acta physica slovaca vol. 57 No. 3, 407 – 564 **June 2007**

LIGHT FRONT FIELD THEORY: AN ADVANCED PRIMER

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Received 6 June 2007, accepted 20 June 2007

We present an elementary introduction to quantum field theory formulated in terms of Dirac's light front variables. In addition to general principles and methods, a few more specific topics and approaches based on the author's work will be discussed. Most of the discussion deals with massive two-dimensional models formulated in a finite spatial volume starting with a detailed comparison between quantization of massive free fields in the usual field theory and the light front (LF) quantization. We discuss basic properties such as relativistic invariance and causality. After the LF treatment of the soluble Federbush model, a LF approach to spontaneous symmetry breaking is explained and a simple gauge theory – the massive Schwinger model in various gauges is studied. A LF version of bosonization and the massive Thirring model are also discussed. A special chapter is devoted to the method of discretized light cone quantization and its application to calculations of the properties of quantum solitons. The problem of LF zero modes is illustrated with the example of the two-dimensional Yukawa model. Hamiltonian perturbation theory in the LF formulation is derived and applied to a few simple processes to demonstrate its advantages. As a byproduct, it is shown that the LF theory cannot be obtained as a "light-like" limit of the usual field theory quantized on an initial space-like surface. A simple LF formulation of the Higgs mechanism is then given. Since our intention was to provide a treatment of the light front quantization accessible to postgradual students, an effort was made to discuss most of the topics pedagogically and a number of technical details and derivations are contained in the appendices.

PACS: 11.10.Ef, 11.15.Tk, 11.25DB

KEYWORDS: Light front field theory, Discretized light cone quantization, Relativistic two-dimensional models, Spontaneous symmetry breaking, Quantum solitons, Hamiltonian methods, Light front perturbation theory

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Contents

1 Introduction

Quantum field theory (QFT) was invented as a conceptual and computational tool for calculating processes of elementary particles (originally electrons and photons) moving with (almost) speed of light. In this kinematical regime, transmutation of different species of particles, i.e. their creation and destruction, was experimentally observed and the adequate theoretical scheme had to include a mathematical apparatus capable to decribe these features. A natural path was to try to generalize the nonrelativistic quantum mechanics to the many-body situation and to include the relativistic kinematics by working with operators satisfying the Poincaré algebra. The physical systems where these ideas were successfully tested were electrons interacting with quanta of the electromagnetic fields, photons. Two immediate problems were encountered: the lack of computational methods beyond perturbation theory (where by construction only small corrections to the properties of a free system can be evaluated in powers of the coupling constant and where a piece of physical information may simply be overlooked by the limitations of the method itself) and emergence of infinite quantities in the course of higher-order calculations. Although technical tools have been invented during the subsequent development for removing the infinite pieces of amplitudes, it is probably fair to say that there is no general consensus until today if these problems are due to the wrong formulation of the fundaments of QFT or they just reflect inadequate mathematical techniques of the perturbation method.

What was taken as a self-evident basis in the development of QFT was the parametrization of the space-time in the manner known from nonrelativistic quantum mechanics. That is, the use of the time variable t measured in the laboratory frame of an observer and three space coordinates x, y, z, all four united into one four-vector x^μ with a specific transformation law under translations, rotations and boosts in space and time, that collectively form the group of Poincaré transformations. It was only in 1949 when P. A. M. Dirac [1] noticed that field theory can be actually formulated in three independent "languages" depending on the parametrization of the space-time and the corresponding definition of the "initial surface" where the initial data for the fields (equal-time commutation relations in the quantum version) are prescribed. The conventional and most widely used formulation was called the "instant form" by Dirac reflecting the fact that the initial surface is simply a fixed moment $t = const$ of the usual time variable. More generally, the quantization surface can be extended by Lorentz transformations to an arbitrary space-like surface. The choice of field variables follows the parametrization of the four-vector $x^\mu = (t, \mathbf{x})$, i.e. the gauge field is $A^\mu(x) = (A^0, \mathbf{A})$, etc. We will simply refer to this conventional form known from textbooks as the space-like (SL) field theory.

The second possibility of the relativistic Hamiltonian dynamics is the *point form* which initializes fields at a hyperboloid $x^2 = a^2$. It is a rarely used formulation which amplifies covariance but has too many dynamical Poincaré generators, namely, containing interaction terms. In other words, this scheme has the least number of kinematical, i.e. interaction-free generators and this feature makes it less appealing.

The topic of the present introductory review is the quantum version of the Dirac's front form of relativistic Hamiltonian dynamics. We will call it simply the light front (LF) field theory. A more adequate characterization would be a quantum field theory expressed in terms of the LF space-time and field variables. The former are $x^\mu = (x^+, x^-, x^1, x^2),\ x^\pm = t \pm z$, the latter are fields expressed in terms of x^μ and satisfying field equations different from conventional ones. In addition, vector field $A^\mu = (A^+, A^-, A^1, A^2)$. The LF fields are quantized at the initial surface

$x^+ = const$. This surface defines a plane tangent to the light cone. As has been gradually realized in a rather unsystematic and disconnected development of the LF theory [2,3,4,5,6,7,8, 9], it possesses a few unique properties which simplify the formalism and make it physically more transparent and closer in spirit to quantum mechanics. For example, a probabilistic interpretation and notion of relativistic wave functions is well defined here. Consequently, one has a consistent Fock expansion of relativistic bound states. In fact, the deeper reason for this feature is the striking character of the LF theory vacuum: it can be derived from kinematical considerations, i.e. independently of dynamics. In yet another words, the state without particles (the Fock vacuum) corresponds to the minimum of energy and it is an eigenstate of the full, that is free plus interacting, LF Hamiltonian. This is in a sharp contrast to the usual SL theory, where the Fock vacuum state is the lowest-energy eigenstate of only the free part of the Hamiltonian. To find the eigenstate of the full Hamiltonian is a very difficult dynamical problem which generally cannot be solved even approximately.

One may ask a question what is the reason for the different status of the vacuum state in the LF approach. Actually, the answer is quite simple: it is the choice of the variables $x^\pm$ which are better suited for the relativistic kinematics. In a qualitative sense one can say that they correspond to a reference frame of an observer (physically unrealizable) moving with the speed of light. This seemingly innocent change of variables has profound consequences for the mathematical structure of the LF field theory and for the form of representation of physical mechanisms and phenomena. Here we are talking for example about the structure of field equations and their different division into dynamical and constrained ones, that leads to a different number of independent (dynamical) field variables, about a necessity to fix ambiguities in the definitions of some inverse operators (Green's functions) by a choice of boundary conditions and also about the fact that the LF momentum P^+ has a positive spectrum of eigenvalues $p^+ = p^0 + p^3$ (the spectrum has a "bottom", it is bounded from below) which is the reason for the possibility to find the LF vacuum as a state with the minimal value of P^+ or, equivalently, with particle number zero. One should make a remark here that the latter feature appears sometimes confusing since at least at the first sight it seems to forbid any vacuum processes. This looks as a serious difficulty and induces questions related to the overall consistency of the LF scheme. One of the key premises of the present text is the opposite idea: since there is no a priori reason for doubts concerning validity and consistency of the LF form of the field theory (if treated in a mathematically correct manner), the theory itself should contain a correct physical information in the form following from its intrinsic structure and properties. One should develop fresh and independent ideas and techniques to be able to extract this physical content of the front form of the relativistic dynamics.

The light front version of the field theory, also known as the light-cone theory, the light-cone, light front or null-plane quantization, has a complicated history and an unsystematic development. On one hand, it has been studied from the axiomatic point of view [7, 10], on the other hand, used in numerous phenomenological applications (see for example [11, 12]), sometimes in a rather heuristic or at least pragmatic manner. In our review, we will adhere to the definition of the LF quantization as the canonical (Hamiltonian) version of the QFT, predominantly formulated in a finite space volume with properly chosen boundary conditions. It is the strong conviction of the author that the advantages of the LF approach are best reflected in the Hamiltonian language where one can study properties of quantum states, calculate (at least in principle) relativistic wave functions and related observables, implement symmetries on the quantum level in terms of (regularized) unitary operators, etc. Some subtleties of the LF theory are more clearly displayed in the discrete finite-volume form, with its denumerable Fock basis and the discrete spectrum of momenta. This is particularly true for infrared aspects of the dynamics, related to the so called (Fourier) zero-momentum modes and to vacuum properties. The formulation of the LF theory in a finite volume with fields obeying periodicity or antiperiodicity in spatial coordinates is known as the discretized light cone quantization – DLCQ [13, 14, 15, 16]. Because of presence of discrete momenta and of consistent Fock expanions for composite states, it leads to quantum Hamiltonians in the form of large matrices which can be diagonalized numerically. In this way, spectrum of physical states, their low-lying masses and other observables have been determined for numerous low-dimensional models. The "discretized" formulation of the light front field theory can be viewed also as a convenient analytical framework. Admittedly, prefering the finite-volume formulation may appear a bit subjective and there exist different opinions in literature. In addition to the standard continuum formulation (i.e., infinite volume, continuous spectrum of momenta, Feynman integrals in perturbation theory, etc.), for some applications accompanied by an infrared cutoff [17], a LF version of the Epstein-Glaser non-perturbative regularization and renormalization of the continuum theory [18] has been developed recently [19, 20, 21] to give a mathematically sound formulation of the LF field theory in an axiomatic spirit [22]. It is based on a consistent interpretation of quantum fields as operator-valued distributions defined with the help of test functions. The attitude of the present author is that a consistent and careful analysis of field-theory models formulated canonically in a finite volume in terms of light front variables actually yields a mathematically well defined, physically complete and yet relatively simple version of the LF dynamics. We will make an attempt to justify this perhaps a little bit intuitive attitude in course of the present notes by analyzing a few examples and explaining additional supportive arguments.

In addition to the review papers cited in the main text of the present notes, a very readable and pedagogical introduction to the light front field theory is due to A. Harindranath [23].

In order to make this review accessible to graduate students, a large portion of the discussion will focus on two-dimensional models, starting from the LF quantization of free *massive* fields.[2] We will try to explain the differences between the SL and LF formulations already at this simple level and simultaneously to emphasize the necessity to work with mathematically well defined objects in a careful way using fully and consistently intrinsic properties of the front form of relativistic dynamics.

2 Free massive space like fields in $\mathbf{D = 1 + 1}$

2.1 Quantum scalar field

It will be instructive to compare a few basic properties of quantum fields in light front theory with those quantized in the usual space-like scheme. For this reason, we recall some simple facts about the conventional quantization of the massive scalar and Fermi fields in a finite volume.

The free massive scalar field satisfies the Klein-Gordon equation, which in the covariant form

[2] The massless LF fields in two dimensions are a subtle subject. Their consistent treatment and relationship to the conformal field theory (which is a powerful method in the conventional quantization) remains to be clarified.

reads ($\partial_\mu \equiv \partial/\partial x^\mu$)

$$\Big(\partial_\mu \partial^\mu + \mu^2\Big)\phi(x) = 0 \tag{2.1}$$

and which in terms of the usual space-time variables is of the second order in the time derivative:

$$\Big(\partial_0 \partial_0 - \partial_1 \partial_1 + \mu^2\Big)\phi(x) = 0. \tag{2.2}$$

Its solution is expressed as a superposition of plane waves with the operator coefficients which have a meaning of the operators creating and destroying a scalar-field quantum:

$$\phi(x) = \frac{1}{\sqrt{2L}} \sum_p \frac{1}{\sqrt{2\omega(p)}} \big[a(p)e^{-ip.x} + a^\dagger(p)e^{ip.x}\big]. \tag{2.3}$$

Here $p.x = p^0 x^0 - p^1 x^1$ and $p^0 \equiv \omega(p) = +\sqrt{p_1^2 + \mu^2}$. If there will be no danger of a confusion, we will use the symbols x, p etc. instead of x^1, p^1. The field $\phi(x)$ is defined on a line $-L \leq x^1 \leq L$. Imposing the periodic boundary condition $\phi(t, -L) = \phi(t, L)$, we easily find that the momentum p is discrete,

$$p \equiv p_n = \frac{2\pi}{L} n,\ n = 0, \pm 1, \pm 2, \dots\ . \tag{2.4}$$

So, the summation in Eq.(2.3) runs over the integers n. Note that the mode with $n = 0$ is also included. It is an independent degree of freedom with the time dependence $\exp\big(\pm imt\big)$. Of course, upon inserting the solution (2.3) into the Klein-Gordon equation, one recovers the dispersion relation of a free massive quantum $\omega^2(p_n) = p_n^2 + \mu^2$, appropriate for the discrete values of the energy and momentum.

The creation and annihilation operators of the scalar field obey the commutation relation

$$\big[a(p), a^\dagger(q)\big] = \delta_{p,q}, \tag{2.5}$$

which is a direct consequence of the assumed canonical equal-time commutation relation (ETCR)

$$\big[\phi(0, x), \partial_0 \phi(0, y)\big] = i\delta_P(x - y). \tag{2.6}$$

In the above expressions, $\delta_{p,q} = \delta_{m,n}$ is the Kronecker symbol for the momenta p and q, or equivalently, for the integers m,n which parametrize their discrete values. The delta function $\delta_P(x - y)$, given by the infinite series

$$\delta_P(x - y) = \frac{1}{2L} + \frac{1}{2L} \sum_n e^{ip_n(x-y)} \equiv \delta_0 + \delta_N(x - y) \tag{2.7}$$

is periodic in both space variables x and y (hence the subscript P) as is required by the periodicity of the scalar field itself. The two parts of δ_P are called the zero-Fourier mode and the normal-Fourier mode pieces of the full delta function. The first one obviously corresponds to the plane wave with $n = 0$, the second one to the sum of all plane waves with non-zero n.

Let us also remind the rest of the usual canonical formalism in this very simple theory. Actually, (the classical version of) the Klein-Gordon equation (2.1) is obtained from the variational principle $\delta S = 0$ with fixed variations at the boundaries. Here S is the classical action of the system, $S = \int_{-\infty}^{\infty} dt L(\phi, \partial_\mu \phi)$ and $L(\phi, \partial_\mu \phi) = \int_{-L}^{L} dx \mathcal{L}(\phi, \partial_\mu \phi)$ is the Lagrangian of the system defined as the space integral over the density $\mathcal{L}$:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}\mu^2 \phi^2 = \frac{1}{2}\big(\partial_0 \phi\big)^2 - \frac{1}{2}\big(\partial_1 \phi\big)^2 - \frac{1}{2}\mu^2 \phi^2. \tag{2.8}$$

The quantization conditions (2.6) transform the classical theory into the quantum one. They have been postulated in a full analogy with the classical mechanics, defining the momentum conjugate to the field "coordinate" $\phi(x)$:

$$\big[\phi(x), \Pi_\phi(y)\big] = i\delta_P(x-y), \quad \Pi_\phi(x) = \frac{\delta \mathcal{L}}{\delta \partial_0 \phi} = \partial_0 \phi(x). \tag{2.9}$$

Here, the Poisson brackets of the classical mechanics, generalized to the classical field theory (see the Appendix D), have been replaced by the commutator $-i[A,B] = -i\big(AB - BA\big)$ according to the rules of the canonical quantization. Next, one goes over to the Hamiltonian, corresponding to the above Lagrangian density:

$$H \equiv P^0 = \int_{-L}^{+L} \mathrm{d}x T^{00}(x). \tag{2.10}$$

The components of the energy-momentum tensor (or density) are

$$T^{\mu\nu}(x) = \Pi^\mu \partial^\nu \phi - g^{\mu\nu}\mathcal{L}, \;\; \Pi^\mu = \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi}, \;\; \Pi^0 \equiv \Pi_\phi. \tag{2.11}$$

and follow from the Noether theorem for the case of space-time translations. The quantity $g^{\mu\nu}$ is the metric tensor with $g^{00} = -g^{11} = 1, g^{12} = g^{21} = 0$. The other two Poincaré generators [3] are the momentum operator P^1 and the operator of the boost M^{01}:

$$P^1 = \int_{-L}^{+L} \mathrm{d}x T^{01}(x), \;\; M^{01} = tH - \int_{-L}^{+L} \mathrm{d}x x T^{01}(x). \tag{2.12}$$

Explicitly, one easily finds

$$\begin{aligned} H &= \int_{-L}^{+L} \mathrm{d}x \Big[\frac{1}{2}\Pi_\phi^2 + \frac{1}{2}\big(\partial_1 \phi\big)^2 + \frac{1}{2}\mu^2\phi^2\Big], \\ P^1 &= -\int_{-L}^{+L} \mathrm{d}x \Pi_\phi \partial_1 \phi, \;\; M^{01} = Ht - \int_{-L}^{+L} \mathrm{d}x x \Pi_\phi \partial_1 \phi. \end{aligned} \tag{2.13}$$

[3] Since there are no transverse dimensions, no transverse translations, rotations and transverse boosts are possible, hence the number of Poincaré generators in one spatial dimension is equal to three.

The Fock representation of the above operators is obtained by inserting the field expansion (2.3) into their definitions (2.13) with the results (see the Appendix B for details)

$$H = \sum_p \omega(p) a^\dagger(p) a(p), \quad P = \sum_p p a^\dagger(p) a(p). \tag{2.14}$$

In the next step, one should check the overall relativistic consistency of the quantized theory by verifying that the abstract Poincaré algebra is satisfied by the three quantum generators (2.13) by using the ETCR (2.6). Neglecting the surface terms that appear due to the integration by parts, this is indeed the case for our simple non-interacting theory. Similarly, the relativistic covariance requires that the scalar field transforms in a particular way under the action of the two translational generators and of the boost M^{01} along the one space direction (see Sec. 5). In the infinitezimal form, this leads to the familiar Heisenberg equations

$$-i\partial^\mu \phi(x) = \big[P^\mu, \phi(x)\big], \;\; P^\mu = (P^0, P^1). \tag{2.15}$$

Again, it is easily checked by applying repeatidly the ETCR (2.6) that the Heisenberg equations reproduce the Euler-Lagrange field equations obtained from the action principle in the Lagrangian formulation. Both properties (validity of Poincaré algebra and correct transformation laws of quantum fields) together establish the relativistic invariance of the field theory on quantum level.

A very important object in any quantum field theory model is the vacuum state $|0\rangle$. Sometimes it is simply defined as

$$a(p)|0\rangle = 0, \tag{2.16}$$

but actually the above relation is a consequence of a more profound property: positivity of the spectrum of the eigenvalues of the Hamiltonian operator H. The positivity of H spectrum indeed implies the existence of a state with the above property which moreover is translationally invariant:

$$e^{ia_\mu P^\mu}|0\rangle = |0\rangle \rightarrow P^\mu|0\rangle = 0. \tag{2.17}$$

The first statement (positivity $\Rightarrow$ vacuum) follows from the most general definition of the vacuum as a state that minimizes the energy. This minimum can be always adjusted to zero, even in the case of diverging zero-point energy (see the Appendix B) because the physically measurable energies are differences in which the infinite constant parts cancel.

One of the basic principles of relativistic quantum field theory assumes that the spectrum of physical states is contained in the forward light cone $P_\mu P^\mu \geq 0, P^0 \geq 0$ (see the Appendix A). To prove the statement discussed in the previous paragraph, one uses the translational invariance (2.17) of the assumed vacuum as well as the Heisenberg equation (2.15) of a scalar field $\phi(x)$, $-i\partial_\mu \phi(x) = [P_\mu, \phi(x)]$. This together with a general Fourier decomposition, valid for an arbitrary scalar field with a priori unknown time dependence

$$\phi(x) = \sum_p \frac{1}{\sqrt{4L\omega(p)}}\big[a(p,t)e^{ip^1x^1} + a^\dagger(p,t)e^{-ip^1x^1}\big] \tag{2.18}$$

implies

$$\begin{aligned}
-\,i \quad \textstyle\sum_k \quad & \frac{1}{\sqrt{4L\omega(p)}}\Big(\partial_0 a(p,t)e^{ip^1x^1} + \partial_0 a^\dagger(p,t)e^{-ip^1x^1}\Big) = \\
= \quad \textstyle\sum_p \quad & \frac{1}{\sqrt{4L\omega(p)}}\Big(\big[P^0, a(p,t)\big]e^{ip^1x^1} + \big[P^0, a^\dagger(p,t)\big]e^{-ip^1x^1}\Big). \qquad (2.19)
\end{aligned}$$

By comparing the coefficients at the plane waves, it follows from this equation that

$$\partial_0 a(p,t) = i\big[P^0, a(p,t)\big]. \tag{2.20}$$

To be able to proceed, we need to know the time dependence of the Fock operators a and $a^\dagger$, which is equivalent to solving the dynamics and thus generally impossible. The exception is a free theory, with the time dependence of the form $\exp(\pm i\omega(p)t), \omega(p) = (m^2 + p^2)^{1/2}$. Then after letting the equation (2.20) act on the state $|0\rangle$, we find

$$-a(p,t)P^0|0\rangle + P^0 a(p,t)|0\rangle = -\omega(p)a(p,t)|0\rangle. \tag{2.21}$$

Due to translational invariance of the vacuum $|0\rangle$, $\exp\left(ia_\mu P^\mu|0\right\rangle = |0\rangle$, i.e. $P^0|0\rangle = 0$, and positivity of P^0 eigenvalues, the only non-contradictory solution of the above relation (2.21) is $a(p,t)|0\rangle = 0$ which defines the vacuum state.

