)] = \sum_{j>k} v(x_j - x_k) - E_0. \tag{6.28}$$

This equation can be satisfied only if

$$\phi(x_3-x_2)\phi(x_3-x_1)-\phi(x_3-x_2)\phi(x_2-x_1)+\phi(x_3-x_1)\phi(x_2-x_1) \\ =f(x_3-x_1)+f(x_3-x_2)+f(x_2-x_1). \tag{6.29}$$

Since the lhs of this equation is invariant with respect to an arbitrary interchange of coordinates, the function f must be even: $f(x)=f(-x)$. Using the notation $x=x_2-x_1$ and $y=x_3-x_2$, so that $x+y=x_3-x_1$, we obtain

$$[\phi(x)+\phi(y)]\,\phi(x+y)-\phi(x)\phi(y)=f(x)+f(y)+f(x+y). \tag{6.30}$$

Solving this functional equation for the two functions $\phi(x)$ and $f(x)$, odd and even respectively, from Eq. (6.28) the pair potential $v(x)$ is given by

$$\frac{1}{2}v(x)=\phi'(x)+\phi^2(x)+(N-2)f(x)+\frac{E_0}{N(N-1)}. \tag{6.31}$$

The functional Eq. (6.30) must hold for all x and y. We shall expand it for small y around the point $y=0$. The even function $f(y)$ is assumed to be analytic everywhere, its expansion around $y=0$ reads $f(y)=f(0)+f''(0)y^2/2!+\cdots$. The odd function $\phi(y)$ is singular at $y=0$; this becomes evident by considering the free-fermion ground-state wavefunction (2.104) with $\psi(y)=\sin(\pi y/L)$, implying

$$\phi_{\text{free fermions}}(y)=\frac{\pi}{L}\cot\left(\frac{\pi y}{L}\right)\underset{y\to 0}{\sim}\frac{1}{y}. \tag{6.32}$$

This is why we assume that

$$\phi(y)\underset{y\to 0}{\sim}\frac{\lambda}{y}-ay-by^3-cy^5-\cdots. \tag{6.33}$$

Based on this assumption, the functional Eq. (6.30) becomes

$$\lambda\phi'(x)+\phi^2(x)-2f(x)-f(0)+y\left[\frac{\lambda}{2}\phi''(x)+\phi'(x)\phi(x)-f'(x)\right] \\ +y^2\left[\frac{\lambda}{6}\phi'''(x)-a\phi'(x)+\frac{1}{2}\phi''(x)\phi(x)-\frac{1}{2}f''(x)-\frac{1}{2}f''(0)\right]+\cdots=0. \tag{6.34}$$

The requirement of the nullity of the zeroth-order term leads to

$$2f(x)=\lambda\phi'(x)+\phi^2(x)-f(0). \tag{6.35}$$

Inserting here the expansion (6.33) with $y=x$, performing the derivative and at the end taking the $x\to 0$ limit, we obtain $f(0)=-\lambda a$ (we shall apply this procedure also below). The nullity of the first-order term in Eq. (6.34),

$$f'(x)=\frac{\lambda}{2}\phi''(x)+\phi'(x)\phi(x), \tag{6.36}$$

is ensured by the previous relation (6.35). The differentiation of this equation leads to

$$f''(x)=\frac{\lambda}{2}\phi'''(x)+\phi''(x)\phi(x)+\left[\phi'(x)\right]^2, \tag{6.37}$$

from which we find that $f''(0) = -5\lambda b + a^2$. Setting to zero the second-order term in Eq. (6.34), after some simple algebra we arrive at

$$\lambda\phi'''(x) + 6\left[\phi'(x) + a\right]^2 - 30\lambda b = 0. \tag{6.38}$$

This equation can be multiplied by $\phi''(x)$ and then integrated, with the result

$$\frac{\lambda}{2}\left[\phi''(x)\right]^2 + 2\left[\phi'(x) + a\right]^3 - 30\lambda b\left[\phi'(x) + a\right] = -70\lambda^2 c. \tag{6.39}$$

The most general solution of this differential equation is

$$\phi'(x) + a = -\lambda\wp(x, q), \tag{6.40}$$

where $\wp(x, q)$ is the Weierstrass elliptic function (see Appendix B) with the complex periods $2\omega_1$ and $2\omega_2$, and the "nome"

$$q = \exp\left[\mathrm{i}\pi\left(\frac{\omega_1}{\omega_2}\right)\right], \qquad \mathrm{Im}\left(\frac{\omega_1}{\omega_2}\right) \neq 0. \tag{6.41}$$

The periods are related to our constants as follows

$$\frac{b}{\lambda} = \sum_{m,n}{}' \frac{1}{(m\omega_1 + n\omega_2)^4}, \qquad \frac{c}{\lambda} = \sum_{m,n}{}' \frac{1}{(m\omega_1 + n\omega_2)^6}, \tag{6.42}$$

where the symbol $\sum'_{m,n}$ means the summation over all combinations of integers m and n, except for $m = n = 0$. There exists another Weierstrass elliptic function $\zeta(x, q)$, such that

$$\frac{\partial\zeta(x, q)}{\partial x} = -\wp(x, q). \tag{6.43}$$

It follows from Eq. (6.40) that

$$\phi(x) = -ax + \lambda\zeta(x, q). \tag{6.44}$$

For $|q| < 1$, $\zeta(x, q)$ can be represented as the series

$$\begin{aligned}\zeta(x, q) \quad = \quad & \frac{\zeta(\omega_1, q)}{\omega_1}x + \frac{\pi}{2\omega_1}\cot\left(\frac{\pi x}{2\omega_1}\right) \\ & + \frac{2\pi}{\omega_1}\sin\left(\frac{\pi x}{\omega_1}\right)\sum_{n=1}^{\infty}\frac{q^{2n}}{1 - 2q^{2n}\cos(\pi x/\omega_1) + q^{4n}}.\end{aligned} \tag{6.45}$$

The requirement of the periodicity $\phi(x+L) = \phi(x)$ leads to $\lambda\zeta(\omega_1, q)/\omega_1 - a = 0$ and $2\omega_1 = L$. Thus, the most general solution of the functional equation (6.30) for $\phi(x)$ reads

$$\frac{\phi(x)}{\lambda} = \frac{\pi}{L}\cot\left(\frac{\pi x}{L}\right) + \frac{4\pi}{L}\sin\left(\frac{2\pi x}{L}\right)\sum_{n=1}^{\infty}\frac{q^{2n}}{1 - 2q^{2n}\cos(2\pi x/L) + q^{4n}}. \tag{6.46}$$

Based on the formula (6.27), the building element $\psi(x)$ of the ground-state wavefunction takes the form

$$\psi(x) = \vartheta_1^{\lambda}\left(\frac{\pi x}{L}, q\right), \tag{6.47}$$

where ϑ_1 is a Jacobi theta function.

In what follows, we shall restrict ourselves to the trigonometric case when the nome $q \to 0$,

$$\phi(x) = \lambda \frac{\pi}{L} \cot\left(\frac{\pi x}{L}\right), \qquad \psi(x) = \sin^{\lambda}\left(\frac{\pi x}{L}\right). \tag{6.48}$$

Note that the $\lambda = 1$ case corresponds to free fermions or hard-core bosons. It is easy to verify that the functional equation (6.30) is indeed satisfied and that

$$f(x) = -\frac{1}{3}\left(\frac{\pi\lambda}{L}\right)^2. \tag{6.49}$$

The pair potential, which turns out to be the expected periodic version of the $1/x^2$ potential, and the ground-state energy can be obtained from Eq. (6.31),

$$v(x) = \frac{2\lambda(\lambda-1)\pi^2}{L^2 \sin^2(\pi x/L)}, \qquad E_0 = \frac{1}{3}\left(\frac{\pi\lambda}{L}\right)^2 N(N^2-1). \tag{6.50}$$

The division of the constant terms between the potential and E_0 is determined by the requirement that for an infinite line $v(x) \to 0$ as $x \to \infty$.

6.3 Excited states for the trigonometric case

For the Hamiltonian (6.25) with the trigonometric potential (6.50), we shall look for the excited states Ψ in the ansatz form

$$\Psi(x_1, \ldots, x_N) = \Psi_0(x_1, \ldots, x_N)\tilde{\Psi}(x_1, \ldots, x_N), \tag{6.51}$$

i.e. we build the elementary excitations on the "background" of the ground-state wavefunction Ψ_0 of the product form (6.24). Let us consider the Schrödinger equation for the excited states, $H\Psi = E\Psi$, and divide it by the nodeless Ψ_0. In this way, we obtain the Schrödinger equation obeyed by $\tilde{\Psi}$:

$$\mathcal{H}\tilde{\Psi} = (E - E_0)\tilde{\Psi}, \tag{6.52}$$

where $\mathcal{H}$ is defined as a similarity transformation of the Hamiltonian H,

$$\Psi_0^{-1} H \Psi_0 = \mathcal{H} + E_0. \tag{6.53}$$

For the ground-state wavefunction of the product form (6.24), the Hamiltonian $\mathcal{H}$ is expressible as $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$ with

$$\mathcal{H}_1 = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}, \qquad \mathcal{H}_2 = -2 \sum_{j>k=1}^{N} \phi(x_j - x_k)\left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k}\right). \tag{6.54}$$

Since the transformation (6.53) is not unitary, $\mathcal{H}$ is not hermitian, however, the energies in (6.52) are real.

Let us consider particles with Bose statistics and express $\mathcal{H}$ in a basis of (unnormalized) free boson eigenstates

$$\Psi_b(x|n) = \sum_{P\in S_N} \exp\left(\frac{2\pi \mathrm{i}}{L}\sum_{j=1}^{N} n_{Pj}x_j\right). \tag{6.55}$$

Here, $n = (n_1, n_2, \ldots, n_N)$ represents a set of N ordered integers $n_1 \le n_2 \le \cdots \le n_N$ and the first summation runs over $N!$ permutations of numbers $(1, 2, \ldots, N)$.

The first term of $\mathcal{H}$, the free boson Hamiltonian $\mathcal{H}_1$, is diagonal in the chosen basis,

$$\mathcal{H}_1\Psi_b(x|n) = E_1(n)\Psi_b(x|n), \qquad E_1(n) = \left(\frac{2\pi}{L}\right)^2 \sum_{j=1}^{N} n_j^2. \tag{6.56}$$

The second term of $\mathcal{H}$, $\mathcal{H}_2$, acts on $\Psi_b(x|n)$ as follows

$$\mathcal{H}_2\Psi_b(x|n) = -\frac{2\pi\lambda}{L}\left[\cot\left(\pi\frac{x_2-x_1}{L}\right)\left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1}\right) + \text{all pairs}\right]\Psi_b(x|n). \tag{6.57}$$

Let us analyze in detail the result of the action of the particle $(1, 2)$ term on $\Psi_b(x|n)$,

$$-\frac{2\pi\lambda}{L}\cot\left(\pi\frac{x_2-x_1}{L}\right)\sum_{P}\frac{2\pi\mathrm{i}}{L}(n_{P2}-n_{P1})\exp\left(\frac{2\pi\mathrm{i}}{L}\sum_{j=1}^{N} n_{Pj}x_j\right). \tag{6.58}$$

For each permutation $P = (P1, P2, P3, \ldots, PN)$, there exists a conjugate one $P_{1,2}$ which differs from P only by exchange of $P1$ and $P2$ elements, $P_{1,2} = (P2, P1, P3, \ldots, PN)$. We group such pairs of permutations, introduce the notation $\varphi = 2\pi x/L$ and rewrite the expression (6.58) as

$$\sum_{\text{pairs of } P}\left(\frac{2\pi}{L}\right)^2 \lambda(n_{P1}-n_{P2})\frac{\mathrm{e}^{\mathrm{i}\varphi_1}+\mathrm{e}^{\mathrm{i}\varphi_2}}{\mathrm{e}^{\mathrm{i}\varphi_1}-\mathrm{e}^{\mathrm{i}\varphi_2}}\left[\mathrm{e}^{\mathrm{i}(n_{P1}\varphi_1+n_{P2}\varphi_2)}\right.$$
$$\left.-\mathrm{e}^{\mathrm{i}(n_{P2}\varphi_1+n_{P1}\varphi_2)}\right]\exp\left(\mathrm{i}\sum_{j=3}^{N} n_{Pj}\varphi_j\right). \tag{6.59}$$

If $n_{P1} = n_{P2}$, the contribution is equal to zero. Assuming that $n_{P1} > n_{P2}$ (the case $n_{P1} < n_{P2}$ can be treated analogously), we have

$$\begin{aligned}&\frac{\mathrm{e}^{\mathrm{i}\varphi_1}+\mathrm{e}^{\mathrm{i}\varphi_2}}{\mathrm{e}^{\mathrm{i}\varphi_1}-\mathrm{e}^{\mathrm{i}\varphi_2}}\left[\mathrm{e}^{\mathrm{i}(n_{P1}\varphi_1+n_{P2}\varphi_2)} - \mathrm{e}^{\mathrm{i}(n_{P2}\varphi_1+n_{P1}\varphi_2)}\right]\\ &\quad= \mathrm{e}^{\mathrm{i}(n_{P1}\varphi_1+n_{P2}\varphi_2)} + 2\mathrm{e}^{\mathrm{i}(n_{P1}-1)\varphi_1}\mathrm{e}^{\mathrm{i}(n_{P2}+1)\varphi_2} + \cdots\\ &\qquad\cdots + 2\mathrm{e}^{\mathrm{i}(n_{P2}+1)\varphi_1}\mathrm{e}^{\mathrm{i}(n_{P1}-1)\varphi_2} + \mathrm{e}^{\mathrm{i}(n_{P2}\varphi_1+n_{P1}\varphi_2)}.\end{aligned} \tag{6.60}$$

Because of the presence of the first and last terms in this equation we see that $\mathcal{H}_2$ has a diagonal element

$$E_2(n) = \left(\frac{2\pi}{L}\right)^2 \lambda \sum_{j>k=1}^{N} |n_j - n_k|. \tag{6.61}$$

The remaining elements are off-diagonal.

The general structure of off-diagonal elements of $\mathcal{H}_2$ is evident from the expansion (6.60). The nonzero off-diagonal elements of $\mathcal{H}_2$ connect the original state $n = (n_1, n_2, \ldots, n_N)$ with any other state $n' = (n'_1, n'_2, \ldots, n'_N)$ provided that n' can be reached from n by squeezing together a single pair (n_j, n_k). More precisely, assuming $n_j < n_k$, there exists a pair $n'_r \leq n'_s$ such that $n'_r = n_j + \delta$, $n'_s = n_k - \delta$ with $0 < \delta \leq (n_k - n_j)/2$; all other $N - 2$ elements of n and n' are identical. If n connects to n' then n' cannot connect to n, and we are allowed to write $n \to n'$. If there exists a sequence $n \to n'^1 \to n'^2 \to \cdots \to n'$ then we write $n > n'$; note that if $n \to n'$ then also $n > n'$. The operator $\mathcal{H}_2$ can be represented in the invariant Hilbert subspace of $\Psi_b(n)$ and all $\Psi_b(n')$ with $n' < n$ as the triangular matrix

$$\mathcal{H}_2(n)\Psi_b(n') = E_2(n)\delta_{nn'}\Psi_b(n) + \sum_{n''<n'} [\mathcal{H}_2]_{n'n''}\,\Psi_b(n''). \tag{6.62}$$

The eigenvalues of a triangular matrix are the diagonal elements. We therefore conclude that

$$\begin{aligned} E(n) - E_0 &= E_1(n) + E_2(n) \\ &= \left(\frac{2\pi}{L}\right)^2 \left[\sum_{j=1}^{N} n_j^2 + \lambda \sum_{j>k=1}^{N} |n_j - n_k|\right]. \end{aligned} \tag{6.63}$$

For the ordering of integers $n_1 \leq n_2 \leq \cdots \leq n_N$, after some algebra the energy $E(n)$ can be expressed as

$$E(n) = \sum_{j=1}^{N} k_j^2, \qquad k_j = \frac{2\pi}{L}\left[n_j - \frac{\lambda}{2}(N + 1 - 2j)\right]. \tag{6.64}$$

The integers $\{n_j\}$ are related to the usual quantum numbers $\{I_j\}$ via the relation (2.129). Thus, the ordered momenta $k_1 < k_2 < \cdots < k_N$ are given by

$$\begin{aligned} k_j L &= 2\pi\left[I_j - \frac{(\lambda - 1)}{2}(N + 1 - 2j)\right] \\ &= 2\pi I_j + \pi(\lambda - 1)\sum_{\substack{l=1 \\ (l\neq j)}}^{N} \mathrm{sign}(k_j - k_l). \end{aligned} \tag{6.65}$$

We recognize the Bethe equations with the phase shift $\theta(k) = \pi(\lambda - 1)\mathrm{sign}(k)$. The same phase shift was derived for the $1/x^2$ potential within framework of the asymptotic two-body scattering, see formula (6.23).

6.4 Ground-state energy and zero-temperature thermodynamics

The functional Eq. (3.38) for the probability distribution $\rho(k)$, taken with $\theta'(k) = 2\pi(\lambda - 1)\delta(k)$, reads

$$\rho(k) = \frac{1}{2\pi} - (\lambda - 1)\int_{-q}^{q} \mathrm{d}k'\, \delta(k - k')\rho(k'). \tag{6.66}$$

The δ-function is outside of the integration range for $|k| > q$, so that

$$\rho(k) = \begin{cases} 1/(2\pi\lambda), & |k| \le q \\ 1/(2\pi), & |k| > q. \end{cases} \tag{6.67}$$

The normalization condition for the particle density implies

$$q = \pi\lambda n. \tag{6.68}$$

The ground-state energy per unit length is given by

$$e_0 = \int_{-q}^{q} \mathrm{d}k\, k^2 \rho(k) = \frac{1}{3}(\pi\lambda)^2 n^3. \tag{6.69}$$

This result reproduces the thermodynamic limit of the previous one (6.50).

The chemical potential is calculated as follows

$$\mu = \frac{\partial e_0}{\partial n} = (\pi\lambda n)^2 \qquad (= q^2). \tag{6.70}$$

The distribution $\epsilon(k)$ satisfies the integral equation

$$\epsilon(k) = k^2 - \mu - (\lambda - 1)\int_{-q}^{q} \mathrm{d}k'\, \delta(k - k')\epsilon(k'). \tag{6.71}$$

The solution of this equation

$$\epsilon(k) = \begin{cases} (k^2 - q^2)/\lambda, & |k| < q \\ k^2 - q^2, & |k| > q. \end{cases} \tag{6.72}$$

fulfills the condition $\epsilon(\pm q) = 0$.

The group velocity of excitations, which is continuous function of k, and the sound velocity read

$$v(k) = 2k, \qquad v_s = 2q = 2\pi\lambda n. \tag{6.73}$$

To obtain dispersion relations for the elementary excitations, we first evaluate the function $f(k)$, for simplicity with $k > 0$,

$$f(k) = \int_0^k \mathrm{d}k'\, \rho(k') = \begin{cases} k/(2\pi\lambda), & 0 \le k \le q \\ q/(2\pi\lambda) + (k - q)/(2\pi), & k > q. \end{cases} \tag{6.74}$$

For particle excitations with $k_p > q$, we have

$$\Delta E(k_p) = k_p^2 - q^2, \qquad K(k_p) = k_p - q. \tag{6.75}$$

Eliminating k_p from these relation, we obtain the dispersion relation for particle excitations

$$\Delta E_p(K) = K^2 + 2\pi\lambda n|K|, \qquad -\infty < K < \infty. \tag{6.76}$$

Similarly, for hole excitations with $0 < k_h < q$, we have

$$\Delta E(k_h) = \frac{1}{\lambda}(q^2 - k_h^2), \qquad K(k_h) = \frac{1}{\lambda}(q - k_h), \tag{6.77}$$

which gives the dispersion relation for hole excitations

$$\Delta E_h(K) = \lambda(2\pi n|K| - K^2), \qquad -\pi n < K < \pi n. \tag{6.78}$$

The pressure is calculated as follows

$$P = \mu n - e_0 = \frac{2}{3}(\pi\lambda)^2 n^3. \tag{6.79}$$

The same result is obtained by substituting $\epsilon(k)$ (6.72) in the formula (3.25) for the pressure.

6.5 Finite-temperature thermodynamics

Considering $\theta'(k) = 2\pi(\lambda - 1)\delta(k)$ in the integral Eq. (4.24), the distribution function $\epsilon(k)$ is determined by

$$k^2 - \mu = \epsilon(k) - \frac{\lambda - 1}{\beta} \ln\left[1 + \mathrm{e}^{-\beta\epsilon(k)}\right] \tag{6.80}$$

or, equivalently, by

$$\mathrm{e}^{-\beta\epsilon(k)} \left[1 + \mathrm{e}^{-\beta\epsilon(k)}\right]^{\lambda-1} = z\mathrm{e}^{-\beta k^2}. \tag{6.81}$$

The temperature serves only as a scale factor in this equation. The scaling function $w(k, z) = \exp[-\beta\epsilon(k/\sqrt{\beta})]$ satisfies the following equation

$$w(k, z)[1 + w(k, z)]^{\lambda-1} = z\mathrm{e}^{-k^2}. \tag{6.82}$$

The expression for the pressure (4.31) takes the scaling form

$$P = \frac{1}{\beta^{3/2}} p(z, \lambda), \qquad p(z, \lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k \ \ln[1 + w(k, z)]. \tag{6.83}$$

The formula for the equilibrium distribution ρ_{eq} follows from Eq. (4.25):

$$\rho_{\mathrm{eq}}(k) = \frac{1}{2\pi} \frac{\mathrm{e}^{-\beta\epsilon(k)}}{1 + \lambda\mathrm{e}^{-\beta\epsilon(k)}}. \tag{6.84}$$

Consequently,

$$n = \frac{1}{2\pi\beta^{1/2}} \int_{-\infty}^{\infty} \mathrm{d}k \ \frac{w(k, z)}{1 + \lambda w(k, z)}, \tag{6.85}$$

$$e = \frac{1}{2\pi\beta^{3/2}} \int_{-\infty}^{\infty} \mathrm{d}k \ k^2 \frac{w(k, z)}{1 + \lambda w(k, z)}. \tag{6.86}$$

Performing in (6.83) an integration by parts and using the relation

$$\frac{\partial w}{\partial k} = -2k \frac{w(1 + w)}{1 + \lambda w} \tag{6.87}$$

obtained from the definition (6.82) of w, we get the Bernoulli equation

$$P = 2e, \tag{6.88}$$

valid for all temperatures and fugacities.

The transcendental equation (6.82) cannot be explicitly inverted, except for special values of $\lambda = 0, 1, 1/2, 2$. The inversion of (6.82) can be accomplished for general λ in the fugacity series format [25], with the following result for $p(z, \lambda)$,

$$p(z,\lambda) = \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}\Gamma(j\lambda)}{\sqrt{j}j!\Gamma[j(\lambda-1)+1]} z^j. \tag{6.89}$$

QUANTUM INVERSE SCATTERING METHOD

7 QISM: Yang-Baxter equation

We shall now pass to systems of identical particles possessing internal degrees of freedom, sometimes called "colours", $\sigma = 1, \ldots, l$. For example, in the case of an electron σ denotes one of two possible spin states $\{\uparrow, \downarrow\}$. Before solving explicitly some of the models with coloured particles, we describe the method based on the concept of the scattering S-matrix, known as the algebraic Bethe ansatz or the Quantum Inverse Scattering Method (QISM). Although QISM looks a bit abstract, it represents a powerful mean for studying integrable models. I recommend the reader to read carefully this and the next two sections because a deep understanding of the QISM will enable him to deal with complicated physical systems on universal ground.

7.1 Generalized Bethe ansatz

In section 2, we studied one-dimensional systems of spinless particles with δ-function interactions. In both fermion and boson cases, the wavefunction in the fundamental ordering sector I: $x_1 < x_2 < \cdots < x_N$ was expressible as the superposition of plane waves

$$\psi_I(x_1, x_2, \ldots, x_N) = \sum_{P \in S_N} \mathrm{sign}(P) A(k_{P1}, k_{P2}, \ldots, k_{PN}) \exp\left(\mathrm{i} \sum_{j=1}^{N} k_{Pj} x_j \right), \quad (7.1)$$

where the permutations P distribute the given set of wave numbers $(k_1, k_2, \ldots, k_N)$ among the particle coordinates. The system of N coloured particles on a line is characterized by the couples of data $(\sigma_1, x_1), (\sigma_2, x_2), \ldots, (\sigma_N, x_N)$. Like in the case of spinless particles, one can define the ordering sector of particle coordinates Q according to the prescription (2.5). If the wavefunction of coloured particles in the ordering sector I is a superposition of plane waves, its most general form reads

$$\begin{aligned} \psi_I(\sigma_1, x_1; \sigma_2, x_2; \ldots; \sigma_N, x_N) \quad = \quad & \sum_{P \in S_N} \mathrm{sign}(P) A_{\sigma_1 \sigma_2 \ldots \sigma_N}(k_{P1}, k_{P2}, \ldots, k_{PN}) \\ & \times \exp\left(\mathrm{i} \sum_{j=1}^{N} k_{Pj} x_j \right). \end{aligned} \quad (7.2)$$

Here, the A-coefficients depend, besides the permuted wave numbers, on the ordered sequence of particle states $\sigma_1, \sigma_2, \ldots, \sigma_N$. The formula (7.2) reduces to the previous one (7.1) in the case of spinless one-state particles since then the subscripts $\sigma_1 \sigma_2 \ldots \sigma_N \equiv 11 \ldots 1$ are superfluous and can be omitted.

In the ordering sector defined by the permutation Q, the wavefunction ψ_Q is expressible in terms of ψ_I as follows

$$\psi_Q(\sigma_1, x_1; \sigma_2, x_2; \ldots; \sigma_N, x_N) = (\pm 1)^{\eta_Q} \psi_I(\sigma_{Q1}, x_{Q1}; \sigma_{Q2}, x_{Q2}; \ldots; \sigma_{QN}, x_{QN}), \quad (7.3)$$

where the $+/-$ sign corresponds to Boson/Fermi statistics. We respect in this formula the rule that an exchange of particle coordinates must be accompanied by the corresponding exchange of particle states.

Eqs. (7.2) and (7.3) imply the form of the generalized Bethe ansatz. For fermions, in the ordering Q-sector we have

$$\psi_Q(\sigma_1, x_1; \sigma_2, x_2; \ldots; \sigma_N, x_N) = \sum_{P \in S_N} [Q, P]_f \exp\left(\mathrm{i} \sum_{j=1}^{N} k_{Pj} x_{Qj} \right), \tag{7.4}$$

where

$$[Q, P]_f = \mathrm{sign}(Q)\mathrm{sign}(P) A_{\sigma_{Q1}\sigma_{Q2}\ldots\sigma_{QN}}(k_{P1}, k_{P2}, \ldots, k_{PN}). \tag{7.5}$$

For bosons, we have

$$\psi_Q(\sigma_1, x_1; \sigma_2, x_2; \ldots; \sigma_N, x_N) = \sum_{P \in S_N} [Q, P]_b \exp\left(\mathrm{i} \sum_{j=1}^{N} k_{Pj} x_{Qj} \right), \tag{7.6}$$

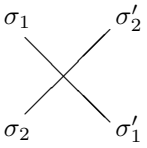
where

$$[Q, P]_b = \mathrm{sign}(P) A_{\sigma_{Q1}\sigma_{Q2}\ldots\sigma_{QN}}(k_{P1}, k_{P2}, \ldots, k_{PN}). \tag{7.7}$$

In both fermion and boson cases, the A-amplitudes in different (P, Q) permutation sectors are related via the scattering matrix. Namely, for $N = 2$ particles we have

$$A_{\sigma_j \sigma_i}(k_v, k_u) = \sum_{\sigma_i' \sigma_j'} S^{\sigma_i \sigma_j}_{\sigma_i' \sigma_j'}(k_u, k_v) A_{\sigma_i' \sigma_j'}(k_u, k_v), \tag{7.8}$$

where $(i, j), (u, v) \in \{(12); (21)\}$ and S denotes the two-particle scattering matrix of dimension l^2. The scattering is elastic, i.e. not only the total momentum but also both individual momenta are conserved. Note however that particles can change their σ-colours in the scattering process. The two-particle S-matrix is usually represented graphically as follows

$S^{\sigma_1 \sigma_2}_{\sigma_1' \sigma_2'}(k_1, k_2) =$  . (7.9)

The following consistency conditions for the S-matrix are required:

- Normalization: Setting $k_u = k_v = k$ in (7.8) yields

$$S(k, k) = \mathcal{P}, \qquad \mathcal{P}^{\sigma_1 \sigma_2}_{\sigma_1' \sigma_2'} = \delta(\sigma_1, \sigma_2') \delta(\sigma_2, \sigma_1'). \tag{7.10}$$

  Here, $\mathcal{P}$ is the transposition or permutation operator, such that $\mathcal{P}^2 = I$ where I is the unity matrix of dimension l^2.

- Unitarity: Applying twice the operation (7.8) implies

$$S(k_1, k_2) S(k_2, k_1) = I. \tag{7.11}$$

- We shall also assume the T and P invariance of $S(k_1, k_2)$:

$$S^{\sigma_1\sigma_2}_{\sigma_1'\sigma_2'}(k_1,k_2) = S^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(k_1,k_2) = S^{\sigma_2\sigma_1}_{\sigma_2'\sigma_1'}(k_1,k_2). \tag{7.12}$$

For an arbitrary number of particles N, the two-particle S-matrix relates the A-amplitudes in the permutation sectors (Q, P) and $(\tilde{Q}, \tilde{P})$ which differ from one another only by the transposition of a pair of nearest neighbors, i.e. $\tilde{Q} = Q_{j,j+1}$ and $\tilde{P} = P_{j,j+1}$ $(j = 1, 2, \ldots, N-1)$. This can be expressed schematically as follows

$$A_{\ldots\sigma_j\sigma_i\ldots}(\ldots k_v, k_u \ldots) = \sum_{\sigma_i'\sigma_j'} S^{\sigma_i\sigma_j}_{\sigma_i'\sigma_j'}(k_u,k_v) A_{\ldots\sigma_i'\sigma_j'\ldots}(\ldots k_u, k_v \ldots). \tag{7.13}$$

Applying successively the nearest-neighbor transposition rule, one can convert an arbitrary amplitude $A_{\sigma_{Q1}\sigma_{Q2}\ldots\sigma_{QN}}(k_{P1}, k_{P2}, \ldots, k_{PN})$ to the one with $Q = I$. In this way, the scattering of N particles factorizes into a product of two-particle scatterings which is the fundamental property of integrable systems.

7.2 Derivation of the Yang-Baxter equation

Possible forms of the S-matrix are very limited for integrable systems. There exists a general constraint among the elements of the S-matrix which can be deduced from the scattering of three particles.

Let us study the scattering process of three particles which starts from the initial state $x_3 \le x_2 \le x_1$, corresponding to $Q = (3, 2, 1)$, and ends in the final state $x_1 \le x_2 \le x_3$, corresponding to $Q = I \equiv (1, 2, 3)$. There are two possible realizations of this three-particle scattering in terms of the two-particle scatterings:

$$(a) \qquad (3,2,1) \to (3,1,2) \to (1,3,2) \to (1,2,3);$$
$$(b) \qquad (3,2,1) \to (2,3,1) \to (2,1,3) \to (1,2,3).$$

Using the prescription (7.13), the (a) sequence of two-particle scatterings is expressible as

$$\begin{aligned} A_{\sigma_3\sigma_2\sigma_1}(k_3,k_2,k_1) \;=\; & \sum_{\sigma_1'\sigma_2'} S^{\sigma_1\sigma_2}_{\sigma_1'\sigma_2'}(k_1,k_2)\, \underbrace{A_{\sigma_3\sigma_1'\sigma_2'}(k_3,k_1,k_2)} \\ & \sum_{\sigma_3'\sigma_1''} S^{\sigma_1'\sigma_3}_{\sigma_1''\sigma_3'}(k_1,k_3)\, \underbrace{A_{\sigma_1''\sigma_3'\sigma_2'}(k_1,k_3,k_2)} \\ & \sum_{\sigma_2''\sigma_3''} S^{\sigma_2'\sigma_3'}_{\sigma_2''\sigma_3''}(k_2,k_3) A_{\sigma_1''\sigma_2''\sigma_3''}(k_1,k_2,k_3), \end{aligned} \tag{7.14}$$

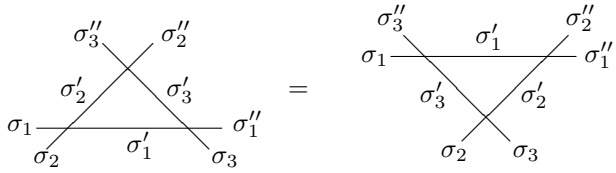
while the (b) sequence is identified with

$$\begin{aligned} A_{\sigma_3\sigma_2\sigma_1}(k_3,k_2,k_1) \;=\; & \sum_{\sigma_2'\sigma_3'} S^{\sigma_2\sigma_3}_{\sigma_2'\sigma_3'}(k_2,k_3)\, \underbrace{A_{\sigma_2'\sigma_3'\sigma_1}(k_2,k_3,k_1)} \\ & \sum_{\sigma_1'\sigma_3''} S^{\sigma_1\sigma_3'}_{\sigma_1'\sigma_3''}(k_1,k_3)\, \underbrace{A_{\sigma_2'\sigma_1'\sigma_3''}(k_2,k_1,k_3)} \\ & \sum_{\sigma_1''\sigma_2''} S^{\sigma_1'\sigma_2'}_{\sigma_1''\sigma_2''}(k_1,k_2) A_{\sigma_1''\sigma_2''\sigma_3''}(k_1,k_2,k_3). \end{aligned} \tag{7.15}$$

The final result must be the same for both sequences of two-particle scatterings, which implies the following constraint for the elements of the S-matrix:

$$\sum_{\sigma_1'\sigma_2'\sigma_3'} S^{\sigma_1\sigma_2}_{\sigma_1'\sigma_2'}(k_1,k_2) S^{\sigma_1'\sigma_3}_{\sigma_1''\sigma_3'}(k_1,k_3) S^{\sigma_2'\sigma_3'}_{\sigma_2''\sigma_3''}(k_2,k_3)$$
$$= \sum_{\sigma_1'\sigma_2'\sigma_3'} S^{\sigma_2\sigma_3}_{\sigma_2'\sigma_3'}(k_2,k_3) S^{\sigma_1\sigma_3'}_{\sigma_1'\sigma_3''}(k_1,k_3) S^{\sigma_1'\sigma_2'}_{\sigma_1''\sigma_2''}(k_1,k_2). \tag{7.16}$$

This overcomplete set amounts to l^6 homogeneous equations for l^4 unknowns scattering-matrix elements. Eq. (7.16) is the Yang-Baxter equation (YBE). It can be represented graphically as follows



. (7.17)

The importance of the YBE as the integrability condition consists in the fact that its validity ensures the equivalence of all possible multiparticle scattering processes, independently of the order in which the two-particle scatterings are performed. This is due to the fact that every multiparticle scattering can be decomposed onto elementary three-particle scatterings.

There exist particle systems which scattering S-matrix depends only on the difference of quasi-momenta, $S(k_u,k_v) \equiv S(k_u - k_v)$. If it is not so, there always exists a parametrization of wave numbers in terms of the spectral parameters (rapidities) λ, $k = k(\lambda)$, such that $S(k_u,k_v) = S(\lambda_u - \lambda_v)$. In terms of the spectral parameters, the YBE (7.16) can be written as

$$\sum_{\sigma_1'\sigma_2'\sigma_3'} S^{\sigma_1\sigma_2}_{\sigma_1'\sigma_2'}(\lambda-\mu) S^{\sigma_1'\sigma_3}_{\sigma_1''\sigma_3'}(\lambda) S^{\sigma_2'\sigma_3'}_{\sigma_2''\sigma_3''}(\mu) = \sum_{\sigma_1'\sigma_2'\sigma_3'} S^{\sigma_2\sigma_3}_{\sigma_2'\sigma_3'}(\mu) S^{\sigma_1\sigma_3'}_{\sigma_1'\sigma_3''}(\lambda) S^{\sigma_1'\sigma_2'}_{\sigma_1''\sigma_2''}(\lambda-\mu). \tag{7.18}$$

We would like emphasize that the S-matrices in this equation differ from each other only by the value of the spectral parameter, all other parametrizations by some variables must be taken at the same point in the parametric space.

Let V be an l-dimensional complex vector space. Upon taking a basis $\{e_\sigma, \sigma = 1,2,\ldots,l\}$ of V and writing

$$S(\lambda)\left(e_{\sigma_1} \otimes e_{\sigma_2}\right) = \sum_{\sigma_1'\sigma_2'} S^{\sigma_1\sigma_2}_{\sigma_1'\sigma_2'}(\lambda)\left(e_{\sigma_1'} \otimes e_{\sigma_2'}\right), \tag{7.19}$$

the YBE (7.18) can be reexpressed as a three-site equation

$$S_{12}(\lambda-\mu) S_{13}(\lambda) S_{23}(\mu) = S_{23}(\mu) S_{13}(\lambda) S_{12}(\lambda-\mu). \tag{7.20}$$

Here, S_{ij} signifies the matrix on $V_1 \otimes V_2 \otimes V_3$, acting as $S(\lambda)$ on the ith and jth components and as identity operator on the remaining component, e.g. $S_{23}(\lambda) = I \otimes S(\lambda)$. The YBE is completed by the *initial condition*

$$S(0) = \mathcal{P} \tag{7.21}$$

having its origin in the normalization (7.10). Under this condition, the YBE (7.20) is identically satisfied for $\lambda = \mu = 0$ since for permutation operators it holds

$$\mathcal{P}_{12}\mathcal{P}_{13}\mathcal{P}_{23} = \mathcal{P}_{23}\mathcal{P}_{13}\mathcal{P}_{12} = \mathcal{P}_{13}. \tag{7.22}$$

In terms of the rapidities, the unitarity condition (7.11) reads

$$S(\lambda)S(-\lambda) = I. \tag{7.23}$$

The conditions (7.21) and (7.23) have to be satisfied by the scattering matrices of true physical systems. When we look for the mathematical solutions of the YBE (7.18), these are determined by this equation up to a multiplication function $\rho(\lambda)$. Then, the initial condition (7.21) will be equivalent to

$$S(0) = \rho(0)\mathcal{P} \tag{7.24}$$

and the unitarity condition (7.23) will be equivalent to

$$S(\lambda)S(-\lambda) = \rho(\lambda)\rho(-\lambda)I. \tag{7.25}$$

It is useful to introduce a permuted R matrix as follows[2]

$$R(\lambda) = \mathcal{P}S(\lambda); \qquad R(0) = I. \tag{7.26}$$

The permutation operator $\mathcal{P}$ maps $(e_{\sigma_1} \otimes e_{\sigma_2}) \to (e_{\sigma_2} \otimes e_{\sigma_1})$. Thus,

$$R(\lambda)\,(e_{\sigma_1} \otimes e_{\sigma_2}) = \sum_{\sigma_1'\sigma_2'} S^{\sigma_1\sigma_2}_{\sigma_1'\sigma_2'}(\lambda)\left(e_{\sigma_2'} \otimes e_{\sigma_1'}\right) \tag{7.27}$$

and the entries of the S and R matrices are related by $R^{\sigma_1\sigma_2}_{\sigma_1'\sigma_2'} = S^{\sigma_1\sigma_2}_{\sigma_2'\sigma_1'}$. The YBE (7.18) can be rewritten in terms of the R-matrix as

$$(I \otimes R(\lambda-\mu))(R(\lambda) \otimes I)(I \otimes R(\mu)) = (R(\mu) \otimes I)(I \otimes R(\lambda))(R(\lambda-\mu) \otimes I). \tag{7.28}$$

This form of the YBE is closely related to a braid group B_N [67] which is a generalization of the symmetric group S_N of all permutations of N objects. Braid groups have many applications in mathematics (knot theory [68]) and physics (statistical mechanics, two-dimensional conformal field theory). An element of the braid group B_N is a system of N strings joining two sets of N points located on two parallel, top and bottom, lines. If n and $n+1$ are two consecutive points on the top and bottom lines, the string starting at n on the top line can reach $n+1$ on the bottom line by either under-crossing or over-crossing the string starting at $n+1$ on the top line and reaching n on the bottom line; the corresponding elements of the braid group are denoted by σ_n and σ_n^{-1}, respectively (see Fig. 7.1). The generators of the braid group σ_n $(n = 1, 2, \ldots, N-1)$ satisfy the relations

$$\begin{aligned} \sigma_n\sigma_{n+1}\sigma_n &= \sigma_{n+1}\sigma_n\sigma_{n+1}, \\ \sigma_n\sigma_m &= \sigma_m\sigma_n \quad \text{for } |n-m| \geq 2, \\ \sigma_n\sigma_n^{-1} &= \sigma_n^{-1}\sigma_n = I. \end{aligned} \tag{7.29}$$

[2] in some works the scattering matrix S is denoted as R and our R matrix is denoted as $\check{R}$

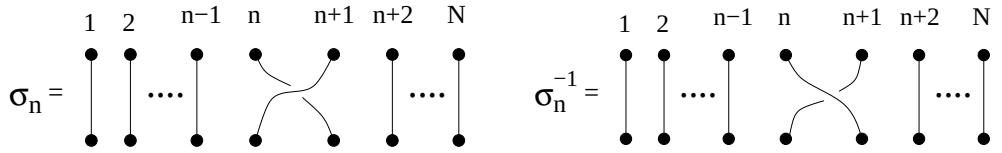


Fig. 7.1. Generators of the braid group

These relations reflect an equivalent topology of the objects generated by the given sequences of operations. In order to make a connection between the YBE (7.28) and the braid relations (7.29), we introduce the operators $R_n(\lambda)$ $(n = 1, 2, \ldots, N-1)$ defined on the Hilbert space $\otimes_{n=1}^{N} V_n$, which act as $R(\lambda)$ on the spaces $V_n \otimes V_{n+1}$ and as the identity operator elsewhere:

$$R_n(\lambda) = I \otimes \cdots \otimes I \otimes \underbrace{R(\lambda)}_{(n,n+1)} \otimes I \otimes \cdots \otimes I. \tag{7.30}$$

Equation (7.28) then becomes

$$R_{n+1}(\lambda - \mu) R_n(\lambda) R_{n+1}(\mu) = R_n(\mu) R_{n+1}(\lambda) R_n(\lambda - \mu), \tag{7.31}$$

while

$$R_n(\lambda) R_m(\mu) = R_m(\mu) R_n(\lambda) \quad \text{for } |n - m| \geq 2. \tag{7.32}$$

The identification of the YBE (7.31) and (7.32) with the braid group relations (7.29) is not yet possible due to the presence of rapidity variables. This problem is avoided when we set $\lambda = \mu = \lambda - \mu$. These equalities have two solutions:

- $\lambda = \mu = 0$,
- $\lambda = 2\mu, |\mu| = \infty$.

The first solution is trivial since $R(\lambda = 0) = I$. The second solution, known as the braid limit, is nontrivial. This solution depends on the particular form of the R-matrix and will be discussed later.

For an arbitrary number of internal particle states l, the simplest S-matrix can be searched in the ansatz form

$$S(\lambda) = b(\lambda) I + c(\lambda) \mathcal{P}, \tag{7.33}$$

where the functions $b(\lambda)$ and $c(\lambda)$ are as-yet unspecified. The initial condition (7.21) fixes the $\lambda = 0$ values of these functions as follows

$$b(0) = 0, \qquad c(0) = 1. \tag{7.34}$$

The unitarity condition (7.23) implies two constraints

$$b(\lambda) b(-\lambda) + c(\lambda) c(-\lambda) = 1, \qquad b(\lambda) c(-\lambda) + b(-\lambda) c(\lambda) = 0, \tag{7.35}$$

which take place for an arbitrary value of λ. Substituting the ansatz (7.33) into the YBE (7.20) and using for the permutation operators the equalities (7.22) and

$$\mathcal{P}_{12}\mathcal{P}_{13} = \mathcal{P}_{13}\mathcal{P}_{23} = \mathcal{P}_{23}\mathcal{P}_{12}, \qquad \mathcal{P}_{12}\mathcal{P}_{23} = \mathcal{P}_{23}\mathcal{P}_{13} = \mathcal{P}_{13}\mathcal{P}_{12}, \tag{7.36}$$

we obtain the only equation

$$\frac{b(\lambda)}{c(\lambda)} = \frac{b(\mu)}{c(\mu)} + \frac{b(\lambda-\mu)}{c(\lambda-\mu)}. \tag{7.37}$$

The solution of this equation is $b(\lambda)/c(\lambda) = \lambda/\alpha$, where α is a complex constant which will be set to unity for simplicity. With regard to the conditions (7.34) and (7.35), we arrive at $b(\lambda) = \lambda/\sqrt{1-\lambda^2}$ and $c(\lambda) = 1/\sqrt{1-\lambda^2}$ $(\lambda < 1)$, so that, with the proper normalization,

$$S(\lambda) = \frac{1}{\sqrt{1-\lambda^2}}\left(\mathcal{P} + \lambda I\right), \qquad R(\lambda) = \frac{1}{\sqrt{1-\lambda^2}}\left(I + \lambda\mathcal{P}\right). \tag{7.38}$$

7.3 Lax operators, transfer and monodromy matrices

Let us now forget for a while about the origin of the "small" scattering S-matrix of dimension l^2 and use it as the building element of "large" matrices, via tensor products formulated on a chain of N sites $n = 1, 2, \ldots, N$ plus two auxiliary sites denoted as ξ and η (we could denote them as 1 and 2, as is usual, but this might interfere with the notation of chain sites). In this part, we introduce a hierarchy of the large matrices and derive for them relations which have the origin in the YBE (7.18).

• *Lax operators* $L_{\xi n}$ $(n = 1, 2, \ldots, N)$ are defined as follows

$$L_{\xi n}(\lambda)^{\gamma_\xi \sigma_1 \ldots \sigma_N}_{\gamma'_\xi \sigma'_1 \ldots \sigma'_N} = S^{\gamma_\xi \sigma_n}_{\gamma'_\xi \sigma'_n}(\lambda)\delta_{\sigma_1\sigma'_1}\ldots\delta_{\sigma_{n-1}\sigma'_{n-1}}\delta_{\sigma_{n+1}\sigma'_{n+1}}\ldots\delta_{\sigma_N\sigma'_N}, \tag{7.39}$$

where the state indices $(\gamma_\xi, \gamma'_\xi)$ of the auxiliary site ξ also run over l possible values $1, \ldots, l$. Since each of the column $\{\sigma_1, \ldots, \sigma_N\}$ or row $\{\sigma'_1, \ldots, \sigma'_N\}$ indices can take l^N values, the dimension of the Lax operator is l^{N+1}. The YBE (7.18) for the S-matrix can be transcribed in terms of Lax operators as follows

$$S_{\xi\eta}(\lambda-\mu)L_{\xi n}(\lambda)L_{\eta n}(\mu) = L_{\eta n}(\mu)L_{\xi n}(\lambda)S_{\xi\eta}(\lambda-\mu). \tag{7.40}$$

Here, the ordinary products of L matrices are performed in the space of N chain sites $1, 2, \ldots, N$ and two auxiliary ξ, η sites; like for example, the matrix element on the lhs has to be understood in the following way:

$$\begin{aligned}[S_{\xi\eta}(\lambda-\mu)L_{\xi n}(\lambda)L_{\eta n}(\mu)]^{\gamma_\xi\gamma_\eta\{\sigma_1\ldots\sigma_N\}}_{\gamma'_\xi\gamma'_\eta\{\sigma'_1\ldots\sigma'_N\}} = \sum_{\gamma''_\xi,\gamma''_\eta}\sum_{\{\sigma''_1,\ldots,\sigma''_N\}} & S_{\xi\eta}(\lambda-\mu)^{\gamma_\xi\gamma_\eta}_{\gamma''_\xi\gamma''_\eta} \\ & \times L_{\xi n}(\lambda)^{\gamma''_\xi\{\sigma_1\ldots\sigma_N\}}_{\gamma'_\xi\{\sigma''_1\ldots\sigma''_N\}} L_{\eta n}(\mu)^{\gamma''_\eta\{\sigma''_1\ldots\sigma''_N\}}_{\gamma'_\eta\{\sigma'_1\ldots\sigma'_N\}}.\end{aligned} \tag{7.41}$$

Eq. (7.40) can be understood as the standard matrix equation if we define $L_{\xi n}(\lambda) = L_n(\lambda) \otimes I$ and $L_{\eta n}(\mu) = I \otimes L_n(\mu)$; since the elements of the matrix L_n are themselves operators, it holds

that $L_{\xi n}(\lambda)L_{\eta n}(\mu) \neq L_{\eta n}(\mu)L_{\xi n}(\lambda)$. Applying the permutation operator $\mathcal{P}_{\xi\eta}$ to both sides of Eq. (7.41), in terms of the R matrix (7.26) we get

$$R(\lambda-\mu)\,[L_n(\lambda)\otimes L_n(\mu)] = [L_n(\mu)\otimes L_n(\lambda)]\,R(\lambda-\mu). \tag{7.42}$$

Here, the ordinary and tensor products are considered in the (ξ,η) space; like for example, the lhs of Eq. (7.42) has to be understood in the following way:

$$\sum_{\gamma''_\xi,\gamma''_\eta} R(\lambda-\mu)^{\gamma_\xi\gamma_\eta}_{\gamma''_\xi\gamma''_\eta} L_n(\lambda)^{\gamma''_\xi}_{\gamma'_\xi} L_n(\mu)^{\gamma''_\eta}_{\gamma'_\eta}, \tag{7.43}$$

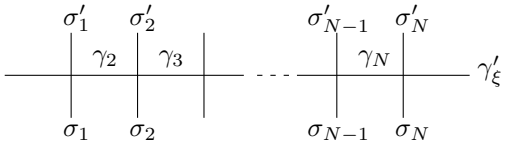
where the Lax operators have dimension l^N in the auxiliary space of sites ξ or η.

• *Monodromy matrix* $\mathcal{T}_\xi$ of dimension l^{N+1} is defined by

$$\mathcal{T}_\xi(\lambda)^{\gamma_\xi\sigma_1\ldots\sigma_N}_{\gamma'_\xi\sigma'_1\ldots\sigma'_N} = \sum_{\gamma_2,\ldots,\gamma_N} S^{\sigma_1\gamma_\xi}_{\sigma'_1\gamma_2}(\lambda)S^{\sigma_2\gamma_2}_{\sigma'_2\gamma_3}(\lambda)\cdots S^{\sigma_N\gamma_N}_{\sigma'_N\gamma'_\xi}(\lambda). \tag{7.44}$$

It can be represented graphically as follows

$\mathcal{T}_\xi(\lambda)^{\gamma_\xi\sigma_1\ldots\sigma_N}_{\gamma'_\xi\sigma'_1\ldots\sigma'_N} =$



. (7.45)

By an explicit evaluation of matrix products it can be shown that the monodromy matrix is expressible as the following product of $L_{\xi n}$-matrices

$$\mathcal{T}_\xi(\lambda) = L_{\xi 1}(\lambda)L_{\xi 2}(\lambda)\cdots L_{\xi N}(\lambda). \tag{7.46}$$

There exists an analogy of the relation (7.40) for the monodromy matrix:

$$S_{\xi\eta}(\lambda-\mu)\mathcal{T}_\xi(\lambda)\mathcal{T}_\eta(\mu) = \mathcal{T}_\eta(\mu)\mathcal{T}_\xi(\lambda)S_{\xi\eta}(\lambda-\mu). \tag{7.47}$$

To prove this relation, we take advantage of the fact that the matrices $\mathcal{L}_{\xi n}$ and $\mathcal{L}_{\eta m}$ commute for $n \neq m$ and write down

$$\mathcal{T}_\xi(\lambda)\mathcal{T}_\eta(\mu) = L_{\xi 1}(\lambda)L_{\eta 1}(\mu)\cdots L_{\xi N}(\lambda)L_{\eta N}(\mu). \tag{7.48}$$

Multiplying this equation from the left by $S_{\xi\eta}(\lambda-\mu)$ and then commuting successively $S_{\xi\eta}$ by using Eq. (7.40) leads to the relation (7.47). In terms of the R-matrix, Eq. (7.47) reads

$$R(\lambda-\mu)\,[\mathcal{T}(\lambda)\otimes\mathcal{T}(\mu)] = [\mathcal{T}(\mu)\otimes\mathcal{T}(\lambda)]\,R(\lambda-\mu). \tag{7.49}$$

As before, the ordinary and tensor products are performed in the auxiliary (ξ,η) space.

• *Transfer matrix* T of dimension l^N is defined as follows

$$T(\lambda)^{\sigma_1\ldots\sigma_N}_{\sigma'_1\ldots\sigma'_N} = \sum_{\gamma_1,\ldots,\gamma_N} S^{\sigma_1\gamma_1}_{\sigma'_1\gamma_2}(\lambda)S^{\sigma_2\gamma_2}_{\sigma'_2\gamma_3}(\lambda)\cdots S^{\sigma_N\gamma_N}_{\sigma'_N\gamma_1}(\lambda). \tag{7.50}$$

The transfer matrix is obtained from the graphical representation of the monodromy matrix (7.45) by connecting the free ends, i.e. setting $\gamma_\xi = \gamma'_\xi \equiv \gamma_1$ and summing over γ_1, creating in this way the circle. Algebraically, we have

$$T(\lambda) = \mathrm{Tr}_\xi \, \mathcal{T}_\xi(\lambda), \tag{7.51}$$

where $\mathrm{Tr}_\xi \cdots \equiv \sum_{\gamma_\xi, \gamma'_\xi} \delta_{\gamma_\xi \gamma'_\xi} \cdots$ denotes the trace in the auxiliary ξ-space.

We shall finally prove the commutation property of the set of transfer matrices $\{T(\lambda)\}$ labeled by the spectral parameter λ. Multiplying both sides of Eq. (7.49) from the right by the inverse matrix $R^{-1}(\lambda - \mu)$ results in

$$R(\lambda - \mu)\,[\mathcal{T}(\lambda) \otimes \mathcal{T}(\mu)]\, R^{-1}(\lambda - \mu) = \mathcal{T}(\mu) \otimes \mathcal{T}(\lambda). \tag{7.52}$$

Let us trace both sides of this equation in the auxiliary ξ and η spaces. The lhs then reads

$$\begin{aligned} \sum_{\gamma_\xi, \gamma_\eta} \sum_{\substack{\alpha, \alpha' \\ \beta, \beta'}} & R(\lambda - \mu)^{\gamma_\xi \gamma_\eta}_{\alpha\beta} \mathcal{T}(\lambda)^{\alpha}_{\alpha'} \mathcal{T}(\mu)^{\beta}_{\beta'} R^{-1}(\lambda - \mu)^{\alpha'\beta'}_{\gamma_\xi \gamma_\eta} \\ & = \sum_{\substack{\alpha, \alpha' \\ \beta, \beta'}} \mathcal{T}(\lambda)^{\alpha}_{\alpha'} \mathcal{T}(\mu)^{\beta}_{\beta'} \delta_{\alpha\alpha'} \delta_{\beta\beta'} = T(\lambda) T(\mu), \end{aligned} \tag{7.53}$$

while the rhs is expressible as

$$\sum_{\gamma_\xi, \gamma_\eta} \{\mathcal{T}(\mu) \otimes \mathcal{T}(\lambda)\}^{\gamma_\xi \gamma_\eta}_{\gamma_\xi \gamma_\eta} = \sum_{\gamma_\xi, \gamma_\eta} \mathcal{T}(\mu)^{\gamma_\xi}_{\gamma_\xi} \mathcal{T}(\lambda)^{\gamma_\eta}_{\gamma_\eta} = T(\mu) T(\lambda). \tag{7.54}$$

We conclude that the commutator

$$[T(\lambda), T(\mu)] = 0 \qquad \text{for arbitrary } \lambda \text{ and } \mu. \tag{7.55}$$

The existence of an infinite family of commuting transfer matrices is of primary importance: *the eigenvectors of transfer matrices $\{T(\lambda)\}$ are common, they do not depend on the spectral parameter λ*. This fact makes the explicit diagonalisation of transfer matrices possible.

7.4 Two-state solutions of the YBE

Let us restrict ourselves to the first nontrivial case $l = 2$ and search for possible forms of the S-matrix which satisfy the YBE. It is convenient to consider the S-matrix in an ansatz form [9, 16]

$$S(\lambda) = \sum_{j=0}^{3} w_j(\lambda)\, \sigma^j \otimes \sigma^j, \tag{7.56}$$

where $\sigma^0 = I$, $\{\sigma^1 \equiv \sigma^x, \sigma^2 \equiv \sigma^y, \sigma^3 \equiv \sigma^z\}$ are Pauli matrices and the w-functions are as-yet unspecified. The S-matrix can be explicitly written as

$$S(\lambda) = \begin{pmatrix} a(\lambda) & 0 & 0 & d(\lambda) \\ 0 & b(\lambda) & c(\lambda) & 0 \\ 0 & c(\lambda) & b(\lambda) & 0 \\ d(\lambda) & 0 & 0 & a(\lambda) \end{pmatrix}, \qquad \begin{aligned} a(\lambda) &= w_0(\lambda) + w_3(\lambda) \\ b(\lambda) &= w_0(\lambda) - w_3(\lambda) \\ c(\lambda) &= w_1(\lambda) + w_2(\lambda) \\ d(\lambda) &= w_1(\lambda) - w_2(\lambda) \end{aligned} \quad . \tag{7.57}$$

The permutation operator $\mathcal{P}$ and the R-matrix read

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad R(\lambda) = \mathcal{P}S(\lambda) = \begin{pmatrix} a(\lambda) & 0 & 0 & d(\lambda) \\ 0 & c(\lambda) & b(\lambda) & 0 \\ 0 & b(\lambda) & c(\lambda) & 0 \\ d(\lambda) & 0 & 0 & a(\lambda) \end{pmatrix}. \quad (7.58)$$

In the auxiliary ξ-space, the Lax operator $L_{\xi n}$ takes the form

$$\begin{aligned} L_n(\lambda) &= \sum_{j=0}^{3} w_j(\lambda)\, \sigma^j \otimes \boldsymbol{\sigma}_n^j \\ &= \begin{pmatrix} w_0(\lambda)\boldsymbol{\sigma}_n^0 + w_3(\lambda)\boldsymbol{\sigma}_n^3 & w_1(\lambda)\boldsymbol{\sigma}_n^1 - \mathrm{i}w_2(\lambda)\boldsymbol{\sigma}_n^2 \\ w_1(\lambda)\boldsymbol{\sigma}_n^1 + \mathrm{i}w_2(\lambda)\boldsymbol{\sigma}_n^2 & w_0(\lambda)\boldsymbol{\sigma}_n^0 - w_3(\lambda)\boldsymbol{\sigma}_n^3 \end{pmatrix}. \end{aligned} \quad (7.59)$$

Having the explicit forms of the R-matrix and the $L_{\xi n}$-matrix, we can look for the solution of the YBE (7.42) in the auxiliary (ξ, η) space. Using the standard relations for the Pauli matrices $\sigma^z\sigma^- = -\sigma^-$, $\sigma^-\sigma^z = \sigma^-$, we find that the YBE is satisfied provided that

$$w_m w_l' w_j'' - w_l w_m' w_k'' + w_k w_j' w_l'' - w_j w_k' w_m'' = 0 \quad (7.60)$$

holds for an arbitrary permutation (j, k, l, m) of $(0, 1, 2, 3)$. Here, we use the notation

$$w_j \equiv w_j(\lambda), \quad w_j' \equiv w_j(\mu), \quad w_j'' \equiv w_j(\lambda - \mu). \quad (7.61)$$

In the most general case of unequal w-functions, there are six independent equations in the system (7.60). Regarding them as linear homogeneous equations for the four unknowns w_0'', w_1'', w_2'' and w_3'', they provide a nontrivial solution only if the equality

$$\frac{w_j^2 - w_k^2}{w_l^2 - w_m^2} = \frac{w'^2_j - w'^2_k}{w'^2_l - w'^2_m} \quad (7.62)$$

holds for an arbitrary permutation (j, k, l, m) of $(0, 1, 2, 3)$. An obvious parametrization of the condition (7.62) is

$$\begin{aligned} w_j^2 &= p(u - u_j), \\ w'^2_j &= p'(u' - u_j), \end{aligned} \qquad j = 0, 1, 2, 3, \quad (7.63)$$

where p, p' are the normalization factors. When equalities (7.62) are satisfied, it follows from (7.60) that also

$$\frac{w''^2_j - w''^2_k}{w''^2_l - w''^2_m} = \frac{w_j^2 - w_k^2}{w_l^2 - w_m^2} \quad (7.64)$$

must hold for all permutations (j, k, l, m). Thus,

$$w_j'' = p''(u'' - u_j). \quad (7.65)$$

The normalization factors p, p', p'' are irrelevant due to the homogeneity of the system of equations (7.60). For fixed constants u_1, u_2, u_3, u_4, each of the equations (7.60) can be regarded

as the definition of u'' as a function of u and u'. Differentiating with respect to u and u', we obtain

$$\frac{1}{g(u)}\frac{\partial u''}{\partial u}+\frac{1}{g(u')}\frac{\partial u''}{\partial u'}=0, \tag{7.66}$$

where

$$g(u)=\prod_{j=0}^{3}(u-u_j)^{-1/2}. \tag{7.67}$$

When we introduce, instead of u and u', the new variables v and v' such that

$$\frac{\mathrm{d}v}{\mathrm{d}u}=g(u), \qquad \frac{\mathrm{d}v'}{\mathrm{d}u'}=g(u'), \tag{7.68}$$

Eq. (7.66) tells us that u'' is a function of $v-v'$.

The differential equations (7.68) can be integrated explicitly using elliptic functions. The Jacobi elliptic sn function with modulus l, $y=\mathrm{sn}(v,l)$, satisfies the differential equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}v}\right)^2=(1-y^2)(1-l^2y^2). \tag{7.69}$$

If we set

$$\mathrm{sn}^2(v,l) = \frac{(u-u_0)(u_1-u_3)}{(u-u_3)(u_1-u_0)}, \tag{7.70}$$

$$l^2 = \frac{(u_1-u_0)(u_2-u_3)}{(u_1-u_3)(u_2-u_0)}, \tag{7.71}$$

it is a simple task to verify that Eq. (7.69) reduces to the first equation in (7.68). Substituting the expression for u,

$$u=\frac{u_0(u_3-u_1)-u_3(u_0-u_1)\mathrm{sn}^2(v,l)}{u_3-u_1-(u_0-u_1)\mathrm{sn}^2(v,l)} \tag{7.72}$$

into the representation of w_j^2 (7.63) and defining a parameter ζ by

$$\mathrm{sn}^2(\zeta,l)=\frac{u_1-u_3}{u_1-u_0}, \tag{7.73}$$

the w-functions turn out to be parametrized in terms of the Jacobi elliptic functions as follows

$$w_0:w_1:w_2:w_3=\frac{\mathrm{sn}(v,l)}{\mathrm{sn}(\zeta,l)}:\frac{\mathrm{cn}(v,l)}{\mathrm{cn}(\zeta,l)}:\frac{\mathrm{dn}(v,l)}{\mathrm{dn}(\zeta,l)}:1. \tag{7.74}$$

The w'-functions can be obtained in the same way and their parametrization coincides, after the substitution $v\to v'$, with (7.74). To find the functions w'', we substitute the expressions (7.74) for w_j and the analogous expressions for w_j' in (7.60). Using the addition theorems for the Jacobi elliptic functions, we find that

$$w_0'':w_1'':w_2'':w_3''=\frac{\mathrm{sn}(v-v'+\zeta,l)}{\mathrm{sn}(\zeta,l)}:\frac{\mathrm{cn}(v-v'+\zeta,l)}{\mathrm{cn}(\zeta,l)}:\frac{\mathrm{dn}(v-v'+\zeta,l)}{\mathrm{dn}(\zeta,l)}:1. \tag{7.75}$$

As is clear from the definition (7.61), the functions w_j, w'_j, w''_j correspond to the only function taken at different values of the spectral parameter. To be consistent with Eqs. (7.74) and (7.75), we must put $v = \lambda + \zeta$ and $v' = \mu + \zeta$. Then,

$$w_0(\lambda) : w_1(\lambda) : w_2(\lambda) : w_3(\lambda) = \frac{\mathrm{sn}(\lambda+\zeta,l)}{\mathrm{sn}(\zeta,l)} : \frac{\mathrm{cn}(\lambda+\zeta,l)}{\mathrm{cn}(\zeta,l)} : \frac{\mathrm{dn}(\lambda+\zeta,l)}{\mathrm{dn}(\zeta,l)} : 1. \quad (7.76)$$

There exists another parametrization of the functions w_j which is obtained from (7.76) applying first the imaginary Jacobi transformation (B.61) and then Landen's transformation (B.62), (B.63). This implies a transition from the modulus l to the modulus $k = (1-l)/(1+l)$. Introducing the new parameter $\eta = \mathrm{i}2\zeta/(1+k)$ and taking $\mathrm{i}\lambda/(1+k)$ as λ, for the coefficients a, b, c, d of the S-matrix representation (7.57) or the R-matrix representation (7.58), we get an elliptic parametrization

$$\begin{aligned} a(\lambda) : b(\lambda) : c(\lambda) : d(\lambda) &= \mathrm{sn}(\lambda+\eta,k) : \mathrm{sn}(\lambda,k) : \mathrm{sn}(\eta,k) \\ &\quad : k\,\mathrm{sn}(\lambda,k)\,\mathrm{sn}(\eta,k)\,\mathrm{sn}(\lambda+\eta,k). \end{aligned} \quad (7.77)$$

Using the formula

$$\mathrm{sn}(u,k) = \frac{1}{\sqrt{k}}\frac{H(u)}{\Theta(u)}, \quad (7.78)$$

where $H(u) \equiv H(u,k)$ and $\Theta(u) \equiv \Theta(u,k)$ are the Jacobi theta functions, we obtain from (7.77) the representation

$$\begin{aligned} a(\lambda) &= \rho(\lambda)\,\Theta(\eta)\,\Theta(\lambda)\,H(\lambda+\eta), \\ b(\lambda) &= \rho(\lambda)\,\Theta(\eta)\,H(\lambda)\,\Theta(\lambda+\eta), \\ c(\lambda) &= \rho(\lambda)\,H(\eta)\,\Theta(\lambda)\,\Theta(\lambda+\eta), \\ d(\lambda) &= \rho(\lambda)\,H(\eta)\,H(\lambda)\,H(\lambda+\eta). \end{aligned} \quad (7.79)$$

The normalization function is determined by the initial condition (7.21) and the unitarity condition (7.23). In many problems, the normalization is not important and one simply puts $\rho(\lambda) = 1$.