It is important to remember that this construction was done in a free theory. It is extremely difficult to repeat it in the case of an interacting theory because there the time dependence of the creation and annihilation operators is unknown (and even their form is unclear [24]) since its knowledge would be equivalent to knowing the solution to the coupled field equations. Thus to find the vacuum of an interacting theory is a complicated dynamical problem. As we will demonstrate in the next section, the situation in the LF theory is quite different.

Let us derive a solution of the Klein-Gordon equation in an alternative way, considering for a moment the continuum theory. This will turn out to be useful for the introduction of the so-called Pauli-Jordan commutator function and its relationship to the initial data specifying the time development of a dynamical system.

The general solution of the Klein-Gordon equation (2.1) in the covariant form is given by

$$\phi(x) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}^2 p}{2\pi}\delta(p^2 - \mu^2)\chi(p)e^{-ip.x}. \tag{2.22}$$

The two-dimensional integration measure is $\mathrm{d}p^0\mathrm{d}p^1$. Integrating over p^0 with the help of the identity $\delta(p^2 - \mu^2) = \frac{1}{2\omega(p)}\big[\delta\big(p^0 - \omega(p)\big) + \delta\big(p^0 + \omega(p)\big)\big]$, where $\omega(p) = +\sqrt{p^2 + \mu^2}$, we find

$$\phi(x) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}p}{4\pi\omega(p)}\Big[\chi\big(\omega(p), p\big)e^{-i\omega(p)x^0 + ipx} + \chi\big(-\omega(p), p\big)e^{i\omega(p)x^0 - ipx}\Big]. \tag{2.23}$$

For $t = 0$, this equation together with its time derivate represents a system of two equations for two amplitudes $\chi\big(\omega(p), p\big)$ and $\chi\big(-\omega(p), p\big)$:

$$\begin{aligned}
f(x) \quad &\equiv \quad \phi(0,x) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}p}{4\pi\omega(p)}\Big[\chi\big(\omega(p), p\big) + \chi\big(-\omega(p), p\big)\Big]e^{ipx}, \\
g(x) \quad &\equiv \quad \partial_0\phi(0,x) = -i\int_{-\infty}^{+\infty} \frac{\mathrm{d}p}{4\pi}\Big[\chi\big(\omega(p), p\big) - \chi\big(-\omega(p), p\big)\Big]e^{ipx}. \qquad (2.24)
\end{aligned}$$

Performing the inverse Fourier transformation, i.e. multiplying both sides by e^{-iqx} and integrating over x with $\int\limits_{-\infty}^{+\infty} \mathrm{d}x e^{i(p-q)x} = 2\pi\delta(p-q)$, one obtains

$$\chi\big(\omega(p),p\big) = \big[\omega(p)f(p) + ig(p)\big], \ \ \chi\big(-\omega(p),p\big) = \big[\omega(p)f(p) - ig(p)\big]. \tag{2.25}$$

These relations show that one indeed needs information about both the field and its time derivative at the initial surface $t = 0$ to know the field at arbitrary t. In quantum theory, $\chi\big(\omega(p),p\big)$ and $\chi\big(-\omega(p),p\big)$ are interpreted as annihilation and creation operators for a massive quantum with momentum p:

$$\phi(x) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}p}{4\pi\omega(p)} \Big[a(p)e^{-i\omega(p)x^0+ipx} + a^\dagger(p)e^{i\omega(p)x^0+ipx}\Big]. \tag{2.26}$$

With

$$\big[a(p), a^\dagger(q)\big] = 4\pi\omega(p)\delta(p-q) \tag{2.27}$$

we find

$$\Big[\phi(0,x), \partial_0\phi(0,y)\Big] = i\delta(x-y). \tag{2.28}$$

Let us derive the Pauli-Jordan commutator function. With the expansion (2.26), we have

$$\big[\phi(x),\phi(y)\big] = \frac{1}{4\pi}\int_{-\infty}^{+\infty} \frac{\mathrm{d}p^1}{\omega(p^1)} \big[e^{-ip.(x-y)} - e^{ip.(p-x)}\big], \tag{2.29}$$

where $p^0 = \omega(p^1)$. In the above integral, we have changed $p^1 \to -p^1$ in the second term. This results only in the reverse of sign in the exponent since the changes in $\mathrm{d}p^1$ and the integral limits compensate. Returning to the covariant form and combining the two terms with the help of the sign function $\epsilon(x^0)$, we get

$$\big[\phi(x),\phi(y)\big] = i\Delta(x-y;\mu), \tag{2.30}$$

where

$$\Delta(x-y;\mu) = -\frac{i}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}^2p e^{-ip.(x-y)}\delta(p^2-\mu^2)\epsilon(x^0). \tag{2.31}$$

Upon inspection, we can directly find the following properties of the $\Delta(x;\mu)$ function: it obeys the Klein-Gordon equation

$$\big(\partial_\mu\partial^\mu + \mu^2\big)\Delta(x-y;\mu) = 0 \tag{2.32}$$

with the initial conditions ($\epsilon(0) = 0$)

$$\Delta(0,x^1;\mu) = 0, \ \ \partial_0\Delta(x;\mu)_{x^0=0} = -\delta(x^1). \tag{2.33}$$

The latter condition follows directly from the representation (2.29). The Pauli-Jordan function also satisfies $\Delta(-x) = -\Delta(x)$ and the relation $\Delta(x-y) = 0$ for $(x-y)^2 < 0$ called locality or causality. The PJ function thus incorporates the important property of causality. Its vanishing of the PJ function outside the light cone, that is, for space-like separations $(x-y)^2 < 0$ of the points x, y, is the manifestation of the fact that for these intervals measurements of physical quantities at the two points are causally independent. This of course just reflects the finite value c of the speed of light (set to one here) which implies that no signal can connect such two points.

For the free theory, we can calculate additional quantities that are important for determining properties of the quantum field theoretical models generally. These are the correlation functions defined as the vacuum expectations of the (unordered) products of field operators in different space-time points. For example, we have for the two-point function

$$\begin{aligned} D(x-y) &= \langle 0|\phi(x)\phi(y)|0\rangle = \frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}p^1}{\omega(p^1)}\big[e^{-ip^0(x^0-y^0)+ip^1(x^1-y^1)} = \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}^2p e^{-ip.(x-y)}\delta(p^2-\mu^2). \end{aligned} \tag{2.34}$$

Its explicit form, calculated in the Appendix B, is

$$D(x) = -\frac{1}{4}\theta(x^2)\Big[N_0(\mu\sqrt{x^2}) + i\epsilon(x^0)J_0\big(\mu\sqrt{x^2}\big)\Big] + \theta(-x^2)\frac{1}{2\pi}K_0\big(\mu\sqrt{-x^2}\big), \tag{2.35}$$

where $J_0(x), K_0(x)$ and $N_0(x)$ are Bessel, modified Bessel and Neumann functions [25].

The last property of the theory of free massive scalars that we would like to discuss are inequivalent representations of the commutation relations (2.5). Although it may seem a little bit formal it actually has profound consequences for the very existence of the QFT models and their mathematical sensibility. We will see that this is one of the moments where the conventional SL theory and the LF theory differ drastically. Here we will sketch only the main steps of the derivation, the details can be found in the Appendix B.

Let us study two real scalar fields $\phi_1(x), \phi_2(x)$ (2.3) with masses μ_1 and μ_2. Let us assume that the Fock operator algebra (2.5) is satisfied separately for two species of the creation and annihilation operators which are independent, i.e. $[a_1(p), a_2(q)] = [a_1(p), a_2^\dagger(q)] = 0$, etc. Since the field equation (2.1) is second order in the time derivative, let us choose the boundary conditions at the initial time $t = 0$ as

$$\phi_1(0,x) = \phi_2(0,x), \;\; \partial_0\phi_1(0,x) = \partial_0\phi_2(0,x). \tag{2.36}$$

We then insert the field expansions (2.3) with the same momentum p into the relations (2.36). After some simple algebraic manipulations one finds two algebraic equations for four operators $a_1(p), a_1^\dagger(p), a_2(p), a_2^\dagger(p)$ where the simplified notation $\omega_1(p) \equiv \omega_1 = +\sqrt{p^2+\mu_1^2}, \omega_2(p) \equiv \omega_2 = +\sqrt{p^2+\mu_2^2}$ is used. Their solution expresses the annihilation and creation operators of the first scalar field as a linear combination of the Fock operators of the second field:

$$a_1(p) = \frac{\omega_1+\omega_2}{\sqrt{4\omega_1\omega_2}}a_2(p) + \frac{\omega_1-\omega_2}{\sqrt{4\omega_1\omega_2}}a_2^\dagger(-p) \equiv c_1(p)a_2(p) + c_2(p)a_2^\dagger(-p). \tag{2.37}$$

This relation permits us to express the vacuum of the first scalar field, defined as $a_1(p)|0_1\rangle = 0$, in terms of the vacuum of the second scalar field, defined as $a_2(p)|0_2\rangle = 0$ (the vacua are normalized: $\langle 0_1|0_1\rangle = \langle 0_2|0_2\rangle = 1$) in the following form:

$$|0_1\rangle = K\exp\Big(\sum_p c_3(p)a_2^\dagger(p)a_2^\dagger(-p)\Big)|0_2\rangle \equiv \hat{A}|0_2\rangle, \tag{2.38}$$

where $c_3(p)$ is a simple function and K is the normalization factor (see the Appendix B).

With this result and taking into account (2.37), one indeed verifies

$$a_1(p)|0_1\rangle = 0. \tag{2.39}$$

Thus we can see that the complicated exponential state (2.38) is indeed the vacuum state for the annihilation operator $a_1(p)$.

Now we are ready for an important statement: the vacua $|0_1\rangle$ and $|0_2\rangle$ are orthogonal in the continuum limit. Indeed, we find

$$\langle 0_2|0_1\rangle = \langle 0_2|\hat{A}|0_2\rangle = K, \tag{2.40}$$

because the creation operators in the exponent of $\hat{A}$ annihilate $|0\rangle$ when acting to the left so that only the factor 1 yields a non-zero contribution leading to $\langle 0_2|0_2\rangle = 1$. Now, using the relation $\sum_p \to \frac{2L}{2\pi}\int \mathrm{d}p$ ($2L$ is the "volume" of our one-dimensional space) for a transition from the finite to infinite volume, we get

$$K = \exp\Big\{\frac{L}{4\pi}\int \mathrm{d}p \ln\Big(1 - \frac{(\omega_2-\omega_1)^2}{(\omega_2+\omega_1)^2}\Big)\Big\}. \tag{2.41}$$

The integral is a negative number because its integrand is negative for $-\infty \le p \le \infty$ as it is equal to $-\ln\Big(\frac{1}{2}+\frac{1}{4}\big(\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}\big)\Big)$. Thus K will approach zero as $\exp(-cL)$, where c is a positive number. In other words, in the infinite-volume limit $L \to \infty$, the overlap between the two vacua as well as between arbitrary Fock states vanishes, i.e. the two Fock spaces become orthogonal. This means that there is no unitarity operator connecting these two spaces and one says that they are unitarily inequivalent.

2.2 Quantum Fermi field

The fermion field of the mass m obeys a two-dimensional version of the Dirac equation

$$\big(i\gamma^\mu\partial_\mu - m\big)\psi(x) = 0. \tag{2.42}$$

As in $(3+1)$ dimensions, it can be derived by a factorization of the Klein-Gordon operator $\partial_\mu\partial^\mu - m^2$ into two components with matrix coefficients which have to satisfy special anti-commutation properties to recover the original Klein-Gordon operator. In the two-dimensional space-time, there are two such 2×2 matrices $\gamma^\mu = (\gamma^0, \gamma^1)$ and they satisfy

$$\big\{\gamma^\mu, \gamma^\nu\big\} = 2g^{\mu\nu}\hat{1}. \tag{2.43}$$

Their concrete form is not unique and there exist a few possible representations, for example $\gamma^0 = \sigma_1, \gamma^1 = i\sigma_2$ (σ_1 and σ_2 are the Pauli matrices). More details are given in the Appendix B.

With the choice of the periodic boundary condition $\psi(t,-L) = \psi(t,L)$, the mode expansion of the Fermi field is written as

$$\psi(x) = \frac{1}{\sqrt{2L}} \sum_p \sqrt{\frac{m}{E(p)}} \Big[b(p)u(p)e^{-ip.x} + d^\dagger(p)v(p)e^{ip.x}\Big]. \tag{2.44}$$

As in the case of the scalar field, $E(p) = +\sqrt{p^2+m^2}$ and the momenta $p \equiv p_n$ are discrete, $p_n = \frac{2\pi}{L}n, n = 0, \pm1, \pm2, \ldots$. The coefficients $u(p)$ and $v(p)$ are the two dimensional "spinors" appropriate for the 2 by 2 γ-matrices. For the chosen representation of the gamma matrices, one can check that with the "spinors"

$$u(p) = \sqrt{\frac{E(p)+m}{2m}} \begin{pmatrix} 1 \\ \frac{p}{E(p)+m} \end{pmatrix}, \quad v(p) = \sqrt{\frac{E(p)+m}{2m}} \begin{pmatrix} \frac{p}{E(p)+m} \\ 1 \end{pmatrix} \tag{2.45}$$

the field $\psi(x)$ (2.44) indeed satisfies our Dirac equation. Some useful properties of the spinors $u(p), v(p)$ and their conjugates $\overline{u}(p) = u(p)^\dagger\gamma^0, \overline{v}(p) = v(p)^\dagger\gamma^0$ are listed in the Appendix B. In order to guarantee the necessary condition of the positivity of energy, the annihilation and creation operators for fermions $(b(p), b^\dagger(p))$ and antifermions $(d(p), d^\dagger(p))$ have to satisfy the anticommutation relations (see again the Appendix B)

$$\big\{b(p), b^\dagger(q)\big\} = \big\{d(p), d^\dagger(q)\big\} = \delta_{p,q}, \tag{2.46}$$

the other combinations being equal to zero. From this anticommutators and using the spinor relations (B.32) we directly find the equal-time anticommutators for the Fermi field

$$\big\{\psi(0,x), \psi^\dagger(0,y)\big\} = \delta_P(x-y), \tag{2.47}$$

with the periodic delta function defined in Eq.(2.7). In a complete analogy with the scalar field case, the Dirac equation (2.42) is obtained in the classical theory from the variational principle. The corresponding Lagrangian density is

$$\mathcal{L} = \frac{i}{2}\overline{\psi}\gamma^\mu \overleftrightarrow{\partial_\mu} \psi - m\overline{\psi}\psi = \frac{i}{2}\Big(\psi^\dagger \overleftrightarrow{\partial_0} \psi + \psi^\dagger\alpha^1 \overleftrightarrow{\partial_1} \psi\Big) - m\psi^\dagger\gamma^0\psi, \tag{2.48}$$

where $a \overleftrightarrow{\partial_\mu} b = a(\partial_\mu b) - (\partial_\mu a)b$ and $\overline{\psi} = \psi^\dagger\gamma^0$. The canonical momenta conjugate to the fields ψ and $\psi^\dagger$ are

$$\Pi_\psi(x) = \frac{\delta\mathcal{L}}{\delta\partial_0\psi} = \frac{i}{2}\psi^\dagger(x), \;\; \Pi_{\psi^\dagger}(x) = \frac{\delta\mathcal{L}}{\delta\partial_0\psi^\dagger} = -\frac{i}{2}\psi(x). \tag{2.49}$$

The formula (2.11) then leads to the Hamiltonian, momentum and the boost operators of the form

$$\begin{aligned} H &= \int\limits_{-L}^{+L} \mathrm{d}x \big[-i\psi^\dagger\alpha^1\partial_1\psi + m\psi^\dagger\gamma^0\psi\big], \\ P &= -i\int\limits_{-L}^{+L} \mathrm{d}x \psi^\dagger\partial_1\psi, \quad M^{01} = tH - \int\limits_{-L}^{+L} \mathrm{d}x x\psi^\dagger\partial_1\psi. \end{aligned} \tag{2.50}$$

Their form in the Fock representation is calculated in the Appendix B and reads

$$
\begin{aligned}
H &= \sum_p E(p)\big[b^\dagger(p)b(p) + d^\dagger(p)d(p)\big],\\
P &= \sum_p p\big[b^\dagger(p)b(p) + d^\dagger(p)d(p)\big].
\end{aligned} \tag{2.51}
$$

The fermionic vacuum is the state that minimizes the energy (which is bounded from below and its minimal value is conventionally adjusted to zero) and from this definition we directly have $b(p)|0\rangle = d(p)|0\rangle = 0$ analogously to the scalar field case. The translational invariance of the vacuum state is then expressed by the formula (2.17).

In a complete analogy with the scalar field, the Dirac equation is reproduced at the quantum level as the Heisenberg equation

$$
-i\partial^\mu\psi(x) = \big[P^\mu, \psi(x)\big]. \tag{2.52}
$$

This can be verified in a straighforward manner using the generators in the coordinate or momentum representation and the equal-time anticommutation relation (2.47) or (2.46).

The two-point correlation function is defined as

$$
S_{\alpha\beta}(x-y) = \langle 0|\psi_\alpha(x)\overline{\psi}_\beta(y)|0\rangle \tag{2.53}
$$

and the Pauli-Jordan commutator function is

$$
F_{\alpha\beta}(x-y) = 2\mathrm{Im}S_{\alpha\beta}(x-y) = \big\{\psi(x), \overline{\psi}(y)\big\}_{\alpha\beta}. \tag{2.54}
$$

An interesting question is again the inequivalent representations of the basic anticommutation relation (2.47). Let us consider two massive Fermi fields with masses m_1 and m_2. Since the Dirac equation is linear in the time derivative, we have to specify only one boundary condition:

$$
\psi_1(0,x) = \psi_2(0,x). \tag{2.55}
$$

Inserting the field expansion (2.44) into this relation and solving two algebraic equations, one finds

$$
\begin{aligned}
b_2(p) &= \alpha(p)b_1(p) + \beta(p)d_1^\dagger(-p),\\
d_2(p) &= \alpha(p)d_1(p) - \beta(p)b_1^\dagger(-p).
\end{aligned} \tag{2.56}
$$

The coefficients $\alpha(p), \beta(p)$ are equal to

$$
\begin{aligned}
\alpha(p) &= \sqrt{\frac{E_1+E_2}{2E_1}},\ \ \beta(p) = \sqrt{\frac{E_1-E_2}{2E_1}},\\
E_1 &= \sqrt{p^2+m_1^2},\ \ E_2 = \sqrt{p^2+m_2^2}.
\end{aligned} \tag{2.57}
$$

The vacua are again defined as

$$
\begin{aligned}
b_1(p)|0_1\rangle &= d_1(p)|0_1\rangle = 0,\\
b_2(p)|0_2\rangle &= d_2(p)|0_2\rangle = 0.
\end{aligned} \tag{2.58}
$$

It is easy to check now that the vacuum corresponding to the second Fermi field is expressed in terms of the creation operators of the first Fermi field as

$$|0_2\rangle = \prod_p \Big[\alpha(p) - \beta(p) b_1^\dagger(p) d_1^\dagger(-p)\Big] |0_1\rangle. \tag{2.59}$$

Analogously to the scalar field case, one finds that two vacua are orthogonal in the infinite volume limit $L \to \infty$: $\langle 0_1|0_2\rangle = 0$. The same property holds also for the Fock spaces corresponding to two Fermi fields with different masses.

3 Free massive light front fields in $\mathbf{D = 1 + 1}$

3.1 Light front definitions and notation

The light front formulation of field theory begins by the different choice of the time and space variables. The LF evolution parameter x^+ and the third space coordinate x^- are given as linear superpositions of the usual (non-relativistic) time t and the z coordinate:

$$\begin{aligned} x^\pm &= t \pm z, \;\; x^\mu = (x^+, x^-, x^1, x^2) \equiv (x^+, \underline{x}) \\ p^\pm &= E \pm p_z, \;\; p^\mu = (p^+, p^-, p^1, p^2) \equiv (p^-, \underline{p}). \end{aligned} \tag{3.1}$$

It is sometimes useful to use the notation $x_\perp = (x^1, x^2), p_\perp = (p^1, p^2)$. For two-dimensional theories, the perpendicular components $x^1 \equiv x, x^2 \equiv y, p^1 \equiv p_x, p^2 \equiv p_y$, which are the same as in the conventional field theory, are of course equal to zero.