The outlined procedure provides the sets of "elliptic" S-matrices, with identical values for the modulus k and the parameter η, which satisfy the YBE (7.18). Let us now consider the case when the modulus k tends to zero, so that the elliptic functions $\mathrm{sn}(u)$, $\mathrm{cn}(u)$ and $\mathrm{dn}(u)$ become $\sin(u)$, $\cos(u)$ and 1, respectively. The parametrization (7.77) then becomes the trigonometric one

$$a(\lambda) : b(\lambda) : c(\lambda) : d(\lambda) = \sin(\lambda+\eta) : \sin\lambda : \sin\eta : 0. \quad (7.80)$$

Expressing from (7.57) the w_j-functions in terms of a, b, c and d,

$$\begin{aligned} w_0(\lambda) &= \frac{1}{2}\left[a(\lambda)+b(\lambda)\right], \\ w_1(\lambda) &= \frac{1}{2}\left[c(\lambda)+d(\lambda)\right], \\ w_2(\lambda) &= \frac{1}{2}\left[c(\lambda)-d(\lambda)\right], \\ w_3(\lambda) &= \frac{1}{2}\left[a(\lambda)-b(\lambda)\right], \end{aligned} \quad (7.81)$$

we see that the parametrization (7.80) describes the case with the symmetry $w_1(\lambda) = w_2(\lambda)$.

Rescaling the rapidity λ by the parameter η and then going with η to zero, the trigonometric parametrization (7.80) becomes of the rational type

$$a(\lambda) : b(\lambda) : c(\lambda) : d(\lambda) = \lambda + 1 : \lambda : 1 : 0. \tag{7.82}$$

With regard to (7.81), this parametrization, being the special case of $l = 2$ particle states of the result (7.38), corresponds to the symmetry $w_1(\lambda) = w_2(\lambda) = w_3(\lambda)$.

7.5 Braid-group solution

We shall now construct the braid solution (7.31) in the braid limit $|\lambda| \to \infty$ from the trigonometric R-matrix with elements (7.80). Substituting $\lambda \to -\mathrm{i}\lambda$ and considering afterwards the limit $\lambda \to +\infty$, the elements (7.80) behave asymptotically as follows

$$a \sim \frac{1}{2\mathrm{i}}\mathrm{e}^{\lambda}\mathrm{e}^{\mathrm{i}\eta}, \quad b \sim \frac{1}{2\mathrm{i}}\mathrm{e}^{\lambda}, \quad c \sim \sin\eta. \tag{7.83}$$

It is clear that the information contained in c is lost in the considered limit and the consequent R-matrix would be too simple to describe the braid relations. This is why we diagonally change the basis $\{e_\sigma | \sigma = 1, 2\}$ to $\tilde{e}_\sigma(\lambda) = f_\sigma(\lambda)e_\sigma$. Recalling that

$$R\left(e_{\sigma_1}(\lambda_1) \otimes e_{\sigma_2}(\lambda_2)\right) = \sum_{\sigma_1'\sigma_2'} R^{\sigma_1\sigma_2}_{\sigma_2'\sigma_1'}(\lambda_1 - \lambda_2)\left(e_{\sigma_2'}(\lambda_2) \otimes e_{\sigma_1'}(\lambda_1)\right), \tag{7.84}$$

the elements of the R-matrix in the new basis $\tilde{e}_\sigma$ are given by

$$\tilde{R}^{\sigma_1\sigma_2}_{\sigma_2'\sigma_1'}(\lambda_1, \lambda_2) = \frac{f_{\sigma_1}(\lambda_1)f_{\sigma_2}(\lambda_2)}{f_{\sigma_1'}(\lambda_1)f_{\sigma_2'}(\lambda_2)} R^{\sigma_1\sigma_2}_{\sigma_2'\sigma_1'}(\lambda_1 - \lambda_2). \tag{7.85}$$

In order to preserve the difference property of the R-matrix, the f-functions must be of the form $f_\sigma(\lambda) = \exp(\alpha\sigma\lambda)$ where α is a free parameter. Since $\tilde{R}^{\sigma_1\sigma_2}_{\sigma_2'\sigma_1'}(\lambda_1, \lambda_2) \neq 0$ only if $\sigma_1 + \sigma_2 = \sigma_1' + \sigma_2'$, the rescaled matrix may be written as

$$\tilde{R}^{\sigma_1\sigma_2}_{\sigma_2'\sigma_1'}(\lambda, \alpha) = \mathrm{e}^{\alpha\lambda(\sigma_1 - \sigma_1')} R^{\sigma_1\sigma_2}_{\sigma_2'\sigma_1'}(\lambda). \tag{7.86}$$

We shall consider a special value of the parameter $\alpha = 1$. Let us define, in the braid $+\infty$ limit,

$$R^{(+)} \equiv 2\mathrm{i}\mathrm{e}^{-\mathrm{i}\eta/2} \lim_{\lambda \to +\infty} \mathrm{e}^{-\lambda}\tilde{R}(-\mathrm{i}\lambda, \alpha = 1). \tag{7.87}$$

Denoting $q = \mathrm{e}^{\mathrm{i}\eta}$, we obtain

$$R^{(+)} = \frac{1}{\sqrt{q}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & (q - q^{-1}) & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \tag{7.88}$$

Similarly, defining the braid $-\infty$ limit

$$R^{(-)} \equiv -2\mathrm{i}\mathrm{e}^{\mathrm{i}\eta/2} \lim_{\lambda \to -\infty} \mathrm{e}^{\lambda}\tilde{R}(-\mathrm{i}\lambda, \alpha = 1), \tag{7.89}$$

we obtain

$$R^{(-)} = \sqrt{q}\begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & -(q-q^{-1}) & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}. \tag{7.90}$$

The $R^{(+)}$ and $R^{(-)}$ matrices are related by

$$R^{(+)}R^{(-)} = I \quad \Longrightarrow \quad R^{(+)} = (R^{(-)})^{-1}. \tag{7.91}$$

These are the R-matrices which fulfill the YBE (7.31) and (7.32) without spectral parameters and consequently provide a representation of the braid group.

The corresponding braid S-matrices $S^{(+)} = \mathcal{P}R^{(+)}$ and $S^{(-)} = \mathcal{P}R^{(-)}$ read

$$S^{(+)} = \frac{1}{\sqrt{q}}\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & (q-q^{-1}) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \tag{7.92}$$

$$S^{(-)} = \sqrt{q}\begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -(q-q^{-1}) & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}. \tag{7.93}$$

With regard to the equality (7.91), they satisfy the relation

$$S^{(-)} = \mathcal{P}(S^{(+)})^{-1}\mathcal{P}. \tag{7.94}$$

It also holds that

$$S^{(+)} - S^{(-)} = (q-q^{-1})\mathcal{P}. \tag{7.95}$$

Both $S^{(+)}$ and $S^{(-)}$ fulfill the three-site YBE (7.20) without spectral parameters

$$S_{12}^{(\pm)}S_{13}^{(\pm)}S_{23}^{(\pm)} = S_{23}^{(\pm)}S_{13}^{(\pm)}S_{12}^{(\pm)}. \tag{7.96}$$

In order to explain the role of the braid matrices $S^{(+)}$ and $S^{(-)}$, we shall analyze in detail the trigonometric case of the scattering matrix of the form (7.57) with elements defined by Eq. (7.80). Using the notation $q = \mathrm{e}^{\mathrm{i}\eta}$, $x = \mathrm{e}^{\mathrm{i}\lambda}$ and an appropriate normalization, we have

$$\begin{aligned} a(x) &= qx - q^{-1}x^{-1}, \\ b(x) &= x - x^{-1}, \\ c(x) &= q - q^{-1}, \\ d(x) &= 0. \end{aligned} \tag{7.97}$$

The corresponding w_j-functions are given by Eq. (7.81). With the aid of the obvious relation

$$\mathrm{e}^{\alpha\boldsymbol{\sigma}_n^z} = (\cosh\alpha)I + (\sinh\alpha)\boldsymbol{\sigma}_n^z, \tag{7.98}$$

the Lax operator (7.59) is expressible in the auxiliary ξ-space as follows

$$L_n(x) = \begin{pmatrix} (x\sqrt{q})q^{\mathbf{S}_n^z} - (x\sqrt{q})^{-1}q^{-\mathbf{S}_n^z} & (q-q^{-1})\mathbf{S}_n^- \\ (q-q^{-1})\mathbf{S}_n^+ & (x\sqrt{q})q^{-\mathbf{S}_n^z} - (x\sqrt{q})^{-1}q^{\mathbf{S}_n^z} \end{pmatrix}, \tag{7.99}$$

where $\mathbf{S}_n = \frac{1}{2}\boldsymbol{\sigma}_n$. It is a simple task to check that the YBE (7.40), written as

$$S_{\xi\eta}(x/y)L_{\xi n}(x)L_{\eta n}(y) = L_{\eta n}(y)L_{\xi n}(x)S_{\xi\eta}(x/y), \tag{7.100}$$

is indeed fulfilled. Performing the similarity transformation

$$\tilde{L}_{\xi n}(x) = Q(x)L_{\xi n}(x)Q^{-1}(x), \tag{7.101}$$

$$\tilde{S}_{\xi\eta}(x/y) = [Q(x)\otimes Q(y)]S_{\xi\eta}(x/y)[Q^{-1}(x)\otimes Q^{-1}(y)] \tag{7.102}$$

with the matrix

$$Q(x) = \begin{pmatrix} x^{1/2} & 0 \\ 0 & x^{-1/2} \end{pmatrix}, \tag{7.103}$$

the transformed $\tilde{S}$-matrix is expressible as

$$\tilde{S}(x) = (x\sqrt{q})S^{(+)} - (x\sqrt{q})^{-1}S^{(-)}, \tag{7.104}$$

where $S^{(+)}$ and $S^{(-)}$ are the braid scattering matrices of interest, and the transformed Lax operator $\tilde{L}$ is given by

$$\tilde{L}(x) = (x\sqrt{q})L^{(+)} - (x\sqrt{q})^{-1}L^{(-)} \tag{7.105}$$

with

$$L^{(+)} = \begin{pmatrix} q^{S^z} & q^{-1/2}(q-q^{-1})S^- \\ 0 & q^{-S^z} \end{pmatrix}, \tag{7.106}$$

$$L^{(-)} = \begin{pmatrix} q^{-S^z} & 0 \\ -q^{1/2}(q-q^{-1})S^+ & q^{S^z} \end{pmatrix}. \tag{7.107}$$

The simple x-dependence of $\tilde{S}$ and $\tilde{L}$ is appealing. The origin of the operators $L^{(+)}$ and $L^{(-)}$ will be clearer later after the introduction of quantum groups, i.e. q-deformation of classical Lie groups. From this point of view, the studied trigonometric case is a q-deformation of the rational case (7.82) which results, under the scaling $x = q^{\lambda}$, from the $q \to 1$ limit.

7.6 Quantum groups

Investigations of integrable models within the framework of the QISM led to certain deformations of Lie algebras, called quantum groups, which play an important role in non-commutative geometry [69]. Quantum groups possess a relatively complicated structure, explained e.g. in monograph [55]. As an example, we shall indicate very briefly the structure of the quantum deformation of the classical Lie group SL(2). This part has to be understood as a motivation for mathematically oriented readers for a detailed study of quantum groups in the mentionned monograph.

7.6.1 Classical plane and the SL(2) group

It is instructive to describe the origin of the special linear group SL(2). A "representation space" for SL(2) is a two-dimensional *classical* space of vectors $\binom{x}{y}$, where the coordinates x and y are real variables which commute with one another,

$$xy = yx. \tag{7.108}$$

In the space of functions $f(x,y) = \sum_{m,n} f_{mn} x^m y^n$ defined in the vector space, we consider the partial derivative operations ∂_x, ∂_y which also commute with one another,

$$\partial_x \partial_y = \partial_y \partial_x. \tag{7.109}$$

The coordinates x, y and the derivatives ∂_x, ∂_y satisfy the standard commutation relations

$$[\partial_x, y] = 0, \quad [\partial_y, x] = 0, \quad [\partial_x, x] = 1, \quad [\partial_y, y] = 1. \tag{7.110}$$

To complete the scheme, we have to introduce an *exterior* differential d possessing the property

$$\mathrm{d}^2 = 0 \tag{7.111}$$

and satisfying the Leibnitz rule

$$\mathrm{d}(fg) = (\mathrm{d}f)g + f(\mathrm{d}g). \tag{7.112}$$

The exterior differential is formally expressible as

$$\mathrm{d} = \xi \partial_x + \eta \partial_y, \tag{7.113}$$

where ξ and η are the differentials of the basic variables,

$$\xi = \mathrm{d}x, \qquad \eta = \mathrm{d}y. \tag{7.114}$$

The differentials commute with the coordinates and the derivatives,

$$\begin{array}{l} [\xi, x] = [\xi, y] = [\xi, \partial_x] = [\xi, \partial_y] = 0, \\ [\eta, x] = [\eta, y] = [\eta, \partial_x] = [\eta, \partial_y] = 0. \end{array} \tag{7.115}$$

The condition (7.111) for d, considered in the form (7.113), then implies

$$\xi\eta = -\eta\xi, \qquad \xi^2 = 0, \qquad \eta^2 = 0, \tag{7.116}$$

i.e. the differentials ξ and η are the anticommuting (Grassmann) variables.

Let us consider a linear transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = T \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} \partial_{x'} \\ \partial_{y'} \end{pmatrix} = (T^t)^{-1} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}, \qquad \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = T \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{7.117}$$

where the entries of the matrix

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{7.118}$$

are real numbers and T^t means the transpose of T. The transformation (7.117) leaves the exterior differential (7.113) invariant,

$$\mathrm{d}' = (\xi', \eta') \begin{pmatrix} \partial_{x'} \\ \partial_{y'} \end{pmatrix} = (\xi, \eta) T^t (T^t)^{-1} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = (\xi, \eta) \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \mathrm{d}. \tag{7.119}$$

If the determinant

$$\mathrm{Det}\, T = ad - bc = 1, \tag{7.120}$$

it is easy to check that the new coordinates (x', y'), derivatives $(\partial_{x'}, \partial_{y'})$ and differentials (ξ, η) satisfy the differential calculus of the original variables, defined by Eqs. (7.108)-(7.110), (7.115), (7.116). The transformation (7.117) with the matrix T (7.118) of the unity determinant constitute an element of the Lie group SL(2) (more precisely, SL(2,R)). We say that the differential calculus on the (x, y)-plane is covariant under this group.

The matrix T corresponds to the fundamental two-dimensional irreducible representation of SL(2). Respecting the determinantal condition (7.120), it can be parametrized as follows

$$T = \begin{pmatrix} \mathrm{e}^{\alpha} & \mathrm{e}^{\alpha}\beta \\ \gamma \mathrm{e}^{\alpha} & \mathrm{e}^{-\alpha} + \gamma \mathrm{e}^{\alpha} \beta \end{pmatrix}, \tag{7.121}$$

where α, β and γ are real numbers. The spin-$\frac{1}{2}$ generators

$$S^z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{7.122}$$

of the $sl(2)$ algebra

$$[S^z, S^{\pm}] = \pm S^{\pm}, \qquad [S^+, S^-] = 2S^z. \tag{7.123}$$

satisfy the equalities

$$(S^{\pm})^2 = 0, \qquad \mathrm{e}^{2\alpha S^z} = (\cosh \alpha) I + (\sinh \alpha) 2S^z. \tag{7.124}$$

The matrix T can be thus rewritten as

$$T = \mathrm{e}^{\gamma S^-} \mathrm{e}^{2\alpha S^z} \mathrm{e}^{\beta S^+}. \tag{7.125}$$

The higher-dimensional representations of the group SL(2) are obtained by considering higher-dimensional generators of the $sl(2)$ algebra, like the spin-1 ones (A.14), etc.

7.6.2 Quantum plane and the SL$_q$(2) group

In analogy with the link between the classical and quantum mechanics, we introduce a two-dimensional quantum space of vectors $\binom{x}{y}$ with the coordinates x and y subjected to the commutation property

$$xy = qyx. \tag{7.126}$$

The "quantum deformation" parameter q is a nonzero complex number. Using (7.126), functions of the non-commuting variables x and y can be always brought to the form of power series

$f(x,y) = \sum_{m,n} f_{mn} x^m y^n$. In analogy with the classical plane, we introduce the q-derivatives (∂_x, ∂_y) and the q-differentials $\xi = \mathrm{d}x$, $\eta = \mathrm{d}y$. The exterior differential $\mathrm{d} = \xi\partial_x + \eta\partial_y$ possesses the property $\mathrm{d}^2 = 0$ and satisfies the Leibnitz rule (7.112).

Similarly as in the case of the classical plane, the differential calculus on the quantum space is defined by a set of quadratic relations among all fundamental objects: coordinates (x,y), q-derivatives (∂_x, ∂_y) and q-differentials (ξ, η). These commutation relations were derived in Ref. [70]. An essential requirement was the consistency of the quadratic algebra in the sense that there are no independent higher-order relations. We do not go into details of the derivation and only write down the final commutation relations between: q-derivatives

$$\partial_x \partial_y = q^{-1} \partial_y \partial_x; \tag{7.127}$$

q-differentials

$$\xi\eta = -q^{-1}\eta\xi, \qquad \xi^2 = 0, \quad \eta^2 = 0; \tag{7.128}$$

variables and q-derivatives

$$\begin{aligned} &\partial_x x = 1 + q^2 x\partial_x + (q^2-1) y\partial_y, \quad \partial_x y = qy\partial_x, \\ &\partial_y x = qx\partial_y, \quad \partial_y y = 1 + q^2 y\partial_y; \end{aligned} \tag{7.129}$$

variables and q-differentials

$$\begin{aligned} &x\xi = q^2\xi x, \quad x\eta = q\eta x + (q^2-1)\xi y, \\ &y\xi = q\xi y, \quad y\eta = q^2\eta y; \end{aligned} \tag{7.130}$$

q-derivatives and q-differentials

$$\begin{aligned} &\partial_x \xi = q^{-2}\xi\partial_x, \quad \partial_x \eta = q^{-1}\eta\partial_x, \\ &\partial_y \xi = q^{-1}\xi\partial_y, \quad \partial_y \eta = q^{-2}\eta\partial_y + \left(q^{-2}-1\right)\xi\partial_x. \end{aligned} \tag{7.131}$$

The differential calculus on the classical plane is restored in the classical limit $q \to 1$.

We consider a linear transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = T \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} \partial_{x'} \\ \partial_{y'} \end{pmatrix} = (T^t)^{-1} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}, \quad \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = T \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{7.132}$$

where the entries of the quantum matrix

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{7.133}$$

commute with coordinates (x,y), q-derivatives (∂_x, ∂_y) and q-differentials (ξ, η), but not with each other. Let us first require that after the transformation (7.132) the new coordinates (x', y') also satisfy the commutation relation of type (7.126), the new q-derivatives $(\partial_{x'}, \partial_{y'})$ also satisfy the commutation relation of type (7.127) and the new q-differentials (ξ', η') also satisfy the commutation relations of type (7.128). The commutation properties of the entries of the matrix T are then completely determined:

$$\begin{aligned} &ab = qba, \quad ac = qca, \quad [a,d] = (q-q^{-1})bc, \\ &bc = cb, \quad bd = qdb, \quad cd = qdc. \end{aligned} \tag{7.134}$$

These relations are consistent in the sense that they do not generate higher-order relations and lead to a finitely generated quadratic algebra. The quantity

$$\mathrm{Det}_q\, T = ad - qbc = da - q^{-1}bc \tag{7.135}$$

commutes with all entries a, b, c, d of the quantum matrix T, and so it is a central element of the algebra which defines the q-determinant of T. The differential calculus on the quantum plane (7.126)-(7.131) is covariant under the transformation (7.132) if, in addition to the commutation relations (7.134), it holds

$$\mathrm{Det}_q\, T = \mathbf{1}, \tag{7.136}$$

where $\mathbf{1}$ is the unity operator. This is the quantum counterpart of the classical condition (7.120). Under this constraint,

$$T^{-1} = \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix} \tag{7.137}$$

satisfies

$$TT^{-1} = T^{-1}T = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}. \tag{7.138}$$

The transformation (7.132) with the quantum matrix T, whose elements satisfy the commutation relations (7.134) and whose q-determinant (7.135) is the unity operator, constitute the q-deformation of the Lie group SL(2), called the quantum group $\mathrm{SL}_q(2)$.

For the future purposes, we recall the definition of Heine's q-number

$$[n]_q = \frac{1-q^n}{1-q} \tag{7.139}$$

and, for $n = 1, 2, \ldots$, the q-number factorial

$$[n]_q! = [n]_q[n-1]_q[n-2]_q \cdots [2]_q[1]_q, \qquad [0]_q! = 1. \tag{7.140}$$

We introduce also a q-generalization of the exponential function

$$\mathrm{e}_q^z = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}. \tag{7.141}$$

Notice that in the limit $q \to 1$

$$[n]_q \to n, \qquad [n]_q! \to n!, \qquad \mathrm{e}_q^z \to \mathrm{e}^z. \tag{7.142}$$

We shall need another q-number

$$[\![n]\!]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{7.143}$$

which also becomes n in the limit $q \to 1$, however, in contrast to Heine's $[n]_q$, it is symmetric with respect to the interchange of q and q^{-1}.

To derive the generators of the group $\mathrm{SL}_q(2)$ we parametrize the quantum T-matrix as follows

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\alpha} & \mathrm{e}^{\alpha}\beta \\ \gamma\mathrm{e}^{\alpha} & \mathrm{e}^{-\alpha} + \gamma\mathrm{e}^{\alpha}\beta \end{pmatrix}. \tag{7.144}$$

In order to ensure the matrix elements $\{a, b, c, d\}$ to obey the commutation relations (7.134) and the q-determinant (7.135) to be the unity operator, the noncommuting variables $\{\alpha, \beta, \gamma\}$ have to satisfy a Lie algebra

$$[\alpha, \beta] = (\ln q)\beta, \quad [\alpha, \gamma] = (\ln q)\gamma, \quad [\beta, \gamma] = 0. \tag{7.145}$$

In the derivation of this result we have used that since

$$\mathrm{e}^{U} V \mathrm{e}^{-U} = V + [U, v] + \frac{1}{2!}[U, [U, V]] + \frac{1}{3!}[U, [U, [U, V]]] + \cdots, \tag{7.146}$$

it holds

$$[U, V] = zV \quad \Longleftrightarrow \quad \mathrm{e}^{U} V \mathrm{e}^{-U} = \mathrm{e}^{z} V. \tag{7.147}$$

The generic form of the quantum matrix T is then given by

$$T = \mathrm{e}_{q^{-2}}^{\gamma \mathcal{S}^-} \mathrm{e}^{2\alpha \mathcal{S}^z} \mathrm{e}_{q^2}^{\beta \mathcal{S}^+}, \tag{7.148}$$

where the generators $\{\mathcal{S}^z, \mathcal{S}^+, \mathcal{S}^-\}$ obey the quantum algebra $sl_q(2)$

$$[\mathcal{S}^z, \mathcal{S}^{\pm}] = \pm \mathcal{S}^{\pm}, \qquad [\mathcal{S}^+, \mathcal{S}^-] = \frac{q^{2\mathcal{S}^z} - q^{-2\mathcal{S}^z}}{q - q^{-1}} = [\![2\mathcal{S}^z]\!]_q. \tag{7.149}$$

The $sl(2)$ algebra (7.123) of the generators $\{S^z, S^+, S^-\}$ is recovered in the classical limit $q \to 1$.

The fundamental two-dimensional irreducible representation of the $sl_q(2)$ algebra (7.149) is provided by the spin-$\frac{1}{2}$ generators (7.122) of the $sl(2)$ algebra. This fact follows directly from the relations

$$q^{\pm 2S^z} = \frac{q + q^{-1}}{2} I \pm \frac{q - q^{-1}}{2} (2S^z) \tag{7.150}$$

having their origin in Eq. (7.124). In this case, the quantities $\mathrm{e}_{q^{-2}}^{\gamma S^-}$ and $\mathrm{e}_{q^2}^{\beta S^+}$ are the same as $\mathrm{e}^{\gamma S^-}$ and $\mathrm{e}^{\beta S^+}$, respectively, due to the equalities $(S^{\pm})^2 = 0$. The three-dimensional irreducible representation of the $sl_q(2)$ algebra is provided by q-dependent generators

$$\begin{aligned} \mathcal{S}^z &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mathcal{S}^+ = \sqrt{[\![2]\!]_q} \begin{pmatrix} 0 & 1/\sqrt{q} & 0 \\ 0 & 0 & \sqrt{q} \\ 0 & 0 & 0 \end{pmatrix}, \\ \mathcal{S}^- &= \sqrt{[\![2]\!]_q} \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{q} & 0 & 0 \\ 0 & 1/\sqrt{q} & 0 \end{pmatrix}. \end{aligned} \tag{7.151}$$

The three-dimensional representation of the quantum T-matrix is obtained by substituting these generators into the formula (7.148). For $q = 1$, the generators (7.151) become the ordinary spin-1 matrices (A.14).

7.6.3 Link between quantum groups and the YBE

We now establish the link between the structure of the quantum group $\mathrm{SL}_q(2)$ and the solutions (7.104) and (7.105) of the trigonometric YBE.

Let, for simplicity, the auxiliary sites ξ and η be denoted as 1 and 2, respectively. We define the quantum matrices T_1 and T_2 acting separately on the two-dimensional Hilbert spaces 1 and 2:

$$\begin{aligned} T_1 &= T \otimes 1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ T_2 &= 1 \otimes T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \end{aligned} \tag{7.152}$$

From the explicit forms of T_1 and T_2 we find that

$$T_1 = \mathcal{P} T_2 \mathcal{P}, \tag{7.153}$$

where $\mathcal{P}$ is the 4×4 permutation matrix. Because the matrix elements $\{a, b, c, d\}$ are non-commutative, we have

$$T_1 T_2 = \begin{pmatrix} a^2 & ab & ba & b^2 \\ ac & ad & bc & bd \\ ca & cb & da & db \\ c^2 & cd & dc & d^2 \end{pmatrix} \neq \begin{pmatrix} a^2 & ba & ab & b^2 \\ ca & da & cb & db \\ ac & bc & ad & bd \\ c^2 & dc & cd & d^2 \end{pmatrix} = T_2 T_1. \tag{7.154}$$

It turns out that the relation between the products $T_1 T_2$ and $T_2 T_1$ is mediated by the braid S-matrices $S^{(+)}$ (7.92) and $S^{(-)}$ (7.93),

$$T_1 T_2 S = S T_2 T_1, \qquad S = S^{(+)} \text{ or } S^{(-)}. \tag{7.155}$$

Note that the relation (7.153) implies

$$T_1 T_2 = \mathcal{P}(T_2 T_1)\mathcal{P}, \tag{7.156}$$

so the validity of Eq. (7.155) for $S^{(+)}$ automatically ensures the validity of this equation for $S^{(-)} = \mathcal{P}(S^{(+)})^{-1}\mathcal{P}$. Eq. (7.155) is nothing but a compact way how to describe the commutation relations (7.134) for the entries of the quantum T-matrix.

In order to reflect the commutation relations (7.149) of the $sl_q(2)$ algebra, we introduce the Lax operators

$$L^{(+)} = \begin{pmatrix} q^{\mathcal{S}^z} & q^{-1/2}(q - q^{-1})\mathcal{S}^- \\ 0 & q^{-\mathcal{S}^z} \end{pmatrix} \tag{7.157}$$

$$L^{(-)} = \begin{pmatrix} q^{-\mathcal{S}^z} & 0 \\ -q^{1/2}(q - q^{-1})\mathcal{S}^+ & q^{\mathcal{S}^z} \end{pmatrix}. \tag{7.158}$$

Note that $L^{(\pm)}$ are special realizations of the quantum T-matrices, i.e. their entries satisfy the commutation relations (7.134). As before, we define the Lax operators acting separately on the Hilbert spaces 1 and 2:

$$L_1^{(\pm)} = L^{(\pm)} \otimes 1, \qquad L_2^{(\pm)} = 1 \otimes L^{(\pm)}. \tag{7.159}$$

They are related by the analogy of Eq. (7.153)

$$L_1^{(\pm)} = \mathcal{P} L_2^{(\pm)} \mathcal{P}. \tag{7.160}$$

Then, the $sl_q(2)$ commutation relations (7.149) are equivalent to the equalities

$$S L_1^{(\pm)} L_2^{(\pm)} = L_2^{(\pm)} L_1^{(\pm)} S, \qquad S = S^{(+)} \text{ or } S^{(-)} \tag{7.161}$$

and

$$S^{(+)} L_1^{(+)} L_2^{(-)} = L_2^{(-)} L_1^{(+)} S^{(+)}, \qquad S^{(-)} L_1^{(-)} L_2^{(+)} = L_2^{(+)} L_1^{(-)} S^{(-)}. \tag{7.162}$$

We used extensively the operator formula (7.147) in the derivation of these relations. Many other equivalent relations can be derived by taking into account the equality (7.160) between L_1 and L_2.

We are now ready to explain why the scattering matrix of the form (7.104) and the Lax operator of the form (7.105) fulfill the YBE

$$\tilde{S}_{12}(x/y) \tilde{L}_1(x) \tilde{L}_2(y) = \tilde{L}_2(y) \tilde{L}_1(x) \tilde{S}_{12}(x/y). \tag{7.163}$$

In each side of this equation, seven different powers of x and y occurs:

$$\{x^2, y^2, x^2 y^2, x^2/y^2, y^2/x^2, 1/(x^2 y^2), 1\}.$$

Setting to zero each of the polynomial coefficients, we obtain seven different equations which relate $S^{(\pm)}$ and $L^{(\pm)}$. Besides the previous four equations (7.161) and two equations (7.162), we obtain

$$S^{(+)} L_1^{(-)} L_2^{(+)} + S^{(-)} L_1^{(+)} L_2^{(-)} = L_2^{(+)} L_1^{(-)} S^{(+)} + L_2^{(-)} L_1^{(+)} S^{(-)}. \tag{7.164}$$

This equation is not independent, it can be derived from Eqs. (7.161) and (7.162) by using the relation (7.160). We conclude that the formalism of $sl_q(2)$ algebra is behind the structure of trigonometric solutions of the YBE.

The worked-out example corresponds to the fundamental two-dimensional representation of the $sl_q(2)$ algebra. It is straightforward to construct solutions of the YBE which correspond to three- and higher-dimensional representations of the $sl_q(2)$ algebra.

There exists a systematic method of deformation of any classical Lie group. In specific cases, the quantum deformation with several q-parameters is possible.

8 QISM: Transfer matrix and its diagonalization

In the previous section, we have introduced the scattering matrix S which, if it describes an integrable system of quantum particles with internal degrees of freedom, has to satisfy the YBE. We afterwards forgot about the particle origin of the S-matrix and constructed from it, with the aid of tensor products, large Lax, monodromy and transfer matrices. The fact that the S-matrix fulfills the YBE leads to the commutation property (7.55) of an infinite set of transfer matrices with an arbitrary value of the spectral parameter. In this part, we answer two important questions. Firstly, we show that the transfer matrix describes real physical quantities of some classical statistical systems formulated on a two-dimensional lattice and the associated quantum systems on a chain. Secondly, we document how the commutation property of the set of transfer matrix makes possible their diagonalization within the framework of the algebraic Bethe ansatz.