The scalar product of two LF vectors is ($i = 1, 2$)

$$\begin{aligned} p.x = g^{\mu\nu} p_\mu x_\nu = p_\mu x^\mu &= p_+ x^+ + p_- x^- + p_i x^i = \\ &= \frac{1}{2} p^- x^+ + \frac{1}{2} p^+ x^- - p^1 x^1 - p^2 x^2. \end{aligned} \tag{3.2}$$

The off-diagonal form of the scalar product follows from the light front metric tensor $g^{\mu\nu}$ obtained from the metric tensor of the usual theory $\tilde{g}^{\mu\nu} = diag(1, -1, -1, -1)$:

$$p.x = p_r g_{rs} x_s \equiv pgx = \tilde{p} C^{-1} C \tilde{g} C^{-1} C \tilde{x}, \tag{3.3}$$

where we passed to the matrix notation and the matrix C_{rs} has been chosen in such a way that it transforms the usual four-vector (a column) $\tilde{a}^\mu$ into the LF four-vector a^μ according to $a = C\tilde{a}$. Explicitly, we find

$$C = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \;\; C^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \end{pmatrix}, \;\; CC^{-1} = 1. \tag{3.4}$$

The off-diagonal form of the LF metric tensor is responsible also for the following relations:

$$a^\pm = 2a_\mp, \;\; \frac{\partial}{\partial x^\pm} \equiv \partial_\pm = \frac{1}{2}\partial^\mp \equiv \frac{1}{2}\frac{\partial}{\partial x_\mp}. \tag{3.5}$$

Since for a free massive quantum (particle) its energy $E(p) = +\sqrt{m^2 + p_1^2 + p_2^2 + p_3^2}$ is always greater then the absolute value of the p_3 component, the LF momentum p^+ is always positive. In other words, in the LF form of the relativistic dynamics we have two quantities that take on only positive values (in quantum theory, two operators with a positive spectrum of eigenvalues). One of these quantities that are bounded from below is kinematical, i.e. interaction-independent, and this feature eventually leads to a possibility to define the vacuum irrespectively of the dynamics as a state with minimum LF momentum p^+. In contrast, the third component of the momentum can be negative in the conventional field theory, i.e. it is not bounded from below and the only quantity with a positive spectrum of eigenvalues is the energy which is however unknown in the interacting theory before we solve it. Thus, probably the main advantage of the LF quantization is the possibility to get the vacuum state of any given model without the need to solve its dynamics. The other advantages which will be discussed in a more detail in the subsequent chapters, is the possibility to write down a consistent Fock expansion for a composite system (this property is also related to the kinematical definition of the vacuum state, i.e. to the absence of vacuum fluctuations) and the minimal number - 3 - of Poincaré generators containing interaction, among all three main types of relativistic Hamiltonian dynamics.

There is another simplification in the LF scheme in comparison with the conventional theory. It is the dispersion relation for a free massive quantum, following from $p_\mu p^\mu = m^2$. It reads

$$p^- = \frac{m^2 + p_\perp^2}{p^+} \tag{3.6}$$

and shows that there is no sign ambiguity between the LF energy and momentum. Thus, for positive LF momentum we have only positive energy. The negative- energy solutions of the field equations are conventionally reinterpreted as antiparticles with positive energy and hence we are dealing with only positive values of these two quantities in the LF scheme, in contrast to the usual scheme, where for the given momentum the energy is $\pm\sqrt{\vec{p}^{\,2} + m^2}$.

3.2 Quantum scalar field

The massive light front scalar field obeys the Klein-Gordon equation which has the same covariant form as the Eq.(2.1). However, written explicitly in terms of the LF variables, the D'Alambert operator changes its structure and becomes linear in the LF time derivative:

$$\big(4\partial_+\partial_- + \mu^2\big)\phi(x) = 0. \tag{3.7}$$

Considered as a classical differential equation, this means that it is sufficient to specify only a value of the field itself at some initial time x_0^+. Analogously, in quantum theory, one needs to know only the value of the field, not its time derivative, to prescribe the equal-time commutation relation, which plays the role of initial data for the commutator at an arbitrary time. The latter quantity is the Pauli-Jordan or the Schwinger function that we already discussed. All this becomes evident when one writes down the Lagrangian (2.8)

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2$$

in terms of LF variables:

$$\mathcal{L}_{lf} = 2\partial_+\phi(x)\partial_-\phi(x) + \frac{1}{2}\mu^2\phi^2(x). \tag{3.8}$$

Again, since the Lagrangian is linear in the time derivative of the field, the canonical momentum is the *spatial* derivative, not the time derivative of the field:

$$\Pi_\phi = \frac{\delta\mathcal{L}_{lf}}{\delta\partial_+\phi} = 2\partial_-\phi(x). \tag{3.9}$$

Then the canonical commutation relation $[\phi(x), \Pi_\phi(y)]$ is for $x^+ = y^+ = 0$ proportional to the Dirac delta function $\delta(x^- - y^-)$ and can be integrated to yield the commutator between two scalar fields proportional to the integral of the delta function which is essentially the sign function $\frac{x}{|x|}$ (see below for more details).

In a complete analogy with the conventional field theory discussed in the previous sections, we will consider the system in a spatial (one-dimensional) box, $-L \leq x^- \leq L$. We impose the periodic boundary condition $\phi(-L) = \phi(L)$. The periodicity condition applied to the plane waves yields $e^{\frac{i}{2}p^+L} = e^{-\frac{i}{2}p^+L}$ leading to the discrete momenta $p^+ = p^+_n = 2\pi n/L$. Then the solution of the massive Klein-Gordon equation is given as a superposition of plane waves with the coefficients that incorporate the quantum nature of the field by their non-trivial commutation relations:

$$\begin{aligned}\phi(x^+, x^-) \quad &= \quad \phi_0 + \frac{1}{\sqrt{2L}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{p^+_n}}\Big[a(p^+_n)e^{-ip.x} + a^\dagger(p^+_n, x^+)e^{ip.x}\Big], \\ &\quad [a(p^+_m), a^\dagger(q^+_n)] = \delta_{mn},\end{aligned} \tag{3.10}$$

where $p.x = \frac{1}{2}p^+x^- + \frac{1}{2}\hat{p}^-x^+, \hat{p}^- = \mu^2/p^+$. In the expansion (3.10), we have separated the $n = 0$ Fourier mode corresponding to $p^+_n = 0$, the so-called LF zero mode which is x^-−independent, from the sum of the *normal* modes which have $p^+_n \neq 0$. The latter will be denoted by $\varphi(x)$. Then the relation (3.9) and the equation of motion (3.7) holds only for $\varphi(x)$, because by definition $\partial_-\phi_0 = 0$: $\Pi_\varphi(x) = 2\partial_-\varphi(x), \Pi_{\phi_0} = \partial_-\phi_0 = 0$. The equation for the zero mode following from (3.7) is $\mu^2\phi_0 = 0$ which tells us that in the free massive theory the scalar zero mode vanishes. This will no longer be true in an interacting theory like for example the $\lambda\phi^4$ model or the Yukawa model which we will discuss later. The fact that the canonical momentum of the LF scalar field is not given by the LF time derivative of the field is supplemented by the observation that the equation of motion (3.7) is actually a constraint for $\partial_+\varphi(x)$ which can be easily inverted by means of the normal-mode part of the periodic sign function $\epsilon_N(x^- - y^-)$ (see below):

$$\partial_+\varphi(x) = -\frac{\mu^2}{4}\int_{-L}^{+L}\frac{\mathrm{d}y^-}{2}\frac{1}{2}\epsilon_N(x^- - y^-)\varphi(x^+, y^-). \tag{3.11}$$

The time dependence of the creation and annihilation operators $a^\dagger(p^+_n, x^+)$ and $a(p^+_n, x^+)$ is obtained by inserting the expansion (3.10) into the field equation (3.7) yielding the oscillatory behaviour $e^{\pm\frac{i}{2}\hat{p}^-_n x^+}$, where $\hat{p}^-_n \equiv \mu^2/p^+_n$. The precise form of the equal-time commutator can be found in the simplest way from the Fock commutation relation in (3.10):

$$\begin{aligned}
\big[\varphi(0,x^-),\varphi(0,y^-)\big] &= \frac{1}{2L}\sum_{m,n=1}^{\infty}\frac{1}{\sqrt{p^+_m p^+_n}}\Big\{\big[a(p^+_m),a^\dagger(p^+_n)\big]e^{-\frac{i}{2}p^+_m x^- + \frac{i}{2}p^+_n y^-} + \\
&+ \big[a^\dagger(p^+_m),a(p^+_n)\big]e^{\frac{i}{2}p^+_m x^- - \frac{i}{2}p^+_n y^-}\Big\} = \\
&= \frac{1}{2L}\sum_n \frac{1}{p^+_n}\big[e^{-\frac{i}{2}p^+_n(x^- - y^-)} - e^{\frac{i}{2}p^+_n(x^- - y^-)}\\
&= \frac{1}{8i}\epsilon_N(x^- - y^-). \qquad (3.12)
\end{aligned}$$

Here, we have used the symbol $\epsilon_N(x^-)$ for the normal-mode part of the sign function (which in the continuum theory is given by $2\theta(x)-1$, $\theta(x) = 1(0)$ for positive (negative) x) adapted to the finite interval and periodic boundary conditions, $\epsilon_N(-L) = \epsilon_N(L) = 0$. It has a Fourier-series representation

$$\epsilon_N(x^- - y^-) = \frac{4i}{L}\sum_n \frac{1}{p^+_n}\big[e^{-\frac{i}{2}p^+_n(x^- - y^-)} - e^{\frac{i}{2}p^+_n(x^- - y^-)}\big]. \qquad (3.13)$$

The full periodic sign function is $\epsilon_P(z^-) = z^-/L + \epsilon_N(z^-)$ and satisfies the defining property $\partial_-\epsilon_P(z^-) = 2\delta_P(z^-)$, where

$$\delta_P(x^-) = \frac{1}{L} + \frac{1}{L}\sum_n \big[e^{-\frac{i}{2}p^+_n x^-} + e^{\frac{i}{2}p^+_n x^-}\big] \equiv \delta_0 + \delta_N(x^-). \qquad (3.14)$$

The canonical formalism for the LF theory is completely paralel to the formalism of the conventional theory described previously. However, we would like to emphasize that in addition to replacing the Lorentz indices $0, 3$ by the "LF values" $+, -$ one has to keep in mind that the structure of the theory has been changed: we are dealing with field equations different from the mathematical point of view, some of them being constraints that effectively reduce number of dynamically independent field variables. Correspondingly, LF fields are initialized on a different surface (usually chosen as $x^+ = 0$), the vacuum states have different properties and the Fock operators create and destroy different excitations [4].

Thus, the LF energy-momentum tensor is constructed according to the formula (2.11) with the indices taking on the values $+, -$ $(+, -, 1, 2$ in four dimensions). In particular, the LF energy density is

$$T^{+-}(x) = \Pi_\phi(x)\partial^-\phi(x) - g^{+-}\mathcal{L}_{lf}(x). \qquad (3.15)$$

In this way, one arrives at the LF Hamiltonian

$$P^- = \int_{-L}^{+L}\frac{\mathrm{d}x^-}{2}T^{+-}(x) = \mu^2\int_{-L}^{+L}\frac{\mathrm{d}x^-}{2}\phi^2(x). \qquad (3.16)$$

[4]One could expect existence of a unitary operator providing a connection between these two representations of the physical reality. The only attempt known to the author is a perturbative analysis [26] in the case of the Yukawa model where it has been demonstrated that an operator creating a single particle in the LF scheme appears as a complicated superposition of Fock operators of the conventional theory, the same being true also for the vacuum state.

Note that there is no kinetic term in P^-, just (minus twice) the potential part of the Lagrangian. In a similar way, we find the LF momentum and boost operators:

$$P^+ = \int_{-L}^{+L} \frac{\mathrm{d}x^-}{2} T^{++}(x) = 4 \int_{-L}^{+L} \frac{\mathrm{d}x^-}{2} \partial_- \phi \partial_- \phi,$$

$$M^{+-} = x^+ P^- - \int_{-L}^{+L} \frac{\mathrm{d}x^-}{2} x^- T^{++}(x^-). \tag{3.17}$$

The Fock representation follows in a straightforward way by inserting the field expansion (3.10) into the $P^\pm$ given in (3.16, 3.17):

$$P^- = \sum_n \frac{\mu^2}{p_n^+} a^\dagger(p_n^+) a(p_n^+), \;\; P^+ = \sum_n p_n^+ a^\dagger(p_n^+) a(p_n^+). \tag{3.18}$$

Next, let us consider the vacuum state in the scalar LF field theory [7, 5, 27]. There is one important difference with respect to the usual space-like QFT: in addition to the positivity of the LF energy P^- we also have positive definite spectrum of the kinematical quantity - the LF momentum P^+. From a general Fourier decomposition of an *interacting* LF scalar field

$$\varphi(x^+, x^-) = \frac{1}{\sqrt{2L}} \sum_{n=1}^{\infty} \frac{1}{p_n^+} \Big(a(p_n^+, x^+) e^{-\frac{i}{2} p_n^+ x^-} + a^\dagger(p_n^+, x^+) e^{\frac{i}{2} p_n^+ x^-} \Big) \tag{3.19}$$

and the non-dynamical Heisenberg equation $-2i\partial_- \phi(x) = [P^+, \phi(x)]$ we get in an analogy with the conventional field theory

$$\begin{aligned} -2i \quad & \frac{1}{\sqrt{2L}} \sum_{n=1}^{\infty} \frac{1}{p_n^+} \Big(a(p_n^+, x^+) \partial_- e^{-\frac{i}{2} p_n^+ x^-} + a^\dagger(p_n^+, x^+) \partial_- e^{\frac{i}{2} p_n^+ x^-} \Big) = \\ = \quad & \frac{1}{\sqrt{2L}} \sum_{n=1}^{\infty} \frac{1}{p_n^+} \Big(\big[P^+, a(p_n^+, x^+)\big] e^{-\frac{i}{2} p_n^+ x^-} + \big[P^+, a^\dagger(p_n^+, x^+)\big] e^{\frac{i}{2} p_n^+ x^-} \Big) \end{aligned} \tag{3.20}$$

and consequently also

$$-p_n^+ a(p_n^+, x^+) = \big[P^+, a(p_n^+, x^+)\big] \tag{3.21}$$

since instead of the time derivative acting on the annihilation operator (the situation in the case of conventional field theory, Eq.(2.20)), here the spatial derivative acts on the kinematical part of the plane wave. With $P^+|0\rangle = 0$ we immediately have

$$-a(p_n^+, x^+) P^+ |0\rangle + P^+ a(p_n^+, x^+) |0\rangle = -p_n^+ a(p_n^+, x^+) |0\rangle \Rightarrow a(p_n^+, x^+)|0\rangle = 0. \tag{3.22}$$

Thus, $|0\rangle$ is a state of an interacting theory with minimum possible momentum $p^+ = 0$. It is at the same time a state with minimum possible LF energy p^-, because the vacuum $|0\rangle$ is annihilated also by P^- due to the fact that the latter does not contain terms of the form

$a^\dagger(p_1^+)a^\dagger(p_2^+)a^\dagger(p_3^+)a^\dagger(p_4^+)$ (and their hermite conjugates). This again follows from the positivity of P^+ as well as from its conservation: these terms, after space integration of their plane-wave factors in the Hamiltonian, are multiplied by a Dirac delta function of the form $\delta(p_1^+ + p_2^+ + p_3^+ + p_4^+)$ which is zero because positive momenta cannot add to zero. The state without particles, the Fock vacuum $|0\rangle$ is the physical vacuum ! This is a most remarkable result. But also a troublesome one: where has the complicated vacuum structure gone? We will try to answer this question, at least partially, in the following chapters.

3.3 Quantum LF Fermi field

The Lagrangian of a free massive fermion field

$$\mathcal{L} = \frac{i}{2}\overline{\psi}\gamma^\mu \overleftrightarrow{\partial_\mu} \psi - m\overline{\psi}\psi \tag{3.23}$$

takes in terms of the LF fermionic field components ψ_2 and ψ_1 the form

$$\mathcal{L}_{lf} = i\psi_2^\dagger \overleftrightarrow{\partial_+} \psi_2 + i\psi_1^\dagger \overleftrightarrow{\partial_-} \psi_1 - m(\psi_2^\dagger\psi_1 + \psi_1^\dagger\psi_2). \tag{3.24}$$

We are using the chiral representation where

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \;\; \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{3.25}$$

The other Dirac matrices are

$$\gamma^0 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \;\; \gamma^1 = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.26}$$

A very useful feature of the LF fermionic fields is the projection-operator interpretation of the algebra of Dirac γ matrices:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\hat{1} \Rightarrow \{\gamma^+\gamma^- + \gamma^-\gamma^+\} = 4\hat{1}, \tag{3.27}$$

where $\gamma^\pm = \gamma^0 \pm \gamma^1$. We see that the matrices

$$\Lambda_+ = \frac{1}{4}\gamma^-\gamma^+, \;\; \Lambda = \frac{1}{4}\gamma^+\gamma^- \tag{3.28}$$

have the properties of projection operators:

$$\Lambda_+ + \Lambda_- = \hat{1}, \;\; \Lambda_\pm^2 = \Lambda_\pm, \;\; \Lambda_\pm\Lambda_\mp = 0. \tag{3.29}$$

Writing the matrices explicitly, we have

$$\gamma^+ = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \;\; \gamma^- = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \;\; \Lambda_+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \;\; \Lambda_- = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.30}$$

Using these relations in the covariant Lagrangian (3.23) we find that the projectors indeed separate the upper and lower fermion field components as shown in (3.24). Also the Dirac equation

$$\big(i\gamma^\mu\partial_\mu - m\big)\psi(x) = 0 \tag{3.31}$$

separates into two equations

$$2i\partial_+\psi_2(x) = m\psi_1(x), \;\; 2i\partial_-\psi_1(x) = m\psi_2(x). \tag{3.32}$$

The first equation contains a time derivative and thus it is the true dynamical equation for the ψ_2 Fermi field component. The second one is non-dynamical, i.e. it is a constraint expressing the upper Fermi field component ψ_1 in terms of the dynamical component ψ_2. The advantage of using the chiral representation of the γ-matrices is that in this representation the dynamical component is simply the lower entry of the "spinor" in Eq.(3.25) and the dependent component is the upper entry. In the representation that we used in the case of massive space-like fermions the dynamical and constrained Fermi field components are given as a superposition of the upper and lower "spinor" components. Here we would also like to contrast the description of the fermionic fields in both schemes. We can see that the matrix and "spinor" structure plays only a marginal role in the LF case, simplifying the picture considerably, while the matrix algebra in the case of the space-like Fermi field is in principle as complicated as in 3+1 dimensions (technically, it is of course simpler because the matrices are only 2 by 2). One may say that the LF description is more adequate because there is no spin in one space dimension (no rotations are possible) and the whole machinery of the spinors and matrices seems to be redundant. The only feature (in addition to Fermi statistics incorporated by using anticommutators instead of commutators) that really distinguishes the two-dimensional fermionic field from the scalar field is that the former one has two components described by the "square root" of the Klein-Gordon equation and this is correctly incorporated in the LF formalism. The fact that the LF Fermi field is "almost" a scalar field becomes also evident if one combines two LF Dirac equations into one by eliminating ψ_1 from its constraint

$$\psi_1(x) = \frac{m}{2i}\partial_-^{-1}\psi_2(x), \tag{3.33}$$

and inserting it into the dynamical equation (the precise form of the symbolic inverse derivative ∂_-^{-1}will be given below). This leads to the Klein-Gordon type of equation for the independent component:

$$\big(4\partial_+\partial_- + m^2\big)\psi_2(x) = 0. \tag{3.34}$$

The dynamical field $\psi_2(x)$ can be expanded at $x^+ = 0$ into the Fourier series:

$$\psi_2(0,x^-) = \frac{1}{\sqrt{2L}}\sum_{n=\frac{1}{2}}^{\infty}\big[b(p_n^+)e^{-\frac{i}{2}p_n^+x^-} + d^\dagger(p_n^+)e^{\frac{i}{2}p_n^+x^-}\big]. \tag{3.35}$$

The Fock operators are assumed to satisfy the anticommutation relations

$$\big\{b(p_m^+), b^\dagger(p_n^+)\big\} = \big\{d(p_m^+), d^\dagger(p_n^+)\big\} = \delta_{mn}, \tag{3.36}$$

the other combinations being equal to zero. As it is evident, we again consider a system in a spatial "box" (a finite line $-L \leq x^- \leq L$ in one spatial dimension) and impose *antiperiodic* boundary conditions (BC) $\psi_2(-L) = -\psi_2(L)$ which lead to a discrete set of momenta $p_n^+ = 2\pi n/L$ labeled by half-integers $1/2, 3/2, \ldots$. The reason for the choice of antiperiodic boundary condition is that it avoids the fermionic zero mode in interacting theories where its treatment is a little bit tricky. In the free theory with periodic BC, the ZM vanishes. [5]

With the antiperiodic boundary condition, the inverse derivative is determined uniquely as the integral operation:

$$\psi_1(x) = m \int_{-L}^{+L} \frac{\mathrm{d}y^-}{2} G_a(x^- - y^-)\psi_2(x^+, y^-), \;\; G_a(x^- - y^-) = \frac{1}{4i}\epsilon_a(x^- - y^-). \quad (3.37)$$

The symbol $\epsilon_a(x^-)$ stands for the antiperiodic counterpart of the periodic sign function $\epsilon_P(x^-)$ from the previous Section. Its form is identical to (3.13) with the index running over half-integers instead of integers:

$$\epsilon_a(x^- - y^-) = \frac{4i}{L} \sum_{n=\frac{1}{2}} \frac{1}{p_n^+} \left[e^{-\frac{i}{2}p_n^+(x^- - y^-)} - e^{\frac{i}{2}p_n^+(x^- - y^-)} \right]. \quad (3.38)$$

The Green function $G_a(x^- - y^-)$ that we used to invert the fermionic constraint satisfies the defining property $2i\partial_- G_a(x^- - y^-) = \delta_a(x^- - y^-)$. When applied to Eq.(3.37), one recovers the initial constraint. It is an elegant feature of the LF formalism that for some simple interacting theories like massive QED(1+1) (massive Schwinger model), massive Thirring model (four-fermion current-current interaction) and even four-dimensional quantum electrodynamics, the fermionic constraint is still solvable in a closed form although it contains interacting terms. Its solution will turn out to be given in terms of a Green function generalizing $G_a(x^- - y^-)$ in a natural way.

It is quite simple to obtain $\psi_1(x)$ in Fock representation. Combining Eqs.(3.38) and (3.35) in (3.37), we find

$$\psi_1(0, x^-) = \frac{m}{\sqrt{2L}} \sum_n \frac{1}{p_n^+} \left[b(p_n^+) e^{-\frac{i}{2}p_n^+ x^-} - d^\dagger(p_n^+) e^{\frac{i}{2}p_n^+ x^-} \right]. \quad (3.39)$$

The passage to the Hamilton formalism follows the procedure sketched for the case of the scalar field. The canonical momenta are

$$\Pi_{\psi_2} = i\psi_2^\dagger, \;\; \Pi_{\psi_2^\dagger} = -i\psi_2 \quad (3.40)$$

and the energy-momentum tensor

$$T^{+\nu}(x) = \Pi_{\psi_2}(x)\partial^\nu\psi(x) + \partial^\nu\psi^\dagger(x)\Pi_{\psi_2^\dagger}(x) - g^{+\nu}\mathcal{L}_{lf}(x) \quad (3.41)$$

analogous to Eq.(3.15) yields the LF Hamiltonian and the momentum operator:

$$P^- = m \int_{-L}^{+L} \frac{\mathrm{d}x^-}{2} \Big(\psi_2^\dagger\psi_1 + \psi_1^\dagger\psi_2 \Big), \;\; P^+ = \int_{-L}^{+L} \frac{\mathrm{d}x^-}{2} 2i\psi_2 \overleftrightarrow{\partial_-} \psi_2. \quad (3.42)$$

[5] For the periodic BC, one can decompose ψ_2 into the mode with $n = 0$ and the set of normal modes with $p_n^+ \neq 0$. The ZM is isolated by the "volume" integration: $\psi_0 = \int_{-L}^{+L} \frac{dx^-}{2L} \psi_2(x)$. Eq.(3.34) yields $0 = m^2\psi_0$ showing that for a massive field $\psi_0 = 0$ analogously to the scalar field case.

The Fock representation of these operators reads

$$
\begin{aligned}
P^- &= \sum_n \frac{m^2}{p_n^+}\Big(b^\dagger(p_n^+)b(p_n^+) + d^\dagger(p_n^+)d(p_n^+)\Big), \\
P^+ &= \sum_n p_n^+\Big(b^\dagger(p_n^+)b(p_n^+) + d^\dagger(p_n^+)d(p_n^+)\Big).
\end{aligned} \tag{3.43}
$$

We conclude this section by a simple observation: as in the case of two LF scalar fields, also the mode expansions of two LF Fermi fields with different masses coincide (see Eq. (3.35)), since the mass parameter does not enter into the expansion. This is true also in the continuum formulation where the integration measure is simply (suitably regularized) $\mathrm{d}p^+/p^+$ (it comes from p^- integration of the δ-function $\delta(p_\mu p^\mu - m^2)$ which in terms of the LF variables has the form $\delta(p^+p^- - m^2)$.) Thus the vacua of two fermion fields with different masses coincide in the light front description.

4 Some mathematical subtleties in the continuum LF theory

In the previous chapters, we showed several times that the LF field theory requires specification of boundary conditions in order to have a mathematically well-defined treatment, in particular unique solutions of certain constrained equations. It seems probable that the quantization in a finite spatial volume is then the most appropriate formulation which yields a regularized theory and permits us to analyze consistently the infrared region of the theory in terms of zero modes and certain unitary operators that are well defined in a finite volume. Although some authors [28] have emphasized a necessity to incorporate boundary conditions also in the case of the continuum LF quantization for a consistent treatment, this issue is quite often neglected. It is worth to recall at this place that the continuum field theory is typically applied in a heuristic manner and its mathematically correct treatment actually requires to regard quantum fields as distributions which have to be regularized by means of test functions. This axiomatic or constructive field theoretical approach was developed by Wightman, Jaffe, Haag, Schroer, Strocchi and many others. Although being certainly very valuable from the point of view of mathematical rigour, it was very difficult to apply it for calculations of real physical problems and predictions outside the realm of simple lower-dimensional models. [6]

We shall adhere to the finite-volume (sometimes called discretized) approach to the LF theory also in the rest of these notes. However, we will switch to the continuum formulation in some specific topics. In this section, we want to illustrate one source of difficulties in the LF theory caused by an improper mathematical treatment. We shall work within the continuum form for simplicity since the corresponding formulae involve integrals and continuous functions (Bessel functions) while in the discretized form one works with infinite series which are more difficult to handle. However, some conclusions from the continuum formulation are relevant also for the discretized one.

The main objects here will be the two-point correlation functions which are vacuum expectation values of products of two fields taken at different space-time points. These products are unordered in the time variable, i.e. no T-ordering is applied. They are also called the Wightman

[6] A new promising LF approach based on the Epstein-Glaser non-perturbative regularization and renormalization has been formulated recently and applied also to higher-dimensional theories [29].

functions. Since we will work in an infinite volume, we will present the continuum versions of the field expansions discussed already in the finite-volume form. The conventions, definitions and field equations will coincide with those presented in the previous chapters.

Let us start with the massive LF Fermion field. The independent component ψ_2 can be expanded at $x^+ = 0$ into the Fourier integral with the operator coefficients $b(p^+), d(p^+)$ and their hermite conjugates. The dynamical equation from (3.32) determines its LF time evolution:

$$\psi_2(x) = \int_0^\infty \frac{dp^+}{4\pi\sqrt{p^+}}\big[b(p^+)e^{-\frac{i}{2}p^+x^- - \frac{i}{2}\frac{m^2}{p^+}x^+} + d^\dagger(p^+)e^{\frac{i}{2}p^+x^- + \frac{i}{2}\frac{m^2}{p^+}x^+}\big]. \tag{4.1}$$

The solution of the constraint equation is

$$\psi_1(x) = m\int_0^\infty \frac{dp^+}{4\pi\sqrt{p^+}p^+}\big[b(p^+)e^{-\frac{i}{2}p^+x^- - \frac{i}{2}\frac{m^2}{p^+}x^+} - d^\dagger(p^+)e^{\frac{i}{2}p^+x^- + \frac{i}{2}\frac{m^2}{p^+}x^+}\big]. \tag{4.2}$$

The quantization prescription

$$\big\{\psi_2(0,x^-), \psi_2^\dagger(0,y^-)\big\} = \frac{1}{2}\delta(x^- - y^-) \tag{4.3}$$

is equivalent to the following anticommutation relations for the Fock operators:

$$\big\{b(p^+), b^\dagger(q^+)\big\} = \big\{d(p^+), d^\dagger(q^+)\big\} = 2\pi p^+\delta(p^+ - q^+). \tag{4.4}$$

The two-point Wightman functions $S_{\alpha\beta}(x-y)$ are easily obtained as

$$S_{22}(x-y) = \langle 0|\psi_2(x)\psi_2^\dagger(y)|0\rangle = \int_0^\infty \frac{dp^+}{8\pi}e^{-\frac{i}{2}p^+(x^- - y^- - i\epsilon) - \frac{i}{2}\frac{m^2}{p^+}(x^+ - y^+ - i\delta)},$$

$$S_{11}(x-y) = \langle 0|\psi_1(x)\psi_1^\dagger(y)|0\rangle = \int_0^\infty \frac{dp^+}{8\pi}\frac{m^2}{p^{+2}}e^{-\frac{i}{2}p^+(x^- - y^- - i\epsilon) - \frac{i}{2}\frac{m^2}{p^+}(x^+ - y^+ - i\delta)},$$

$$S_{12}(x-y) = \langle 0|\psi_1(x)\psi_2^\dagger(y)|0\rangle = \int_0^\infty \frac{dp^+}{8\pi}\frac{m}{p^+}e^{-\frac{i}{2}p^+(x^- - y^- - i\epsilon) - \frac{i}{2}\frac{m^2}{p^+}(x^+ - y^+ - i\delta)}. \tag{4.5}$$

Note that we have introduced the small imaginary parts in time and space coordinates. This step is dictated by the mathematical consistency. Without the damping factors the integrals would not exist as mathematical objects. [7] The corresponding formulae and references to the relevant mathematical literature can be found for example in the tables of integrals by Gradshteyn and Ryzhik [25]. It will turn out that this mathematical subtlety has quite remarkable consequences for more physical aspects of the theory.

[7] It is possible to interpret the corresponding damping exponential functions as a specific choice of the test functions [30].

In order to evaluate the correlation functions S_{22}, S_{11} and S_{12} , one has to consider four combinations of the signs of $x^+ - y^+, x^- - y^-$ because the results of the integrations are different for different combinations [25]. We also have to change the sign of $i\epsilon$ accordingly to guarantee the exponential damping of the form $\exp(-\epsilon p^+)$. The results can be read-off directly from [25] and are summarized as follows:

$$\begin{aligned}
S_{22}(z) = &\quad - \quad \theta\big(z^2\big)\frac{m}{8}\sqrt{\frac{z^+}{z^-}}\Big[J_1\big(m\sqrt{z^2}\big) - i\,\mathrm{sgn}(z^+)N_1\big(m\sqrt{z^2}\big)\Big] + \\
&\quad + \quad \theta\big(-z^2\big)\mathrm{sgn}(z^+)\frac{im}{4\pi}\sqrt{-\frac{z^+}{z^-}}K_1\big(m\sqrt{-z^2}\big), \\
S_{11}(z) = &\quad \quad \theta\big(z^2\big)\frac{m}{8}\sqrt{\frac{z^-}{z^+}}\Big[J_1\big(m\sqrt{z^2}\big) - i\,\mathrm{sgn}(z^+)N_1\big(m\sqrt{z^2}\big)\Big] - \\
&\quad - \quad \theta\big(-z^2\big)\mathrm{sgn}(z^+)\frac{im}{4\pi}\sqrt{-\frac{z^-}{z^+}}K_1\big(m\sqrt{-z^2}\big), \\
S_{12}(z) = &\quad - \quad \theta\big(z^2\big)\frac{m}{8}\Big[N_0\big(m\sqrt{z^2}\big) + i\,\mathrm{sgn}(z^+)J_0\big(m\sqrt{z^2}\big)\Big] + \\
&\quad + \quad \theta\big(-z^2\big)\frac{m}{4\pi}K_0\big(m\sqrt{-z^2}\big)
\end{aligned} \tag{4.6}$$

We have set $z = x - y$ and used the "function" $\mathrm{sgn}(z^\pm)$ for the sign function $z^\pm/|z^\pm|$. Note however that the latter has only a symbolical meaning here since we do not require the property $\mathrm{sgn}(0) = 0$. The $x^+ = 0$ limit of the above expressions has to be calculated separately. The implicit small imaginary parts in the arguments with appropriate sign are crucial for this step. Let us also remind that $J_1(z), K_1(z)$ and $N_1(z)$ are the Bessel, modified Bessel and Neumann functions. Note that the calculation of the analogous correlation functions in the conventional theory is more complicated and requires a clever change of variables [31] (see the Appendix B).

Next we will study the equal-time limit of the S-functions. This is a self-consistency check since in that limit $S_{22}(x)$ should reduce to the Eq.(4.3) and S_{12} to the anticommutator

$$\big\{\psi_1(0, x^-), \psi_2^\dagger(0, y^-)\big\} = \frac{m}{8i}\mathrm{sgn}(x^- - y^-) \tag{4.7}$$

which is easily computed from the constraint (4.2) and the basic anticommutator (4.3). We will consider the fields at space-like separations and choose $x^+ - y^+ < 0, x^- - y^- > 0$ for definiteness. With all factors explicitly shown, we have

$$\begin{aligned}
&\Delta(x-y) \equiv \Big\{\psi_2(x), \psi_2^\dagger(y)\Big\} = S_{22}(x-y) + S_{22}^*(x-y) = \\
&= -\frac{im}{4\pi}\Bigg[\sqrt{\frac{|x^+ - y^+| + i\epsilon}{x^- - y^- - i\epsilon}}K_1\Big(m\sqrt{(|x^+ - y^+| + i\epsilon)(x^- - y^- - i\epsilon)}\Big) - \\
&\qquad - \sqrt{\frac{|x^+ - y^+| - i\epsilon}{x^- - y^- + i\epsilon}}K_1\Big(m\sqrt{(|x^+ - y^+| - i\epsilon)(x^- - y^- + i\epsilon)}\Big)\Bigg].
\end{aligned} \tag{4.8}$$

For $-(x^+ - y^+) = \eta << 1$, we can use the expansion $K_1(z) \approx 1/z + O(z^2)$ to obtain

$$\big\{\psi_2(0, x^-), \psi_2^\dagger(0, y^-)\big\} = \frac{-i}{4\pi}\Big[\frac{1}{x^- - y^- - i\epsilon} - \frac{1}{x^- - y^- + i\epsilon}\Big] = \frac{1}{2}\delta(x^- - y^-), \tag{4.9}$$

because the time difference η canceled out. The relation $1/(x-i\epsilon) = \mathcal{P}\frac{1}{x} + i\pi\delta(x)$ was used in the final step ($\mathcal{P}$ stands for the principal value). The same result is obtained for $x^+ - y^+ > 0, x^- - y^- < 0$. The presence of the $i\epsilon$ part was essential for deriving the correct equal-time anticommutator. We would have obtained the wrong result, namely zero, without the convergence factor. In other words, one can directly set $x^+ = y^+$ in the correctly defined anticommutator function. But, generally speaking, the equal-time limit of the anticommutator function is not immediately the Dirac delta function $\delta(x^- - y^-)$ but the expression

$$\begin{aligned}\Delta(0, x^- - y^-) &= -\frac{im}{4\pi}\Bigg[\sqrt{\frac{i\epsilon}{|x^- - y^-| - i\epsilon}}K_1\Big(m\sqrt{i\epsilon(|x^- - y^-| - i\epsilon)}\Big)\\ &- \sqrt{\frac{-i\epsilon}{|x^- - y^-| + i\epsilon}}K_1\Big(m\sqrt{-i\epsilon(|x^- - y^-| + i\epsilon)}\Big)\Bigg] \qquad (4.10)\end{aligned}$$

which for finite $x^- - y^-$ reduces to $1/2\delta(x^- - y^-)$.

A similar conclusion can be reached for the $x^+ - y^+ = 0$ limit of the S_{12} function. Here we have for $x^+ - y^+ < 0,\ x^- - y^- > 0$

$$\begin{aligned}\big\{\psi_1(x), \psi_2^\dagger(y)\big\} &= S_{12}(x-y) - S_{12}^*(x-y) \equiv m\Sigma(x^+ - y^+, x^- - y^-) =\\ &= \frac{m}{4\pi}\Bigg[K_0\Big(m\sqrt{(|x^+ - y^+| + i\epsilon)(x^- - y^- - i\epsilon)}\Big)\\ &- K_0\Big(m\sqrt{(|x^+ - y^+| - i\epsilon)(x^- - y^- + i\epsilon)}\Big)\Bigg]. \qquad (4.11)\end{aligned}$$

The relative minus sign in the first line of the above expression is due to the negative sign of the second term in the Fock representation of $\psi_1(x)$. With the expansion $K_0(z) \approx -\gamma_E - \ln\frac{z}{2}$ we find in the $-(x^+ - y^+) = \eta \to 0$ limit

$$\begin{aligned}\Big\{\psi_1(0, x^-), \psi_2^\dagger(0, y^-)\Big\} &= \frac{m}{4\pi}\Big[-\ln\Big(\frac{m}{2}\sqrt{i\epsilon(x^- - y^- - i\epsilon)}\Big) +\\ &+ \ln\Big(\frac{m}{2}\sqrt{-i\epsilon(x^- - y^- + i\epsilon)}\Big)\Big]\\ &= \frac{m}{4\pi}\Big[-\frac{1}{2}\ln(i) + \frac{1}{2}\ln(-i)\Big] = -i\frac{m}{8}. \qquad (4.12)\end{aligned}$$

For the case $x^- - y^- < 0$, one gets the above result with the opposite sign. Hence we recover the correct anticommutator, Eq.(4.7). The essential role played by the $i\epsilon$ factor is evident. Generally speaking however, the equal time limit of the considered anticommutator function is not the sign function but the expression

$$m\Sigma(0, z^-) = \frac{m}{4\pi}\Bigg[K_0\Big(m\sqrt{i\epsilon(|z^-| - i\epsilon)}\Big) - K_0\Big(m\sqrt{-i\epsilon(|z^-| + i\epsilon)}\Big)\Bigg]. \qquad (4.13)$$

5 Poincaré algebra and surface terms

A necessary condition for the Lorentz invariance of a relativistic quantum field-theory model is that the abstract algebra of the Poincaré generators has to be satisfied when one calculates the corresponding commutators using the equal-time commutation relations of field variables. There are no rotations and only one boost, represented by the generator M^{+-}, in the two-dimensional space-time and the abstract Poincaré algebra one should reproduce is simply

$$\big[P^+, P^-\big] = 0, \quad \big[P^+, M^{+-}\big] = -2iP^+, \quad \big[P^-, M^{+-}\big] = 2iP^-. \tag{5.1}$$

This is a special case of the general abstract algebra of ten Poincaré generators in D=$3+1$:

$$\big[P^\mu, P^\nu\big] = 0, \;\; \big[P^\lambda, M^{\mu\nu}\big] = ig^{\lambda\mu}P^\nu - ig^{\lambda\nu}P^\mu,$$

$$\big[M^{\mu\nu}, M^{\rho\sigma}\big] = -ig^{\mu\rho}M^{\nu\sigma} + ig^{\mu\sigma}M^{\nu\rho} + ig^{\nu\rho}M^{\mu\sigma} - ig^{\nu\sigma}M^{\mu\rho}. \tag{5.2}$$

The above relations can be derived from the basic coordinate representation of a relativistic point particle, $\hat{p}_\mu = -i\partial_\mu, \hat{M}_{\mu\nu} = i\big(x_\mu\hat{p}_\nu - x_\nu\hat{p}_\mu\big)$ by a direct computation.