8.1 Vertex models on the square lattice

We consider a two-dimensional square lattice with M rows and N columns, with torus cyclic boundary conditions row $M+1 \equiv 1$ and column $N+1 \equiv 1$. Each node (vertex) of the lattice is joined with its 4 nearest neighbors by edges. In contrast to spin models where the spin variables are localized at nodes of a lattice, local state variables in vertex models are defined on the edges of a lattice. In two-state vertex models, the edge variable σ takes two values, say ± 1. As an alternative definition of the edge variable we may use an arrow: $\sigma = +1$ corresponds to the arrow going to the right or upwards and $\sigma = -1$ corresponds to the arrow going to the left or downwards. The configuration of four edge states $\{\sigma, \sigma', \gamma, \gamma'\}$ around a vertex is pictured in Fig. 8.1; the variables σ, σ' are attached to the vertical edges and γ, γ' to the horizontal edges. The model is called the general 16-vertex model since there are $2^4 = 16$ distinct configurations of edge states around a vertex. To each of the configuration of the edge states we ascribe an energy ϵ_j $(j = 1, 2, \ldots, 16)$ and the corresponding Boltzmann weight

$$W^{\sigma}_{\sigma'}(\gamma, \gamma') = \exp(-\beta \epsilon_j). \tag{8.1}$$

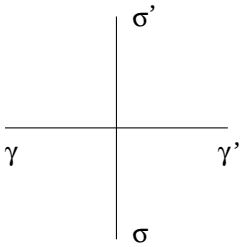


Fig. 8.1. The vertex configuration

To a given configuration of arrows we associate a total energy, which is the sum of the energies of all vertices

$$E = \sum_{j=1}^{16} N_j \epsilon_j, \tag{8.2}$$

where N_j is the number of vertices with the energy ϵ_j. The statistical sum Z is defined as

$$Z = \sum_{\text{conf.}} \exp(-\beta E), \tag{8.3}$$

where the sum goes over all possible configurations of states (arrows) on the lattice edges. The quantity of interest is the thermodynamic limit of the free energy per node f, defined by

$$-\beta f = \lim_{M,N\to\infty} \frac{1}{MN} \ln Z. \tag{8.4}$$

To express the statistical sum Z in a convenient way, we consider a lattice row drawn in Eq. (7.45) with the cyclic boundary condition $\gamma_\xi = \gamma'_\xi = \gamma_1$. The lower N edge arrows are fixed in the state $\{\sigma_1, \sigma_2, \ldots, \sigma_N\}$, the upper N edge arrows are fixed in the state $\{\sigma'_1, \sigma'_2, \ldots, \sigma'_N\}$. The Boltzmann weight of the row is then the sum over γ-variables,

$$T^{\sigma_1\sigma_2\ldots\sigma_N}_{\sigma'_1\sigma'_2\ldots\sigma'_N} = \sum_{\gamma_1,\ldots,\gamma_N} W^{\sigma_1}_{\sigma'_1}(\gamma_1,\gamma_2) W^{\sigma_2}_{\sigma'_2}(\gamma_2,\gamma_3)\cdots W^{\sigma_N}_{\sigma'_N}(\gamma_N,\gamma_1). \tag{8.5}$$

Comparing this expression with the definition of the transfer matrix

$$T(\lambda)^{\sigma_1\ldots\sigma_N}_{\sigma'_1\ldots\sigma'_N} = \sum_{\gamma_1,\ldots,\gamma_N} S^{\sigma_1\gamma_1}_{\sigma'_1\gamma_2}(\lambda) S^{\sigma_2\gamma_2}_{\sigma'_2\gamma_3}(\lambda)\cdots S^{\sigma_N\gamma_N}_{\sigma'_N\gamma_1}(\lambda) \tag{8.6}$$

we see that the two expressions coincide if we identify

$$W^{\sigma}_{\sigma'}(\gamma,\gamma') = S^{\sigma\gamma}_{\sigma'\gamma'}(\lambda). \tag{8.7}$$

To obtain the statistical sum (8.3), we have to sum over all rows:

$$Z = \sum_{\{\sigma^1\}} \cdots \sum_{\{\sigma^M\}} T^{\{\sigma^1\}}_{\{\sigma^2\}} T^{\{\sigma^2\}}_{\{\sigma^3\}} \cdots T^{\{\sigma^M\}}_{\{\sigma^1\}} = \operatorname{Tr} T^M, \tag{8.8}$$

where the lattice periodicity was reflected via the identification $\{\sigma^{M+1}\} = \{\sigma^1\}$. The transfer matrix has the dimension 2^N. Let us denote its eigenvalue of the greatest modulus by $\Lambda_{\max}(N)$. Inserting the representation (8.8) into the definition of the specific free energy (8.4), in the limit $M \to \infty$ we get

$$-\beta f = \lim_{N\to\infty} \frac{1}{N} \ln |\Lambda_{\max}(N)|. \tag{8.9}$$

We see that the problem of finding the free energy reduces to the problem of the diagonalization of the transfer matrix.

We shall restrict ourselves to specific vertex models for which the number of arrows oriented towards each lattice node is even, i.e. 0, 2 or 4. Eight admissible configurations of arrows

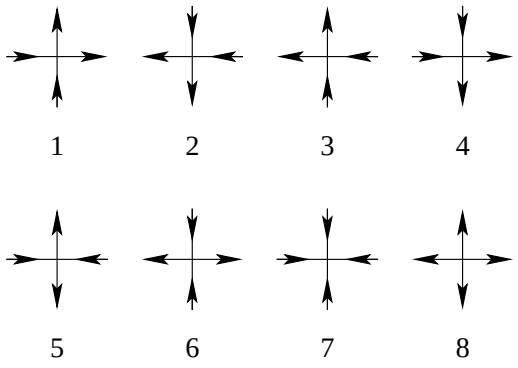


Fig. 8.2. Admissible vertex configurations

around a vertex, which meet this rule, are presented in Fig. 8.2. The forbidden configurations $j = 9, \ldots, 16$ have infinite energies $\epsilon_j \to \infty$ $(j = 9, \ldots, 16)$. We assume that there are no external fields, so the energies are invariant with respect to the simultaneous inversion of arrows on the lattice:

$$\epsilon_1 = \epsilon_2 \equiv \epsilon_a, \quad \epsilon_3 = \epsilon_4 \equiv \epsilon_b, \quad \epsilon_5 = \epsilon_6 \equiv \epsilon_c, \quad \epsilon_7 = \epsilon_8 \equiv \epsilon_d. \tag{8.10}$$

We denote the corresponding Boltzmann weights as follows

$$a(\lambda) = \mathrm{e}^{-\beta\epsilon_a}, \quad b(\lambda) = \mathrm{e}^{-\beta\epsilon_b}, \quad c(\lambda) = \mathrm{e}^{-\beta\epsilon_c}, \quad d(\lambda) = \mathrm{e}^{-\beta\epsilon_d}. \tag{8.11}$$

According to (8.7), the S-matrix related to the vertex model then reads

$$S(\lambda) = \begin{pmatrix} a(\lambda) & 0 & 0 & d(\lambda) \\ 0 & b(\lambda) & c(\lambda) & 0 \\ 0 & c(\lambda) & b(\lambda) & 0 \\ d(\lambda) & 0 & 0 & a(\lambda) \end{pmatrix}. \tag{8.12}$$

This is exactly the S-matrix (7.57) analyzed in the previous section.

If $d(\lambda) \neq 0$, the matrix elements admit the elliptic parametrization (7.79) under which two transfer matrices with different values of the spectral parameter commute. The statistical lattice system is called the 8-vertex model since there are eight admissible vertex configurations of edge arrows.

If $d(\lambda) = 0$, i.e. the configurations 7 and 8 in Fig. 8.2 are forbidden $(\epsilon_7 = \epsilon_8 \to \infty)$, the matrix elements admit the trigonometric parametrization (7.80). The statistical lattice system is known as the 6-vertex model. This model satisfies the ice rule: if the oxygen atoms are located at nodes of a lattice of coordination four, from the four hydrogen ions surrounding each oxygen atom two are bounded to it (two arrows are oriented towards this atom) and two are bounded to its neighbors (two arrows are oriented outwards this atom).

8.2 Connection with quantum models on a chain

We now establish the relationship between the transfer matrix (8.6) and a quantum model formulated on the chain of N sites. The elements of the S-matrix (8.12) are parametrized elliptically, see Eq. (7.77). We shall not require the normalization of the S-matrix and simply set

$$a(\lambda) = \mathrm{sn}(\lambda+\eta), \quad b(\lambda) = \mathrm{sn}\,\lambda, \quad c(\lambda) = \mathrm{sn}\,\eta, \quad d(\lambda) = k\,\mathrm{sn}\,\lambda\,\mathrm{sn}\,\eta\,\mathrm{sn}(\lambda+\eta), \tag{8.13}$$

where the Jacobi elliptic function sn has the modulus k. We recall that, according to Eqs. (7.56) and (7.57), the S-matrix of type (8.12) is expressible in terms of tensor products of the unity 2×2 matrix σ^0 and the Pauli matrices $\sigma^x \equiv \sigma^1$, $\sigma^y \equiv \sigma^2$, $\sigma^z \equiv \sigma^3$ as follows

$$S^{\sigma_1\sigma_2}_{\sigma_1'\sigma_2'}(\lambda) = \sum_{j=0}^{3} w_j(\lambda)(\sigma^j)^{\sigma_1}_{\sigma_1'}(\sigma^j)^{\sigma_2}_{\sigma_2'}, \tag{8.14}$$

where

$$w_0 = \frac{1}{2}(a+b), \quad w_1 = \frac{1}{2}(c+d), \quad w_2 = \frac{1}{2}(c-d), \quad w_3 = \frac{1}{2}(a-b). \tag{8.15}$$

At $\lambda = 0$, the S-matrix is proportional to the permutation matrix,

$$S^{\sigma_1\sigma_2}_{\sigma_1'\sigma_2'}(\lambda=0) = \mathrm{sn}\,\eta\,\delta(\sigma_1,\sigma_2')\delta(\sigma_2,\sigma_1'). \tag{8.16}$$

At the same point, the transfer matrix (8.6) is, up to a prefactor, an operator of cyclic displacement:

$$T(\lambda=0)^{\sigma_1\ldots\sigma_N}_{\sigma_1'\ldots\sigma_N'} = (\mathrm{sn}\,\eta)^N\delta(\sigma_1,\sigma_2')\delta(\sigma_2,\sigma_3')\ldots\delta(\sigma_N,\sigma_1'). \tag{8.17}$$

Thus,

$$T^{-1}(\lambda=0)^{\sigma_1\ldots\sigma_N}_{\sigma_1'\ldots\sigma_N'} = (\mathrm{sn}\,\eta)^{-N}\delta(\sigma_1,\sigma_N')\delta(\sigma_2,\sigma_1')\ldots\delta(\sigma_N,\sigma_{N-1}'). \tag{8.18}$$

We first differentiate the expression for the transfer matrix (8.6) with respect to λ and then set $\lambda = 0$, with the result

$$\begin{aligned}\left\{\frac{\mathrm{d}}{\mathrm{d}\lambda}T(\lambda)\right\}^{\sigma_1\ldots\sigma_N}_{\sigma_1'\ldots\sigma_N'}\Bigg|_{\lambda=0} &= (\mathrm{sn}\,\eta)^{N-1}\sum_{n=1}^{N}\delta(\sigma_1,\sigma_2')\ldots\delta(\sigma_{n-2},\sigma_{n-1}') \\ &\quad\times\frac{\mathrm{d}}{\mathrm{d}\lambda}S^{\sigma_n\sigma_{n-1}}_{\sigma_n'\sigma_{n+1}'}(\lambda)\Big|_{\lambda=0}\delta(\sigma_{n+1},\sigma_{n+2}')\ldots\delta(\sigma_N,\sigma_1').\end{aligned} \tag{8.19}$$

For the logarithmic derivative

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln\,T(\lambda) = T^{-1}(\lambda)\frac{\mathrm{d}}{\mathrm{d}\lambda}T(\lambda) \tag{8.20}$$

taken at $\lambda = 0$ we thus obtain

$$\begin{aligned}\left\{\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln T(\lambda)\right\}^{\sigma_1\ldots\sigma_N}_{\sigma_1'\ldots\sigma_N'}\Bigg|_{\lambda=0} &= \frac{1}{\mathrm{sn}\,\eta}\sum_{n=1}^{N}\delta(\sigma_1,\sigma_1')\ldots\delta(\sigma_{n-1},\sigma_{n-1}') \\ &\quad\times\frac{\mathrm{d}}{\mathrm{d}\lambda}S^{\sigma_{n+1}\sigma_n}_{\sigma_n'\sigma_{n+1}'}(\lambda)\Big|_{\lambda=0}\delta(\sigma_{n+2},\sigma_{n+2}')\ldots\delta(\sigma_N,\sigma_N').\end{aligned} \tag{8.21}$$

It is easy to verify that the representation of the S-matrix (8.14) can be transformed to an equivalent one

$$S^{\sigma_1\sigma_2}_{\sigma_1'\sigma_2'}(\lambda) = \sum_{j=0}^{3} p_j(\lambda)(\sigma^j)^{\sigma_1}_{\sigma_2'}(\sigma^j)^{\sigma_2}_{\sigma_1'}, \tag{8.22}$$

where

$$\begin{aligned} p_0 &= \frac{1}{2}(w_0 + w_1 + w_2 + w_3) = \frac{1}{2}(a + c), \\ p_1 &= \frac{1}{2}(w_0 + w_1 - w_2 - w_3) = \frac{1}{2}(b + d), \\ p_2 &= \frac{1}{2}(w_0 - w_1 + w_2 - w_3) = \frac{1}{2}(b - d), \\ p_3 &= \frac{1}{2}(w_0 - w_1 - w_2 + w_3) = \frac{1}{2}(a - c). \end{aligned} \tag{8.23}$$

Inserting this representation into the relation (8.21) and recalling from Appendix A the definition of the spin operators on the chain, we finally arrive at

$$\operatorname{sn}\eta \, \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln \, T(\lambda)\Big|_{\lambda=0} = \frac{1}{2}\sum_{n=1}^{N} \left(J_x \boldsymbol{\sigma}_n^x \boldsymbol{\sigma}_{n+1}^x + J_y \boldsymbol{\sigma}_n^y \boldsymbol{\sigma}_{n+1}^y + J_z \boldsymbol{\sigma}_n^z \boldsymbol{\sigma}_{n+1}^z\right) + \frac{N}{2} J_z \mathbf{I}, \tag{8.24}$$

where

$$\begin{aligned} J_x &= 2\frac{\partial p_1}{\partial \lambda}\Big|_{\lambda=0} = 1 + k\,\mathrm{sn}^2\eta, \\ J_y &= 2\frac{\partial p_2}{\partial \lambda}\Big|_{\lambda=0} = 1 - k\,\mathrm{sn}^2\eta, \\ J_z &= 2\frac{\partial p_3}{\partial \lambda}\Big|_{\lambda=0} = \mathrm{cn}\,\eta\,\mathrm{dn}\,\eta, \end{aligned} \tag{8.25}$$

$\mathbf{I}$ is the $2^N \times 2^N$ unity matrix and $\boldsymbol{\sigma}_{N+1}^{(x,y,z)} \equiv \boldsymbol{\sigma}_1^{(x,y,z)}$. The quantum model of interacting spins with the chain Hamiltonian as written on the rhs of Eq. (8.24) is the XYZ Heisenberg model. When the modulus $k = 0$ (trigonometric parametrization), we have the XXZ Heisenberg model with coupling constants

$$J_x = J_y = 1, \qquad J_z = \cos\eta. \tag{8.26}$$

In the limit $\eta \to 0$, we obtain the XXX Heisenberg model with $J_x = J_y = J_z$. The relation (8.24) makes a direct link between the eigenvalue spectrum of the transfer matrix and that of the Heisenberg Hamiltonian.

8.3 Diagonalization of the trigonometric transfer matrix

The aim of this part is the construction of an algebraic generalization of the Bethe ansatz for finding the eigenvectors and the energy spectrum of the transfer matrix. The S-matrix with the elliptic parametrization of the matrix elements will be considered later in connection with the

Heisenberg XYZ model. Here, we shall restrict ourselves to the simpler case of the S-matrix (8.12) with the trigonometric parametrization (7.80),

$$a(\lambda) = \sin(\lambda + \eta), \quad b(\lambda) = \sin\lambda, \quad c(\lambda) = \sin\eta, \quad d(\lambda) = 0. \tag{8.27}$$

The coefficients w_j (7.81) are given by

$$w_0 + w_3 = \sin(\lambda + \eta), \quad w_0 - w_3 = \sin\lambda, \quad w_1 = w_2 = \frac{1}{2}\sin\eta. \tag{8.28}$$

The Lax operator (7.59) has in the auxiliary 2×2 ξ-space the form

$$L_n(\lambda) = \begin{pmatrix} w_0(\lambda)\boldsymbol{\sigma}_n^0 + w_3(\lambda)\boldsymbol{\sigma}_n^3 & w_1(\lambda)\boldsymbol{\sigma}_n^- \\ w_1(\lambda)\boldsymbol{\sigma}_n^+ & w_0(\lambda)\boldsymbol{\sigma}_n^0 - w_3(\lambda)\boldsymbol{\sigma}_n^3 \end{pmatrix} \equiv \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix}. \tag{8.29}$$

The monodromy matrix, defined in Eq. (7.46) as the product

$$\mathcal{T}(\lambda) = \prod_{n=1}^{N} L_n(\lambda), \tag{8.30}$$

will be represented in the auxiliary ξ-space as follows

$$\mathcal{T}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \tag{8.31}$$

where each of the operator matrix elements A, B, C and D acts in the 2^N-dimensional space of indices $\{\alpha_1 \ldots \alpha_N\}$. Within the framework of this representation, the transfer matrix is given by

$$T(\lambda) = \mathrm{Tr}_\xi \mathcal{T}_\xi(\lambda) = A(\lambda) + D(\lambda). \tag{8.32}$$

The monodromy matrix $\mathcal{T}$ and the R-matrix $R = \mathcal{P}S$ satisfy the YBE (7.49),

$$R(\lambda - \mu)\,[\mathcal{T}(\lambda) \otimes \mathcal{T}(\mu)] = [\mathcal{T}(\mu) \otimes \mathcal{T}(\lambda)]\,R(\lambda - \mu). \tag{8.33}$$

Considering the monodromy-matrix representation (8.31) in this equation leads to all possible permutation relations between the matrices $\{A, B, C, D\}$ with various values of the spectral parameter. We write down only those which are relevant for our next purposes:

$$[A(\lambda), A(\mu)] = [B(\lambda), B(\mu)] = [C(\lambda), C(\mu)] = [D(\lambda), D(\mu)] = 0, \tag{8.34}$$

$$A(\lambda)B(\mu) = \frac{a(\mu-\lambda)}{b(\mu-\lambda)}B(\mu)A(\lambda) - \frac{c(\mu-\lambda)}{b(\mu-\lambda)}B(\lambda)A(\mu), \tag{8.35}$$

$$D(\lambda)B(\mu) = \frac{a(\lambda-\mu)}{b(\lambda-\mu)}B(\mu)D(\lambda) - \frac{c(\lambda-\mu)}{b(\lambda-\mu)}B(\lambda)D(\mu). \tag{8.36}$$

We choose as a "generating" vector of the 2^N-dimensional Hilbert space the tensor product of spin-up vectors $e^+ = \binom{1}{0}$ on the chain of N sites:

$$\Omega = \underbrace{e^+}_{1} \otimes \underbrace{e^+}_{2} \otimes \cdots \otimes \underbrace{e^+}_{N-1} \otimes \underbrace{e^+}_{N}. \tag{8.37}$$

The Lax operator L_n (8.29) acts on each site as the identity operator, except the nth site where it acts on $e_n^+ = \binom{1}{0}$ as follows

$$\begin{aligned} &\alpha_n(\lambda)e_n^+ = (w_0+w_3)e_n^+ = a(\lambda)e_n^+, \qquad &&\beta_n(\lambda)e_n^+ = 2w_1e_n^- = c(\lambda)e_n^-, \\ &\gamma_n(\lambda)e_n^+ = 0, &&\delta_n(\lambda)e_n^+ = (w_0-w_1)e_n^+ = b(\lambda)e_n^+. \end{aligned} \tag{8.38}$$

These relations can be written in a compact form

$$L_n(\lambda)e_n^+ = \begin{pmatrix} a(\lambda) & [\cdots] \\ 0 & b(\lambda) \end{pmatrix} e_n^+, \tag{8.39}$$

where the symbol $[\cdots]$ denotes an operator matrix element, transforming the vector $e_n^+ = \binom{1}{0}$ to $e_n^- = \binom{0}{1}$, whose explicit form will not be needed. The triangle form of the matrix on the rhs of Eq. (8.39) permits us to express the action of the monodromy matrix (8.30) on the vector Ω as

$$\mathcal{T}(\lambda)\Omega = \begin{pmatrix} a^N(\lambda) & [\cdots] \\ 0 & b^N(\lambda) \end{pmatrix} \Omega. \tag{8.40}$$

From the representation (8.31), we obtain the action of the elements of the monodromy matrix on the generating vector Ω:

$$A(\lambda)\Omega = a^N(\lambda)\Omega, \quad C(\lambda)\Omega = 0, \quad D(\lambda)\Omega = b^N(\lambda)\Omega. \tag{8.41}$$

The action of the operator $B(\lambda)$ on Ω is too complicated to be written explicitly. It is clear from the above scheme that the vector $B(\lambda)\Omega$ is a superposition of N vectors which are obtained from Ω by substituting one e^+-vector by e^--vector, at site 1, or 2,. . ., or N. More generally, the action of the operator $B(\lambda)$ on a chain vector in the sector with $N-M$ spins up and M spins down implies a superposition of chain vectors in the sector with $N-M-1$ spins up and $M+1$ spins down. The action of the operator B will be reflected indirectly via its commutation relations with A (8.35) and D (8.36).

The eigenvectors of the transfer matrix T will be searched in an ansatz form

$$\psi(\lambda_1,\ldots,\lambda_M) = \prod_{\alpha=1}^{M} B(\lambda_\alpha)\Omega, \tag{8.42}$$

where $M = 0, 1, \ldots, N$ and the as-yet unspecified parameters $\{\lambda_1, \lambda_2, \ldots, \lambda_M\}$ will be fixed by some self-consistent condition. The ansatz has two important features. Firstly, since there are just M B-operators acting on Ω, the eigenvector is searched in an invariant sector with $N-M$ spins up and M spins down. Secondly, the eigenvectors are not supposed to depend on the spectral parameter λ, which is in agreement with the previous finding that two transfer matrices with different values of the spectral parameter commute to 0. With regard to the representation of the transfer matrix (8.32), the eigenfunction equation reads

$$\begin{aligned} T(\lambda)\psi(\lambda_1,\ldots,\lambda_M) &\equiv [A(\lambda)+D(\lambda)]\prod_{\alpha=1}^{M} B(\lambda_\alpha)\Omega \\ &= t(\lambda;\lambda_1,\ldots,\lambda_M)\prod_{\alpha=1}^{M} B(\lambda_\alpha)\Omega. \end{aligned} \tag{8.43}$$

Since the action of B on Ω is not known explicitly, it is necessary to commute these operators with A and D, with the aid of the commutation relations (8.35) and (8.36); A and D then act directly on the generating vector Ω according to the relations in (8.41). To document the procedure, we consider the expression in (8.43) containing A and make the first expansion step

$$\begin{aligned} A(\lambda)\prod_{\alpha=1}^{M}B(\lambda_\alpha)\Omega &= \left[\frac{a(\lambda_1-\lambda)}{b(\lambda_1-\lambda)}B(\lambda_1)A(\lambda)-\frac{c(\lambda_1-\lambda)}{b(\lambda_1-\lambda)}B(\lambda)A(\lambda_1)\right] \\ &\quad\times\prod_{\alpha=2}^{M}B(\lambda_\alpha)\Omega. \end{aligned} \tag{8.44}$$

In the next step, the rhs of this equation is expanded as follows

$$\begin{aligned} &\Big[\frac{a(\lambda_1-\lambda)}{b(\lambda_1-\lambda)}\frac{a(\lambda_2-\lambda)}{b(\lambda_2-\lambda)}B(\lambda_1)B(\lambda_2)A(\lambda) \\ &-\frac{a(\lambda_1-\lambda)}{b(\lambda_1-\lambda)}\frac{c(\lambda_2-\lambda)}{b(\lambda_2-\lambda)}B(\lambda_1)B(\lambda)A(\lambda_2) \\ &-\frac{c(\lambda_1-\lambda)}{b(\lambda_1-\lambda)}\frac{a(\lambda_2-\lambda_1)}{b(\lambda_2-\lambda_1)}B(\lambda)B(\lambda_2)A(\lambda_1) \\ &+\frac{c(\lambda_1-\lambda)}{b(\lambda_1-\lambda)}\frac{c(\lambda_2-\lambda_1)}{b(\lambda_2-\lambda_1)}B(\lambda)B(\lambda_1)A(\lambda_2)\Big]\prod_{\alpha=3}^{M}B(\lambda_\alpha)\Omega. \end{aligned} \tag{8.45}$$

Since the operators $\{B(\lambda_\alpha)\}$ commute with each other, the result must be symmetric with respect to the interchange $\lambda_1 \leftrightarrow \lambda_2$. From the point of view of Eq. (8.45), this is equivalent to saying that the following relation

$$\frac{c(\lambda_1-\lambda)}{b(\lambda_1-\lambda)}\frac{c(\lambda_2-\lambda_1)}{b(\lambda_2-\lambda_1)}-\frac{a(\lambda_1-\lambda)}{b(\lambda_1-\lambda)}\frac{c(\lambda_2-\lambda)}{b(\lambda_2-\lambda)}=-\frac{c(\lambda_2-\lambda)}{b(\lambda_2-\lambda)}\frac{a(\lambda_1-\lambda_2)}{b(\lambda_1-\lambda_2)} \tag{8.46}$$

must hold. It is easy to check that within the parametrization (8.27) this equation is indeed fulfilled. Thus, Eq. (8.45) can be reexpressed as follows

$$\begin{aligned} &\Big[\frac{a(\lambda_1-\lambda)}{b(\lambda_1-\lambda)}\frac{a(\lambda_2-\lambda)}{b(\lambda_2-\lambda)}B(\lambda_1)B(\lambda_2)A(\lambda) \\ &-\frac{c(\lambda_1-\lambda)}{b(\lambda_1-\lambda)}\frac{a(\lambda_2-\lambda_1)}{b(\lambda_2-\lambda_1)}B(\lambda)B(\lambda_2)A(\lambda_1) \\ &-\frac{c(\lambda_2-\lambda)}{b(\lambda_2-\lambda)}\frac{a(\lambda_1-\lambda_2)}{b(\lambda_1-\lambda_2)}B(\lambda)B(\lambda_1)A(\lambda_2)\Big]\prod_{\alpha=3}^{M}B(\lambda_\alpha)\Omega. \end{aligned} \tag{8.47}$$

Since $b(\lambda)=\sin\lambda$, it must hold that $\lambda_1 \neq \lambda_2$. Proceeding in this way further, and repeating the procedure also for the expression in (8.43) containing the operator D, one ends up with the result

$$\begin{aligned} T(\lambda)\psi(\lambda_1,\ldots,\lambda_M) &= t(\lambda;\lambda_1,\ldots,\lambda_M)\prod_{\alpha=1}^{M}B(\lambda_\alpha)\Omega \\ &\quad+\sum_{\alpha=1}^{M}t_\alpha(\lambda;\lambda_1,\ldots,\lambda_M)B(\lambda)\prod_{\substack{\beta=1\\(\beta\neq\alpha)}}^{M}B(\lambda_\beta)\Omega, \end{aligned} \tag{8.48}$$

where

$$t(\lambda;\lambda_1,\ldots,\lambda_M) = a^N(\lambda)\prod_{\alpha=1}^{M}\frac{a(\lambda_\alpha-\lambda)}{b(\lambda_\alpha-\lambda)} + b^N(\lambda)\prod_{\alpha=1}^{M}\frac{a(\lambda-\lambda_\alpha)}{b(\lambda-\lambda_\alpha)} \tag{8.49}$$

and

$$\begin{aligned} t_\alpha(\lambda;\lambda_1,\ldots,\lambda_M) &= -\frac{c(\lambda_\alpha-\lambda)}{b(\lambda_\alpha-\lambda)}\left[a^N(\lambda_\alpha)\prod_{\substack{\beta=1\\(\beta\neq\alpha)}}^{M}\frac{a(\lambda_\beta-\lambda_\alpha)}{b(\lambda_\beta-\lambda_\alpha)}\right. \\ &\quad \left. -b^N(\lambda_\alpha)\prod_{\substack{\beta=1\\(\beta\neq\alpha)}}^{M}\frac{a(\lambda_\alpha-\lambda_\beta)}{b(\lambda_\alpha-\lambda_\beta)}\right]. \end{aligned} \tag{8.50}$$

In view of Eq. (8.48), the condition for $\psi(\lambda_1,\ldots,\lambda_M)$ (8.42) to be an eigenvector of the transfer matrix is the nullity of all t_α, i.e. the system of nonlinear Bethe equations

$$\left[\frac{a(\lambda_\alpha)}{b(\lambda_\alpha)}\right]^N = \prod_{\substack{\beta=1\\(\beta\neq\alpha)}}^{M}\frac{a(\lambda_\alpha-\lambda_\beta)}{a(\lambda_\beta-\lambda_\alpha)}\frac{b(\lambda_\beta-\lambda_\alpha)}{b(\lambda_\alpha-\lambda_\beta)} \qquad \alpha=1,2,\ldots,M, \tag{8.51}$$

which determines the set of distinct parameters $\{\lambda_1,\lambda_2,\ldots,\lambda_M\}$. Within the trigonometric parametrization (8.27), this set of equations takes the form

$$\begin{aligned} \left(\frac{\sin(\lambda_\alpha+\eta)}{\sin\lambda_\alpha}\right)^N &= \prod_{\substack{\beta=1\\(\beta\neq\alpha)}}^{M}\frac{\sin(\lambda_\alpha-\lambda_\beta+\eta)}{\sin(\lambda_\alpha-\lambda_\beta-\eta)} \\ &= -\prod_{\beta=1}^{M}\frac{\sin(\lambda_\alpha-\lambda_\beta+\eta)}{\sin(\lambda_\alpha-\lambda_\beta-\eta)} \qquad \alpha=1,2,\ldots,M. \end{aligned} \tag{8.52}$$

The corresponding eigenvalue of $T(\lambda)$ is $t(\lambda;\lambda_1,\ldots,\lambda_M)$ given by Eq. (8.49),

$$t(\lambda) = \sin^N(\lambda+\eta)\prod_{\alpha=1}^{M}\frac{\sin(\lambda_\alpha-\lambda+\eta)}{\sin(\lambda_\alpha-\lambda)} + \sin^N(\lambda)\prod_{\alpha=1}^{M}\frac{\sin(\lambda-\lambda_\alpha+\eta)}{\sin(\lambda-\lambda_\alpha)}. \tag{8.53}$$

It is sometimes useful to perform the shift in rapidities $\lambda_\alpha \to \lambda_\alpha - \eta/2$ to obtain the symmetrized form of the Bethe equations. The described method of the diagonalization of the transfer matrix is called the algebraic Bethe ansatz.

Using the relationship (8.24), the energy eigenvalues E of the XXZ Hamiltonian

$$H_{XXZ} = -\frac{1}{2}\sum_{n=1}^{N}\left(\boldsymbol{\sigma}_n^x\boldsymbol{\sigma}_{n+1}^x + \boldsymbol{\sigma}_n^y\boldsymbol{\sigma}_{n+1}^y + \cos\eta\,\boldsymbol{\sigma}_n^z\boldsymbol{\sigma}_{n+1}^z\right) \tag{8.54}$$

are expressible in terms of $t(\lambda)$ (8.53) as follows

$$E = -\sin\eta\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln t(\lambda)\Big|_{\lambda=0} + \frac{N}{2}\cos\eta. \tag{8.55}$$

After simple algebra, we find that

$$E = -\frac{N}{2}\cos\eta + \sin\eta \sum_{\alpha=1}^{M} \left[\cot(\lambda_\alpha + \eta) - \cot(\lambda_\alpha)\right]. \tag{8.56}$$

The advantage of the outlined QISM consists in its universal applicability to an arbitrary integrable model. At first stage, it is necessary to introduce an S-matrix which fulfills the Yang-Baxter equation. The transfer matrix constructed from this S-matrix is then related to a quantum Hamiltonian on the discrete chain. Finally, the diagonalization of the transfer matrix proceeds along the above lines. We repeat once more that the S-matrix used in the QISM *is not* the S-matrix associated with the resulting quantum model. For example, the S-matrix of the one-dimensional Heisenberg XXZ model is a scalar.

9 QISM: Treatment of boundary conditions

There exists another important application of the QISM in the context of the treatment of boundary conditions for one-dimensional quantum systems of particles with internal degrees of freedom, solvable by using the generalized Bethe ansatz. The S-matrix in this problem is the true scattering matrix of the particle system which satisfies all necessary consistency conditions and is properly normalized.

9.1 Formulation of boundary conditions

We start with fermions on the line $\langle 0, L\rangle$, for which the generalized Bethe ansatz in the ordering sector

$$Q: \qquad 0 \le x_{Q1} < x_{Q2} < \cdots < x_{QN} \le L \tag{9.1}$$

has the form given by (7.4) and (7.5),

$$\begin{aligned}\psi_Q(\sigma_1, x_1; \ldots; \sigma_N, x_N) \;&=\; \sum_{P\in S_N} \operatorname{sign}(Q)\operatorname{sign}(P) A_{\sigma_{Q1}\ldots\sigma_{QN}}(k_{P1},\ldots,k_{PN}) \\ &\quad\times \exp\left(\mathrm{i}\sum_{j=1}^{N} k_{Pj}x_{Qj}\right).\end{aligned} \tag{9.2}$$

Let the smallest of the particle coordinates x_{Q1} be equal to 0. The periodic boundary condition corresponds to the invariance of ψ with respect to the shift of x_{Q1} by L, i.e. the transformation

$$x_{Q1} = 0 \to \tilde{x}_{Q1} = L, \quad \tilde{x}_{Q2} = x_{Q2}, \quad \ldots, \quad \tilde{x}_{QN} = x_{QN}. \tag{9.3}$$

Since after the shift the coordinate $\tilde{x}_{Q1}$ becomes the largest one, the new ordering sector is $\tilde{Q} = (Q2, Q3, \ldots, QN, Q1)$. The invariance of the wavefunction under the shift

$$\psi_Q(\sigma_1, x_1; \ldots; \sigma_N, x_N) = \psi_{\tilde{Q}}(\sigma_1, \tilde{x}_1; \ldots; \sigma_N, \tilde{x}_N) \tag{9.4}$$

is equivalent to the relation

$$\begin{aligned}&\sum_{P\in S_N} \operatorname{sign}(Q)\operatorname{sign}(P) A_{\sigma_{Q1}\sigma_{Q2}\ldots\sigma_{QN}}(k_{P1}, k_{P2}, \ldots, k_{PN}) \\ &\qquad\times \exp\left(\mathrm{i}k_{P1}\cdot 0 + \mathrm{i}k_{P2}x_{Q2} + \cdots + \mathrm{i}k_{PN}x_{QN}\right) \\ =\;&\sum_{P\in S_N} \operatorname{sign}(\tilde{Q})\operatorname{sign}(P) A_{\sigma_{Q2}\ldots\sigma_{QN}\sigma_{Q1}}(k_{P1},\ldots,k_{P(N-1)}, k_{PN}) \\ &\qquad\times \exp\left(\mathrm{i}k_{P1}x_{Q2} + \cdots + \mathrm{i}k_{P(N-1)}x_{QN} + \mathrm{i}k_{PN}L\right).\end{aligned} \tag{9.5}$$

Let us assign to each permutation $P = (P1, P2, \ldots, PN)$ on the rhs of (9.5) the conjugate one $\tilde{P} \equiv (\tilde{P}1, \tilde{P}2, \ldots, \tilde{P}N) = (PN, P1, P2, \ldots, P(N-1))$. Since $\operatorname{sign}(\tilde{Q}) = (-1)^{N-1}\operatorname{sign}(Q)$ and $\operatorname{sign}(P) = (-1)^{N-1}\operatorname{sign}(\tilde{P})$, the rhs of Eq. (9.5) can be rewritten as

$$\begin{aligned}&\sum_{\tilde{P}\in S_N} \operatorname{sign}(Q)\operatorname{sign}(\tilde{P}) A_{\sigma_{Q2}\ldots\sigma_{QN}\sigma_{Q1}}(k_{\tilde{P}2},\ldots,k_{\tilde{P}N}, k_{\tilde{P}1}) \\ &\qquad\times \exp\left(\mathrm{i}k_{\tilde{P}2}x_{Q2} + \cdots + \mathrm{i}k_{\tilde{P}N}x_{QN} + \mathrm{i}k_{\tilde{P}1}L\right).\end{aligned} \tag{9.6}$$

Redefining formally $\tilde{P} \to P$, we thus get

$$A_{\sigma_{Q2}\ldots\sigma_{QN}\sigma_{Q1}}(k_{P2},\ldots,k_{PN},k_{P1}) = \exp(-\mathrm{i}k_{P1}L) \times A_{\sigma_{Q1}\sigma_{Q2}\ldots\sigma_{QN}}(k_{P1},k_{P2},\ldots,k_{PN}). \quad (9.7)$$

Note that this equation holds for an arbitrary choice of the permutations P and Q.

The same procedure can be applied also for Bose particles. In the generalized Bethe ansatz for bosons

$$\psi_Q(\sigma_1,x_1;\ldots;\sigma_N,x_N) = \sum_{P\in S_N} \mathrm{sign}(P) A_{\sigma_{Q1}\ldots\sigma_{QN}}(k_{P1},\ldots,k_{PN}) \times \exp\left(\mathrm{i}\sum_{j=1}^{N} k_{Pj}x_{Qj}\right). \quad (9.8)$$

the factor $\mathrm{sign}(Q)$ is missing. This brings an additional factor $(-1)^{N-1}$ since the factor in $\mathrm{sign}(P) = (-1)^{N-1}\mathrm{sign}(\tilde{P})$ is not compensated. We find that

$$A_{\sigma_{Q2}\ldots\sigma_{QN}\sigma_{Q1}}(k_{P2},\ldots,k_{PN},k_{P1}) = (-1)^{N-1}\exp(-\mathrm{i}k_{P1}L) \times A_{\sigma_{Q1}\sigma_{Q2}\ldots\sigma_{QN}}(k_{P1},k_{P2},\ldots,k_{PN}). \quad (9.9)$$

The two relations (9.7) and (9.9) can be unified in one equation

$$A_{\sigma_{Q2}\ldots\sigma_{QN}\sigma_{Q1}}(k_{P2},\ldots,k_{PN},k_{P1}) = (\mp 1)^{N-1}\exp(-\mathrm{i}k_{P1}L) \times A_{\sigma_{Q1}\sigma_{Q2}\ldots\sigma_{QN}}(k_{P1},k_{P2},\ldots,k_{PN}), \quad (9.10)$$

where the $-/+$ sign corresponds to bosons/fermions.

9.2 Boundary conditions and the inhomogeneous transfer matrix

For the special choice $P = Q = (1,2,\ldots,N)$, Eq. (9.10) takes the form

$$A_{\sigma_2\ldots\sigma_N\sigma_1}(k_2,\ldots,k_N,k_1) = (\mp 1)^{N-1}\exp(-\mathrm{i}k_1L) \times A_{\sigma_1\sigma_2\ldots\sigma_N}(k_1,k_2,\ldots,k_N). \quad (9.11)$$

The A-coefficient on the lhs of this equation differs from the one on the rhs by the presence of the particle state σ_1 and the corresponding wave number k_1 on the right side of the sequence. We would like to have an equation which relates the A-coefficients with the same ordering of state indices and wave numbers. To accomplish this aim, we shall apply the two-particle scattering formula (7.13),

$$A_{\ldots\sigma_j\sigma_i\ldots}(\ldots k_v,k_u\ldots) = \sum_{\sigma_i'\sigma_j'} S^{\sigma_i\sigma_j}_{\sigma_i'\sigma_j'}(k_u,k_v) A_{\ldots\sigma_i'\sigma_j'\ldots}(\ldots k_u,k_v\ldots), \quad (9.12)$$

to the A-coefficient on the lhs of Eq. (9.11) in order to "commute" successively k_1 with all other wave numbers $k_2,k_3,\ldots,k_N$:

$$A_{\sigma_2\ldots\sigma_N\sigma_1}(k_2,\ldots,k_N,k_1) = \sum_{\gamma_2\sigma_N'} S^{\sigma_1\sigma_N}_{\gamma_2\sigma_N'}(k_1,k_N)\, A_{\sigma_2\ldots\gamma_2\sigma_N'}(k_2,\ldots,k_1,k_N)$$

$$= \sum_{\substack{\sigma'_1,\ldots,\sigma'_N \\ \gamma_2,\ldots,\gamma_{N-1}}} S^{\sigma_1\sigma_N}_{\gamma_2\sigma'_N}(k_1,k_N)\, S^{\gamma_2\sigma_{N-1}}_{\gamma_3\sigma'_{N-1}}(k_1,k_{N-1})\cdots \times S^{\gamma_{N-1}\sigma_2}_{\sigma'_1\sigma'_2}(k_1,k_2) A_{\sigma'_1\sigma'_2\ldots\sigma'_N}(k_1,k_2,\ldots,k_N). \quad (9.13)$$

The quantization condition for k_1 (9.11) is thus transformed to the eigenvalue problem

$$T_1\, A = (\mp 1)^{N-1} \exp(-\mathrm{i}k_1 L)\, A \quad (9.14)$$

for the matrix

$$T_1{}^{\sigma_1\sigma_2\ldots\sigma_N}_{\sigma'_1\sigma'_2\ldots\sigma'_N} = \sum_{\gamma_2,\ldots,\gamma_{N-1}} S^{\sigma_1\sigma_N}_{\gamma_2\sigma'_N}(k_1,k_N)\, S^{\gamma_2\sigma_{N-1}}_{\gamma_3\sigma'_{N-1}}(k_1,k_{N-1})\cdots S^{\gamma_{N-1}\sigma_2}_{\sigma'_1\sigma'_2}(k_1,k_2). \quad (9.15)$$

We introduce the rapidities $\{\lambda_j\}_{j=1}^N$ which parametrize the wave numbers $\{k_j = k(\lambda_j)\}_{j=1}^N$ in such a way that $S(k_j,k_l) = S(\lambda_j - \lambda_l)$. Moreover, we assume that the S-matrix possesses the T and P symmetries (7.12). The T_1-matrix can be then reexpressed as follows

$$T_1{}^{\sigma_1\sigma_2\ldots\sigma_N}_{\sigma'_1\sigma'_2\ldots\sigma'_N} = \sum_{\gamma_2,\ldots,\gamma_{N-1}} S^{\sigma_N\sigma_1}_{\sigma'_N\gamma_2}(\lambda_1-\lambda_N)\, S^{\sigma_{N-1}\gamma_2}_{\sigma'_{N-1}\gamma_3}(\lambda_1-\lambda_{N-1}) \cdots S^{\sigma_2\gamma_{N-1}}_{\sigma'_2\sigma'_1}(\lambda_1-\lambda_2). \quad (9.16)$$

Let us define the inhomogeneous transfer matrix

$$T(\lambda;\lambda_1,\ldots,\lambda_N)^{\sigma_1\ldots\sigma_N}_{\sigma'_1\ldots\sigma'_N} = \sum_{\gamma_1,\ldots,\gamma_N} S^{\sigma_N\gamma_1}_{\sigma'_N\gamma_2}(\lambda-\lambda_N)\, S^{\sigma_{N-1}\gamma_2}_{\sigma'_{N-1}\gamma_3}(\lambda-\lambda_{N-1}) \cdots S^{\sigma_2\gamma_{N-1}}_{\sigma'_2\gamma_N}(\lambda-\lambda_2)\, S^{\sigma_1\gamma_N}_{\sigma'_1\gamma_1}(\lambda-\lambda_1). \quad (9.17)$$

At the point $\lambda = \lambda_1$, the initial condition $S^{\sigma_1\gamma_N}_{\sigma'_1\gamma_1}(0) = \delta(\sigma_1,\gamma_1)\delta(\sigma'_1,\gamma_N)$ implies

$$T_1 = T(\lambda = \lambda_1;\lambda_1,\ldots,\lambda_N). \quad (9.18)$$

We conclude that the quantization of the wave number k_1 due to the periodic boundary conditions is determined by the eigenvalue equation

$$T_1\, A = (\mp 1)^{N-1} \exp(-\mathrm{i}k_1 L)\, A, \qquad T_1 = T(\lambda = \lambda_1;\lambda_1,\ldots,\lambda_N). \quad (9.19)$$

The above procedure can be performed for other choices of the permutations P and Q in Eq. (9.10). We obtain the following eigenvalue equations determining the wave numbers k_j $(j = 1,\ldots,N)$:

$$T_j\, A = (\mp 1)^{N-1} \exp(-\mathrm{i}k_j L)\, A, \qquad T_j = T(\lambda = \lambda_j;\lambda_1,\ldots,\lambda_N). \quad (9.20)$$

The original problem of boundary conditions is thus reduced to the problem of the diagonalization of the inhomogeneous tranfer matrix (9.17).

9.3 Diagonalization of the inhomogeneous transfer matrix

The transfer matrix (9.17) is by the structure very similar to the one defined in the QISM by Eq. (7.50). A minor difference is the opposite order of indices σ, σ'. An important difference is that the spectral parameters of the S-matrices are site-dependent. As we shall see, this complication does not prevent from the diagonalization of the inhomogeneous transfer matrix (9.17) by using an inhomogeneous version of the QISM.

The S-matrices in the definition of the transfer matrix (9.17) are considered in the form (8.12) with the trigonometric parametrization (8.27) of the matrix elements a, b, c and $d = 0$. As before, the transfer matrix is the trace of the monodromy matrix $\mathcal{T}$ in the auxiliary ξ-space,

$$T(\lambda; \lambda_1, \ldots, \lambda_N) = \mathrm{Tr}_\xi \, \mathcal{T}_\xi(\lambda; \lambda_1, \ldots, \lambda_N). \tag{9.21}$$

The monodromy matrix is expressible as the product of local Lax $L_{\xi n}$-matrices defined by Eq. (7.39),

$$\mathcal{T}_\xi(\lambda; \lambda_1, \ldots, \lambda_N) = L_{\xi N}(\lambda - \lambda_N) L_{\xi(N-1)}(\lambda - \lambda_{N-1}) \cdots L_{\xi 1}(\lambda - \lambda_1). \tag{9.22}$$

The Yang-Baxter equations, obtained within the homogeneous QISM, now take the following forms. The counterpart of the Yang-Baxter Eq. (7.40) reads

$$S_{\xi\eta}(\lambda - \mu) L_{\xi n}(\lambda - \lambda_n) L_{\eta n}(\mu - \lambda_n) = L_{\eta n}(\mu - \lambda_n) L_{\xi n}(\lambda - \lambda_n) S_{\xi\eta}(\lambda - \mu). \tag{9.23}$$

Here, the local shift by the spectral parameter λ_n is canceled in the subtraction of the spectral parameters $\lambda - \mu$. The analogy of the Yang-Baxter Eq. (7.47) is then

$$\begin{aligned} &S_{\xi\eta}(\lambda - \mu) \mathcal{T}_\xi(\lambda; \lambda_1, \ldots, \lambda_N) \mathcal{T}_\eta(\mu; \lambda_1, \ldots, \lambda_N) \\ &= \mathcal{T}_\eta(\mu; \lambda_1, \ldots, \lambda_N) \mathcal{T}_\xi(\lambda; \lambda_1, \ldots, \lambda_N) S_{\xi\eta}(\lambda - \mu). \end{aligned} \tag{9.24}$$

Introducing $R(\lambda) = \mathcal{P} S_{\xi\eta}(\lambda)$, this equation takes an equivalent form

$$\begin{aligned} &R(\lambda - \mu) \left[\mathcal{T}(\lambda; \lambda_1, \ldots, \lambda_N) \otimes \mathcal{T}(\mu; \lambda_1, \ldots, \lambda_N) \right] \\ &= \left[\mathcal{T}(\mu; \lambda_1, \ldots, \lambda_N) \otimes \mathcal{T}(\lambda; \lambda_1, \ldots, \lambda_N) \right] R(\lambda - \mu). \end{aligned} \tag{9.25}$$

In analogy with the representation (8.29), the local Lax operator $L_{\xi n}$ is expressible in the ξ-space as

$$L_n(\lambda - \lambda_n) = \begin{pmatrix} \alpha_n(\lambda - \lambda_n) & \beta_n(\lambda - \lambda_n) \\ \gamma_n(\lambda - \lambda_n) & \delta_n(\lambda - \lambda_n) \end{pmatrix}. \tag{9.26}$$

The operator matrix elements act on the local vector $e_n^+ = \binom{1}{0}$ in the following way:

$$\begin{aligned} &\alpha_n(\lambda - \lambda_n) e_n^+ = a(\lambda - \lambda_n) e_n^+, \qquad &&\beta_n(\lambda - \lambda_n) e_n^+ = c(\lambda - \lambda_n) e_n^-, \\ &\gamma_n(\lambda - \lambda_n) e_n^+ = 0, &&\delta_n(\lambda - \lambda_n) e_n^+ = b(\lambda - \lambda_n) e_n^+. \end{aligned} \tag{9.27}$$

These relations can be written in a compact form

$$L_n(\lambda - \lambda_n) e_n^+ = \begin{pmatrix} a(\lambda - \lambda_n) & [\cdots] \\ 0 & b(\lambda - \lambda_n) \end{pmatrix} e_n^+. \tag{9.28}$$

The monodromy matrix is expressible in the auxiliary ξ-space as

$$\mathcal{T}(\lambda;\lambda_1,\ldots,\lambda_N)=\begin{pmatrix} A(\lambda;\lambda_1,\ldots,\lambda_N) & B(\lambda;\lambda_1,\ldots,\lambda_N)\\ C(\lambda;\lambda_1,\ldots,\lambda_N) & D(\lambda;\lambda_1,\ldots,\lambda_N)\end{pmatrix}. \tag{9.29}$$

According to (9.21), the transfer matrix is given by

$$T(\lambda;\lambda_1,\ldots,\lambda_N)=A(\lambda;\lambda_1,\ldots,\lambda_N)+D(\lambda;\lambda_1,\ldots,\lambda_N). \tag{9.30}$$

The Yang-Baxter equation (9.25) implies the commutation rules for the matrices A, B, C and D which are the same as the ones (8.34)-(8.36) derived in the homogeneous QISM. Note that the original shifts of spectral parameters λ and μ by $\{\lambda_n\}_{n=1}^N$ are canceled in parameters $a(\mu-\lambda)$, $b(\mu-\lambda)$ and $c(\mu-\lambda)$ and the inhomogeneity does not enter into the commutation relations of the operator B with A or D. When the monodromy matrix acts on the generating vector Ω (8.37), the representation (9.22) and Eq. (9.28) imply

$$\mathcal{T}(\lambda;\lambda_1,\ldots,\lambda_N)\Omega=\begin{pmatrix} \prod_{n=1}^N a(\lambda-\lambda_n) & [\cdots]\\ 0 & \prod_{n=1}^N b(\lambda-\lambda_n)\end{pmatrix}\Omega. \tag{9.31}$$

Comparing Eqs. (9.29) and (9.31) with one another, the elements of the monodromy matrix act on the generating vector Ω as follows

$$A\,\Omega=\prod_{n=1}^N a(\lambda-\lambda_n)\,\Omega,\quad C\,\Omega=0,\quad D\,\Omega=\prod_{n=1}^N b(\lambda-\lambda_n)\,\Omega. \tag{9.32}$$

The eigenvectors of the transfer matrix $T=A+D$ are searched in the ansatz form

$$\psi(\lambda_1,\ldots,\lambda_N;\Lambda_1,\ldots,\Lambda_M)=\prod_{\alpha=1}^M B(\Lambda_\alpha;\lambda_1,\ldots,\lambda_N)\Omega, \tag{9.33}$$

where the parameters $\Lambda_1,\ldots,\Lambda_M$ are as-yet undetermined. Similarly as in the homogeneous case, the eigenvector is a superposition of tensor products of $N-M$ spins up and M spins down. Performing the whole commutation procedure between Eqs. (8.44)-(8.47), we end up with a counterpart of the relation (8.48), where the eigenvalue of the transfer matrix (8.49) is replaced by

$$\begin{aligned} t(\lambda;\lambda_1,\ldots,\lambda_N;\Lambda_1,\ldots,\Lambda_M) &= \prod_{n=1}^N a(\lambda-\lambda_n)\prod_{\alpha=1}^M\frac{a(\Lambda_\alpha-\lambda)}{b(\Lambda_\alpha-\lambda)}\\ &\quad+\prod_{n=1}^N b(\lambda-\lambda_n)\prod_{\alpha=1}^M\frac{a(\lambda-\Lambda_\alpha)}{b(\lambda-\Lambda_\alpha)}\end{aligned} \tag{9.34}$$

and the nullity of t_α for $\alpha=1,2,\ldots,M$ implies the Bethe equations determining the parameters $\Lambda_1,\ldots,\Lambda_M$ as the functions of $\{\lambda_n\}_{n=1}^N$:

$$\prod_{n=1}^N\frac{a(\Lambda_\alpha-\lambda_n)}{b(\Lambda_\alpha-\lambda_n)}=\prod_{\substack{\beta=1\\(\beta\neq\alpha)}}^M\frac{a(\Lambda_\alpha-\Lambda_\beta)}{a(\Lambda_\beta-\Lambda_\alpha)}\frac{b(\Lambda_\beta-\Lambda_\alpha)}{b(\Lambda_\alpha-\Lambda_\beta)}\qquad\alpha=1,2,\ldots,M. \tag{9.35}$$

Eq. (9.20) tells us that the rapidities $\lambda_1, \ldots, \lambda_N$, or equivalently the wave numbers $k_1 = k(\lambda_1), \ldots, k_N = k(\lambda_N)$, are determined by

$$(\mp 1)^{N-1} \exp(-\mathrm{i} k_j L) = t(\lambda = \lambda_j; \lambda_1, \ldots, \lambda_N; \Lambda_1, \ldots, \Lambda_M) \qquad j = 1, \ldots, N. \quad (9.36)$$

Due to $b(0) = 0$, the second term in (9.34) vanishes for $\lambda = \lambda_j$ and we have the quantization conditions

$$(\mp 1)^{N-1} \exp(-\mathrm{i} k_j L) = \prod_{n=1}^{N} a(\lambda_j - \lambda_n) \prod_{\alpha=1}^{M} \frac{a(\Lambda_\alpha - \lambda_j)}{b(\Lambda_\alpha - \lambda_j)}, \qquad j = 1, \ldots, N. \quad (9.37)$$

The $M + N$ coupled equations (9.35) and (9.37) for $M + N$ unknowns $\{\Lambda_\alpha\}$ and $\{k_j\}$ are referred to as the "nested" Bethe equations.

10 Spin-$\frac{1}{2}$ fermions with δ-function interactions: Nested Bethe ansatz

In this section, we apply the QISM to the spin-$\frac{1}{2}$ particles interacting via the δ-function potential, with the Hamiltonian

$$\mathcal{H} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{j>k=1}^{N} \delta(x_j - x_k). \tag{10.1}$$

We first derive the two-body scattering matrices for both Fermi and Bose systems and show that they fulfill the YBE. Then we restrict ourselves to the system of spin-$\frac{1}{2}$ fermions, which is of physical interest, and derive for it the nested Bethe equations. The ground-state energy will be determined from these equations in both repulsive ($c > 0$) and attractive ($c < 0$) regimes.

10.1 The two-body scattering problem

We consider two particles (σ_1, x_1) and (σ_2, x_2) in the ordering sector $Q : x_{Q1} < x_{Q2}$. The generalized Bethe ansatz for the wavefunction reads

$$\psi_Q(\sigma_1, x_1; \sigma_2, x_2) = \sum_{P \in S_2} [Q, P] \exp(\mathrm{i} k_{P1} x_{Q1} + \mathrm{i} k_{P2} x_{Q2}), \tag{10.2}$$

where, for the time being, the same symbol $[Q, P]$ is used for both fermion and boson coefficients. Without any restriction on the ordering of the particle coordinates, the wavefunction is expressible as follows

$$\begin{aligned} \psi(\sigma_1, x_1; \sigma_2, x_2) \;=\; & \theta(x_2 - x_1) \left\{ [12, 12] \mathrm{e}^{\mathrm{i}(k_1 x_1 + k_2 x_2)} + [12, 21] \mathrm{e}^{\mathrm{i}(k_2 x_1 + k_1 x_2)} \right\} \\ & + \theta(x_1 - x_2) \left\{ [21, 12] \mathrm{e}^{\mathrm{i}(k_1 x_2 + k_2 x_1)} + [21, 21] \mathrm{e}^{\mathrm{i}(k_2 x_2 + k_1 x_1)} \right\}. \end{aligned} \tag{10.3}$$

The requirement of continuity of ψ at $x_1 = x_2$ implies the relation

$$[12, 12] + [12, 21] = [21, 12] + [21, 21]. \tag{10.4}$$

Using the formula $\mathrm{d}\theta(x)/\mathrm{d}x = \delta(x)$, it is straightforward to show that

$$\begin{aligned} \left(-\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) \psi \;=\; & (k_1^2 + k_2^2)\psi + \mathrm{e}^{\mathrm{i}(k_1 + k_2)x_1} \mathrm{i} \delta(x_1 - x_2)(k_1 - k_2) \\ & \times \{ [12, 12] + [21, 12] - [12, 21] - [21, 21] \}. \end{aligned} \tag{10.5}$$

The wavefunction ψ then satisfies the $N = 2$ Schrödinger equation

$$\mathcal{H}\psi \equiv \left[-\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + 2c\delta(x_1 - x_2) \right] \psi = E\psi \tag{10.6}$$

if the energy is given by

$$E = k_1^2 + k_2^2 \tag{10.7}$$

and the coefficients are constrained by

$$\mathrm{i}(k_1 - k_2) \{ [12, 12] + [21, 12] - [12, 21] - [21, 21] \} + 2c \{ [12, 12] + [12, 21] \} = 0. \tag{10.8}$$

Eqs. (10.4) and (10.8) represent the formal solution of the problem of the coefficients $[Q, P]$ which is the same for fermions and bosons. We see that the wave numbers k_1 and k_2 occur only as the difference $k_1 - k_2$, so the λ-rapidities are identical to wave numbers,

$$k_1 = \lambda_1, \qquad k_2 = \lambda_2, \qquad \lambda_1 - \lambda_2 = \lambda. \tag{10.9}$$

Let us first consider the fermions for which

$$[Q, P] \equiv [Q, P]_f = \mathrm{sign}(Q)\mathrm{sign}(P) A_{\sigma_{Q1}\sigma_{Q2}}(k_{P1}, k_{P2}). \tag{10.10}$$

When the two fermions have the same spin, say $\sigma_1 = \sigma_2 = +$, Eqs. (10.4) and (10.8) imply the only condition

$$A_{++}(k_2, k_1) = A_{++}(k_1, k_2). \tag{10.11}$$

With regard to the definition of the S-matrix (7.8), we thus have

$$S^{++}_{++}(\lambda) = 1, \qquad S^{++}_{+-}(\lambda) = S^{++}_{-+}(\lambda) = S^{++}_{--}(\lambda) = 0. \tag{10.12}$$

The result for the diagonal element S^{++}_{++} has already been obtained in the treatment of identical (spinless) fermions, see Eq. (2.31). The nullity of the other elements in (10.12) is due to the spin conservation law

$$S^{\sigma_1\sigma_2}_{\sigma_1'\sigma_2'}(\lambda) = 0 \qquad \text{if } \sigma_1 + \sigma_2 \neq \sigma_1' + \sigma_2'. \tag{10.13}$$

For $\sigma_1 = \sigma_2 = -$, we obtain analogously $A_{--}(k_2, k_1) = A_{--}(k_1, k_2)$ which implies

$$S^{--}_{--}(\lambda) = 1. \tag{10.14}$$

For $\sigma_1 = +$ and $\sigma_2 = -$, the solution of Eqs. (10.4) and (10.8) can be written in the matrix form

$$\begin{pmatrix} A_{-+}(k_2, k_1) \\ A_{+-}(k_2, k_1) \end{pmatrix} = \frac{1}{k_1 - k_2 + \mathrm{i}c} \begin{pmatrix} k_1 - k_2 & \mathrm{i}c \\ \mathrm{i}c & k_1 - k_2 \end{pmatrix} \begin{pmatrix} A_{+-}(k_1, k_2) \\ A_{-+}(k_1, k_2) \end{pmatrix}. \tag{10.15}$$

The corresponding nonzero S-matrix elements read

$$S^{+-}_{+-}(\lambda) = S^{-+}_{-+}(\lambda) = \frac{\lambda}{\lambda + \mathrm{i}c}, \qquad S^{+-}_{-+}(\lambda) = S^{-+}_{+-}(\lambda) = \frac{\mathrm{i}c}{\lambda + \mathrm{i}c}. \tag{10.16}$$

From the obtained results, the fermion S-matrix, which will be denoted as S_f, is the one of the form (7.57) with $d(\lambda) = 0$ and the rational elements

$$a(\lambda) = 1, \qquad b(\lambda) = \frac{\lambda}{\lambda + \mathrm{i}c}, \qquad c(\lambda) = \frac{\mathrm{i}c}{\lambda + \mathrm{i}c}. \tag{10.17}$$

Since it is expressible as

$$S_f(\lambda) = \frac{\lambda}{\lambda + \mathrm{i}c} I + \frac{\mathrm{i}c}{\lambda + \mathrm{i}c} \mathcal{P}, \tag{10.18}$$

S_f belongs to the family of the simplest S-matrices (7.33) which fulfill the YBE via the equality (7.37). The matrix $S_f(\lambda)$, being derived for the true particle system, is properly normalized.

Both the initial condition $S_f(\lambda = 0) = \mathcal{P}$ and the unitarity condition $S_f(\lambda)S_f(-\lambda) = I$ are satisfied.

If the particles are spin-$\frac{1}{2}$ bosons, we set

$$[Q, P] \equiv [Q, P]_b = \mathrm{sign}(P) A_{\sigma_{Q1}\sigma_{Q2}}(k_{P1}, k_{P2}) \tag{10.19}$$

in Eqs. (10.4) and (10.8). When the two bosons have the same spin, the nonzero S-matrix elements read

$$S^{++}_{++}(\lambda) = S^{--}_{--}(\lambda) = \frac{\mathrm{i}c - \lambda}{\mathrm{i}c + \lambda}. \tag{10.20}$$

This result has been derived in the treatment of identical (spinless) bosons, see Eq. (2.44). When the two bosons have different spins, we find that

$$\begin{pmatrix} A_{-+}(k_2, k_1) \\ A_{+-}(k_2, k_1) \end{pmatrix} = \frac{1}{k_1 - k_2 + \mathrm{i}c} \begin{pmatrix} k_2 - k_1 & \mathrm{i}c \\ \mathrm{i}c & k_2 - k_1 \end{pmatrix} \begin{pmatrix} A_{+-}(k_1, k_2) \\ A_{-+}(k_1, k_2) \end{pmatrix}. \tag{10.21}$$

The resulting S-matrix, denoted as S_b, has the form

$$S_b(\lambda) = -\frac{\lambda}{\lambda + \mathrm{i}c} I + \frac{\mathrm{i}c}{\lambda + \mathrm{i}c} \mathcal{P}. \tag{10.22}$$

It also fulfills the YBE and both the initial and unitarity conditions.

10.2 Nested Bethe equations for spin-$\frac{1}{2}$ fermions

In what follows, we shall deal with spin-$\frac{1}{2}$ fermions which are of physical interest. The generalization of the above results to the case of N fermions is straightforward. The generalized Bethe ansatz wavefunction (9.2) is by the construction antisymmetric with respect to the simultaneous exchange of two-particle spin and coordinate variables. This fact ensures that the Schrödinger equation is trivially satisfied when three or more fermions interact with each other at the same point. The only non-trivial case to consider is the presence of two fermions with opposite spins at the same point. Requiring the continuity of the wavefunction and solving the matching conditions at the boundaries of each ordering sector Q, we get the expected nearest-neighbor scattering between the amplitudes

$$A_{\ldots\sigma_j\sigma_i\ldots}(\ldots k_v, k_u \ldots) = \sum_{\sigma'_i\sigma'_j} S^{\sigma_i\sigma_j}_{\sigma'_i\sigma'_j}(k_u, k_v) A_{\ldots\sigma'_i\sigma'_j\ldots}(\ldots k_u, k_v \ldots), \tag{10.23}$$

where $S^{\sigma_i\sigma_j}_{\sigma'_i\sigma'_j}(k_u, k_v)$ are the elements of the fermion S_f-matrix (10.18) taken at $\lambda = k_u - k_v$. For a given set of wave numbers $\{k_j\}_{j=1}^N$, the total momentum and energy read

$$K = \sum_{j=1}^{N} k_j, \qquad E = \sum_{k=1}^{N} k_j^2. \tag{10.24}$$

The scattering formula (10.23) confirms the integrability of the fermion system for an arbitrary number of particles and enables us to apply the machinery of the inhomogeneous QISM. With regard to the trivial parametrization of the wave numbers $k = \lambda$ and the explicit forms

(10.17) of the S_f-matrix elements $\{a, b, c\}$, the nested Bethe ansatz equations (9.37) and (9.35) take the form

$$\exp(-\mathrm{i}k_j L) = \prod_{\alpha=1}^{M} \left(\frac{\Lambda_\alpha - k_j + \mathrm{i}c}{\Lambda_\alpha - k_j} \right), \qquad j = 1, \ldots, N; \tag{10.25}$$

$$\begin{aligned} \prod_{j=1}^{N} \left(\frac{\Lambda_\alpha - k_j + \mathrm{i}c}{\Lambda_\alpha - k_j} \right) &= \prod_{\substack{\beta=1 \\ (\beta \neq \alpha)}}^{M} \left(\frac{\Lambda_\alpha - \Lambda_\beta + \mathrm{i}c}{\Lambda_\alpha - \Lambda_\beta - \mathrm{i}c} \right), \\ &= -\prod_{\beta=1}^{M} \left(\frac{\Lambda_\alpha - \Lambda_\beta + \mathrm{i}c}{\Lambda_\alpha - \Lambda_\beta - \mathrm{i}c} \right), \qquad \alpha = 1, \ldots, M. \end{aligned} \tag{10.26}$$

It is useful to symmetrize these equations by shifting the variables $\Lambda_\alpha \to \Lambda_\alpha - \mathrm{i}c/2$, which does not change the possible values of the wave numbers, with the result

$$\exp(\mathrm{i}k_j L) = \prod_{\alpha=1}^{M} \left(\frac{k_j - \Lambda_\alpha + \mathrm{i}c'}{k_j - \Lambda_\alpha - \mathrm{i}c'} \right), \qquad j = 1, \ldots, N; \tag{10.27}$$

$$\prod_{j=1}^{N} \left(\frac{\Lambda_\alpha - k_j + \mathrm{i}c'}{\Lambda_\alpha - k_j - \mathrm{i}c'} \right) = -\prod_{\beta=1}^{M} \left(\frac{\Lambda_\alpha - \Lambda_\beta + \mathrm{i}c}{\Lambda_\alpha - \Lambda_\beta - \mathrm{i}c} \right), \qquad \alpha = 1, \ldots, M, \tag{10.28}$$

where $c' = c/2$.