There is also the second condition for the full relativistic invariance of a quantum field-theoretical model. It is a correct transformation of the quantum field under Lorentz transformations Λ and translations a^μ. As is well known, the former ones are defined by $x'^\mu = \Lambda^\mu_\nu x^\nu$, where the property $g_{\mu\nu}\Lambda^\mu_\rho\Lambda^\nu_\sigma = g_{\rho\sigma}$ guarantees invariance of the interval $x_\mu x^\mu$. In the classical theory, the values of the fields as seen by two observers related by a Lorentz transformation $x' = \Lambda x$ are related according to

$$\phi'(x') = \phi(x), \;\; \psi'(x') = S(\Lambda)\psi(x), \tag{5.3}$$

for the scalar and fermion field, respectively. $S(\Lambda)$ is a matrix transforming the components of the Dirac fields in such a way that the Dirac equation in the transformed coordinate system maintains its form (covariance). The above relations should be generalized to amplitudes of a quantum field in the quantum theory, where one assumes existence of an operator implementing Lorentz transformations for state vectors, $|\Phi'\rangle = U(a,\Lambda)|\Phi\rangle$. The operator $U(a,\Lambda)$ must be unitary to conserve probability. Then the classical relations (5.3) are replaced by

$$\langle\Phi|U^{-1}(\Lambda)\phi(\Lambda x)U(\Lambda)|\Phi\rangle = \langle\Phi|\phi(x)|\Phi\rangle,$$

$$\langle\Phi|U^{-1}(\Lambda)\psi(\Lambda x)U(\Lambda)|\Phi\rangle = S(\Lambda)\langle\Phi|\psi(x)|\Phi\rangle \tag{5.4}$$

and we immediately find the transformation laws for the two fields:

$$\phi(x') = U(\Lambda)\phi(x)U^{-1}(\Lambda), \quad \phi(x+a) = U(a)\phi(x)U^{-1}(a),$$

$$S^{-1}(\Lambda)\psi(\Lambda x) = U(\Lambda)\psi(x)U^{-1}(\Lambda), \quad \psi(x+a) = U(a)\psi(x)U^{-1}(a). \tag{5.5}$$

In course of calculation of the Poincaré algebra one has to perform partial integrations that generate terms at spatial infinity. In the conventional field theory one usually omits these surface terms since the fields themselves (at least classically) are assumed to vanish at spatial infinity. In the light front theory, it was found that even for free massive fermions [32] the corresponding surface terms do not vanish. This fact was interpreted as a failure of relativistic invariance of the LF quantization. Since this problem was identified already for the continuum formulation, we shall temporarily leave the finite volume and will study carefully surface terms arising in the LF Poincaré algebra (5.1) of the free massive fields quantized in an infinite volume.

Let us start with the scalar field. We will first use this opportunity and show yet another derivation of the field expansion in the continuum theory. It is parallel to that shown for the space-like scalar field. The general solution of the Klein-Gordon equation (2.1) can be written in the covariant form as

$$\phi(x) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}p^2}{2(2\pi)^2} \pi\delta(p^2 - \mu^2)\chi(p)e^{-ip.x}. \tag{5.6}$$

In the LF variables, using $\delta(p^+p^- - \mu^2) = \frac{1}{|p^+|}\delta\big(p^- - \frac{\mu^2}{p^+}\big)$ and the notation $f(p) = 2\pi\chi(p)$, we get, with $\hat{p}^- = \frac{\mu^2}{p^+}$,

$$\begin{aligned}\phi(x) \quad &= \quad \int_{-\infty}^{0} \frac{\mathrm{d}p^+}{8\pi|p^+|} f\big(p^+, p^- = \hat{p}^-\big)e^{-\frac{i}{2}p^+x^- - \frac{i}{2}\hat{p}^-x^+} + \\ &+ \quad \int_{0}^{+\infty} \frac{\mathrm{d}p^+}{8\pi p^+} f\big(p^+, p^- = \hat{p}^-\big)e^{-\frac{i}{2}p^+x^- - \frac{i}{2}\hat{p}^-x^+}.\end{aligned} \tag{5.7}$$

Changing $p^+ \to -p^+$ in the first term, we find

$$\phi(x) = \int_{0}^{\infty} \frac{\mathrm{d}p^+}{8\pi p^+}\Big[a(p^+)e^{-\frac{i}{2}p^+x^- - \frac{i}{2}\hat{p}^-x^+} + a^*(p^+)e^{\frac{i}{2}p^+x^- + \frac{i}{2}\hat{p}^-x^+}\Big], \tag{5.8}$$

where we called $a(p^+) \equiv f\big(p^+, p^- = \hat{p}^-\big), a^*(p^+) \equiv f\big(-p^+, p^- = -\hat{p}^-\big)$. As before, one promotes the amplitudes $a,\ a^*$ to operators in quantum theory:

$$\phi(x) = \int\limits_{0}^{\infty} \frac{dp^+}{8\pi p^+}\big[a(p^+)e^{-\frac{i}{2}p^+x^- - \frac{i}{2}\hat{p}^-x^+} + a^\dagger(p^+)e^{\frac{i}{2}p^+x^- + \frac{i}{2}\hat{p}^-x^+}\big]. \tag{5.9}$$

where small imaginary parts for $x^\pm$ are understood as discussed for the case of fermions. The corresponding two-point function

$$D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle = \int\limits_{0}^{\infty} \frac{dp^+}{8\pi p^+} e^{-\frac{i}{2}p^+(x^- - y^- - i\epsilon) - \frac{i}{2}\frac{\mu^2}{p^+}(x^+ - y^+ - i\epsilon)} \tag{5.10}$$

can easily be obtained using the commutation relation

$$\big[a(p^+), a^\dagger(q^+)\big] = 4\pi p^+\delta(p^+ - q^+). \tag{5.11}$$

With the help of known integral formulae [25], we again find

$$D(z) = -\frac{1}{8}\theta\big(z^2\big)\Big(N_0(\mu\sqrt{z^2}) - i\mathrm{sgn}(z^+)J_0(\mu\sqrt{z^2})\Big) + \frac{1}{4\pi}\theta\big(-z^2\big)K_0(\mu\sqrt{-z^2}). \tag{5.12}$$

In a complete analogy with the fermionic case, one can calculate the $x^+ = y^+$ projection of the commutator function $[\phi(x), \phi(y)] = D(x-y) - D^*(x-y)$i:

$$\Sigma(x^- - y^-) = \frac{1}{4\pi}\bigg[K_0\Big(\mu\sqrt{i\epsilon(x^- - y^- - i\epsilon)}\Big) - K_0\Big(\mu\sqrt{-i\epsilon(x^- - y^- + i\epsilon)}\Big)\bigg] \tag{5.13}$$

which for finite $|x^- - y^-|$ reduces to $-\frac{i}{8}\mathrm{sgn}(x^- - y^-)$.

The Hamiltonian, the LF momentum operators and the boost generator M^{+-} (3.16), (3.17) are given in terms of the energy-momentum densities

$$\Theta^{++}(x) = 4\partial_-\phi(x)\partial_-\phi(x), \quad \Theta^{+-}(x) = \mu^2\phi^2(x) \tag{5.14}$$

as

$$P^+ = \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}x^-}{2}\Theta^{++}(x), \quad P^- = \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}x^-}{2}\Theta^{+-}(x),$$

$$M^{+-} = x^+P^- - \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}x^-}{2}x^-\Theta^{++}(x). \tag{5.15}$$

It is now straightforward to calculate the commutators (5.1) using the anticommutation relations (4.3),(4.7). One finds that in the first two commutators the surface terms cancel. However, it was claimed in Ref. [32] that in the latter commutator an unwanted surface term appears. A similar redundant term was found in the four-dimensional theory in the commutator between the LF rotational generator M^{-i} and the scalar field [8]. These difficulties were not clarified in the LF literature so far. If the above-mentioned violations of relativistic invariance were really true, the LF field theory would face a serious consistency problem [33]. A careful mathematical analysis shows however that the claimed violation is an artifact and the surface terms actually vanish. In what follows, we shall justify our statement in the case of free massive fermions.

The problematic surface term is equal to

$$s(L) = -8imL\psi_1^\dagger(L)\psi_2(L), \tag{5.16}$$

where L is a value of x^- tending to infinity (a cutoff). The surface term is obviously non-vanishing in this form and violates the Lorentz invariance. Let us again use our careful definitions of the anticommutators and replace the $\delta(x^-)$ and $\epsilon(x^-)$ functions by $2\Delta(x)$ (4.8) and $8i\Sigma(x)$ (4.11) for $x^+ = 0$. Then we find

$$\begin{aligned}
\left[P^-, M^{+-}\right] &= -\int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}x^-}{2} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}y^-}{2} y^- \left[\Theta^{+-}(x), \Theta^{++}(y)\right] = \\
&= -4im \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}x^-}{2} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}y^-}{2} y^- \Big\{ \Sigma(x-y)\psi_2^\dagger(x)\partial_-\psi_2(y) \\
&- \partial_-^y\Delta(x-y)\psi_2^\dagger(y)\psi_1(x) + \Delta(x-y)\psi_1^\dagger(x)\partial_-\psi_2(y) \\
&- \frac{m}{2i}\Delta(x-y)\psi_2^\dagger(y)\psi_2(x)\Big\}.
\end{aligned} \tag{5.17}$$

[8] This commutator implements correct transformation properties of the quantum scalar field under one type of rotations. It is a particular form of Eq.(5.5) for an infinitesimal value of the rotational parameter.

Performing the partial integration and using the limiting values of the $\Delta(x)$ and $\Sigma(x)$ functions we obtain that the expression (5.17) equals to $2iP^-$ plus the surface term

$$s(i\epsilon, x^- - L) = 4imL \int_{-\infty}^{+\infty} \frac{\mathrm{d}x^-}{2} \Big[\Sigma(i\epsilon, x^- - L)\psi_2^\dagger(L) + \Sigma(i\epsilon, x^- + L)\psi_2^\dagger(-L)\Big]\psi_1(x). \quad (5.18)$$

To proceed we have to look at the large-distance behaviour of the function $\Sigma(i\epsilon, x^- \pm L)$:

$$\Sigma(i\epsilon, x^- - L) = \frac{m}{2\pi}\Big\{K_0\Big(m\sqrt{(-i\epsilon)|x^- - L|}\Big) - \frac{m}{2\pi}\{K_0\Big(m\sqrt{i\epsilon|x^- - L|}\Big)\Big\}. \quad (5.19)$$

Using the value of the modified Bessel function $K_\nu(z)$ for large z

$$K_\nu(z) \approx \sqrt{\frac{\pi}{2z}} \exp(-z), \quad (5.20)$$

we get

$$\begin{aligned} &K_0\big(m\sqrt{\pm i\epsilon L}\big) \approx \sqrt{\frac{\pi}{2m\sqrt{\pm i\epsilon L}}} \exp(-m\sqrt{\pm i\epsilon L}) \\ &\approx L^{-1/4} \exp\big[-\frac{m}{\sqrt{2}}(1 \pm i)\sqrt{\epsilon L}\big] \end{aligned} \quad (5.21)$$

and similarly for other surface terms. Thus, all of them are exponentially suppressed for $L \to \infty$ and the correct Poincaré algebra is recovered. A possibility of this mechanism for higher dimensional LF field theories was mentioned also in [34].

6 Discrete correlation functions and causality in the finite-volume LF theory

As we have already indicated and as will be further demonstrated, for studying non-perturbative problems, it is often desirable to analyze light-front fields in a finite spatial volume where they can be decomposed to a discrete infinity of modes. Such dynamical systems can be regarded as a quantum field theory in its own right. This implies that one should verify all usual well-established properties (such as causality, Poincaré invariance, singularity structure, etc.) in this framework to check its overall consistency. In this section, we will focus on the problem of microcausality or locality. This principle is derived from a finite value of the speed of light which is the limiting speed with which a signal can propagate in the space-time. This implies that two events or measurements separated by a space-like interval cannot influence each other and thus are causally independent. In quantum theory, this property is incorporated by vanishing of a commutator of two fields separated by the ispace-like interval $x_\mu x^\mu < 0$. One may ask a question if this principle holds true in a finite volume with fields (anti)periodic in x^-.

Here one should realize that the introduction of a finite volume generally speaking violates some symmetries present in the continuum or infinite-volume theories. A natural requirement is that this violation should go away in the infinite-volume limit so that for example the relativistic invariance (invariance under boosts, rotations and translations) is restored. Thus, there would be no reason to worry if one would find a violation of causality in a finite volume which would disappear for $L \to \infty$. Since it was claimed by some authors [36] that already a massive scalar field in two dimensions fails to satisfy this property we found it important to clarify the situation. We will present the method and results of our study [35] in this section. If the conclusion that periodic boundary conditions are incompatible with causal behavior of the (infrared-regularized) light-front quantum theory were indeed correct it would cast considerable doubts on the validity of the DLCQ approach.

The method used to demonstrate the causality failure for $L \to \infty$ was a numerical study of the infinite series, represented the LF Pauli-Jordan function in the finite volume, truncated at some value of discretized LF momentum p^+ for which the results stabilized. This method gave a very satisfactory picture of the causality in the usual field theory, where the Pauli-Jordan (PJ) has the finite-volume form

$$\Delta(x-y) = -i\big[\phi(x),\phi(y)\big] = -\frac{1}{2L}\sum_{n=1}^{\infty}\frac{1}{\omega(p)}\sin\Big(\omega(p)(x^0-y^0) - \frac{\pi n}{L}(x^1-y^1)\Big). \quad (6.1)$$

We have used the expansion (2.3) for the periodic scalar field ($p^1 = 2\pi n/L$) as well as the Fock commutators (2.5) to obtain the above discrete representation. Truncation of this series at some moderate value of n lead to vanishing (up to negligible numerical effects) of PJ function for space-like separations between the points x and y and a usual oscillatory behaviour for time-like separations. This picture however failed for a LF system restricted to $-L \leq x^- \leq L$. Not only did the numerical results for the PJ function fail to vanish for $x^2 < 0$, but it was even found not to converge to the correct continuum expression. One has to conclude that either the discretized light-front theory has some fundamental difficulty or the numerical computations have some internal difficulty and are not reliable. To clarify this issue it would be preferable to find a method for analytical evaluation of the infinite series expansion of the PJ function, corresponding to the integral form of the PJ function in the continuum formulation.

In the following, we will briefly discuss such an analytical formalism for evaluation of infinite series corresponding to various correlation functions of the discretized LF theory. This formalism is very well adapted to the form of LF kinematics and dispersion relation and it uses some properties of polylogarithm functions. As a result, an integral representation can be given for these infinite series expansions. This representation explicitly depends on the box length L. Then one can use analytical methods to study the large L dependence of the integral representation of the PJ function.

The expression for the Pauli-Jordan function of a massive LF scalar field quantized in a finite volume and satisfying periodic boundary condition is obtained by commuting two scalar fields (3.10) at different space-time points (x^+, x^-) and (y^+, y^-) using the corresponding Fock commutation relations. The result is expressed in terms of the correlation function

$$\hat{D}(x) = \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-\frac{i}{2}p_n^+ x^- - \frac{i}{2}\frac{\mu^2}{p_n^+}x^+}, \quad p_n^+ = \frac{2\pi}{L}n \quad (6.2)$$

as

$$\hat{\Delta} = -i[\varphi(x),\varphi(y)] = -i\Big(\hat{D}(x) - \hat{D}^*(x)\Big) = 2\mathrm{Im}\hat{D}(x). \quad (6.3)$$

To demonstrate the general nature of the present approach, we will derive corresponding formulae also for the fermionic case although the implications for causality will be discussed only for the scalar field. We will work with periodic fermi field $\psi(-L) = \psi(L)$ to keep the discussion close to the scalar field case,

The field expansions (3.35), (3.39) yield for the correlation functions

$$\hat{S}_{22}(x) = \frac{1}{2L} \sum_{n=1}^{\infty} e^{-\frac{i}{2}p_n^+ x^- - \frac{i}{2}\frac{m^2}{p_n^+}x^+},$$

$$\hat{S}_{11}(x) = \frac{m^2 L}{8\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-\frac{i}{2}p_n^+ x^- - \frac{i}{2}\frac{m^2}{p_n^+}x^+},$$

$$\hat{S}_{12}(x) = \frac{m}{4\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{-\frac{i}{2}p_n^+ x^- - \frac{i}{2}\frac{m^2}{p_n^+}x^+}. \tag{6.4}$$

The key question now is: do the discrete representations (6.2) and (6.4) lead to the continuum results (5.12) and (4.6) for $L \to \infty$? More specifically, does 2 Im$\hat{D}$ reproduce the continuum Pauli-Jordan function in this limit?

To answer this question, we replace the infinite series (6.2) by an integrals using an integral representation of the polylogarithm functions. Consider the function of two independent complex variables defined by

$$F_1(z,q) \quad = \quad \sum_{n=1}^{\infty} \frac{z^n}{n} e^{q/n}. \tag{6.5}$$

For any finite q it can be shown that the power series converges only within the unit circle $|z| < 1$. Expanding $e^{q/n}$ in powers of its argument we obtain

$$F_1(z,q) = \sum_{k=0}^{\infty} \frac{(q)^k}{k!} \sum_{n=1}^{\infty} \frac{z^n}{n^{k+1}} = \sum_{k=0}^{\infty} \frac{q^k}{k!} Li_{k+1}(z). \tag{6.6}$$

Here we have used the definition [37] of the polylogarithm function Li_m,

$$Li_m(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^m}. \tag{6.7}$$

Note that this series representation (6.7) of Li_m converges only if $|z| < 1$. Its analytic continuation to the rest of the complex z plane is provided by the integral representation

$$Li_m(z) = \frac{1}{(m-1)!} \int_0^{\infty} du \frac{u^{m-1}}{z^{-1}e^u - 1}, (\ m \geq 1), \tag{6.8}$$

which shows that $Li_m(z)$ is actually analytic throughout the z-plane except for a cut on the positive real z axis, linking $+1$ to $+\infty$. Substituting this formula into (6.6) and interchanging integration on u with the summation on k, we arrive at the integral representation of the series (6.5),

$$F_1(z,q) = \int_0^\infty du \frac{1}{z^{-1}e^u - 1} I_0(2\sqrt{qu}). \tag{6.9}$$

To obtain this result we have used the identity

$$\sum_{k=0}^{\infty} \frac{(v)^k}{(k!)^2} = I_0(2\sqrt{v}), \tag{6.10}$$

where $I_0(z)$ is the modified Bessel function [25]. Comparing (6.2) and (6.5) we note that

$$\hat{D}(x) = \frac{1}{4\pi} F_1(e^{i\xi/Q}, iQ), \tag{6.11}$$

where

$$\xi = \mu^2 x^+ x^- /4, \;\; Q = \frac{\mu^2 L}{4\pi} x^+. \tag{6.12}$$

This is the first step of the analysis whereby the infinite series (6.2) has been rewritten in integral form via (6.9) and (6.11).

There are four distinct cases to consider each associated with a quadrant of the x^+, x^- plane. Consider first the case where both $x^\pm$ are positive ($Q > 0, \xi > 0$). We may rewrite Eqs.(6.9) and (6.11) as

$$\hat{D}(x^+ > 0, x^- > 0) = \frac{1}{4\pi Q} \int_0^\infty du \frac{1}{\exp\left(\frac{1}{Q}(u - i\xi)\right) - 1} I_0(2e^{i\pi/4}\sqrt{u}). \tag{6.13}$$

The continuum limit is then obtained by considering the limiting value of the RHS of Eq.(6.13) for $Q \to +\infty$. The result is the reduced version of Eq.(5.12) appropriate to the regime $x^\pm > 0$. Before demonstrating this we note that $I_0(2e^{i\pi/4}\sqrt{u})$ is an analytic function of u throughout the entire finite complex plane since its Taylor series expansion in powers of the argument has an infinite radius of convergence. The remaining factor in the integrand of (6.13) is analytic throughout the u plane with the exception of simple poles at the discrete set of points $u_n = i(2n\pi Q + \xi)$ on the imaginary axis, where n is any integer. In view of these analytic properties of the integrand we can alter the integration contour without changing the value of the integral in (6.13) as long as we avoid crossing through any of the singular points u_n and maintain the given endpoints. The simplest choice of a preferable contour has a rectangular shape and consists of the straight-line segments (C_1) on the imaginary axis, the semi-circle (C_2) centered on the pole u_0, the line (C_3) parallel to the positive real axis, and finally the straight-line segment (C_4) parallel to the imaginary axis. Individual contributions are calculated in the Appendix C. The overall result is

$$\lim_{Q\to\infty} \hat{D}(x^+ > 0, x^- > 0) = -\frac{1}{4}\Big(N_0(\mu\sqrt{x^2}) - iJ_0(\mu\sqrt{x^2})\Big), \tag{6.14}$$

in agreement with Eq.(5.12). The first contribution comes from the segment C_1 and the second one correspond to the residuum of the integral at the pole. Contributions from the segments C_3 and C_4 vanish in the continuum result.