Although the above Bethe equations determine uniquely the energy spectrum of the spin-$\frac{1}{2}$ fermions with δ-function interactions, it is instructive to mention the formal structure of the eigenfunctions, in particular, the A-coefficients. For $N = 2$ particles, the matrix equation (10.15) is solved by

$$A_{-+}(k_{P1}, k_{P2}) = k_{P2} - \Lambda - \mathrm{i}c', \qquad A_{+-}(k_{P1}, k_{P2}) = k_{P1} - \Lambda + \mathrm{i}c', \tag{10.29}$$

where Λ is a free parameter. Inserting these expressions into the boundary condition (9.11),

$$A_{-+}(k_2, k_1) = \exp(-\mathrm{i}k_1 L) A_{+-}(k_1, k_2), \tag{10.30}$$

we obtain that Λ is just equal to the variable Λ_1 obeying the Bethe equation (10.27). This result was extended to the case of $N - 1$ fermions with spin up and $M = 1$ fermion with spin down by McGuire [71, 72]. For the down-spin being at the position $y(= 1, 2, \ldots, N)$ in the N-state sequence, he proved that

$$A_{+\ldots+\underbrace{-}_{y}+\ldots+}(k_{P1}, \ldots, k_{PN}) \equiv A_P(y) = F_P(y, \Lambda = \Lambda_1), \tag{10.31}$$

where

$$F_P(y, \Lambda) = \prod_{j=1}^{y-1} (k_{Pj} - \Lambda + \mathrm{i}c') \prod_{l=y+1}^{N} (k_{Pl} - \Lambda - \mathrm{i}c'). \tag{10.32}$$

The case $M = 2$ was solved in Ref. [73], the solution for a general number of M down-spins in an N-particle system was derived in Refs. [26, 27]. If the M down-spins are at the integer

positions $1 \leq y_1 < y_2 < \ldots < y_M \leq N$ in the N-state sequence, the corresponding A-coefficient is expressible in terms of $\{\Lambda_j\}_{j=1}^{M}$ as follows

$$A_P(y_1, y_2, \ldots, y_M) = \sum_{R \in S_M} A(R) F_P(y_1, \Lambda_{R1}) F_P(y_2, \Lambda_{R2}) \cdots F_P(y_M, \Lambda_{RM}), \quad (10.33)$$

where R denotes permutations of numbers $(12 \ldots M)$ and

$$A(R) = \mathrm{sign}(R) \prod_{j<l} (\lambda_{Rj} - \lambda_{Rl} - \mathrm{i}c). \quad (10.34)$$

10.3 The ground state

The character of the ground state depends on whether we are in the repulsive region $c > 0$ or the attractive region $c < 0$. In order to simplify the formalism, we always assume that the number of fermions N is even and the number of down-spin fermions M is odd. This restriction does not mean any loss of generality since we are interested in the thermodynamic limit $L, N, M \to \infty$, with the fixed fermion density $n = N/L$ and the fixed density of down-spin fermions $n_- = M/L$. The density of up-spin fermions is $n_+ = n - n_-$. Due to the invariance of the energy spectrum with respect to the flip $+ \leftrightarrow -$ of spins of all particles, it is sufficient to study the case $M \leq N/2$, i.e. $n_- \leq n/2$.

10.3.1 Repulsive regime $c > 0$

In the repulsive region $c > 0$, all k-roots and Λ-roots of the Bethe equations (10.27) and (10.28) are real in the ground state. We can therefore take the logarithm of the Bethe equations, with the result

$$k_j L = 2\pi I_j - \sum_{\alpha=1}^{M} \theta\left(2(k_j - \Lambda_\alpha)\right), \qquad j = 1, \ldots, N; \quad (10.35)$$

$$\sum_{j=1}^{N} \theta\left(2(\Lambda_\alpha - k_j)\right) = 2\pi J_\alpha + \sum_{\beta=1}^{M} \theta(\Lambda_\alpha - \Lambda_\beta), \qquad \alpha = 1, \ldots, M. \quad (10.36)$$

Here, I_j is an integer (half-odd integer) for even (odd) M and J_α is an integer (half-odd integer) for odd (even) $N - M$; only solutions with distinct quantum numbers $\{I_j\}$ and distinct quantum numbers $\{J_\alpha\}$ are allowed in order to avoid the nullity of the wavefunction. The phase-shift function θ is now defined by

$$\theta(x) = 2 \arctan\left(\frac{x}{c}\right). \quad (10.37)$$

The total momentum is given by

$$K = \sum_{j=1}^{N} k_j = \frac{2\pi}{L} \left(\sum_{j=1}^{N} I_j + \sum_{\alpha=1}^{M} J_\alpha \right). \quad (10.38)$$

Eqs. (10.35) and (10.36) will also be used in an alternative form

$$k = 2\pi\frac{I(k)}{L} - \frac{1}{L}\sum_{\Lambda}\theta\left(2(k-\Lambda)\right), \tag{10.39}$$

$$\frac{1}{L}\sum_{k}\theta\left(2(\Lambda-k)\right) = 2\pi\frac{J(\Lambda)}{L} + \frac{1}{L}\sum_{\Lambda'}\theta(\Lambda-\Lambda'). \tag{10.40}$$

Studying the ground state in the (pointlike) hard-core limit $c \to \infty$ we find that, as usual, it corresponds to quantum numbers I_j and J_α symmetrically distributed around zero, with the unity step:

$$I_j = \frac{N+1}{2} - j \quad (j = 1, \ldots, N), \quad J_\alpha = \frac{M+1}{2} - \alpha \quad (\alpha = 1, \ldots, M). \tag{10.41}$$

In the thermodynamic limit $L, N, M \to \infty$, we can perform a continualization procedure analogous to that for spinless bosons. We introduce the state density of I's and J's,

$$f(k) = \frac{I(k)}{L}, \qquad g(\Lambda) = \frac{J(\Lambda)}{L}, \tag{10.42}$$

which are related to the particle density in k-space $\rho(k)$ and the down-spin particle density in Λ-space $\sigma(\Lambda)$ as follows

$$f(k) = \int_0^k \mathrm{d}k'\,\rho(k'), \qquad g(\Lambda) = \int_0^\Lambda \mathrm{d}\Lambda'\,\sigma(\Lambda'). \tag{10.43}$$

We expect the real k's and Λ's to be distributed symmetrically around zero, $\rho(k) = \rho(-k)$ and $\sigma(\Lambda) = \sigma(-\Lambda)$, between some limits $\pm q$ and $\pm Q$, respectively. The normalization requires that

$$n \equiv \frac{N}{L} = \int_{-q}^{q}\mathrm{d}k\,\rho(k), \qquad n_- \equiv \frac{M}{L} = \int_{-Q}^{Q}\mathrm{d}\Lambda\,\sigma(\Lambda). \tag{10.44}$$

The ground-state momentum (10.38) per unit length vanishes,

$$\frac{K}{L} = \int_{-q}^{q}\mathrm{d}k\,k\rho(k) + \int_{-Q}^{Q}\mathrm{d}\Lambda\,\Lambda\sigma(\Lambda) = 0. \tag{10.45}$$

The ground-state energy per unit length is expressible as

$$e_0 \equiv \frac{E_0}{L} = \int_{-q}^{q}\mathrm{d}k\,k^2\rho(k). \tag{10.46}$$

The equations for the distributions $\rho(k)$ and $\sigma(\Lambda)$ are obtained by the continualization of the Bethe equations (10.39) and (10.40):

$$k = 2\pi\int_0^k \mathrm{d}k'\,\rho(k') - \int_{-Q}^{Q}\mathrm{d}\Lambda\,\sigma(\Lambda)\theta\left(2(k-\Lambda)\right), \tag{10.47}$$

$$\int_{-q}^{q}\mathrm{d}k\,\rho(k)\theta\left(2(\Lambda-k)\right) = 2\pi\int_0^\Lambda \mathrm{d}\Lambda'\,\sigma(\Lambda') + \int_{-Q}^{Q}\mathrm{d}\Lambda'\,\sigma(\Lambda')\theta(\Lambda-\Lambda'). \tag{10.48}$$

The differentiation of Eq. (10.47) with respect to k and the differentiation of Eq. (10.48) with respect to Λ, together with the formula $\partial\theta(nk)/\partial k = 2nc/(n^2k^2 + c^2)$, lead to

$$\rho(k) = \frac{1}{2\pi} + \frac{2c}{\pi}\int_{-Q}^{Q} \mathrm{d}\Lambda \frac{\sigma(\Lambda)}{4(k-\Lambda)^2 + c^2}, \tag{10.49}$$

$$\sigma(\Lambda) = \frac{2c}{\pi}\int_{-q}^{q} \mathrm{d}k \frac{\rho(k)}{4(k-\Lambda)^2 + c^2} - \frac{c}{\pi}\int_{-Q}^{Q} \mathrm{d}\Lambda' \frac{\sigma(\Lambda')}{(\Lambda-\Lambda')^2 + c^2}. \tag{10.50}$$

These coupled equations, supplemented by the normalization conditions (10.44), determine the distributions $\rho(k)$ and $\sigma(\Lambda)$. Their explicit solution is accessible only in special cases.

In the limit $Q \to 0$, we obtain $\rho(k) = 1/(2\pi)$ bellow the Fermi level $|k| \le q = \pi n$, which is the expected result valid for spinless fermions.

In the limit $Q \to \infty$, Eq. (10.50) can be treated by the technique of the Fourier transformation for the distribution σ. Using the integral formula

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\Lambda \frac{\mathrm{e}^{-\mathrm{i}\Lambda\xi}}{1+\Lambda^2} = \mathrm{e}^{-|\xi|} \tag{10.51}$$

available due to the residuum theorem, we get

$$\sigma(\Lambda) = \frac{1}{2c}\int_{-q}^{q} \mathrm{d}k \frac{\rho(k)}{\cosh[\pi(k-\Lambda)/c]}. \tag{10.52}$$

Since

$$n_- = \int_{-\infty}^{\infty} \mathrm{d}\Lambda\, \sigma(\Lambda) = \frac{1}{2}\int_{-q}^{q} \mathrm{d}k\, \rho(k) = \frac{n}{2}, \tag{10.53}$$

the limit $Q \to \infty$ corresponds to $n_+ = n_-$. This is the absolute ground state, with the energy minimized over all possible magnetizations for the fixed particle number N.

Another interesting case is the limit $c \to 0^+$. With regard to the representation of the δ-function as the limit

$$\lim_{c\to 0} \frac{1}{\pi}\frac{c}{(x-x')^2 + c^2} = \delta(x - x'), \tag{10.54}$$

Eqs. (10.49) and (10.50) lead to the uniform distributions

$$\rho(k) = \begin{cases} 1/\pi & \text{for } |k| \le Q, \\ 1/(2\pi) & \text{for } Q < |k| \le q, \end{cases} \qquad \sigma(\Lambda) = \frac{1}{2\pi} \quad \text{for } |\Lambda| \le Q, \tag{10.55}$$

where the limits $Q = \pi n_-$ and $q = \pi n_+$. At $c = 0$, we have

$$\Lambda_\alpha = \frac{\pi}{L}(M + 1 - 2\alpha), \qquad \alpha = 1, \ldots, M. \tag{10.56}$$

Since the density of k's is twice larger than the density of Λ's in the interval $\langle -Q, Q\rangle$, for each Λ_α there exist two k's with the same value,

$$k_{\alpha,1} = k_{\alpha,2} = \Lambda_\alpha, \qquad \alpha = 1, \ldots, M. \tag{10.57}$$

The remaining $N - 2M$ k's are uniformly distributed over the interval $\pi n_- < |k| < \pi n_+$.

10.3.2 Attractive regime $c < 0$

In the limit $c \to 0^-$, each Λ_α of the ground state remains to lie on the real axis and forms a bound state with the corresponding wave numbers $k_{\alpha,1}$ and $k_{\alpha,2}$, which become complex conjugates of one another. Namely, in the thermodynamic limit $L \to \infty$,

$$k_{\alpha,1} = \Lambda_\alpha + \mathrm{i}c' + O(\mathrm{e}^{-L/L_0}), \qquad k_{\alpha,2} = \Lambda_\alpha - \mathrm{i}c' + O(\mathrm{e}^{-L/L_0}). \tag{10.58}$$

We can document this fact by analyzing the nested Bethe equations (10.27) and (10.28) for the simplest case of $N = 2$ and $M = 1$:

$$\mathrm{e}^{\mathrm{i}k_1 L} = \frac{k_1 - \Lambda + \mathrm{i}c'}{k_1 - \Lambda - \mathrm{i}c'}, \qquad \mathrm{e}^{\mathrm{i}k_2 L} = \frac{k_2 - \Lambda + \mathrm{i}c'}{k_2 - \Lambda - \mathrm{i}c'}, \tag{10.59}$$

$$\frac{\Lambda - k_1 + \mathrm{i}c'}{\Lambda - k_1 - \mathrm{i}c'} \frac{\Lambda - k_2 + \mathrm{i}c'}{\Lambda - k_2 - \mathrm{i}c'} = 1. \tag{10.60}$$

The multiplication of the first two equations implies

$$\mathrm{e}^{\mathrm{i}(k_1+k_2)L} = 1 \quad \Longrightarrow \quad k_1 = k + \mathrm{i}\kappa, \quad k_2 = k - \mathrm{i}\kappa \tag{10.61}$$

with k and κ being real numbers. The relation for κ reads

$$\mathrm{e}^{2\kappa L} = \frac{(k-\Lambda)^2 + (\kappa - c')^2}{(k-\Lambda)^2 + (\kappa + c')^2}. \tag{10.62}$$

For $c > 0$ and $L \to \infty$, the solution of this equation is trivial $\kappa = 0$. For $c < 0$ and $L \to \infty$, we obtain $\kappa = c'$ and $k = \Lambda$, in accordance with formula (10.58). The remaining $N - 2M$ k's, which are not in pairs bounded with some Λ, are real. With regard to Eq. (10.35), they are determined by

$$k_j L = 2\pi I_j - \sum_{\alpha=1}^{M} \theta(2(k_j - \Lambda_\alpha)), \qquad j = 1, \ldots, N - 2M. \tag{10.63}$$

Our aim is to obtain an equation for Λ_α which does not involve the complex-conjugate wave numbers. We first multiply with one another the Bethe equations (10.27) for $k_{\alpha,1}$ and $k_{\alpha,2}$,

$$\exp(\mathrm{i}k_{\alpha,1}L) = \frac{k_{\alpha,1} - \Lambda_\alpha + \mathrm{i}c'}{k_{\alpha,1} - \Lambda_\alpha - \mathrm{i}c'} \prod_{\beta \neq \alpha} \left(\frac{\Lambda_\alpha - \Lambda_\beta + \mathrm{i}c}{\Lambda_\alpha - \Lambda_\beta} \right), \tag{10.64}$$

$$\exp(\mathrm{i}k_{\alpha,2}L) = \frac{k_{\alpha,2} - \Lambda_\alpha + \mathrm{i}c'}{k_{\alpha,2} - \Lambda_\alpha - \mathrm{i}c'} \prod_{\beta \neq \alpha} \left(\frac{\Lambda_\alpha - \Lambda_\beta}{\Lambda_\alpha - \Lambda_\beta - \mathrm{i}c} \right), \tag{10.65}$$

with the result

$$\exp(2\mathrm{i}\Lambda_\alpha L) = \frac{k_{\alpha,1} - \Lambda_\alpha + \mathrm{i}c'}{k_{\alpha,1} - \Lambda_\alpha - \mathrm{i}c'} \frac{k_{\alpha,2} - \Lambda_\alpha + \mathrm{i}c'}{k_{\alpha,2} - \Lambda_\alpha - \mathrm{i}c'} \prod_{\beta \neq \alpha} \left(\frac{\Lambda_\alpha - \Lambda_\beta + \mathrm{i}c}{\Lambda_\alpha - \Lambda_\beta - \mathrm{i}c} \right). \tag{10.66}$$

For the couple of complex conjugates $k_{\beta,1}$ and $k_{\beta,2}$ with the index $\beta \neq \alpha$ it is easy to show that

$$\frac{\Lambda_\alpha - k_{\beta,1} + \mathrm{i}c'}{\Lambda_\alpha - k_{\beta,1} - \mathrm{i}c'} \frac{\Lambda_\alpha - k_{\beta,2} + \mathrm{i}c'}{\Lambda_\alpha - k_{\beta,2} - \mathrm{i}c'} = \frac{\Lambda_\alpha - \Lambda_\beta + \mathrm{i}c}{\Lambda_\alpha - \Lambda_\beta - \mathrm{i}c}. \tag{10.67}$$

Eq. (10.28) can be thus rewritten as

$$\frac{\Lambda_\alpha - k_{\alpha,1} + \mathrm{i}c'}{\Lambda_\alpha - k_{\alpha,1} - \mathrm{i}c'} \frac{\Lambda_\alpha - k_{\alpha,2} + \mathrm{i}c'}{\Lambda_\alpha - k_{\alpha,2} - \mathrm{i}c'} \prod_{j=2M+1}^{N} \frac{\Lambda_\alpha - k_j + \mathrm{i}c'}{\Lambda_\alpha - k_j - \mathrm{i}c'} = 1. \tag{10.68}$$

Comparing this equation with (10.66), we arrive at the equality

$$\exp(2\mathrm{i}\Lambda_\alpha L) = \prod_{j=1}^{N-2M} \frac{\Lambda_\alpha - k_j + \mathrm{i}c'}{\Lambda_\alpha - k_j - \mathrm{i}c'} \prod_{\beta\neq\alpha} \left(\frac{\Lambda_\alpha - \Lambda_\beta + \mathrm{i}c}{\Lambda_\alpha - \Lambda_\beta - \mathrm{i}c}\right) \tag{10.69}$$

which includes only real wave numbers. Taking the logarithm, we end up with

$$2\Lambda_\alpha L = 2\pi J_\alpha - \sum_{\beta=1}^{M} \theta(\Lambda_\alpha - \Lambda_\beta) - \sum_{j=1}^{N-2M} \theta(2(\Lambda_\alpha - k_j)). \tag{10.70}$$

The energy is given by

$$E = \sum_{\alpha=1}^{M} \left[(\Lambda_\alpha + \mathrm{i}c')^2 + (\Lambda_\alpha - \mathrm{i}c')^2\right] + \sum_{j=1}^{N-2M} k_j^2. \tag{10.71}$$

The quantum numbers in the crucial Eqs. (10.63) and (10.70), which correspond to the ground state, are

$$\begin{aligned} I_j &= (N - 2M + 1 - 2j)/2 \quad j = 1, \ldots, N-2M, \\ J_\alpha &= (M + 1 - 2\alpha)/2 \qquad\quad \alpha = 1, \ldots, M. \end{aligned} \tag{10.72}$$

In the continuum limit, we can introduce the distribution function $\rho(k)$ of only real k's and the distribution function $\sigma(\Lambda)$ of Λ's or, equivalently, of complex conjugate pairs of k's. Differentiating the continuous versions of Eqs. (10.63) and (10.70) with respect to k and Λ, respectively, we obtain

$$\rho(k) = \frac{1}{2\pi} - \frac{2|c|}{\pi} \int_{-Q}^{Q} \mathrm{d}\Lambda \frac{\sigma(\Lambda)}{4(k-\Lambda)^2 + c^2}, \tag{10.73}$$

$$\sigma(\Lambda) = \frac{1}{\pi} - \frac{2|c|}{\pi} \int_{-q}^{q} \mathrm{d}k \frac{\rho(k)}{4(k-\Lambda)^2 + c^2} - \frac{|c|}{\pi} \int_{-Q}^{Q} \mathrm{d}\Lambda' \frac{\sigma(\Lambda')}{(\Lambda - \Lambda')^2 + c^2}. \tag{10.74}$$

These integral equations have to be supplemented by the expressions for the particle number densities

$$n = 2\int_{-Q}^{Q} \mathrm{d}\Lambda\, \sigma(\Lambda) + \int_{-q}^{q} \mathrm{d}k\, \rho(k), \tag{10.75}$$

$$n_+ - n_- = \int_{-q}^{q} \mathrm{d}k\, \rho(k), \tag{10.76}$$

and for the ground-state energy per unit length

$$e_0 = \int_{-Q}^{Q} \mathrm{d}\Lambda \left(2\Lambda^2 - \frac{c^2}{2}\right) \sigma(\Lambda) + \int_{-q}^{q} \mathrm{d}k\, k^2 \rho(k). \tag{10.77}$$

In contrast to attractive bosons (spinless or with an arbitrary spin), there exist bound states of only two fermions and the system has the well behaved thermodynamic limit.

In the special case $n_- = n_+ = n/2$ we have $\rho(k) = 0$, i.e. a fluid composed of only bounded pairs with the density

$$\frac{n}{2} = \int_{-Q}^{Q} \mathrm{d}\Lambda\, \sigma(\Lambda). \tag{10.78}$$

The simple integral equation

$$\sigma(\Lambda) = \frac{1}{\pi} - \frac{|c|}{\pi} \int_{-Q}^{Q} \mathrm{d}\Lambda' \frac{\sigma(\Lambda')}{(\Lambda - \Lambda')^2 + c^2} \tag{10.79}$$

for the distribution function of pairs $\sigma(\Lambda)$ is treatable by techniques for one-component systems.

11 Thermodynamics of spin-$\frac{1}{2}$ fermions with δ-function interactions

Thermodynamics of the δ-function fermions was derived in Refs. [28–30]. The basic concept of particles and holes, developed for one-component systems in Section 4, is in many aspects applicable to the present case and we shall not repeat details of the method.

11.1 Repulsive regime $c > 0$

11.1.1 TBA equations

To construct the thermodynamic Bethe ansatz (TBA), we have to understand the nature of k-roots and Λ-roots of the Bethe equations (10.27) and (10.28). In the repulsive region $c > 0$, all k-roots are real. The Λ-roots organize themselves into a collection of strings of various lengths $n = 1, 2, \ldots$. A particular solution of the Bethe equations is thus characterized by a set of non-negative integers $\{M_n\}_{n=1}^{\infty}$, where M_n is the number of strings of length n. Since the total number of Λ-roots is equal to M, the possible numbers of strings are constrained by $\sum_{n=1}^{\infty} nM_n = M$. For a given n, real numbers Λ_α^n ($\alpha = 1, \ldots, M_n$) denote the string centers. The Λ-roots, which belong to the given n-string α, are distributed symmetrically and equidistantly around the real axis:

$$\Lambda_\alpha^{(n,r)} = \Lambda_\alpha^n + \mathrm{i}c'(n + 1 - 2r), \qquad r = 1, 2, \ldots, n. \tag{11.1}$$

We introduce the function

$$e_n(\Lambda) \equiv \frac{\Lambda + \mathrm{i}nc'}{\Lambda - \mathrm{i}nc'} \tag{11.2}$$

and present for it some important product relations over the strings. The product relation for a string of length n (11.1) reads

$$\prod_{r=1}^{n} e_m(\Lambda_\alpha^{(n,r)}) = \prod_{l=1}^{\min(n,m)} e_{n+m+1-2l}(\Lambda_\alpha^n). \tag{11.3}$$

The product relation for two strings of length n and m reads

$$\prod_{r=1}^{n}\prod_{s=1}^{m} e_2(\Lambda_\alpha^{(n,r)} - \Lambda_\beta^{(m,s)}) = E_{nm}(\Lambda_\alpha^n - \Lambda_\beta^m), \tag{11.4}$$

where

$$E_{nm}(\Lambda) \equiv e_{|n-m|}(\Lambda) e_{|n-m|+2}^2(\Lambda) \cdots e_{n+m-2}^2(\Lambda) e_{n+m}(\Lambda). \tag{11.5}$$

Finally, for any real Λ and $n > 0$ we have

$$\ln e_n(\Lambda) = \mathrm{i}\,[\pi - \theta_n(\Lambda)] \pmod{2\pi\mathrm{i}}, \qquad \theta_n(\Lambda) \equiv 2\arctan\left(\frac{\Lambda}{nc'}\right). \tag{11.6}$$

Note that from the definition (10.37) it holds $\theta \equiv \theta_2$.

The Bethe equations (10.27) can be rewritten in terms of e-functions as follows

$$\begin{aligned}\exp(\mathrm{i}k_jL) &= \prod_{\alpha=1}^{M} e_1(k_j-\Lambda_\alpha) = \prod_{n=1}^{\infty}\prod_{\alpha=1}^{M_n}\prod_{r=1}^{n} e_1(k_j-\Lambda_\alpha^{(n,r)}) \\ &= \prod_{n=1}^{\infty}\prod_{\alpha=1}^{M_n} e_n(k_j-\Lambda_\alpha^n), \end{aligned} \tag{11.7}$$

so that only string centers enter into the representation of k-roots. The Bethe equations (10.28) are expressible as

$$\begin{aligned}\prod_{j=1}^{N} e_1(\Lambda_\alpha-k_j) &= -\prod_{\beta=1}^{M} e_2(\Lambda_\alpha-\Lambda_\beta) \\ &= -\prod_{m=1}^{\infty}\prod_{\beta=1}^{M_m}\prod_{s=1}^{m} e_2(\Lambda_\alpha-\Lambda_\beta^{(m,s)}). \end{aligned} \tag{11.8}$$

Let Λ_α belongs to the n-string (11.1) and let us make a product over all Λ-roots from this string:

$$\prod_{j=1}^{N}\prod_{r=1}^{n} e_1(\Lambda_\alpha^{(n,r)}-k_j) = -\prod_{m=1}^{\infty}\prod_{\beta=1}^{M_m}\prod_{r=1}^{n}\prod_{s=1}^{m} e_2(\Lambda_\alpha^{(n,r)}-\Lambda_\beta^{(m,s)}). \tag{11.9}$$

Then, according to Eqs. (11.3)-(11.5), for string centers it holds

$$\prod_{j=1}^{N} e_n(\Lambda_\alpha^n-k_j) = -\prod_{m=1}^{\infty}\prod_{\beta=1}^{M_m} E_{nm}(\Lambda_\alpha^n-\Lambda_\beta^m). \tag{11.10}$$

Taking the logarithm of of Eqs. (11.7) and (11.10), we obtain

$$k_jL = 2\pi I_j - \sum_{n=1}^{\infty}\sum_{\alpha=1}^{M_n}\theta_n(k_j-\Lambda_\alpha^n), \tag{11.11}$$

$$\sum_{j=1}^{N}\theta_n(\Lambda_\alpha^n-k_j) = 2\pi J_\alpha^n + \sum_{m=1}^{\infty}\sum_{\beta=1}^{M_m}\Theta_{nm}(\Lambda_\alpha^n-\Lambda_\beta^m), \tag{11.12}$$

where

$$\Theta_{nm}(\Lambda)=(1-\delta_{nm})\theta_{|n-m|}(\Lambda)+2\theta_{|n-m|+2}(\Lambda)+\cdots+2\theta_{n+m-2}(\Lambda)+\theta_{n+m}(\Lambda). \tag{11.13}$$

I_j are distinct integers (half-odd integers) for even (odd) $M_1+M_2+\cdots$. J_α^n are distinct integers (half-odd integers) for odd (even) $N-M_n$, constrained by $-J_{\max}^n \le J_\alpha^n \le J_{\max}^n$. The value of the bound $J_{\max}^n$ is found from the condition

$$\Lambda_\alpha^n \to \infty \quad \text{for} \quad J_\alpha^n = J_{\max}^n + \frac{1}{2}, \tag{11.14}$$

which tells us that the string momentum has to reach its maximum value just one elementary step beyond $J^n_{\max}$. Since $\theta_n(\Lambda \to \infty) = \pi$ $(n > 0)$, the condition (11.14) is equivalent to the constraint

$$|J^n_\alpha| \leq \frac{1}{2}\left(N - 1 - \sum_{m=1}^{\infty} t_{nm} M_m\right), \qquad t_{nm} = 2\min(n,m) - \delta_{nm}. \tag{11.15}$$

Similarly as in the case of one-component systems, for any set of admissible quantum numbers $\{I_j, J^n_\alpha\}$ there exists a unique set of the particle Bethe solutions $\{k_j, \Lambda^n_\alpha\}$. The set of admissible quantum numbers $\{\tilde{I}, \tilde{J}^n\}$, which are not in $\{I_j, J^n_\alpha\}$, define the hole Bethe solutions $\{\tilde{k}, \tilde{\Lambda}^n\}$. In terms of the function $f(k)$ defined by

$$2\pi L f(k) \equiv Lk + \sum_{n=1}^{\infty}\sum_{\alpha=1}^{M_n} \theta_n(k - \Lambda^n_\alpha), \tag{11.16}$$

the particle and hole wave numbers are given by

$$Lf(k_j) = I_j, \quad j = 1, \ldots, N; \qquad Lf(\tilde{k}) = \tilde{I}. \tag{11.17}$$

In terms of the n-string function $g_n(\Lambda)$ defined by

$$2\pi L g_n(\Lambda) = \sum_{j=1}^{N} \theta_n(\Lambda - k_j) - \sum_{m=1}^{\infty}\sum_{\beta=1}^{M_m} \Theta_{nm}(\Lambda - \Lambda^m_\beta), \tag{11.18}$$

the particle and hole n-string Λ's are given by

$$Lg_n(\Lambda^n_\alpha) = J^n_\alpha, \quad \alpha = 1, \ldots, M_n; \qquad Lg_n(\tilde{\Lambda}^n) = \tilde{J}^n. \tag{11.19}$$

In the thermodynamic limit, we can replace the summations in Eqs. (11.16) and (11.18) by integrals in the standard way

$$\sum_{j=1}^{N} \cdots \to L\int_{-\infty}^{\infty} \mathrm{d}k\, \rho(k) \cdots, \qquad \sum_{\alpha=1}^{M_n} \cdots \to L\int_{-\infty}^{\infty} \mathrm{d}\Lambda\, \sigma_n(\Lambda) \cdots, \tag{11.20}$$

where $\rho(k)$ and $\sigma_n(\Lambda)$ are the particle distribution functions of k-roots and n-string Λ centers, respectively. The continualized equations read

$$2\pi f(k) = k + \sum_{n=1}^{\infty} \theta_n * \sigma_n(k), \tag{11.21}$$

$$2\pi g_n(\Lambda) = \theta_n * \rho(\Lambda) - \sum_{m=1}^{\infty} \Theta_{nm} * \sigma_m(\Lambda), \tag{11.22}$$

where the symbol $*$ denotes the convolution

$$f * g(x) = \int_{-\infty}^{\infty} \mathrm{d}y\, f(x-y) g(y). \tag{11.23}$$

By the definition, the hole distribution functions $\tilde{\rho}(k)$ and $\tilde{\sigma}_n(\Lambda)$ are related to their particle counterparts via

$$\rho(k) + \tilde{\rho}(k) = \frac{\mathrm{d}f(k)}{\mathrm{d}k}, \qquad \sigma_n(\Lambda) + \tilde{\sigma}_n(\Lambda) = \frac{\mathrm{d}g(\Lambda)}{\mathrm{d}\Lambda}. \tag{11.24}$$

These equations can be put into the form

$$\frac{1}{2\pi} = \rho(k) + \tilde{\rho}(k) - \sum_{n=1}^{\infty} a_n * \sigma_n(k), \tag{11.25}$$

$$a_n * \rho(\Lambda) = \tilde{\sigma}_n(\Lambda) + \sum_{m=1}^{\infty} A_{nm} * \sigma_m(\Lambda), \tag{11.26}$$

where

$$a_n(k) \equiv \frac{1}{2\pi}\frac{\mathrm{d}\theta_n(k)}{\mathrm{d}k} = \frac{1}{\pi}\frac{nc'}{k^2 + (nc')^2} \quad n = 1, 2, \ldots \tag{11.27}$$

and

$$\begin{aligned} A_{nm}(\Lambda) &= \delta(\Lambda)\delta_{nm} + \frac{1}{2\pi}\frac{\mathrm{d}\Theta_{nm}(\Lambda)}{\mathrm{d}\Lambda} = \delta(\Lambda)\delta_{nm} + (1-\delta_{nm})a_{|n-m|}(\Lambda) \\ &\quad + 2a_{|n-m|+2}(\Lambda) + \cdots + 2a_{n+m-2}(\Lambda) + a_{n+m}(\Lambda). \end{aligned} \tag{11.28}$$

It is easy to derive the Fourier transforms of the quantities a_n and A_{nm}:

$$\hat{a}_n(\omega) = \mathrm{e}^{-nc'|\omega|}, \qquad \hat{A}_{nm}(\omega) = \coth(c'|\omega|)\left[\mathrm{e}^{-|n-m|c'|\omega|} - \mathrm{e}^{-(n+m)c'|\omega|}\right]. \tag{11.29}$$

For reasons which will be clear later, we introduce the "inverse" matrix function $A^{-1}_{nm}(\Lambda)$ defined by the relation

$$\sum_{n'=1}^{\infty} \left(A^{-1}_{nn'} * A_{n'm}\right)(\Lambda) = \delta(\Lambda)\delta_{nm}. \tag{11.30}$$

By using the convolution theorem

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}\omega x}\,(f * g)\,(x) = \hat{f}(\omega)\hat{g}(\omega) \tag{11.31}$$

and after some algebra, we get

$$\hat{A}^{-1}_{nm}(\omega) = \delta_{nm} - \hat{s}(\omega)\left(\delta_{n,m+1} + \delta_{n,m-1}\right), \tag{11.32}$$

where

$$\hat{s}(\omega) = \frac{1}{2\cosh(c'\omega)}, \qquad s(x) = \frac{1}{4c'}\mathrm{sech}\left(\frac{\pi x}{2c'}\right). \tag{11.33}$$