From the expression (6.14), it is not difficult to find the results corresponding to other three combinations of signs of x^+, x^-. First, we conclude from the formula (6.9) that results for finite L for the case $x^+ < 0, x^- < 0$ ($\xi > 0, Q < 0$) can be obtained from those for $x^+ > 0, x^- > 0$ by complex conjugation:

$$F_1(e^{-i\xi/|Q|}, -i|Q|) = \big[F_1(e^{i\xi/Q}, iQ)\big]^*. \tag{6.15}$$

Likewise, we have for $x^+ > 0, x^- < 0$ ($Q > 0, \xi < 0$)

$$\hat{D}(x) = \frac{1}{4\pi} F_1(e^{-i|\xi|/Q}, iQ), \tag{6.16}$$

and this can be used to generate the results also for the regime $x^+ < 0, x^- > 0$ ($\xi < 0, Q < 0$) according to

$$\hat{D}(x) = \frac{1}{4\pi} F_1(e^{i|\xi|/|Q|}, -i|Q|) = \big[\frac{1}{4\pi} F_1(e^{-i|\xi|/Q}, iQ)\big]^*. \tag{6.17}$$

The evaluation of F_1 in Eq.(6.16) for large L proceeds as above using the same multi-component contour except that the semicircle C_2 is not applicable since the pole u_0 is situated at $-i|\xi|$. The final result is

$$\lim_{Q \to +\infty} \hat{D}(x^+ > 0, x^- < 0) = \frac{1}{4\pi} \int_0^\infty dv \frac{J_0(2\sqrt{v})}{v + \xi} = \frac{1}{2\pi} K_0(2\sqrt{|\xi|}), \tag{6.18}$$

in agreement with the continuum formula (5.12). In particular, this result means that since the imaginary part of the function $\hat{D}(x)$ vanishes for spacelike x^2, the causality is restored in the infinite-volume limit. For large L the leading L-dependent terms are of the order $L^{-1/4}$. The results for the PJ function $2\mathrm{Im}\hat{D}(x)$ are consistent with the results of Ref. [38] although their intepretation is somewhat different. Also, as will be shown next, the method that we desribed in this chapter can be extended to calculate the complete Wightman functions of two-dimensional free massive bosons and fermions quantized in a finite volume.

Let us therefore briefly discuss the correlation functions $\hat{S}_{22}$ and $\hat{S}_{11}$ of (6.4). The first of these is a special case of the function

$$F_0(z, q) = \sum_{n=1}^{\infty} z^n e^{q/n} = \frac{z}{1 - z} + q \int_0^\infty du \frac{1}{z^{-1} e^u - 1} \frac{I_1(2\sqrt{qu})}{\sqrt{qu}}, \tag{6.19}$$

while $\hat{S}_{11}$ is a special case of

$$F_2(z, q) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} e^{q/n} = \int_0^\infty du \frac{u}{z^{-1} e^u - 1} \frac{I_1(2\sqrt{qu})}{\sqrt{qu}}. \tag{6.20}$$

In the regime $x^{\pm} > 0$ we find that

$$\begin{aligned}\lim_{Q\to+\infty}\hat{S}_{22}(x) &= 4e^{i\frac{\pi}{4}}P\int_0^\infty dw\frac{J_1(w)}{w^2-4\xi}-\frac{\pi}{\sqrt{\xi}}e^{-i\frac{\pi}{4}}J_1(2\sqrt{\xi})\\ &= -\frac{m}{4}\sqrt{\frac{x^+}{x^-}}\Big(J_1(m\sqrt{x^2})+iN_1(m\sqrt{x^2})\Big). \qquad (6.21)\end{aligned}$$

Likewise, in the same regime, we have

$$\lim_{Q\to+\infty}\hat{S}_{11}(x) = -\frac{2\pi^2}{m}\sqrt{\frac{x^-}{x^+}}\Big(J_1(m\sqrt{x^2})+iN_1(m\sqrt{x^2})\Big). \qquad (6.22)$$

These results are in agreement with the continuum formulas (4.6).

In principle, one may try to evaluate the integral (6.9) representing the Pauli-Jordan function in finite volume numerically for increasing values of L to examine the rate of convergence towards the continuum result. In practice, this is rather difficult to achieve since the integrand of the representation (6.9) oscillates rapidly due to the presence of the Bessel function I_0. Already for relatively small values of Q the amplitudes of these oscillations are so large and the spacings of successive zeros are so small that it is very difficult to reliably evaluate the integral by standard numerical routines. The numerical computations of the integral were therefore restricted to a few relatively small values using an integration method based on Chebyshev polynomials as well as by a Clenshaw-Curtis method. The results for $Q = 4$ and $Q = 18$ are displayed in Fig.(6.1). For definiteness one can set $\mu^2 x^+ = 1$ so that the corresponding box lengths given by $L = 4\pi Q$ are approximately $L = 50$ and 226. A qualitative difference in the behavior of the Pauli-Jordan commutator function between the space-like region (negative values of x^-/L) and time-like region (positive x^-/L) is obvious already for the smallest value $Q = 4$. It is also evident that for larger Q the oscillatory behavior of the continuum curve in the time-like region is resolved with increased accuracy. Although the Pauli-Jordan function for finite volumes is not zero in the space-like region, it is fairly close to it. We recall that for our choice of Q values we are still very far from the infinite-volume limit so the obtained behavior of the Pauli-Jordan function is plausible and consistent with our analytical results. It would be desirable however to be able to compute the behaviour of the PJ function numerically for much higher values of Q and verify that the curve in the space-like region indeed converges to zero.

The physical conclusions of this mathematical treatment are summarized as follows. With the infinite number of field modes the violation of microcausality in a LF finite volume with periodic scalar field is only a marginal effect and continuum results including the causal behavior are restored in the $L \to \infty$ limit. In practice, the DLCQ calculations of mass spectra and wavefunctions are always performed for finite L and with a finite number of Fourier modes. At this step, the causality may seem to be violated [36] (see also [39] for a treatment that averages over some range of L values and restores the causal behavior in a finite volume). However, as physical quantities calculated with the DLCQ method have to be extrapolated to the continuum limit, there is no inconsistency, since, as we have shown, the causality is restored there.

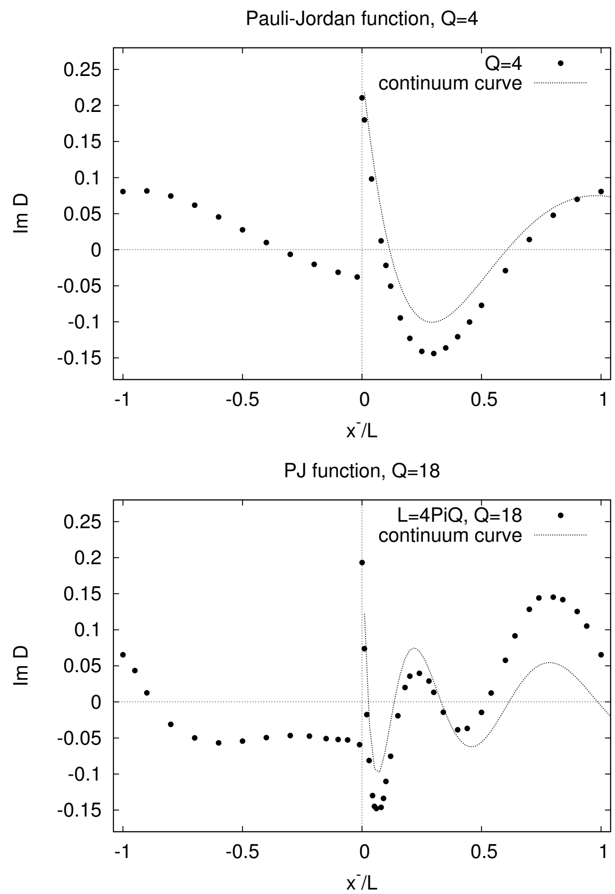

Fig. 6.1. The commutator function $2\mathrm{Im}D(x^+, x^-)$ evaluated numerically is plotted for $Q = 4$ and $Q = 18$ as a function of $s = x^-/L$ and compared with the continuum function $1/4J_0(\sqrt{Ls})$ (solid line) in the box of the length $L = 4\pi Q$. For simplicity, we chose $\mu^2 x^+ = 1$. The time-like region corresponds to positive values of s, the space-like region to negative ones.

7 Soluble interacting model

As a first example of interacting models which are the main focus of the quantum field theory, let us study a very simple model with two species of the Fermi field with masses m and M interacting via a current-current coupling, proposed by Federbush a long time ago [40]. The correct mathematical formulation was later given by Wightman [41] and Schroer and Truong [42]. The model can be considered as a toy version of the massive Thirring model which is a surprisingly rich, non-trivial but still relatively simple relativistic quantum model. The Federbush model was not discussed in the light front formulation so far and we will use it here to demonstrate the LF approach (and simplifications it offers) to an interacting theory in the simplest yet non-trivial setting. The Lagrangian of the model in the covariant form reads

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^\mu \overleftrightarrow{\partial_\mu} \Psi - M\overline{\Psi}\Psi + \frac{i}{2}\overline{\chi}\gamma^\mu \overleftrightarrow{\partial_\mu} \chi - m\overline{\chi}\chi - g\epsilon_{\mu\nu}J^\mu j^\nu. \tag{7.1}$$

The vector currents are $J^\mu =: \overline{\Psi}\gamma^\mu\Psi :$ and $j^\mu =: \overline{\chi}\gamma^\mu\chi :$, where the colon denotes the normal-ordered expressions, i.e. the operators where all creation operators stand to the left of all annihilation operators. This is dictated by the C-parity requirements of the electromagnetic current, namely the current should be odd under the charge conjugation on physical grounds (opposite charges of a fermion and an antifermion). The antisymmetric symbol $\epsilon^{\mu\nu}$ is defined by $\epsilon^{01} = 1$. From $\epsilon_{\mu\nu}\epsilon^{\mu\nu} = -2$ we obtain its LF components $\epsilon^{+-} = 2,\ \epsilon_{+-} = 1/2$. The light front version of the Lagrangian (7.1) is

$$\begin{aligned}\mathcal{L}_{lf} &= i\Psi_2^\dagger \overleftrightarrow{\partial_+} \Psi_2 + i\Psi_1^\dagger \overleftrightarrow{\partial_-} \Psi_1 - M\big(\Psi_2^\dagger\Psi_1 + \Psi_1^\dagger\Psi_2\big) + \\ &+ i\chi_2^\dagger \overleftrightarrow{\partial_+} \chi_2 + i\chi_1^\dagger \overleftrightarrow{\partial_-} \chi_1 - m\big(\chi_2^\dagger\chi_1 + \chi_1^\dagger\chi_2\big) + \frac{g}{2}J^+j^- - \frac{g}{2}J^-j^+. \end{aligned} \tag{7.2}$$

The LF currents are conserved and their components are

$$\begin{aligned} J^+(x) &= 2 : \Psi_2^\dagger(x)\Psi_2(x) :,\ \ J^-(x) = 2 : \Psi_1^\dagger(x)\Psi_1(x) :, \\ j^+(x) &= 2 : \chi_2^\dagger(x)\chi_2(x) :,\ \ \ j^-(x) = 2 : \chi_1^\dagger(x)\chi_1(x) : . \end{aligned} \tag{7.3}$$

Here we are using our previous convention

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix},\ \ \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \tag{7.4}$$

where the upper components are the constraint fields while the lower ones are dynamical. This is obvious from the (classical) equations of motion which are derived from the action principle as in the usual field theory:

$$\begin{aligned} 2i\partial_+\Psi_2(x) &= M\Psi_1 - gj^-\Psi_2,\ \ 2i\partial_-\Psi_1 = M\Psi_2 + gj^+\Psi_1, \\ 2i\partial_+\chi_2(x) &= m\chi_1 + gJ^-\chi_2,\ \ \ 2i\partial_-\chi_1 = m\chi_2 - gJ^+\chi_1. \end{aligned} \tag{7.5}$$

Note that the equations are coupled in two ways: equation for the upper component depends on the lower one and vice versa and moreover the current components of one Fermi field enters into the equation for the other Fermi field. However, due to the specific form of the coupling term, the set of field equations is explicitly solvable. On the quantum level, one has a rare situation that one knows the operator solution of the field equations and can calculate various correlation functions, test their singularity structure [42] etc.

Using the equations of motion (7.5) one easily establishes that the currents $J^\mu(x)$ and $j^\mu(x)$ are conserved:

$$\partial_\mu j^\mu(x) = \partial_+ j^+(x) + \partial_- j^-(x) = 0,\ \ \partial_\mu J^\mu(x) = 0. \tag{7.6}$$

Since there exists an independent gamma matrix γ^5, one can define also the axial-vector currents $j_5^\mu(x), J_5^\mu(x)$ according to

$$j_5^\mu(x) = \overline{\chi}(x)\gamma^\mu\gamma^5\chi(x),\ \ J_5^\mu(x) = \overline{\Psi}(x)\gamma^\mu\gamma^5\Psi(x). \tag{7.7}$$

Inserting the components of the Fermi fields and the gamma matrices (3.26) and (3.25), we find

$$
\begin{aligned}
j_5^+(x) &= j^+(x),\ j_5^-(x) = -j^-(x),\\
J_5^+(x) &= J^+(x),\ J_5^-(x) = -J^-(x).
\end{aligned} \tag{7.8}
$$

The axial-vector currents are not conserved:

$$
\partial_\mu j_5^\mu(x) = 2im\overline{\chi}(x)\gamma^5\chi(x),\ \ \partial_\mu J_5^\mu(x) = 2iM\overline{\Psi}(x)\gamma^5\Psi(x). \tag{7.9}
$$

Again, these equations are easily verified by means of the equations of motion. Their explicit LF form reads

$$
\begin{aligned}
\partial_+ j_5^+(x) + \partial_- j_5^-(x) &= 2im\big(\chi_1^\dagger(x)\chi_2(x) - \chi_2^\dagger(x)\chi_1(x)\big),\\
\partial_+ J_5^+(x) + \partial_- J_5^-(x) &= 2iM\big(\Psi_1^\dagger(x)\Psi_2(x) - \Psi_2^\dagger(x)\Psi_1(x)\big).
\end{aligned} \tag{7.10}
$$

We will consider the Federbush model on a finite line $-L \leq x^- \leq L$ and will choose antiperiodic boundary conditions for the Fermi field:

$$
\Psi(-L) = -\Psi(L),\ \ \chi(-L) = -\chi(L). \tag{7.11}
$$

All considerations in the previous paragraphs are valid irrespectively of this choice. Defining the ”integrated currents” $\Sigma(x)$ and $\sigma(x)$ [41]

$$
\begin{aligned}
\Sigma(x) &= \frac{\sqrt{\pi}}{4}\int_{-L}^{+L}\frac{\mathrm{d}z^-}{2}\epsilon_N(x^- - z^-)J^+(x^+, z^-),\\
\sigma(x) &= \frac{\sqrt{\pi}}{4}\int_{-L}^{+L}\frac{\mathrm{d}z^-}{2}\epsilon_N(x^- - z^-)j^+(x^+, z^-),
\end{aligned} \tag{7.12}
$$

we can convince ourselves that the constraints in (7.5) are solved by

$$
\Psi_1(x) = e^{-\frac{ig}{\sqrt{\pi}}\sigma(x)}\psi_1(x),\ \ \chi_1(x) = e^{\frac{ig}{\sqrt{\pi}}\Sigma(x)}\varphi_1(x), \tag{7.13}
$$

where $\psi_1(x)$ and $\varphi_1(x)$ are the solutions of the free constraint equation (3.32). Actually, contrary to the massive Thirring model which will be discussed later, the dynamical equations admit the solutions in the same form:

$$
\Psi_2(x) = e^{-\frac{ig}{2}\sigma(x)}\psi_2(x),\ \ \chi_2(x) = e^{\frac{ig}{2}\Sigma(x)}\varphi_2(x), \tag{7.14}
$$

where again $\psi_2(x)$ and $\varphi_2(x)$ are the solutions of the free dynamical equation (3.32). To verify that the expressions (7.13, 7.14) represent the solutions of the classical field equations, take simply the derivative of the assumed solution for Ψ_2 to obtain

$$
2i\partial_+\Psi_2(x) = e^{-\frac{ig}{2}\sigma(x)}2i\partial_+\psi_2(x) + g\partial_+\sigma(x) = M\Psi_1(x) - gj^-(x)\Psi_2(x). \tag{7.15}
$$

In the second step, we used the dynamical equation $2i\partial_+\psi_2 = m\psi_1$ and the current conservation

$$\begin{aligned}
\partial_+\sigma(x) \quad & \frac{\sqrt{\pi}}{4}\int\limits_{-L}^{+L}\frac{\mathrm{d}z^-}{2}\epsilon_N(x^- - z^-)\partial_+ j^+(x^+,z^-) = \\
&= \quad -\frac{\sqrt{\pi}}{4}\int\limits_{-L}^{+L}\frac{\mathrm{d}z^-}{2}\epsilon_N(x^- - z^-)\partial_- j^-(x^+,z^-) = \\
&= \quad -\frac{\sqrt{\pi}}{2}\int\limits_{-L}^{+L}\frac{\mathrm{d}z^-}{2}\delta_N(x^- - z^-)j^-(x^+,z^-) = -\frac{\sqrt{\pi}}{2}j^-(x),
\end{aligned} \tag{7.16}$$

where the sign function $\epsilon_N(x^- - y^-)$ was defined in (3.13) and the partial integration was preformed in the third step. The same derivation is valid also for the solution of the $\chi_2(x)$ field. The relation (7.16) together with an analogous one for $\partial_\mu\sigma(x)$ shows that the integrated currents satisfy the conditions

$$\partial_\mu\Sigma(x) = \sqrt{\pi}\epsilon_{\mu\nu}J^\nu(x),\ \partial_\mu\sigma(x) = \sqrt{\pi}\epsilon_{\mu\nu}j^\nu(x). \tag{7.17}$$

Let us quantize the model. At $x^+ = 0$, we assume the standard anticommutation relations for the free fermion fields

$$\left\{\psi_2(0,x^-),\psi_2(0,y^-)\right\} = \left\{\varphi_2(0,x^-),\varphi_2(0,y^-)\right\} = \frac{1}{2}\delta_a(x^- - y^-). \tag{7.18}$$

The field expansions read

$$\Psi_2(x) = \frac{1}{\sqrt{2L}}\sum_{n=\frac{1}{2}}^{\infty}\left[a(p_n^+)e^{-\frac{i}{2}p_n^+x^- - \frac{i}{2}\frac{M^2}{p_n^+}x^+} + c^\dagger(p_n^+)e^{\frac{i}{2}p_n^+x^- + \frac{i}{2}\frac{M^2}{p_n^+}x^+}\right],$$

$$\chi_2(0,x^-) = \frac{1}{\sqrt{2L}}\sum_{n=\frac{1}{2}}^{\infty}\left[b(p_n^+)e^{-\frac{i}{2}p_n^+x^- - \frac{i}{2}\frac{m^2}{p_n^+}x^+} + d^\dagger(p_n^+)e^{\frac{i}{2}p_n^+x^- + \frac{i}{2}\frac{m^2}{p_n^+}x^+}\right] \tag{7.19}$$

and the corresponding constraint components are

$$\begin{aligned}
\Psi_1(0,x^-) \quad &= \quad \frac{M}{\sqrt{2L}}\sum_n\frac{1}{p_n^+}\left[a(p_n^+)e^{-ip.x} - c^\dagger(p_n^+)e^{ip.x}\right], \\
\chi_1(0,x^-) \quad &= \quad \frac{m}{\sqrt{2L}}\sum_n\frac{1}{p_n^+}\left[b(p_n^+)e^{-ip.x} - d^\dagger(p_n^+)e^{ip.x}\right].
\end{aligned} \tag{7.20}$$

where $ip.x$ stands for $\frac{i}{2}p_n^+x^- + \frac{i}{2}\frac{\mu^2}{p_n^+}x^+$ with $\mu = M$ or $\mu = m$. The non-vanishing anticommutators of the Fock operators are

$$\left\{a_m,a_n^\dagger\right\} = \left\{c_m,c_n^\dagger\right\} = \left\{b_m,b_n^\dagger\right\} = \left\{d_m,d_n^\dagger\right\} = \delta_{mn}, \tag{7.21}$$

where we have introduced the simplified notation $b_n \equiv b(p_n^+)$, etc.