We assume that the system of spin-$\frac{1}{2}$ fermions is in an external magnetic field $h \geq 0$ whose presence changes the energy by $-h(N - 2M)$. In the grand canonical ensemble characterized

by the inverse temperature β and the chemical potential μ, the Gibbs free energy per unit length is given by

$$g = -\frac{1}{\beta}\frac{S}{L} + \frac{E}{L} - \mu\frac{N}{L}, \tag{11.34}$$

where

$$\frac{E}{L} = \int \mathrm{d}k\, k^2 \rho(k) - h\left(\frac{N}{L} - 2\frac{M}{L}\right), \tag{11.35}$$

$$\frac{N}{L} = \int \mathrm{d}k\, \rho(k), \qquad \frac{M}{L} = \sum_{n=1}^{\infty} n \int \mathrm{d}\Lambda\, \sigma_n(\Lambda), \tag{11.36}$$

$$\begin{aligned}\frac{S}{L} = {} & \int \mathrm{d}k\, [(\rho+\tilde{\rho})\ln(\rho+\tilde{\rho}) - \rho\ln\rho - \tilde{\rho}\ln\tilde{\rho}] \\ & + \sum_{n=1}^{\infty} \int \mathrm{d}\Lambda\, [(\sigma_n+\tilde{\sigma}_n)\ln(\sigma_n+\tilde{\sigma}_n) - \sigma_n\ln\sigma_n - \tilde{\sigma}_n\ln\tilde{\sigma}_n]\,.\end{aligned} \tag{11.37}$$

The Gibbs free energy is a functional of the particle distributions $\{\rho, \sigma_n\}$ and the hole distributions $\{\tilde{\rho}, \tilde{\sigma}_n\}$. In the thermodynamic equilibrium, it attains its minimum under the constraints (11.25) and (11.26). The variational condition $\delta g = 0$ is equivalent to

$$\begin{aligned} 0 = {} & \int \mathrm{d}k\, \beta(k^2 - \mu - h)\delta\rho(k) + \sum_{n=1}^{\infty} 2n\beta h \int \mathrm{d}\Lambda\, \delta\sigma_n(\Lambda) \\ & - \int \mathrm{d}k \left[\ln\left(\frac{\rho+\tilde{\rho}}{\rho}\right)\delta\rho + \ln\left(\frac{\rho+\tilde{\rho}}{\tilde{\rho}}\right)\delta\tilde{\rho}\right] \\ & - \sum_{n=1}^{\infty} \int \mathrm{d}\Lambda \left[\ln\left(\frac{\sigma_n+\tilde{\sigma}_n}{\sigma_n}\right)\delta\sigma_n + \ln\left(\frac{\sigma_n+\tilde{\sigma}_n}{\tilde{\sigma}_n}\right)\delta\tilde{\sigma}_n\right]. \end{aligned} \tag{11.38}$$

From the constraints (11.25) and (11.26), we have

$$\delta\tilde{\rho} = -\delta\rho + \sum_{n=1}^{\infty} a_n * \delta\sigma_n, \qquad \delta\tilde{\sigma}_n = a_n * \delta\rho - \sum_{m=1}^{\infty} A_{nm} * \delta\sigma_m. \tag{11.39}$$

Substituting these relations into Eq. (11.38) and setting to zero the coefficients of $\delta\rho(k)$ and $\{\delta\sigma_n(\Lambda)\}_{n=1}^{\infty}$, we obtain a set of integral equations for the ratios

$$\mathrm{e}^{\beta\epsilon(k)} \equiv \frac{\tilde{\rho}(k)}{\rho(k)}, \qquad \eta_n(\Lambda) \equiv \frac{\tilde{\sigma}_n(\Lambda)}{\sigma_n(\Lambda)} \quad n = 1, 2, \ldots. \tag{11.40}$$

Taking into account the symmetries $A_{nm} = A_{mn}$ and $a_n(k) = a_n(-k)$, these integral equations read

$$\beta\epsilon = \beta(k^2 - \mu - h) - \sum_{n=1}^{\infty} a_n * \ln(1+\eta_n^{-1}), \tag{11.41}$$

$$\ln(1+\eta_n) = 2n\beta h - a_n * \ln\left(1+\mathrm{e}^{-\beta\epsilon}\right) + \sum_{m=1}^{\infty} A_{nm} * \ln(1+\eta_m^{-1}). \tag{11.42}$$

The knowledge of $\epsilon(k)$ and $\{\eta_n(\Lambda)\}_{n=1}^{\infty}$ enables us to determine the particle distribution functions $\rho(k)$ and $\{\sigma_n(\Lambda)\}_{n=1}^{\infty}$ from the constraints (11.25) and (11.26), rewritten as

$$[1+\mathrm{e}^{\beta\epsilon(k)}]\rho(k) = \frac{1}{2\pi}+\sum_{n=1}^{\infty} a_n * \sigma_n(k), \tag{11.43}$$

$$a_n * \rho(\Lambda) = \eta_n(\Lambda)\sigma_n(\Lambda)+\sum_{m=1}^{\infty} A_{nm} * \sigma_m(\Lambda). \tag{11.44}$$

Using Eqs. (11.34)-(11.37), (11.41) and (11.42), the pressure $P=-g$ is given by

$$\beta P = \int \frac{\mathrm{d}k}{2\pi} \ln\left[1+\mathrm{e}^{-\beta\epsilon(k)}\right]. \tag{11.45}$$

At a given temperature, $\epsilon(k)$ and $\beta^{-1}\ln\eta_n(\Lambda)$ can be interpreted as the energy of excitations related to the distribution functions $\rho(k)$ and $\sigma_n(\Lambda)$, respectively [29]. The result for the pressure thus coincides with the one valid for one-component systems.

The resulting equations (11.41) and (11.42) can be further simplified. The convolution of the $n=1$ Eq. (11.42) with the s-function (11.33), when combined with Eq. (11.41) and the formula $s * A_{1n} \equiv s * (a_{n-1}+a_{n+1}) = a_n$, yields

$$\beta\epsilon = \beta(k^2-\mu) - R * \ln\left(1+\mathrm{e}^{-\beta\epsilon}\right) - s * \ln(1+\eta_1), \tag{11.46}$$

where $R = a_1 * s$. Forming the convolution of Eq. (11.42) with the inverse matrix function A^{-1} and using the relations

$$\sum_{m=1}^{\infty} A^{-1}_{nm} * a_m = s\delta_{n1}, \qquad \sum_{m=1}^{\infty} A^{-1}_{nm} * m = 0, \tag{11.47}$$

we arrive at

$$\ln\eta_1 = s * \left[\ln(1+\eta_2) - \ln\left(1+\mathrm{e}^{-\beta\epsilon}\right)\right], \tag{11.48}$$

$$\ln\eta_n = s * \ln\left[(1+\eta_{n-1})(1+\eta_{n+1})\right], \quad n \geq 2. \tag{11.49}$$

These equations are not complete to determine all η_n because they do not contain the magnetic field h. We return to the generic Eq. (11.42) to deduce its leading $n \to \infty$ asymptotic. Since $\lim_{n\to\infty} a_n \to 0$, the leading asymptotic is

$$\lim_{n\to\infty} \frac{\ln\eta_n}{n} = 2\beta h. \tag{11.50}$$

Eqs. (11.46) and (11.48)-(11.50) are known as the TBA equations. They can always be treated numerically. Explicit solutions of the TBA equations are possible only in special limits. These equations can also serve as a systematic tool for developing series expansions.

11.1.2 Special cases

• **The limit** $c \to 0^+$: The functions $s(x)$ and $R(x)$ have the zero width in this limit and they can be replaced by $\frac{1}{2}\delta(x)$. The TBA equations then take the form

$$\begin{aligned} \beta\epsilon &= \beta(k^2-\mu) - \frac{1}{2}\left[\ln\left(1+\mathrm{e}^{-\beta\epsilon}\right) + \ln(1+\eta_1)\right], && (11.51)\\ \ln\eta_1 &= \frac{1}{2}\left[\ln(1+\eta_2) - \ln\left(1+\mathrm{e}^{-\beta\epsilon}\right)\right], && (11.52)\\ \ln\eta_n &= \frac{1}{2}\ln\left[(1+\eta_{n-1})(1+\eta_{n+1})\right], \quad n \geq 2, && (11.53)\\ \lim_{n\to\infty}\frac{\ln\eta_n}{n} &= 2\beta h. && (11.54) \end{aligned}$$

Eq. (11.53) is a difference equation whose general solution reads

$$\eta_n = \left(\frac{a^n b - a^{-n}b^{-1}}{a-a^{-1}}\right)^2 - 1. \tag{11.55}$$

The free parameters a and b are determined by the asymptotic condition (11.54), $a = \exp(\beta|h|)$, and by the relation (11.52),

$$b^2 = \frac{1+\exp[\beta(k^2-\mu+h)]}{1+\exp[\beta(k^2-\mu-h)]}. \tag{11.56}$$

After obtaining $\epsilon(k)$ from (11.51), the pressure (11.45) is given by

$$\beta P = \int \frac{\mathrm{d}k}{2\pi}\ln\left[\left(1+\mathrm{e}^{-\beta(k^2-\mu+h)}\right)\left(1+\mathrm{e}^{-\beta(k^2-\mu-h)}\right)\right]. \tag{11.57}$$

This is the well-known expression for noninteracting $(c=0)$ Fermi gas in a magnetic field.

• **The limit** $c \to \infty$: The functions $s(x)$ and $R(x)$ are infinitely wide in this limit. This means that the contributions $R * \ln(1+\mathrm{e}^{-\beta\epsilon})$ and $s * \ln(1+\mathrm{e}^{-\beta\epsilon})$ vanish in the TBA equations and the functions $\eta_n(\Lambda)$ become Λ-independent. The TBA equations are simplified to

$$\ln\eta_1 = \frac{1}{2}\ln(1+\eta_2), \qquad \ln\eta_n = \frac{1}{2}\ln\left[(1+\eta_{n-1})(1+\eta_{n+1})\right] \quad n \geq 2, \tag{11.58}$$

with $\lim_{n\to\infty}(\ln\eta_n)/n = 2\beta h$. Their solution is

$$\eta_n = \left(\frac{\sinh[(n+1)\beta h]}{\sinh(\beta h)}\right)^2 - 1. \tag{11.59}$$

Thus,

$$\epsilon(k) = k^2 - \mu - \frac{1}{2\beta}\ln(1+\eta_1) = k^2 - \mu - \ln[2\cosh(\beta h)]. \tag{11.60}$$

The pressure is given by

$$\beta P = \int \frac{\mathrm{d}k}{2\pi}\ln\left[1+2\cosh(\beta h)\mathrm{e}^{\beta(\mu-k^2)}\right]. \tag{11.61}$$

The system looks like to be composed of free fermions where each energy level k^2 can only be occupied by either spin-up or spin-down.

• **The fugacity expansion**: To obtain the expansion of thermodynamic quantities in the fugacity $z = \exp(\beta\mu) \ll 1$, we set

$$\begin{aligned} \mathrm{e}^{-\beta\epsilon(k)} &= \sum_{n=1}^{\infty} A_n(k,\beta) z^n, && (11.62) \\ 1+\eta_n(\Lambda) &= b_n(\Lambda,\beta) + c_n(\Lambda,\beta) + d_n(\Lambda,\beta) + \cdots. && (11.63) \end{aligned}$$

Substituting these expansions into the TBA equations (11.46)-(11.50) and comparing the terms of the order z^0 and z^1, we get the conditions

$$\ln A_1 + \beta k^2 = s * \ln b_1, \qquad \frac{A_2}{A_1} = R * A_1 + s * \frac{c_1}{b_1} \tag{11.64}$$

and

$$\begin{aligned} \ln(b_n - 1) &= s * (\ln b_{n-1} + \ln b_{n+1}), \qquad b_0 = 1; && (11.65) \\ \frac{c_n}{b_n - 1} &= s * \left(\frac{c_{n-1}}{b_{n-1}} + \frac{c_{n+1}}{b_{n+1}} \right), \qquad \frac{c_0}{b_0} = -A_1. && (11.66) \end{aligned}$$

The set of equations (11.65), complemented by the asymptotic $\lim_{n\to\infty}(\ln b_n)/n = 2\beta h$, has the solution

$$b_n = f_n^2, \qquad f_n = \frac{\sinh[(n+1)\beta h]}{\sinh(\beta h)}. \tag{11.67}$$

The Fourier transform of Eq. (11.66),

$$\frac{\hat{c}_n}{f_{n-1} f_{n+1}} = \frac{1}{2\cosh(c'\omega)} \left(\frac{\hat{c}_{n-1}}{f_{n-1}^2} + \frac{\hat{c}_{n+1}}{f_{n+1}^2} \right), \qquad n \geq 1, \tag{11.68}$$

is the difference equation whose converging solution is

$$\hat{c}_n = C(\omega) \left(f_n f_{n-1} \mathrm{e}^{-(n+2)c'|\omega|} - f_n f_{n+1} \mathrm{e}^{-nc'|\omega|} \right). \tag{11.69}$$

The initial condition $c_0/b_0 = -A_1$ implies

$$\hat{c}_1 = \hat{A}_1 \left(\mathrm{e}^{-3c'|\omega|} - f_2 \mathrm{e}^{-c'|\omega|} \right). \tag{11.70}$$

Let us set $h \to 0$, for simplicity, i.e. $f_n = n + 1$. The relations in Eq. (11.64) then imply

$$A_1 = 2\mathrm{e}^{-\beta k^2}, \qquad A_2 = \mathrm{e}^{-\beta k^2} \int \frac{\mathrm{d}p}{\pi} \frac{c}{(k-p)^2 + c^2} \mathrm{e}^{-\beta p^2} \tag{11.71}$$

and the pressure is given by

$$\begin{aligned} \beta P &= \int \frac{\mathrm{d}k}{2\pi} \left[A_1 z + \left(A_2 - \frac{1}{2} A_1^2 \right) z^2 + \cdots \right] \\ &= \frac{\sqrt{\pi\beta}}{2\pi} \left[2z + z^2 \left(-\sqrt{2} + \frac{1}{\sqrt{2}} \int \frac{\mathrm{d}p}{\pi} \frac{c'}{p^2 + (c')^2} \mathrm{e}^{-2\beta p^2} \right) + \cdots \right]. \end{aligned} \tag{11.72}$$

11.2 Attractive regime $c < 0$

If $c < 0$, the general structure of N k-roots and M Λ-roots of the Bethe equations (10.27) and (10.28) is more complicated. In particular, there can be $M' \leq M$ real Λ'-roots accompanied by two complex k-roots which form a bounded pair of up-spin and down-spin fermions:

$$k_{\alpha,1} = \Lambda'_\alpha + \mathrm{i}|c'|, \qquad k_{\alpha,2} = \Lambda'_\alpha - \mathrm{i}|c'|, \quad \alpha = 1, 2, \ldots, M'. \tag{11.73}$$

We have seen that in the ground state all available Λ-roots belong to this category. The remaining $M - M'$ Λ-roots form the strings of length $n = 1, 2 \ldots$, which are analogous to those in the repulsive case,

$$\Lambda_\alpha^{(n,r)} = \Lambda_\alpha^n + \mathrm{i}|c'|(n + 1 - 2r), \quad r = 1, 2, \ldots, n; \quad \alpha = 1, 2, \ldots, M_n. \tag{11.74}$$

There must hold $M = M' + \sum_{n=1}^{\infty} nM_n$. The remaining $N - 2M'$ unpaired k-roots $\{k_j\}_{j=1}^{N-2M'}$ lie on the real axis.

The derivation of the TBA equations goes basically along the same lines as in the repulsive case. We shall not repeat the derivation procedure; the original paper [30] or Section 13 of the monograph [56] are recommended to those who are interested in details. The novelty is that, besides the particle and hole densities of the unpaired k-roots, $\{\rho, \tilde{\rho}\}$, and n-string centers $\{\sigma_n, \tilde{\sigma}_n\}_{n=1}^{\infty}$, we need to introduce these quantities also for the bounded Λ'-roots, $\{\sigma', \tilde{\sigma}'\}$. Defining

$$\mathrm{e}^{\beta\tau} \equiv \frac{\tilde{\sigma}'}{\sigma'}, \tag{11.75}$$

the TBA equations read

$$\begin{aligned}
\beta\tau &= 2\beta\left[k^2 - (c')^2 - \mu\right] + a_2 * \ln\left(1 + \mathrm{e}^{-\beta\tau}\right) + a_1 * \ln\left(1 + \mathrm{e}^{-\beta\epsilon}\right), && (11.76)\\
\beta\epsilon &= s * \left[\ln\left(1 + \mathrm{e}^{\beta\tau}\right) - \ln(1 + \eta_1)\right], && (11.77)\\
\ln\eta_1 &= s * \left[\ln\left(1 + \mathrm{e}^{-\beta\epsilon}\right) + \ln(1 + \eta_2)\right], && (11.78)\\
\ln\eta_n &= s * \left[\ln(1 + \eta_{n-1}) + \ln(1 + \eta_{n+1})\right], \quad n \geq 2, && (11.79)
\end{aligned}$$

where

$$a_n(k) = \frac{1}{\pi}\frac{n|c'|}{k^2 + (nc')^2}. \tag{11.80}$$

The asymptotic condition is

$$\lim_{n\to\infty}\frac{\ln\eta_n}{n} = 2\beta h. \tag{11.81}$$

The pressure is given by

$$\beta P = \int \frac{\mathrm{d}k}{2\pi}\ln\left[1 + \mathrm{e}^{-\beta\tau(k)}\right] + \int \frac{\mathrm{d}k}{2\pi}\ln\left[1 + \mathrm{e}^{-\beta\epsilon(k)}\right]. \tag{11.82}$$

The thermodynamics is explicitly available in special case, like the $c \to 0^-$ limit, zero temperature, etc. [56].

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Appendix A: Spin and spin operators on chain

Spin of a particle

In quantum mechanics, spin operator $\mathbf{S} = (S^x, S^y, S^z)$ is a non-classical kind of angular momentum intrinsic to an elementary particle. Spin components obey commutation relations analogous to those valid for components of the orbital angular momentum,

$$[S^\alpha, S^\beta] = \mathrm{i}\hbar \sum_{\gamma=x,y,z} \epsilon_{\alpha\beta\gamma} S^\gamma, \tag{A.1}$$

where $\epsilon_{\alpha\beta\gamma}$ is the antisymmetric tensor ($\epsilon_{xyz} = 1, \epsilon_{yxz} = -1$, etc.). The operator $\mathbf{S}^2 = (S^x)^2 + (S^y)^2 + (S^z)^2$ commutes to zero with each of the spin components. In the common basis of $\mathbf{S}^2$ and say S^z, the eigenvectors are given by

$$\mathbf{S}^2|s,\sigma\rangle = \hbar^2 s(s+1)|s,\sigma\rangle, \tag{A.2}$$

$$S^z|s,\sigma\rangle = \hbar\sigma|s,\sigma\rangle, \tag{A.3}$$

where s and σ are quantum numbers related to the operators $\mathbf{S}^2$ and S^z, respectively. The raising and lowering spin operators $S^\pm = S^x \pm \mathrm{i}S^y$ act on these eigenvectors as follows

$$S^\pm|s,\sigma\rangle = \hbar\sqrt{s(s+1) - \sigma(\sigma\pm 1)}|s,\sigma\pm 1\rangle. \tag{A.4}$$

To obtain a finite-dimensional representation of the spin algebra, the possible values of σ must include $\sigma = s$, for which $S^+|s,s\rangle = 0$, and $\sigma = -s$, for which $S^-|s,-s\rangle = 0$. This means that the component of spin angular momentum measured along the z-axis can only take the values $\hbar\sigma$ with $\sigma \in \{-s, -s+1, \ldots, s\}$. Since the total number $2s+1$ of σ-values must be a positive integer, the allowed values of the quantum number s are $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$, etc.

The wave function of a quantum particle with spin s depends on its position in space $\mathbf{r}$ as well as its spin state $\sigma \in \{-s, -s+1, \ldots, s\}$, $\psi \equiv \psi(\sigma, \mathbf{r})$. For systems of N identical particles with spin s, the value of s determines the statistics of particles. Namely, interchanging any two of the particles we must have

$$\psi(\cdots;\sigma_j,\mathbf{r}_j;\cdots;\sigma_k,\mathbf{r}_k;\cdots) = (-1)^{2s}\psi(\cdots;\sigma_k,\mathbf{r}_k;\cdots;\sigma_j,\mathbf{r}_j;\cdots). \tag{A.5}$$

For bosons with s=0 or 1 or 2 etc., the prefactor $(-1)^{2s}$ reduces to $+1$ and so the wavefunction is symmetric with respect to any interchange of two particles. For fermions with $s = \frac{1}{2}$ or $\frac{3}{2}$ or $\frac{5}{2}$ etc., the prefactor $(-1)^{2s}$ reduces to -1 and so the wavefunction is antisymmetric with respect to any interchange of two particles. As a consequence, no two fermions in the same spin state can occur at the same point of the space (Pauli exclusion principle).

It is often useful to have at one's disposal explicit matrix representations of spin operators. In units of $\hbar = 1$, the spin operator of a spin-$\frac{1}{2}$ Fermi particle is given by $\mathbf{S} = (1/2)(\sigma^x, \sigma^y, \sigma^z)$, where

$$\sigma^x \equiv \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^y \equiv \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma^z \equiv \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.6}$$

are the Pauli matrices. The Pauli matrices fulfill the obvious product relations

$$(\sigma^\alpha)^2 = I, \qquad \sigma^\alpha\sigma^\beta = \mathrm{i}\epsilon_{\alpha\beta\gamma}\sigma^\gamma \qquad \text{for } \alpha \neq \beta \tag{A.7}$$

and their trace vanishes. In the basis formed by the operator S^z

$$S^z e^+ = +\frac{1}{2}e^+, \qquad e^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \tag{A.8}$$

$$S^z e^- = -\frac{1}{2}e^-, \qquad e^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{A.9}$$

the eigenvector e^+ corresponds to the "up" spin state and the eigenvector e^- corresponds to the "down" spin state. The raising and lowering combinations of the spin operators

$$S^+ \equiv S^x + \mathrm{i}S^y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad S^- \equiv S^x - \mathrm{i}S^y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{A.10}$$

act on the eigenvectors as follows

$$S^+e^+ = S^-e^- = 0; \qquad S^+e^- = e^+, \quad S^-e^+ = e^-. \tag{A.11}$$

The 2×2 matrices

$$S^z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{A.12}$$

correspond to the two-dimensional (spin-$\frac{1}{2}$) irreducible representation of the generators of the $sl(2)$ algebra

$$[S^z, S^\pm] = \pm S^\pm, \qquad [S^+, S^-] = 2S^z. \tag{A.13}$$

For spin-1 particles, the spin matrices obeying the $sl(2)$ algebra (A.13) read

$$S^z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad S^+ = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad S^- = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{A.14}$$

The basis is formed by the eigenvectors of the operator S^z

$$e^+ = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e^0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e^- = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{A.15}$$

with the eigenvalues $+1$, 0, -1, respectively.

Spin operators on a chain

Operators are usually defined in quantum mechanics implicitly via operator relations. It is sometimes useful to have at one's disposal an explicit representation of an operator as a matrix in some basis forming the Hilbert space. The standard way how to increase the dimension of the Hilbert space is the tensor (direct) product of two or more subspaces.

Let us consider general matrices (operators) $\mathbf{A}$ of arbitrary dimension $a \times a'$ with elements $A_{\sigma\sigma'} \equiv A^{\sigma}_{\sigma'}$, where $\sigma = 1, \ldots, a$ numerates rows and $\sigma' = 1, \ldots, a'$ numerates columns. The "standard" product of two matrices $\mathbf{A}$ and $\mathbf{B}$, with the dimension constraint $a' = b$, is the matrix of dimension $a \times b'$ defined by

$$(\mathbf{A} \cdot \mathbf{B})^{\sigma}_{\sigma'} = \sum_{\sigma''} A^{\sigma}_{\sigma''} B^{\sigma''}_{\sigma'} . \tag{A.16}$$

The tensor product of two matrices $\mathbf{A}$ and $\mathbf{B}$ is defined as follows

$$(\mathbf{A} \otimes \mathbf{B})^{\sigma_1\sigma_2}_{\sigma_1'\sigma_2'} = A^{\sigma_1}_{\sigma_1'} B^{\sigma_2}_{\sigma_2'}. \tag{A.17}$$

The resulting matrix is indexed by $\sigma = (\sigma_1, \sigma_2)$ and $\sigma' = (\sigma_1', \sigma_2')$, and as such has the dimension $ab \times a'b'$. Explicitly,

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} A_{11}\mathbf{B} & A_{12}\mathbf{B} & \ldots & A_{1a'}\mathbf{B} \\ A_{21}\mathbf{B} & A_{22}\mathbf{B} & \ldots & A_{2a'}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ A_{a1}\mathbf{B} & A_{a2}\mathbf{B} & \ldots & A_{aa'}\mathbf{B} \end{pmatrix}. \tag{A.18}$$

The generalization of the tensor product to more than two matrices is straightforward,

$$(\mathbf{A} \otimes \mathbf{B} \otimes \mathbf{C})^{\sigma_1\sigma_2\sigma_3}_{\sigma_1'\sigma_2'\sigma_3'} = A^{\sigma_1}_{\sigma_1'} B^{\sigma_2}_{\sigma_2'} C^{\sigma_3}_{\sigma_3'}, \tag{A.19}$$

etc.

As concerns combinations of standard and tensor products of matrices, there exists a simple rule. Let us consider the product $(\mathbf{A} \otimes \mathbf{B}) \cdot (\mathbf{C} \otimes \mathbf{D})$ with the matrix dimensions $a' = c$ and $b' = d$. Via a sequence of rearrangements

$$\begin{aligned} [(\mathbf{A} \otimes \mathbf{B}) \cdot (\mathbf{C} \otimes \mathbf{D})]^{\sigma_1\sigma_2}_{\sigma_1'\sigma_2'} &= \sum_{\sigma_1''\sigma_2''} (\mathbf{A} \otimes \mathbf{B})^{\sigma_1\sigma_2}_{\sigma_1''\sigma_2''} (\mathbf{C} \otimes \mathbf{D})^{\sigma_1''\sigma_2''}_{\sigma_1'\sigma_2'} \\ &= \sum_{\sigma_1''\sigma_2''} A^{\sigma_1}_{\sigma_1''} B^{\sigma_2}_{\sigma_2''} C^{\sigma_1''}_{\sigma_1'} D^{\sigma_2''}_{\sigma_2'} \\ &= (\mathbf{A} \cdot \mathbf{C})^{\sigma_1}_{\sigma_1'} (\mathbf{B} \cdot \mathbf{D})^{\sigma_2}_{\sigma_2'} \\ &= [(\mathbf{A} \cdot \mathbf{C}) \otimes (\mathbf{B} \cdot \mathbf{D})]^{\sigma_1\sigma_2}_{\sigma_1'\sigma_2'} \end{aligned} \tag{A.20}$$

we see that

$$(\mathbf{A} \otimes \mathbf{B}) \cdot (\mathbf{C} \otimes \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C}) \otimes (\mathbf{B} \cdot \mathbf{D}). \tag{A.21}$$

Similarly,

$$(\mathbf{A}_1 \otimes \mathbf{B}_1 \otimes \mathbf{C}_1) \cdot (\mathbf{A}_2 \otimes \mathbf{B}_2 \otimes \mathbf{C}_2) = (\mathbf{A}_1 \cdot \mathbf{A}_2) \otimes (\mathbf{B}_1 \cdot \mathbf{B}_2) \otimes (\mathbf{C}_1 \cdot \mathbf{C}_2) \tag{A.22}$$

provided that the matrix dimensions $a'_1 = a_2$, $b'_1 = b_2$, $c'_1 = c_2$, and so on.

Let us consider a chain of N lattice sites $n = 1, 2, \ldots, N$. For each site n, we introduce the chain spin-$\frac{1}{2}$ operators of dimension $2^N \times 2^N$

$$\mathbf{S}_n^\alpha = \underbrace{I \otimes \ldots \otimes I}_{n-1} \otimes \underbrace{S^\alpha}_{n} \otimes I \otimes \ldots \otimes \underbrace{I}_{N}, \tag{A.23}$$

where I is the unity 2×2 matrix and S^α ($\alpha = x, y, z$) is one of the three SL(2) generators (A.12). The explicit matrix representation of the chain spin operator reads

$$(S_n^\alpha)^{\sigma_1 \ldots \sigma_n \ldots \sigma_N}_{\sigma'_1 \ldots \sigma'_n \ldots \sigma'_N} = \delta(\sigma_1, \sigma'_1) \cdots (S^\alpha)^{\sigma_n}_{\sigma'_n} \cdots \delta(\sigma_N, \sigma'_N). \tag{A.24}$$

Based on the multiplication rule (A.22), there exists an important rule concerning the ordinary product of two chain spin operators: first one has to perform the matrix products of every two 2×2 matrices at the corresponding $n = 1, 2, \ldots, N$ sites and the resulting $2^N \times 2^N$ matrix is simply the tensor product of these N matrices. As an example, the product of two spin operators $\mathbf{S}_n^\alpha$ and $\mathbf{S}_{n'}^{\alpha'}$ for two different sites $n \neq n'$ reads

$$\mathbf{S}_n^\alpha \mathbf{S}_{n'}^{\alpha'} = \underbrace{I}_{1} \otimes \ldots \otimes I \otimes \underbrace{S^\alpha}_{n} \otimes I \otimes \ldots \otimes \underbrace{S^{\alpha'}}_{n'} \otimes I \otimes \ldots \otimes \underbrace{I}_{N}. \tag{A.25}$$

It stands to reason that

$$[\mathbf{S}_n^\alpha, \mathbf{S}_{n'}^{\alpha'}] = 0 \qquad \text{for } n \neq n', \tag{A.26}$$

i.e., for two different sites the chain spin operators resemble Bose operators.

Let V be two-dimensional vector space isomorphic to C^2. We shall choose as its basis orthonormal vectors e^+ (spin up) and e^- (spin down) defined in Eqs. (A.8) and (A.9), respectively. The Hilbert space for the chain is $V_1 \otimes V_2 \otimes \ldots \otimes V_N$. The vector basis for chain spin operators is generated by all possible 2^N tensor products of the the basis site vectors on the chain. We adopt the convention according to which a basis vector $|n_1, n_2, \ldots, n_M\rangle$ corresponds to the tensor product of M vectors e^- put on the set of ordered lattice sites

$$n_1 < n_2 < \ldots < n_M \tag{A.27}$$

and $(N - M)$ vectors e^+ put on all remaining sites:

$$|n_1, n_2, \ldots, n_M\rangle = \underbrace{e^+}_{1} \otimes e^+ \ldots \otimes e^+ \otimes \underbrace{e^-}_{n_1} \otimes e^+ \otimes \ldots e^+ \otimes \underbrace{e^-}_{n_2} \otimes \ldots \otimes \underbrace{e^+}_{N}. \tag{A.28}$$

Equivalently,

$$|n_1, n_2, \ldots, n_M\rangle = \mathbf{S}^-_{n_1} \mathbf{S}^-_{n_2} \ldots \mathbf{S}^-_{n_M} |0\rangle, \tag{A.29}$$

where $|0\rangle$ is the tensor product of N spin-up vectors e^+,

$$|0\rangle = \underbrace{e^+}_{1} \otimes e^+ \otimes \ldots e^+ \otimes \underbrace{e^+}_{N}. \tag{A.30}$$

Since the total number of these orthogonal vectors is

$$\binom{N}{0} + \binom{N}{1} + \cdots + \binom{N}{N} = 2^N,$$

they form a complete basis of the 2^N-dimensional Hilbert space.

We can proceed analogously in the case of spin-s operators in order to construct an orthogonal basis of the $(2s+1)^N$-dimensional Hilbert space.

Appendix B: Elliptic functions

A single-valued nonconstant function f of a complex variable z is said to be elliptic if it has two periods $2\omega_1$ and $2\omega_2$,

$$f(z + 2m\omega_1 + 2n\omega_2) = f(z), \qquad m, n \text{ integers.} \tag{B.1}$$

The ratio of the periods must not be purely real,

$$\mathrm{Im}\left(\frac{\omega_1}{\omega_2}\right) \neq 0, \tag{B.2}$$

since if it is, the function reduces to either a singly periodic function for rational ω_2/ω_1 or a constant for irrational ω_2/ω_1. The z-plane can be partitioned into period parallelograms the vertices of which are the points $z_0 + 2m\omega_1 + 2n\omega_2$; if the function $f(z)$ is known within and on a period parallelogram, its value can be determined at any point of the complex plane by applying relation (B.1). Let a and b be the sides of the period parallelogram and α be the angle between the sides. The nome q is then defined as

$$q = \mathrm{e}^{\mathrm{i}\pi\tau}, \qquad \tau = \frac{\omega_1}{\omega_2} = \frac{a}{b}\mathrm{e}^{\mathrm{i}\alpha}. \tag{B.3}$$

The periods are labeled such that $\mathrm{Im}(\tau) > 0$, i.e. $|q| < 1$. Although elliptic functions depend on two complex variables, the argument z and the nome q, for brevity the parameter q will be omitted in the notation whenever possible.

The standard Liouville theorem is generalized to elliptic functions as follows: An elliptic function which is analytic inside and on a period parallelogram, is a constant. An elliptic function has no singularities, except for poles in the finite part of the complex plane. Elliptic functions with a single pole of order 2 per a period parallelogram are called Weierstrass elliptic functions; they arise as solutions to differential equations of the form

$$\frac{\partial^2}{\partial z^2} f = A + Bf + Cf^2. \tag{B.4}$$

Elliptic functions with simple poles are called Jacobi elliptic functions; they arise as solutions to differential equations of the form

$$\frac{\partial^2}{\partial z^2} f = A + Bf + Cf^2 + Df^3. \tag{B.5}$$

Any elliptic function is expressible in terms of either Weierstrass or Jacobi elliptic functions.