It is straightforward to find the currents j^+, j^- in Fock representation:

$$j^+(x) = \frac{1}{L}\sum_{m,n}\Big[b_m^\dagger b_n e^{i(p-q).x} + b_m^\dagger d_n^\dagger e^{i(p+q).x} + d_m b_n e^{-i(p+q).x} - d_n^\dagger d_m e^{-i(p-q).x}\Big], \quad (7.22)$$

$$j^-(x) = -\frac{1}{L}\sum_{m,n}\frac{m^2}{p_m^+ p_n^+}\Big[b_m^\dagger b_n e^{i(p-q).x} - b_m^\dagger d_n^\dagger e^{i(p+q).x} - d_m b_n e^{-i(p+q).x} - d_n^\dagger d_m e^{-i(p-q).x}\Big]. \quad (7.23)$$

The currents J^+, J^- have the same form with $a_n, a_n^\dagger, c_n, c_n^\dagger$ instead of $b_n, b_n^\dagger, d_n, d_n^\dagger$. Now it is simple to check that the correct form of the "potentials", satisfying (7.17) is

$$\sigma(x) = -\frac{i\sqrt{\pi}}{L}\sum_{m,n}\Big[b_m^\dagger b_n \frac{e^{i(p-q).x}}{p_m^+ - q_n^+} + b_m^\dagger d_n^\dagger \frac{e^{i(p+q).x}}{p_m^+ + p_n^+} - d_m b_n \frac{e^{-i(p+q).x}}{p_m^+ + q_n^+} + d_m^\dagger d_m \frac{e^{-i(p-q).x}}{p_m^+ - q_n^+}\Big], \quad (7.24)$$

$$\Sigma(x) = -\frac{i\sqrt{\pi}}{L}\sum_{m,n}\Big[a_m^\dagger a_n \frac{e^{i(p-q).x}}{p_m^+ - q_n^+} + a_m^\dagger c_n^\dagger \frac{e^{i(p+q).x}}{p_m^+ + p_n^+} - c_m a_n \frac{e^{-i(p+q).x}}{p_m^+ + q_n^+} + c_m^\dagger c_m \frac{e^{-i(p-q).x}}{p_m^+ - q_n^+}\Big]. \quad (7.25)$$

The appearance of $\sqrt{\pi}$ in the above definitions will be explained in a moment. In the sums in Eqs.(7.24), (7.25), the $m = n$ terms should be omitted for the $b^\dagger b$ and $d^\dagger d$ terms because these correspond to the zero-mode part of e.g. the j^+ current, $j_0^+ = Q/L, Q = \sum_n \big(b_n^\dagger b_n - d_n^\dagger d_n\big)$. Its inclusion would violate periodicity of $\sigma(x^-)$. The same remark applies for the $\Sigma(x^-)$ potential. The Fock representation of $\sigma(x), \Sigma(x)$ is obtained by an "educated guess" from their definitions because the momentum dependence of the currents is very simple (this is not the case for the conventional space-like treatment of the model [41]). The same result follows from the formal inversions (7.12) after inserting the expansions (7.23).

Some calculations get simplified if one uses instead of the currents (7.23) their bosonized form obtained after taking the Fourier transform of the expressions (7.23). Writing

$$\begin{aligned} j^+(x) &= j_0^+ + \frac{1}{L}\sum_{n=1}\Big[A_n e^{-ip.x} + A_n^\dagger e^{ip.x}\Big] \\ J^+(x) &= J_0^+ + \frac{1}{L}\sum_{n=1}\Big[C_n e^{-ip.x} + C_n^\dagger e^{ip.x}\Big] \end{aligned} \quad (7.26)$$

and taking the inverse Fourier transformation

$$A_n = \int_{-L}^{+L}\frac{\mathrm{d}x^-}{2} j^+(x) e^{\frac{i}{2}q_n^+ x^-}, \quad C_n = \int_{-L}^{+L}\frac{\mathrm{d}x^-}{2} J^+(x) e^{\frac{i}{2}q_n^+ x^-} \quad (7.27)$$

we find

$$A_l = \sum_{k=\frac{1}{2}}^{\infty} \Big(b_k^\dagger b_{l+k} + d_k^\dagger d_{l+k} + d_{l-k} b_k \Big),$$

$$A_l^\dagger = \sum_{k=\frac{1}{2}}^{\infty} \Big(b_{l+k}^\dagger b_k + d_{l+k}^\dagger d_k + b_k^\dagger d_{l-k}^\dagger \Big). \tag{7.28}$$

and an analogous expressions for $C_n,\ C_n^\dagger$ in terms of a, c operators. Note that the summation in the last terms runs only up to $l-1/2$ because the index of the Fock operator corresponding to the LF momentum can only be positive. Also, being fermion bilinears, the "fusion" operators $A_l, A_l^\dagger$ are labeled by integers, although the fermionic index in the sums (7.28) runs over half-integers. The time dependence of the current (7.26) is found to be very simple, $\exp\big(\pm \frac{i}{2}\frac{m^2}{p_n^+}\big)$, as a result of cancellations in the exponents of (7.27) caused by the momentum Kronecker symbols.

A straightforward but lengthy calculation based on a multiple use of the basic Fock anticommutation relation reveals the commutation relation

$$\big[A_k, A_l^\dagger\big] = \big[C_k, C_l^\dagger\big] = k\delta_{kl} \tag{7.29}$$

obeyed by these composite Fock operators which also satisfy $A_l|0\rangle = 0,\ C_l|0\rangle = 0$, as follows from their representation (7.28). This means that the original fermionic Fock vacuum is also the vacuum of the composite bosonic operators.

Inserting now the expressions (7.26) into the definitions (7.12), we also have

$$\begin{aligned}
\sigma(x) &= \frac{i}{L}\sqrt{\pi}\sum_{n=1}\frac{1}{p_n^+}\Big[A_n e^{-\frac{i}{2}p_n^+x^- - \frac{i}{2}\frac{m^2}{p_n^+}x^+} - A_n^\dagger e^{\frac{i}{2}p_n^+x^- + \frac{i}{2}\frac{m^2}{p_n^+}x^+}\Big],\\
\Sigma(x) &= \frac{i}{L}\sqrt{\pi}\sum_{n=1}\frac{1}{p_n^+}\Big[C_n e^{-\frac{i}{2}p_n^+x^- - \frac{i}{2}\frac{m^2}{p_n^+}x^+} - C_n^\dagger e^{\frac{i}{2}p_n^+x^- + \frac{i}{2}\frac{m^2}{p_n^+}x^+}\Big].
\end{aligned} \tag{7.30}$$

The commutators for the composite scalar field $\sigma(x)$ at $x^+ = 0$ is then equal to

$$\big[\sigma(x^-), \sigma(y^-)\big] = \frac{\pi}{4}\int_{-L}^{+L}\frac{\mathrm{d}u^-}{2}\int_{-L}^{+L}\frac{\mathrm{d}v^-}{2}\epsilon_N(x^- - u^-)\epsilon_N(y^- - v^-)\big[j^+(u^-), j^+(v^-)\big],$$

and analogously for $\Sigma(x)$. If one calculates the current-current comutator naively in the x-representation, one obtains zero. This result is however wrong because the latter commutator is actually equal to the so-called Schwinger term proportional to a derivative of the delta function:

$$\big[j^+(x^-), j^+(y^-)\big] = \frac{i}{\pi}\partial_-^x \delta_N(x^- - y^-). \tag{7.31}$$

The corresponding momentum-space calculation is lengthy but straightforward. Then we get

$$\big[\Sigma(x^-), \Sigma(y^-)\big] = \big[\sigma(x^-), \sigma(y^-)\big] = -\frac{i}{8}\epsilon_N(x^- - y^-). \tag{7.32}$$

These relations show that the composite fields have been rescaled above by a factor $\sqrt{\pi}$ in order to satisfy the canonical commutation relation of a scalar field, Eq.(3.12).

Actually, it is quite easy to calculate the full Pauli-Jordan function of the $\sigma(x), \Sigma(x)$ fields by means of their bosonized form (7.30) because we know their time dependence. One has

$$\big[\sigma(x), \sigma(y)\big] = \frac{\pi}{L^2}\sum_{m,n}\frac{1}{\sqrt{p_m^+ p_n^+}}\Big\{\big[A_m^\dagger, A_n\big]e^{ip.x}e^{-iq.y} - \big[A_m, A_n^\dagger\big]e^{-ip.x}e^{iq.y}\Big\} =$$

$$= \frac{1}{2L}\sum_n \frac{1}{p_n^+}\Big[e^{-\frac{i}{2}p_n^+(x^- - y^-) - \frac{i}{2}\frac{m^2}{p_n^+}(x^+ - y^+)} - e^{\frac{i}{2}p_n^+(x^- - y^-) + \frac{i}{2}\frac{m^2}{p_n^+}(x^+ - y^+)}\Big]. \tag{7.33}$$

The same result holds also for $\Sigma(x)$.

One should give a precise mathematical meaning to the operators $\exp\big(-\frac{ig}{\sqrt{\pi}}\Sigma(x)\big)$ and $\exp\big(-\frac{ig}{\sqrt{\pi}}\sigma(x)\big)$ in quantum theory. These operators are singular due to the infinity of field modes without a proper renormalization. Here we will follow the approach of Coleman [43] in which one subtracts the two-point correlation function at the origin.

The exponential operators are then well-defined and one can calculate the equal-time commutators of the interacting Fermi fields as well as the same commutators for arbitrary unequal times and various n-point correlation functions.

The LF Hamiltonian of the Federbush model has a surprisingly simple structure. Performing the Legendre transformation as in the case of the free massive fermions and inserting the fermionic constraint (7.5), we find

$$P^- = m\int_{-L}^{+L}\frac{\mathrm{d}x^-}{2}\big[\chi_2^\dagger\chi_1 + \chi_1^\dagger\chi_2\big] + M\int_{-L}^{+L}\frac{\mathrm{d}x^-}{2}\big[\Psi_1^\dagger\Psi_2 + \Psi_2^\dagger\Psi_1\big]. \tag{7.34}$$

But since the exponential factors cancel in the product of interacting fields $\chi(x)$ and $\Psi(x)$, the Hamiltonian is the free one,

$$P^- = m\int_{-L}^{+L}\frac{\mathrm{d}x^-}{2}\big[\varphi_2^\dagger\varphi_1 + \varphi_1^\dagger\varphi_2\big] + M\int_{-L}^{+L}\frac{\mathrm{d}x^-}{2}\big[\psi_1^\dagger\psi_2 + \psi_2^\dagger\psi_1\big]. \tag{7.35}$$

In Fock representation, it has the form presented in Eq.(3.43). Despite the fact that the LF Hamiltonian of the LF Federbush model coincides with the free Hamiltonian, it generates the correct dynamical equations of motion (7.5) (the Heisenberg equations) when commuted with the interacting fields $\chi_2(x)$ and $\Psi_2(x)$.

It is instructive to compare the Hamiltonians of the Federbush model in the LF and SL formalisms.

The interacting part of the Fock Hamiltonian in the usual space-like theory is obtained by inserting the fermion field expansions (2.44) into the definitions of the currents in the Hamiltonian

$$H_{int} = g\int_{-L}^{+L}\mathrm{d}x\big(j^0(x)J^1(x) - j^1(x)J^0(x)\big). \tag{7.36}$$

The computation is straightforward and the main steps involve the space integration which yields a Kronecker symbol reducing number of momenta to three, and the use of spinor identities for different momenta. They are listed in the Appendix B.

The resultant interacting Hamiltonian has a rather complicated four-fermion form

$$
\begin{aligned}
H_{int} = -\frac{1}{2L} g \sum_{p,q,r} \\
&\Big\{ a^\dagger(p)a(q)b^\dagger(r)b(p+r-q)\Big[f_1(p,q)f_3(r,p+r-q) - f_1(r,p+r-q)f_3(p,q)\Big] \\
&+a^\dagger(p)a(q)b^\dagger(r)d^\dagger(q-p-r)\Big[f_1(p,q)f_4(q-p-r) - f_2(r,q-p-r)f_3(p,q)\Big] \\
&-a^\dagger(p)a(q)d(r)b(p-q-r)\Big[f_1(p,q)f_4(r,p-q-r) - f_3(p,q)f_2(r,p-q-r)\Big] \\
&+a^\dagger(p)a(q)d^\dagger(q+r-p)d(r)\Big[f_1(p,q)f_3(r,q-p+r) + f_3(p,q)f_1(r,q+r-p)\Big] \\
&+a^\dagger(p)c^\dagger(q)b^\dagger(r)b(p+q+r)\Big[f_2(p,q)f_3(r,p+q+r) - f_4(p,q)f_1(r,p+q+r)\Big] \\
&-a^\dagger(p)c^\dagger(q)d(r)b(p+q-r)\Big[f_2(p,q)f_4(r,p+q-r) + f_4(p,q)f_2(r,p+q-r)\Big] \\
&+a^\dagger(p)c^\dagger(q)d^\dagger(r-p-q)d(r)\Big[f_2(p,q)f_3(r,r-p-q) + f_4(p,q)f_1(r,r-p-q)\Big] \\
&+c(p)a(q)b^\dagger(r)b(r-p-q)\Big[f_2(p,q)f_3(r,r-p-q) + f_4(p,q)f_1(r,r-p-q)\Big] \\
&+c(p)a(q)b^\dagger(r)d^\dagger(p+q-r)\Big[f_2(p,q)f_4(r,p+q-r) + f_4(p,q)f_2(r,p+q-r)\Big] \\
&+c(p)a(q)d^\dagger(p+q+r)d(r)\Big[f_2(p,q)f_3(r,p+q+r) - f_4(p,q)f_1(r,p+q+r)\Big] \\
&-c^\dagger(q)c(p)b^\dagger(r)b(q+r-p)\Big[f_1(p,q)f_3(r,q+r-p) + f_3(p,q)f_1(r,q+r-p)\Big] \\
&-c^\dagger(q)c(p)b^\dagger(r)d^\dagger(p-q-r)\Big[f_1(p,q)f_4(r,p-q-r) + f_3(p,q)f_2(r,p-q-r)\Big] \\
&+c^\dagger(q)c(p)d(r)b(q-p-r)\Big[f_1(p,q)f_4(r,q-p-r) - f_3(p,q)f_2(r,q-p-r)\Big]
\end{aligned}
$$

$$-c^\dagger(q)c(p)d^\dagger(p+r-q)d(r)\Big[f_1(p,q)f_3(r,p+r-q)-f_3(p,q)f_1(r,p+r-q)$$

$$+a^\dagger(p)c^\dagger(q)b^\dagger(r)d^\dagger(-P)\Big[f_2(p,q)f_4(r,-P)-f_4(p,q)f_2(r,-P)\Big]$$

$$-c(p)a(q)d(r)b(-P)\Big[f_2(p,q)f_4(r,-P)-f_4(p,q)f_2(r,-P)\Big]\Big\}, \tag{7.37}$$

where $P = p + r + q$. We have also used the notation

$$f_1(p,q) = \sqrt{\frac{E_p+M}{2E_p}}\sqrt{\frac{E_q+M}{2E_q}}\left(1+\frac{pq}{(E_p+M)(E_q+M)}\right),$$
$$f_2(p,q) = \sqrt{\frac{E_p+M}{2E_p}}\sqrt{\frac{E_q+M}{2E_q}}\left(\frac{p}{E_p+M}+\frac{q}{(E_q+M)}\right),$$
$$f_3(p,q) = -\sqrt{\frac{E_p+m}{2E_p}}\sqrt{\frac{E_q+m}{2E_q}}\left(1-\frac{pq}{(E_p+m)(E_q+m)}\right),$$
$$f_4(p,q) = \sqrt{\frac{E_p+m}{2E_p}}\sqrt{\frac{E_q+m}{2E_q}}\left(\frac{p}{E_p+m}-\frac{q}{(E_q+m)}\right). \tag{7.38}$$

In comparison with this a bit cumbersome Hamiltonian, the simplicity of the LF Hamiltonian is quite remarkable. The main reason for the striking difference between the two representations is the simplified LF description of the "spinor" structure of two-dimensional massive fermions.

Additional aspects of the LF Federbush model, its regularization by using the so-called triple-dot ordering, calculation of the various correlation functions and comparison with the space-like treatment [42] requires further studies.

8 LF picture of spontaneous symmetry breaking

Spontaneous symmetry breaking (SSB) is a fundamental non-perturbative phenomenon of quantum field theory. It occurs when the Hamiltonian of a theory is symmetric under a group of transformations while the ground state is non-invariant. This implies that the vacuum is not unique since its non-invariance means that it is transformed to another vacuum state. Hence there must be a family of ground states which all correspond to the same energy, i.e. they are degenerate. The degeneracy follows from a simple consideration. Let $U = \exp\big(i\alpha Q\big)$ is the unitary operator that implements the symmetry, $U(\alpha)HU^{-1}(\alpha) = H$, where Q is the generator - i.e. the charge equal to the volume integral of the symmetry current. If $|0\rangle$ is a vacuum state, corresponding to the energy E_0, then

$$U(\alpha)H|0\rangle = E_0U(\alpha)|0\rangle = U(\alpha)HU^{-1}(\alpha)U(\alpha)|0\rangle = HU(\alpha)|0\rangle. \tag{8.1}$$

We can see that the new vacuum state, $|\alpha\rangle \equiv U(\alpha)|0\rangle$, is the eigenstate of the Hamiltonian H with the same eigenvalue E_0. It can be shown that there exists also a field operator (elementary or composite) with non-zero expectation value in our vacuum state. For continuous symmetries, if there exists a corresponding conserved current and the property of locality is satisfied, it follows that the spectrum of such a theory contains a massless state, the Nambu-Goldstone boson [44,45, 46], if the space dimension is greater than one [47]. This is the content of the Goldstone theorem.

This overall picture of the broken phase is well understood in the conventional field theory. On the other hand, SSB still remains a bit mysterious in the light-front field theory. The main reason for difficulties with obtaining a clear picture of SSB in the LF theory is the simplicity of the LF vacuum discussed previously. In the following, we will develop a simple LF picture of the symmetry breaking. It is based on a concept which is close to the scheme known from the conventional field theory. Let us first illustrate the idea of the SSB in the simplest setting of the $\lambda\phi^4$ theory in two dimensions. Since the field $\phi(x)$ is real, there is only a discrete symmetry $\phi(x) \to -\phi(x)$ following from the fact that only even powers of the field or its derivative are present in the Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\lambda}{4}\big(\phi - v\big)^4. \tag{8.2}$$

For positive value of the parameter v, the quadratic term has negative sign and the classical potential has a form of double well with minima at $\phi = \pm v$. In quantum description, we have $\langle\Omega_\pm|\phi(x)|\Omega_\pm\rangle = \pm v, \langle 0|\phi(x)|0\rangle = 0$. Here, $|\Omega_\pm\rangle$ denotes the two "non-trivial" vacua which correspond to the minimum of the potential in (8.2) and $|0\rangle$ is the "false" vacuum corresponding to the local maximum of the potential. To have the usual physical situation, we should build the theory on the state corresponding to the true minimum of the energy, i.e. we should shift the field $\phi(x)$ to $\varphi(x) = \phi(x) - v$ so that $\langle\Omega_+|\varphi(x)|\Omega_+\rangle = 0$. This is accomplished by a unitary operator $U(v)$:

$$\langle 0|\phi(x)|0\rangle = 0 = \langle 0|U^{-1}(v)\varphi(x)U(v)|0\rangle = \langle\Omega_+|\varphi(x)|\Omega_+\rangle. \tag{8.3}$$

The "physical" vacuum $|\Omega_+\rangle$ is obviously a shifted state

$$|\Omega_+\rangle = U(v)|0\rangle, \;\; U(v) = \exp\big(-iv\int \mathrm{d}x\Pi(x)\big), \tag{8.4}$$

where $\Pi(x)$ is the conjugate momentum, $\big[\phi(t,x), \Pi(0,y)\big]_{t=0} = i\delta(x-y)$. The operator identity $e^A B e^{-A} = B + [A,B]$, valid if the commutator $[A,B]$ is a c-number, is useful in showing that $U(v)$ shifts the field $\phi(x)$ by the constant value v. It is possible to derive a Fock representation of the vacua $|\Omega\rangle$ if one considers the system in a box with periodic boundary conditions. With the Fock expansion (2.3), we find

$$U(v) = \exp\Big[-iv\sqrt{mL}i[a_0^+ - a_0]\Big],$$

$$|\Omega_+\rangle = \exp\big[-\frac{mL}{2}v^2\big]\exp\big[\sqrt{mL}va_0^+\big]|0\rangle, \tag{8.5}$$

because the x-integration leads to $\delta(p)$ which separates just the zero mode from the infinite number of Fourier modes in the field expansion. Thus, in this semiclassical description, the physical vacuum is a coherent state of the scalar-field zero mode which is a dynamical degree of freedom. The whole derivation could have been based alternatively on the second vacuum $|\Omega_-\rangle$

with the identical result. Both vacua are connected by the "parity" transformation $a_0^\dagger \rightarrow -a_0^\dagger$ which follows from $\phi(x) \rightarrow -\phi(x)$. Finally, it is simple to establish the orthogonality of the two vacua in the continuum limit $L \rightarrow \infty$:

$$\langle \Omega_+ | \Omega_- \rangle = e^{-mLv^2} \langle 0 | e^{\sqrt{mL} a_0} e^{\sqrt{mL} a_0^+} | 0 \rangle = e^{-2mLv^2} \tag{8.6}$$

in agreement with the discussion concerning the inequivalent representations of the Fock commutation relations, see Eqs.(2.40) and (2.41). Note also that the state $|0\rangle$ which was transformed to the "physical" vacua $|\Omega_\pm\rangle$ is not an eigenstates of the full interacting Hamiltonian

$$H = \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}x^-}{2} \Big[\big(\partial_0 \phi\big)^2 + \big(\partial_1 \phi\big)^2 + \frac{\lambda}{4} \big(\phi - v\big)^2 \Big], \tag{8.7}$$

because the last term contains besides the quadratic and the constant term also the quartic self-interaction. The latter will generate operator structure $a^\dagger(p_1) a^\dagger(p_2) a^\dagger(p_3) a^\dagger(p_4)$ which will not annihilate the Fock vacuum $|0\rangle$. Hence the true vacuum even before its transformation to $|\Omega_\pm\rangle$ is unknown and working with $|0\rangle$ which is the eigenstate of H_0, is just an approximation.