The definitions of elliptic functions, their basic properties and relationships are presented in monographs [74, 75] and partially in Refs. [16, 51, 76]. Here, we list and indicate proofs of only those formulae which are used in the main text.

The Weierstrass functions

The Weierstrass elliptic function $\wp(z)$ is defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n}{}' \left[\frac{1}{(z - 2m\omega_1 - 2n\omega_2)^2} - \frac{1}{(2m\omega_1 + 2n\omega_2)^2}\right], \tag{B.6}$$

where the symbol $\sum'_{m,n}$ means the summation over all combinations of integers m and n, except for the one $m = n = 0$. Obviously, it holds

$$\wp(z + 2m\omega_1 + 2n\omega_2) = \wp(z), \qquad m, n \text{ integers}, \tag{B.7}$$

i.e. $\wp(z)$ is doubly periodic with half-periods ω_1 and ω_2. $\wp(z)$ has second-order poles at points $z = 2m\omega_1 + 2n\omega_2$. It is an even function,

$$\wp(z) = \wp(-z). \tag{B.8}$$

To obtain the differential equation from which $\wp(z)$ arises, we expand the function $f(z) \equiv \wp(z) - z^{-2}$ around the origin:

$$f(z) \equiv \wp(z) - z^{-2} = \frac{1}{2!} f''(0) z^2 + \frac{1}{4!} f^{(4)}(0) z^4 + \cdots, \tag{B.9}$$

where we have used that $f(0) = 0$ and $f(z)$ is even, i.e. $f'(0) = f'''(0) = \cdots = 0$. Writing $\Omega_{mn} \equiv 2m\omega_1 + 2n\omega_2$, from (B.6) we obtain

$$f''(0) = 6 \sum_{m,n}{}' \Omega_{mn}^{-4}, \qquad f^{(4)}(0) = 120 \sum_{m,n}{}' \Omega_{mn}^{-6}. \tag{B.10}$$

We introduce the so-called elliptic invariants

$$g_2 = 60 \sum_{m,n}{}' \Omega_{mn}^{-4}, \qquad g_3 = 140 \sum_{m,n}{}' \Omega_{mn}^{-6}, \tag{B.11}$$

in terms of which

$$\wp(z) = z^{-2} + \frac{g_2}{20} z^2 + \frac{g_3}{28} z^4 + O(z^6), \tag{B.12}$$

$$\frac{\partial}{\partial z} \wp(z) = -2z^{-3} + \frac{g_2}{10} z + \frac{g_3}{7} z^3 + O(z^5). \tag{B.13}$$

Taking into account that $\wp(z) = z^{-2} + O(z^2)$ and after simple algebra, these equations imply the relation

$$\left[\frac{\partial}{\partial z} \wp(z) \right]^2 - 4\wp^3(z) + g_2 \wp(z) + g_3 = O(z^2). \tag{B.14}$$

The function on the lhs of (B.14) is an elliptic function, with no singularities (the poles of order 2 at $z = 0$ and at points $z = 2m\omega_1 + 2n\omega_2$ were subtracted). By Liouville's elliptic-function theorem, it is therefore a constant. Taking $z \to 0$ we have $O(z^2) \to 0$, so that

$$\left[\frac{\partial}{\partial z} \wp(z) \right]^2 = 4\wp^3(z) - g_2 \wp(z) - g_3. \tag{B.15}$$

Differentiating this equation with respect to z, we obtain the second-order differential equation

$$\frac{\partial^2}{\partial z^2} \wp(z) = 6\wp^2(z) - \frac{g_2}{2}, \tag{B.16}$$

which is of type (B.4).

There exists another Weierstrass function $\zeta(z)$ defined by

$$\zeta(z) = \frac{1}{z} - \int_0^z \mathrm{d}u \, \left[\wp(u) - \frac{1}{u^2}\right]. \tag{B.17}$$

The function is expressible as the sum over (m, n)-integers,

$$\zeta(z) = \frac{1}{z} + \sum_{m,n}{}' \left[\frac{1}{z - 2m\omega_1 - 2n\omega_2} + \frac{1}{2m\omega_1 + 2n\omega_2} + \frac{z}{(2m\omega_1 - 2n\omega_2)^2}\right]. \tag{B.18}$$

The shift of the argument z by $2\omega_1$ and $2\omega_2$ results in

$$\zeta(z + 2\omega_1) = \zeta(z) + 2\zeta(\omega_1), \qquad \zeta(z + 2\omega_2) = \zeta(z) + 2\zeta(\omega_2). \tag{B.19}$$

For $|q| < 1$, $\zeta(z)$ can be represented as the convergent series

$$\zeta(z) = \frac{\zeta(\omega_1)}{\omega_1} z + \frac{\pi}{2\omega_1} \cot\left(\frac{\pi z}{2\omega_1}\right) + \frac{2\pi}{\omega_1} \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - q^{2n}} \sin\left(\frac{\pi n z}{\omega_1}\right). \tag{B.20}$$

The theta functions

The theta function $\vartheta_4(z, q)$ with $|q| < 1$ is defined as the sum of the series

$$\vartheta_4(z, q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \mathrm{e}^{\mathrm{i}2\pi n z}, \tag{B.21}$$

which converges absolutely for all complex z and represents an entire (i.e. analytic everywhere) function of z. This series representation is the Fourier transform of an infinite-product form

$$\vartheta_4(z, q) = \prod_{n=1}^{\infty} \left[1 - 2q^{2n-1} \cos(2\pi z) + q^{2(2n-1)}\right] (1 - q^{2n}). \tag{B.22}$$

Since it holds

$$\vartheta_4(z + 1) = \vartheta_4(z), \qquad \vartheta_4(z + \tau) = -\frac{1}{q} \mathrm{e}^{-\mathrm{i}2\pi z} \vartheta_4(z) \tag{B.23}$$

[see the definition of τ in Eq. (B.3)], $\vartheta_4(z)$ is the quasi-doubly-periodic function of z, with quasi-periods 1 and τ. The remaining three theta functions are defined as follows

$$\begin{aligned}
\vartheta_1(z, q) &= -\mathrm{i} q^{1/4} \mathrm{e}^{\mathrm{i}\pi z} \vartheta_4\left(z + \frac{\tau}{2}, q\right), \\
\vartheta_2(z, q) &= q^{1/4} \mathrm{e}^{\mathrm{i}\pi z} \vartheta_4\left(z + \frac{1}{2} + \frac{\tau}{2}, q\right), \\
\vartheta_3(z, q) &= \vartheta_4\left(z + \frac{1}{2}, q\right).
\end{aligned} \tag{B.24}$$

Also these functions are quasi-doubly-periodic in z, with quasi-periods 1 and τ. Whereas $\vartheta_1(z)$ is an odd function of z, the functions $\vartheta_2(z), \vartheta_3(z), \vartheta_4(z)$ are even:

$$\vartheta_1(z) = -\vartheta_1(-z), \quad \vartheta_2(z) = \vartheta_2(-z), \quad \vartheta_3(z) = \vartheta_3(-z), \quad \vartheta_4(z) = \vartheta_4(-z). \tag{B.25}$$

The theta functions have simple zeros:

$$\begin{array}{l} \vartheta_1(z)=0 \quad \text{at } z=m+n\tau, \\ \vartheta_2(z)=0 \quad \text{at } z=m+\frac{1}{2}+n\tau, \\ \vartheta_3(z)=0 \quad \text{at } z=m+\frac{1}{2}+\left(n+\frac{1}{2}\right)\tau, \\ \vartheta_4(z)=0 \quad \text{at } z=m+\left(n+\frac{1}{2}\right)\tau, \end{array} \tag{B.26}$$

where m and n are integers. There exist many important addition formulae involving four theta functions, their derivatives, multiples and sums of their arguments. For example, the theta functions ϑ_1 and ϑ_4 fulfill the functional relations

$$\begin{aligned} &\vartheta_1(u+x)\vartheta_1(u-x)\vartheta_1(v+y)\vartheta_1(v-y) \\ &\quad -\vartheta_1(u+y)\vartheta_1(u-y)\vartheta_1(v+x)\vartheta_1(v-x) \\ &\qquad =\vartheta_1(u+v)\vartheta_1(u-v)\vartheta_1(x+y)\vartheta_1(x-y), \\ &\vartheta_4(u+x)\vartheta_4(u-x)\vartheta_4(v+y)\vartheta_4(v-y) \\ &\quad -\vartheta_4(u+y)\vartheta_4(u-y)\vartheta_4(v+x)\vartheta_4(v-x) \\ &\qquad =-\vartheta_1(u+v)\vartheta_1(u-v)\vartheta_1(x+y)\vartheta_1(x-y), \\ &\vartheta_4(u+x)\vartheta_4(u-x)\vartheta_1(v+y)\vartheta_1(v-y) \\ &\quad -\vartheta_4(u+y)\vartheta_4(u-y)\vartheta_1(v+x)\vartheta_1(v-x) \\ &\qquad =\vartheta_4(u+v)\vartheta_4(u-v)\vartheta_1(x+y)\vartheta_1(x-y). \end{aligned} \tag{B.27}$$

A class of identities involving the squares of Jacobi theta functions reads

$$\begin{aligned} \vartheta_1^2(z)\vartheta_4^2(0) &= \vartheta_3^2(z)\vartheta_2^2(0)-\vartheta_2^2(z)\vartheta_3^2(0), \\ \vartheta_2^2(z)\vartheta_4^2(0) &= \vartheta_4^2(z)\vartheta_2^2(0)-\vartheta_1^2(z)\vartheta_3^2(0), \\ \vartheta_3^2(z)\vartheta_4^2(0) &= \vartheta_4^2(z)\vartheta_3^2(0)-\vartheta_1^2(z)\vartheta_2^2(0), \\ \vartheta_4^2(z)\vartheta_4^2(0) &= \vartheta_3^2(z)\vartheta_3^2(0)-\vartheta_2^2(z)\vartheta_2^2(0). \end{aligned} \tag{B.28}$$

The special $z=0$ case of the last identity is

$$\vartheta_4^4(0)=\vartheta_3^4(0)-\vartheta_2^4(0). \tag{B.29}$$

To prove these identities, it is sufficient to note that the ratio of the lhs and the rhs of each identity is an entire and doubly periodic function. According to the Liouville elliptic-function theorem, this function is a constant which can be readily determined by choosing a special point in the complex plane.

The modulus k and the conjugate (supplementary) modulus k' are defined by

$$k = \frac{\vartheta_2^2(0)}{\vartheta_3^2(0)} = 4\sqrt{q}\prod_{n=1}^{\infty}\left(\frac{1+q^{2n}}{1+q^{2n-1}}\right)^4, \tag{B.30}$$

$$k' = \frac{\vartheta_4^2(0)}{\vartheta_3^2(0)} = \prod_{n=1}^{\infty}\left(\frac{1-q^{2n-1}}{1+q^{2n-1}}\right)^4. \tag{B.31}$$

Due to the equality (B.29), they are constrained by

$$k^2+k'^2=1. \tag{B.32}$$

The half-period magnitudes K and K' are defined by

$$K = \frac{\pi}{2}\vartheta_3^2(0) = \frac{\pi}{2}\prod_{n=1}^{\infty}\left(\frac{1+q^{2n-1}}{1-q^{2n-1}}\cdot\frac{1-q^{2n}}{1+q^{2n}}\right)^2, \tag{B.33}$$

$$K' = -\frac{\ln q}{\pi}K = -\mathrm{i}\tau K. \tag{B.34}$$

K and K' are also expressible as the complete elliptic integrals of the first kind of modulus k and k', respectively:

$$K = \int_0^{\pi/2}\frac{\mathrm{d}\phi}{\sqrt{1-k^2\sin^2\phi}}, \qquad K' = \int_0^{\pi/2}\frac{\mathrm{d}\phi}{\sqrt{1-k'^2\sin^2\phi}}. \tag{B.35}$$

The Jacobi theta functions are related to the theta functions as follows

$$\begin{aligned} &H(u) \equiv \vartheta_1\left(\tfrac{u}{2K}, q\right), \quad H_1(u) \equiv \vartheta_2\left(\tfrac{u}{2K}, q\right), \\ &\Theta_1(u) \equiv \vartheta_3\left(\tfrac{u}{2K}, q\right), \quad \Theta(u) \equiv \vartheta_4\left(\tfrac{u}{2K}, q\right). \end{aligned} \tag{B.36}$$

They are entire functions of the complex variable u. Their simple zeros are located:

$$\begin{aligned} &H(u) = 0 \quad \text{at } u = 2mK + 2\mathrm{i}nK', \\ &H_1(u) = 0 \quad \text{at } u = (2m+1)K + 2\mathrm{i}nK', \\ &\Theta_1(u) = 0 \quad \text{at } u = (2m+1)K + \mathrm{i}(2n+1)K', \\ &\Theta(u) = 0 \quad \text{at } u = 2mK + \mathrm{i}(2n+1)K', \end{aligned} \tag{B.37}$$

where m and n are any integers. The reflection properties of the functions read

$$H(u) = -H(-u), \quad H_1(u) = H_1(-u), \quad \Theta_1(u) = \Theta_1(-u), \quad \Theta(u) = \Theta(-u). \tag{B.38}$$

The function H satisfies the quasi-periodic relations

$$H(u+2K) = -H(u), \qquad H(u+2\mathrm{i}K') = -\frac{1}{q}\exp\left(-\frac{\mathrm{i}\pi u}{K}\right)H(u) \tag{B.39}$$

with the quasi-periods $2K$ and $2\mathrm{i}K'$. The remaining Jacobi theta functions are related to H as follows

$$\begin{aligned} H_1(u) &= H(u+K), \\ \Theta(u) &= -\mathrm{i}q^{1/4}\exp\left(\frac{\mathrm{i}\pi u}{2K}\right)H(u+\mathrm{i}K'), \\ \Theta_1(u) &= \Theta(u+K) \;=\; q^{1/4}\exp\left(\frac{\mathrm{i}\pi u}{2K}\right)H(u+K+\mathrm{i}K'). \end{aligned} \tag{B.40}$$

They are also the doubly quasi-periodic functions of u, with the same quasi-periods $2K$ and $2\mathrm{i}K'$. The Jacobi theta functions fulfill the addition formulae analogous to (B.27)-(B.29),

$$\begin{aligned} &H(u+x)H(u-x)H(v+y)H(v-y) \\ &\quad -H(u+y)H(u-y)H(v+x)H(v-x) \\ &\qquad = H(u+v)H(u-v)H(x+y)H(x-y) \end{aligned} \tag{B.41}$$

and so on.

The Jacobi elliptic functions

The Jacobi elliptic functions $\mathrm{sn}\,u \equiv \mathrm{sn}(u,k)$, $\mathrm{cn}\,u \equiv \mathrm{cn}(u,k)$ and $\mathrm{dn}\,u \equiv \mathrm{dn}(u,k)$ are defined by

$$\begin{aligned}
\mathrm{sn}\,u &= \frac{1}{\sqrt{k}}\frac{H(u)}{\Theta(u)} = \frac{1}{\sqrt{k}}\frac{\vartheta_1(v)}{\vartheta_4(v)}, && \text{(B.42)}\\
\mathrm{cn}\,u &= \sqrt{\frac{k'}{k}}\frac{H_1(u)}{\Theta(u)} = \sqrt{\frac{k'}{k}}\frac{\vartheta_2(v)}{\vartheta_4(v)}, && \text{(B.43)}\\
\mathrm{dn}\,u &= \sqrt{k'}\frac{\Theta_1(u)}{\Theta(u)} = \sqrt{k'}\frac{\vartheta_3(v)}{\vartheta_4(v)}, && \text{(B.44)}
\end{aligned}$$

where $v = u/(2K)$. These functions are meromorphic since their only singularities are simple poles at $u = 2mK + \mathrm{i}(2n+1)K'$ (m, n are any integers). Their reflection properties are

$$\mathrm{sn}\,u = -\mathrm{sn}(-u), \qquad \mathrm{cn}\,u = \mathrm{cn}(-u), \qquad \mathrm{dn}\,u = \mathrm{dn}(-u). \tag{B.45}$$

It follows from the relations (B.40) that the Jacobi elliptic functions are periodic/anti-periodic with periods $2K$ and $2\mathrm{i}K'$:

$$\begin{aligned}
&\mathrm{sn}(u+2K) = -\mathrm{sn}\,u, && \mathrm{sn}(u+2\mathrm{i}K') = \mathrm{sn}\,u;\\
&\mathrm{cn}(u+2K) = -\mathrm{cn}\,u, && \mathrm{cn}(u+2\mathrm{i}K') = -\mathrm{cn}\,u;\\
&\mathrm{dn}(u+2K) = \mathrm{dn}\,u, && \mathrm{dn}(u+2\mathrm{i}K') = -\mathrm{dn}\,u.
\end{aligned} \tag{B.46}$$

The half-period shifts are described by the relations

$$\begin{aligned}
&\mathrm{sn}(u+K) = \frac{\mathrm{cn}\,u}{\mathrm{dn}\,u}, && \mathrm{sn}(u+\mathrm{i}K') = \frac{1}{k\,\mathrm{sn}\,u};\\
&\mathrm{cn}(u+K) = -k'\frac{\mathrm{sn}\,u}{\mathrm{dn}\,u}, && \mathrm{cn}(u+\mathrm{i}K') = -\mathrm{i}\frac{\mathrm{dn}\,u}{k\,\mathrm{sn}\,u};\\
&\mathrm{dn}(u+K) = k'\frac{1}{\mathrm{dn}\,u}, && \mathrm{dn}(u+\mathrm{i}K') = -\mathrm{i}\frac{\mathrm{cn}\,u}{\mathrm{sn}\,u}.
\end{aligned} \tag{B.47}$$

The addition theorems for the Jacobi elliptic functions have the form

$$\begin{aligned}
\mathrm{sn}(u+v) &= \frac{\mathrm{sn}\,u\,\mathrm{cn}\,v\,\mathrm{dn}\,v + \mathrm{cn}\,u\,\mathrm{dn}\,u\,\mathrm{sn}\,v}{1-k^2\mathrm{sn}^2u\,\mathrm{sn}^2v},\\
\mathrm{cn}(u+v) &= \frac{\mathrm{cn}\,u\,\mathrm{cn}\,v - \mathrm{sn}\,u\,\mathrm{dn}\,u\,\mathrm{sn}\,v\,\mathrm{dn}\,v}{1-k^2\mathrm{sn}^2u\,\mathrm{sn}^2v}, && \text{(B.48)}\\
\mathrm{dn}(u+v) &= \frac{\mathrm{dn}\,u\,\mathrm{dn}\,v - k^2\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{sn}\,v\,\mathrm{cn}\,v}{1-k^2\mathrm{sn}^2\,u\,\mathrm{sn}^2v}.
\end{aligned}$$

The Jacobi elliptic functions also arise from the inversion of the incomplete elliptic integral of the first kind

$$u = F(\xi,k) = \int_0^\xi \frac{\mathrm{d}\phi}{\sqrt{1-k^2\sin^2\phi}} \qquad (0<k^2<1), \tag{B.49}$$

giving the Jacobi amplitude

$$\xi \equiv \mathrm{am}(u,k) = F^{-1}(u,k). \tag{B.50}$$

In terms of the Jacobi amplitude,

$$\mathrm{sn}(u,k) = \sin\xi, \quad \mathrm{cn}(u,k) = \cos\xi, \quad \mathrm{dn}(u,k) = \sqrt{1-k^2\sin^2\xi}. \tag{B.51}$$

According to this representation, it holds

$$\mathrm{sn}^2 u + \mathrm{cn}^2 u = 1, \qquad k^2\mathrm{sn}^2 u + \mathrm{dn}^2 u = 1. \tag{B.52}$$

The Jacobi elliptic functions are doubly periodic generalizations of the trigonometric functions. Indeed, if $k \to 0$, the supplementary modulus $k' \to 1$, the half-periods $K \to \pi/2$, $K' \to \infty$ and $\xi = u$, so that

$$\mathrm{sn}(u,0) = \sin u, \qquad \mathrm{cn}(u,0) = \cos u, \qquad \mathrm{dn}(u,0) = 1. \tag{B.53}$$

The integral representation (B.49) together with the relations (B.51) allow us to derive explicit expressions for the derivatives of the Jacobi elliptic functions with respect to the argument. From (B.49) we get

$$\frac{\mathrm{d}u}{\mathrm{d}\xi} = \frac{1}{\sqrt{1-k^2\sin^2\xi}} = \frac{1}{\mathrm{dn}(u,k)}. \tag{B.54}$$

Consequently,

$$\frac{\partial}{\partial u}\mathrm{sn}(u,k) = \frac{\partial \sin\xi}{\partial\xi}\frac{\partial\xi}{\partial u} = \mathrm{cn}\,u\,\mathrm{dn}\,u\,. \tag{B.55}$$

Similarly,

$$\frac{\partial}{\partial u}\mathrm{cn}(u,k) = -\mathrm{sn}\,u\,\mathrm{dn}\,u\,, \qquad \frac{\partial}{\partial u}\mathrm{dn}(u,k) = -k^2\mathrm{sn}\,u\,\mathrm{cn}\,u\,. \tag{B.56}$$

With regard to the equalities (B.52), the Jacobi elliptic functions can be defined as solutions to the differential equations

$$\begin{aligned}
\frac{\partial y}{\partial u} &= \sqrt{(1-y^2)(1-k^2y^2)}, \qquad y = \mathrm{sn}(u,k);\\
\frac{\partial y}{\partial u} &= -\sqrt{(1-y^2)(k'^2+k^2y^2)}, \qquad y = \mathrm{cn}(u,k);\\
\frac{\partial y}{\partial u} &= -\sqrt{(1-y^2)(y^2-k'^2)}, \qquad y = \mathrm{dn}(u,k).
\end{aligned} \tag{B.57}$$

Differentiating these equations with respect to u, we obtain

$$\begin{aligned}
\frac{\partial^2 y}{\partial u^2} &= -(1+k^2)y + 2k^2y^3, \qquad y = \mathrm{sn}(u,k);\\
\frac{\partial^2 y}{\partial u^2} &= -(1-2k^2)y - 2k^2y^3, \qquad y = \mathrm{cn}(u,k);\\
\frac{\partial^2 y}{\partial u^2} &= (2-k^2)y - 2y^3, \qquad y = \mathrm{dn}(u,k).
\end{aligned} \tag{B.58}$$

These equations are of type (B.5). Higher-order derivatives are calculable by repeated application of the relations (B.57) and (B.58).

For $u = 0$ we have $\xi = 0$, which implies

$$\mathrm{sn}\,0 = 0, \qquad \mathrm{cn}\,0 = \mathrm{dn}\,0 = 1, \tag{B.59}$$

independently of modulus k. The Jacobi elliptic functions are regular in the neighborhood of $u = 0$. Using MacLaurin's theorem, their power series expansions can be found by calculating the higher-order derivatives at $u = 0$ according to the above prescription. The final result is:

$$\begin{aligned}
\mathrm{sn}\,u &= u - \frac{1}{3!}(1+k^2)u^3 + \frac{1}{5!}(1+14k^2+k^4)u^5 - \cdots, \\
\mathrm{cn}\,u &= 1 - \frac{1}{2!}u^2 + \frac{1}{4!}(1+4k^2)u^4 - \cdots, \\
\mathrm{dn}\,u &= 1 - \frac{1}{2!}k^2u^2 + \frac{1}{4!}(4k^2+k^4)u^4 - \cdots.
\end{aligned} \tag{B.60}$$

The imaginary Jacobi transformation makes a bridge between the Jacobi elliptic functions of argument u and modulus k and the ones of argument $\mathrm{i}u$ and supplementary modulus k':

$$\begin{aligned}
\mathrm{sn}(u,k) &= \frac{1}{\mathrm{i}}\frac{\mathrm{sn}(\mathrm{i}u,k')}{\mathrm{cn}(\mathrm{i}u,k')}, \\
\mathrm{cn}(u,k) &= \frac{1}{\mathrm{cn}(\mathrm{i}u,k')}, \\
\mathrm{dn}(u,k) &= \frac{\mathrm{dn}(\mathrm{i}u,k')}{\mathrm{cn}(\mathrm{i}u,k')}.
\end{aligned} \tag{B.61}$$

The connection between the Jacobi elliptic functions of argument u and modulus k with those of argument $\tilde{u}$ and modulus l given by

$$\tilde{u} = (1+k')u, \qquad l = \frac{1-k'}{1+k'}, \tag{B.62}$$

follows from the Landen transformation:

$$\begin{aligned}
\mathrm{sn}(\tilde{u},l) &= (1+k')\frac{\mathrm{sn}(u,k)\mathrm{cn}(u,k)}{\mathrm{dn}(u,k)}, \\
\mathrm{cn}(\tilde{u},l) &= \frac{1-(1+k')\mathrm{sn}^2(u,k)}{\mathrm{dn}(u,k)}, \\
\mathrm{dn}(\tilde{u},l) &= \frac{1-(1-k')\mathrm{sn}^2(u,k)}{\mathrm{dn}(u,k)}.
\end{aligned} \tag{B.63}$$

The half-period magnitudes K_l and K_l', associated with modulus l, are related to the ones K_k and K_k', associated with modulus k, via

$$K_l = \frac{1+k'}{2}K_k, \qquad K_l' = (1+k')K_k', \qquad \tau_l = 2\tau_k. \tag{B.64}$$

References

[1] W. Heisenberg: *Z. Phys.* **49** (1928) 619
[2] H. Bethe: *Z. Phys.* **71** (1931) 205
[3] L. Hultén: *Ark. Mat. Astron. Fys. B.* **26A** (1938) 1
[4] J. des Cloizeaux, J. J. Pearson: *Phys. Rev.* **128** (1962) 2131
[5] C. N. Yang, C. P. Yang: *Phys. Rev.* **150** (1966) 321
[6] C. N. Yang, C. P. Yang: *Phys. Rev.* **150** (1966) 327
[7] R. J. Baxter: *Phys. Rev. Lett.* **26** (1971) 832
[8] R. J. Baxter: *Phys. Rev. Lett.* **26** (1971) 834
[9] R. J. Baxter: *Ann. Phys. (NY)* **70** (1972) 193
[10] R. J. Baxter: *Ann. Phys. (NY)* **70** (1972) 323
[11] J. D. Johnson, S. Krinsky, B. M. McCoy: *Phys. Rev. A* **8** (1973) 2526
[12] E. K. Sklyanin, L. D. Faddeev: *Dokl. Acad. Nauk SSSR* **243** (1978) 1430
[13] E. K. Sklyanin, L. D. Faddeev: *Dokl. Acad. Nauk SSSR* **244** (1978) 1337
[14] C. S. Gardner, J. M. Greene, M. D. Kruskal, R. M. Miura: *Phys. Rev. Lett.* **19** (1967) 1095
[15] B. A. Dubrovin, V. B. Matveev, S. P. Novikov: *Russian Math. Surveys* **31** (1976) 59
[16] L. A. Takhtajan, L. D. Faddeev: *Russian Math. Surveys* **34** (1979) 11
[17] L. D. Faddeev, L. A. Takhtajan: *J. Soviet. Math* **24** (1984) 241
[18] P. P. Kulish, N. Yu. Reshetikhin, E. K. Sklyanin: *Lett. Math. Phys.* **5** (1981) 393
[19] V. G. Drinfel'd: *Dokl. Acad. Nauk SSSR* **283** (1985) 1060
[20] M. Jimbo: *Lett. Math. Phys.* **10** (1985) 63 .
[21] E. H. Lieb, W. Liniger: *Phys. Rev.* **130** (1963) 1605
[22] E. H. Lieb: *Phys. Rev.* **130** (1963) 1616
[23] C. N. Yang, C. P. Yang: *J. Math. Phys.* **10** (1969) 1115
[24] B. Sutherland: *J. Math. Phys.* **12** (1971) 246
[25] B. Sutherland: *J. Math. Phys.* **12** (1971) 251
[26] C. N. Yang: *Phys. Rev. Lett.* **19** (1967) 1312
[27] M. Gaudin: *Phys. Lett. A* **24** (1967) 55
[28] C. K. Lai: *Phys. Rev. Lett.* **26** (1971) 1472
[29] C. K. Lai: *Phys. Rev. A* **8** (1973) 2567
[30] M. Takahashi: *Prog. Theor. Phys.* **46** (1971) 1388
[31] M. Takahashi: *Prog. Theor. Phys.* **46** (1971) 401
[32] M. Gaudin: *Phys. Rev. Lett.* **26** (1971) 1302
[33] C. Destri, H. J. de Vega: *Phys. Rev. Lett.* **69** (1992) 2313
[34] C. Destri, H. J. de Vega: *Nucl. Phys. B* **438** (1995) 413
[35] E. H. Lieb, F. Y. Wu: *Phys. Rev. Lett.* **20** (1968) 1445
[36] N. Andrei: *Phys. Rev. Lett.* **45** (1980) 379
[37] P. B. Wiegmann: *J. Phys. C* **14** (1981) 1463
[38] S. Coleman: *Phys. Rev. D* **11** (1975) 2088
[39] M. Karowski, H.-J. Thun, T. T. Truong, P. H. Weisz: *Phys. Lett. B* **67** (1977) 321
[40] A. B. Zamolodchikov, Al. B. Zamolodchikov: *Ann. Phys. (N.Y.)* **120** (1979) 253

[41] C. Destri, H. J. de Vega: *Nucl. Phys. B* **358** (1991) 251
[42] Al. B. Zamolodchikov: *Int. J. Mod. Phys. A* **10** (1995) 1125
[43] L. Onsager: *Phys. Rev.* **65** (1944) 117
[44] E. H. Lieb: *Phys. Rev. Lett.* **18** (1967) 1046
[45] E. H. Lieb: *Phys. Rev. Lett.* **19** (1967) 108
[46] E. H. Lieb: *Phys. Rev.* **162** (1967) 162
[47] B. Sutherland: *Phys. Rev. Lett.* **19** (1967) 103
[48] L. Šamaj, I. Travěnec: *J. Stat. Phys.* **101** (2000) 713
[49] P. Kalinay, L. Šamaj: *J. Stat. Phys.* **106** (2002) 857
[50] L. Šamaj: *J. Stat. Phys.* **111** (2003) 261
[51] R. J. Baxter: *Exactly Solved Models in Statistical Mechanics* (Academic Press, London, 1982)
[52] M. Gaudin: *La Fonction d'Onde de Bethe* (Masson, Paris, 1983)
[53] D. C. Mattis: *The Many-Body Problem (An Encyclopedia of Exactly Solved Models in One Dimension* (World Scientific,Singapore,1993)
[54] V. E. Korepin, F. H. L. Essler: *Exactly Solvable Models of Strongly Correlated Electrons* (World Scientific,Singapore,1994)
[55] C. Gómez, M. Ruiz-Altaba, G. Sierra: *Quantum Groups in Two-Dimensional Physics* (Cambridge University Press, Cambridge, 1996)
[56] M. Takahashi: *Thermodynamics of One-dimensional Solvable Models* (Cambridge University Press, Cambridge, 1999)
[57] B. Sutherland: *Beautiful Models: 70 Years of Exactly Solved Quantum Many-Body Problems* (World Scientific, Singapore, 2004)
[58] F. H. L. Essler, H. Frahm, F. Göhmann, A. Klümper, V. E. Korepin: *The One-Dimensional Hubbard Model* (Cambridge University Press, Cambridge, 2005)
[59] J. B. McGuire: *J. Math. Phys.* **5** (1964) 622
[60] L. N. Sneddon: *Mixed Boundary Value Problem in Potential Theory* (North-Holland, Amsterdam, 1966)
[61] M. Gaudin: *Phys. Rev. A* **4** (1971) 386
[62] C. P. Yang: *Phys. Rev. A* **2** (1970) 154
[63] G. Kato, M. Wadati: *Phys. Rev. E* **63** (2001) 036106
[64] G. Kato, M. Wadati: *J. Phys. Soc. Jpn.* **70** (2001) 1924
[65] M. Wadati, G. Kato: *Chaos, Solitons & Fractals* **14** (2002) 23
[66] L. Tonks: *Phys. Rev.* **50** (1936) 955
[67] J. Birman: *Ann. Math. Studies* **82** (1974) 1
[68] J. Birman: *Bull. Am. Math. Soc.* **28** (1983) 253
[69] A. Connes: *Noncommutative geometry* (Academic Press, San Diego, 1994)
[70] J. Wess, B. Zumino: *Nucl. Phys. B (Proc. Suppl.)* **18** (1990) 302
[71] J. B. McGuire: *J. Math. Phys.* **6** (1965) 432
[72] J. B. McGuire: *J. Math. Phys.* **7** (1965) 123
[73] M. Flicker, E. H. Lieb: *Phys. Rev.* **161** (1967) 179
[74] K. G. Jacobi, *Fundamenta Nova Theoriae Functionum Ellipticarum* (Königsberg, 1829)
[75] E. T. Whittaker, G. N. Watson: *A Course of Modern Analysis*, 4th edn. (Cambridge University Press, Cambridge, 1927)
[76] I. S. Gradshteyn, I. M. Ryzhik: *Tables of Integrals, Series, and Products,*, 5th edn. (Academic Press, London, 1994)