It is generally believed that this type of description of spontaneous symmetry breaking is impossible in the LF theory because the Fock vacuum, i.e. the state without particles, is the "final" physical vacuum, since it is an eigenstate of the complete Hamiltonian, not only its free part. Hence there is just one unique vacuum, and no degenerate set, necessary for the standard picture of SSB, can be constructed. The detailed reason is that charges, i.e. the symmetry generators in a scalar theory always annihilate the LF Fock vacuum because due to positivity of the momentum p^+ they cannot contain terms composed of purely creation operators [48, 49] if there are no dynamical zero modes, i.e. independent degrees of freedom corresponding to $p^+ = 0$, in the theory. Put in the simplest way, these terms are absent because they are multiplied by a delta function expressing the momentum conservation whose argument can never vanish for positive momenta. Without such terms it is not possible to transform the LF Fock vacuum into a more complex object and one cannot construct multiple vacua.

The prevailing opinion in the LF literature is that the the vacuum physics is encoded in the zero-mode constraint [50, 51, 38]. For example, starting from the symmetric phase of the two-dimensional $\lambda\phi^4$ with periodic boundary condition, an approximative non-perturbative solution of the operator zero mode constraint

$$\phi_0 = -\frac{\lambda}{6\mu^2} \int\limits_{-L}^{+L} \frac{\mathrm{d}x^-}{2L} (\phi_0 + \varphi)^2 \tag{8.8}$$

showed two branches above certain critical coupling replacing the picture with two vacua. Inserting these two solutions into the LF Hamiltonian one found two Hamiltonians corresponding to these doubly-degenerate vacuum states.

In four dimensions, it is usually assumed that the scalar zero mode contains a constant piece. As a consequence, symmetry breaking is found to manifest itself in a rather unusual way by a non-conservation of the current even in the symmetry limit while the physical vacuum is identified with the Fock vacuum [52, 53].

As already indicated, here we will develop a different scenario for the description of the broken phase. The "trivial" LF vacuum, being a simple but rigorously defined non-perturbative state, will be viewed as an intermediate object, not the ultimate physical vacuum. This is possible due to a simple observation that for scalar theories with polynomial self-interaction and negative quadratic term the LF Fock vacuum is not the state of minimum LF energy [54]. The energy is minimized by a specific coherent state and this state is *not* annihilated by the symmetry generators. Hence, the unitary operators implementing the discrete or continuous symmetry will generate, when applied to this state, a discrete or continuous set of new (semiclassical) vacuum states. One might expect that a unitary operator could be constructed which would shift the scalar field $\phi(x)$ to the true minimum of the LF energy in the way we discussed for the space-like theory. Unfortunately, for $\phi(x^+, x^-)$ periodic in the space coordinate x^-, such a construction is very difficult. This is due to the complicated non-linear operator zero-mode constraint. On the other hand, choosing antiperiodic boundary conditions in x^- [55] (which is a consistent choice for polynomial interactions with even powers of fields) allows one to define shift operators which transform the Fock vacuum to new states that correspond to lower LF energy. They are coherent states of large but finite number of Fourier modes. We will illustrate this mechanism for two models [56]. The first one is the two-dimensional $\lambda\phi^4$ theory in broken phase (8.2) possessing classically two degenerate ground states. The second model is a three-dimensional O(2)-symmetric linear sigma model. It has a continuum of degenerate vacuum states and one can expect the Goldstone phenomenon to take place. Both models are superrenormalizable. Renormalization can be performed by normal ordering the Hamiltonian or equivalently by adding a mass counterterm (an operator with two fields contracted to a point, called a "tadpole") in the first case and a tadpole together with the second-order self-energy counterterm in the second case [57, 58].

A separate problem related to the symmtery breaking is a transition from the symmetric phase of the theory to the broken phase. To calculate characteristic of this phase transition like the value of the critical coupling and critical indices, is a challenge for the DLCQ method because these properties are associated with the behaviour of systems at large distances. The first encouraging results in have [59]. These quantities have been computed in the continuum LF theory from the zero-mode dynamics [19].

Let us look closer at two two-dimensional $\lambda\phi^4$ theory in the broken phase. We will use a version of the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4}\phi^4,\ \mu^2 > 0, \tag{8.9}$$

differing from (8.2) by a term $\lambda v^2/4!$ which only shifts energy levels by an irrelevant constant. Lagrangian (8.9) is invariant under the discrete transformation of the real scalar field $\phi(x) \to -\phi(x)$. Classically, the potential energy in (8.9) has two minima at $\phi_c = \pm\mu/\sqrt{\lambda}$. As was already shown, in the tree-level analysis, one usually shifts the field by $\pm\phi_c$ and obtains two Lagrangians which reveal the particle spectrum of the theory in terms of "small" oscillations above ϕ_c. The original symmetry becomes hidden in the sense that the two Lagrangians are individually not symmetric under $\phi(x) \to -\phi(x)$ but the symmetry operation transforms one to the other. We have also seen that due to the existence of more than one minimum of the potential, the model exhibits in addition to symmetry breaking also nontrivial topological properties [60]. There exist solutions of the classical equations of motion with finite energy which interpolate between the minima. They carry a conserved topological charge, proportional to the difference of the field values at the boundaries, and corresponding to the conserved topological current $k^\rho = \frac{\sqrt{\lambda}}{\mu}\epsilon^{\rho\nu}\partial_\nu\phi$.

The Lagrangian (8.9) is expressed in terms of the LF variables as

$$\mathcal{L}_{lf} = 2\partial_+\phi\partial_-\phi + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4}\phi^4, \tag{8.10}$$

where $\partial_\pm = \partial/\partial^\pm$. We restrict the spatial coordinate by $-L \leq x^- \leq L$. In order to obtain a clear physical picture of SSB we wish to avoid the difficult zero-mode problem present in the case of periodic BC. The point is that it is not clear how one could solve the corresponding ZM operator constraint in a non-perturbative manner (for approximative solution in the continuum theory, see [19]). We impose therefore the antiperiodic boundary condition $\phi(L) = -\phi(-L)$ which results in discrete Fourier modes

$$p_n^+ = \frac{2\pi}{L}n, \quad n = 1/2, 3/2, \dots\infty. \tag{8.11}$$

The antiperiodic BC also implies that in the quantum theory we can define the operator of the topological charge $Q = \frac{\sqrt{\lambda}}{\mu}[\phi(L) - \phi(-L)] = 2\frac{\sqrt{\lambda}}{\mu}\phi(L)$.

The standard canonical treatment yields the energy-momentum tensor components T^{+-} and T^{++} which define the LF Hamiltonian P^-

$$P^- = \frac{1}{2}\int_{-L}^{+L}\frac{\mathrm{d}x^-}{2}T^{+-}(x^-) = \frac{1}{2}\int_{-L}^{+L}\frac{\mathrm{d}x^-}{2} : \big[-\mu^2\phi^2 + \frac{\lambda}{2}\phi^4\big] : . \tag{8.12}$$

as well as the LF momentum operator

$$P^+ = \frac{1}{2}\int_{-L}^{+L}\frac{\mathrm{d}x^-}{2}T^{++}(x^-) = \frac{1}{2}\int_{-L}^{L} dx^- 4 : \big[\partial_-\phi\partial_-\phi\big] : . \tag{8.13}$$

The field expansion at $x^+ = 0$ in terms of the Fourier modes reads

$$\phi(0, x^-) = \frac{1}{\sqrt{2L}}\sum_{n=1/2}^{\infty}\frac{1}{\sqrt{p_n^+}}\big[a_n e^{-\frac{i}{2}p_n^+ x^-} + a_n^\dagger e^{\frac{i}{2}p_n^+ x^-}\big]. \tag{8.14}$$

The annihilation and creation operators are required to satisfy the quantization condition

$$[a_m, a_n^\dagger] = \delta_{mn}. \tag{8.15}$$

As a consequence, one recovers the familiar commutator at equal LF times,

$$\big[\phi(0, x^-), \phi(0, y^-)\big] = -\frac{i}{8}\epsilon_a(x^- - y^-), \tag{8.16}$$

where $\epsilon_a(x^-)$ is the antiperiodic sign function

$$\epsilon_a(x^-) = \frac{4i}{L}\sum_{n=1/2}^{\infty}\frac{1}{p_n^+}\big[e^{-\frac{i}{2}p_n^+ x^-} - e^{\frac{i}{2}p_n^+ x^-}\big], \tag{8.17}$$

defined in terms of the discrete momenta (8.11). Recall that the conjugate momentum Π_ϕ is not equal to the time derivative of the scalar field in the LF theory. It is a dependent variable, determined by $\phi(x)$ itself, $\Pi_\phi = 2\partial_-\phi$ [7]. Hence, the alternative form of the basic commutation relation, following from Eq.(8.16), is

$$\big[\phi(0,x^-),\Pi_\phi(0,y^-)\big] = \frac{i}{2}\delta_a(x^- - y^-), \tag{8.18}$$

where $\delta_a(x^-)$ is the antiperiodic delta function, $\delta_a(x^-) = 1/2\partial_-\epsilon_a(x^-)$. The same quantization rules can be obtained more rigorously by the Dirac-Bergmann method [61] for constrained systems (see the Appendix).

Consider now a unitary operator

$$U(b) = \exp\big[-2ib\int\limits_{-L}^{+L}\frac{\mathrm{d}x^-}{2}\Pi_\phi(x^-)\big]. \tag{8.19}$$

For antiperiodic boundary conditions, it reduces to

$$U(b) = e^{-8ib\phi(L)} \tag{8.20}$$

and translates the field $\phi(x^-)$ by a constant b as can be again shown by means of the operator relation $\exp(A)B\exp(-A) = B + [A,B] + \dots$:

$$\begin{aligned} U(b)\phi(x^-)U^{-1}(b) &= \phi(x^-) - 8ib\big[\phi(L),\phi(x^-)\big] \\ &= \phi(x^-) - b\epsilon_a(L-x^-). \end{aligned} \tag{8.21}$$

Thus, the antiperiodic scalar field can be shifted by a constant without violating its antiperiodicity. The reason for that is the simple property of the sign function $\epsilon_a(L-x^-)$: it is equal to 1 for all x^- in the box except for the endpoints where it drops to zero. This is of course a direct consequence of the basic property $\epsilon_a(0) = \epsilon_a(2L) = 0$. It is much more difficult to perform a similar shift of the field in the case of periodic boundary condition because of the presence of the a priori unknown operator zero mode. As demonstrated in Eq.(8.5), the volume integration in the shift operator analogous to Eq.(8.19) projects out only its zero-mode component in the usual theory.

We should note however that the above considerations were a bit formal and the actual situation is slightly more complicated. The point is that the operator $U(b)$ (8.20) exists (is non-zero) only if we impose a cutoff on the number of modes (see Eq.(8.26) and the discussion after Eq.(8.32)). Consequently, the sign function in (8.21) is replaced by a truncated series $\epsilon_\Lambda(x^-)$ defined by Eq.(8.17) with $n \le \Lambda$.

We may use $U(b)$ to generate a family of shifted vacuum states $|b\rangle = U(b)|0\rangle$, where $|0\rangle$ is the Fock vacuum, $a_n|0\rangle = 0$. Can one of these states be a better candidate for the true physical vacuum? To determine this, let us minimize the expectation value of the LF Hamiltonian,

$$\langle b|P^-|b\rangle = \langle 0|U^{-1}(b)P^-U(b)|0\rangle = \langle 0|\frac{1}{2}\int\limits_{-L}^{+L}\frac{\mathrm{d}x^-}{2}T_b^{+-}(x^-)|0\rangle \tag{8.22}$$

where

$$T_b^{+-}(x^-) =: \big[-\mu^2\big(\phi + b\epsilon_\Lambda(L - x^-)\big)^2 + \frac{\lambda}{2}\big(\phi + b\epsilon_\Lambda(L - x^-)\big)^4\big] : \tag{8.23}$$

As illustrated in the figures of the Appendix, for sufficiently large value of Λ the powers of $\epsilon_\Lambda(L - x^-)$ in (8.23) differ only negligibly from 1 at the interval $-L \le x^- \le L$ and we will therefore suppress them henceforth in the formulae where they appear. Thus, we find $\langle b|P^-|b\rangle = Lb^2(\frac{\lambda}{2}b^2 - \mu^2)$ which has a non-trivial minimum for $b^2 = \frac{\mu^2}{\lambda} \equiv v^2$. The LF energy density is lower in the new vacuum $|v\rangle$:

$$\langle v|\frac{P^-}{2L}|v\rangle = -\frac{\mu^4}{4\lambda} \quad < \quad \langle 0|\frac{P^-}{2L}|0\rangle = 0. \tag{8.24}$$

The vacuum expectation value (VEV) of the scalar field in this state coincides with the position of the minimum of the classical potential:

$$\langle v|\phi(x^-)|v\rangle = \langle 0|U^{-1}(v)\phi(x^-)U(v)|0\rangle = \frac{\mu}{\sqrt{\lambda}}\epsilon_\Lambda(x^- - L) = \frac{\mu}{\sqrt{\lambda}}. \tag{8.25}$$

The last equality holds in the approximative sense described above. Due to the finite number of Fourier modes, the function $\epsilon_\Lambda(L-x^-)$ does not have an exactly rectangular shape but is smooth in the neighborhood of the points $x^- = \pm L$ (see the Figure).

Inserting the field expansion (8.14) into the definition of $U(v)$, we get a coherent state representing the physical vacuum of the model in the semi-quantum approximation:

$$|v\rangle = \exp\big\{v\sum_{n=1/2}^{\Lambda}\tilde{c}_n\big(a_n^\dagger - a_n\big)\big\}|0\rangle = \mathcal{N}\exp\big\{v\sum_{n=1/2}^{\Lambda}\tilde{c}_n a_n^\dagger\big\}|0\rangle, \tag{8.26}$$

where

$$\tilde{c}_n = 4(-1)^{n-1/2}/\sqrt{\pi n}, \quad \mathcal{N} = \exp\big\{-\frac{v^2}{2}\sum_{n=1/2}^{\Lambda}\tilde{c}_n^2\big\} \approx \exp\big\{-\frac{8v^2}{\pi}\ln\Lambda\big\}. \tag{8.27}$$

Notice that the coherent states (8.26) are L-independent and also correctly normalized, $\langle v|v\rangle = 1$. Further, the scalar product $\langle -v|v\rangle = \mathcal{N}^4 = \Lambda^{-32v^2/\pi}$ and thus the overlap between the two vacua vanishes in the limit $\Lambda \to \infty$. This means that, in contrast to the space-like theory, the two vacua are orthogonal even in the finite volume as long as the number of degrees of freedom is infinite. The corresponding multiparticle spaces are also orthogonal. They can be generated by applying creation operators $a_n^\dagger$ on $|v\rangle$. Alternatively, one can transform just the original Fock states, built on $|0\rangle$, by means of $U(v)$ [62]. The Hamiltonian matrix elements will be (up to normalization) of the form

$$\langle 0|a_{m_1}a_{m_2}...a_{m_i}U^{-1}(v)P^-U(v)a_{n_j}^\dagger...a_{n_2}^\dagger a_{n_1}^\dagger|0\rangle. \tag{8.28}$$

In both cases the physically relevant Hamiltonian is the transformed ("effective") one, equal to $P^-_{(v)} = U^{-1}(v)P^-U(v)$ which for $\Lambda \to \infty$ approaches the form

$$P^-_{(v)} = \frac{1}{2}\int_{-L}^{+L}\frac{\mathrm{d}x^-}{2} : \big[2\mu^2\phi^2 + \frac{\lambda}{2}\phi^4 + 2\lambda v\phi^3 - \frac{\mu^4}{2\lambda}\big] : . \tag{8.29}$$

It has a correct sign of the term quadratic in ϕ and thus describes a massive scalar field with mass equal to $\sqrt{2}\mu$. However, it has lost the symmetry of the original Hamiltonian under $\phi(x) \to -\phi(x)$ – this symmetry has been broken by choosing $|v\rangle$ as the vacuum state. Actually, the theory originally had also the second ground state. This can be demonstrated by considering a unitary operator that implements the original discrete symmetry,

$$V(\pi) = \exp\big[-i\pi \sum_{n=1/2}^{\Lambda} a_n^\dagger a_n\big]. \tag{8.30}$$

It acts correctly on the creation and annihilation operators,

$$V(\pi)a_n V^-(\pi) = -a_n,\ V(\pi)a_n^\dagger V^-(\pi) = -a_n^\dagger \tag{8.31}$$

and hence leaves P^- invariant, $V(\pi)P^-V^-(\pi) = P^-$. The operator $V(\pi)$ generates the second vacuum:

$$V(\pi)|v\rangle = |-v\rangle. \tag{8.32}$$

Indeed, using the operator identity $\exp(A)\exp(B) = \exp(e^\rho B)\exp(A)$, valid if $[A, B] = \rho B$ (ρ = real parameter), we get $V(\pi)U(v) = U(-v)V(\pi)$. We also easily find $\langle -v|\phi(x^-)|-v\rangle = -v$. The corresponding "effective" Hamiltonian $P^-_{(-v)}$ in the space sector built on $|-v\rangle$ coincides with the expression (8.29) up to the opposite sign of the cubic term. Although both Hamiltonians are individually not invariant, they are connected by the parity transformation: $P^-_{(-v)} = V(\pi)P^-_{(v)}V^{-1}(\pi)$ and vice versa. Of course, we can choose any of the two vacua and the corresponding "effective" Hamiltonian to describe the physical system.

An alternative way of obtaining the coherent state vacuum (8.26) is to minimize the expectation value of the Hamiltonian in the coherent states $|\alpha\rangle$, $|\alpha\rangle \sim \exp\big(\sum \alpha_n a_n^\dagger\big)|0\rangle$, imposing the condition that the expectation value of the iantiperiodic field is constant. If one requires instead of a constant value for $\langle\alpha|\phi(x^-)|\alpha\rangle$ the value $-v$ for $-L \le x^- \le 0$ and v for $0 \le x^- \le L$, i.e. a step-like shape, one obtains a configuration that also minimizes the LF energy and qualitatively approximates a kink [63]:

$$|\alpha\rangle = \exp\big[v \sum_{n=1/2}^{\Lambda} \alpha_n\big(a_n^\dagger - a_n\big)\big]|0\rangle, \quad \alpha_n = \frac{4i}{\sqrt{\pi}}\frac{1}{\sqrt{n}}. \tag{8.33}$$

In x-representation, the state $|\alpha\rangle$ can be expressed in terms of the unitary operator $W(v)$ as

$$|\alpha\rangle = W(v)|0\rangle,\ \ W(v) = e^{i8v\phi(0)} \tag{8.34}$$

and one easily obtains

$$\langle\alpha|\phi(x^-)|\alpha\rangle = \langle 0|W^{-1}(v)\phi(x^-)W(v)|0\rangle = v\epsilon_\Lambda(x^-), \tag{8.35}$$

which indeed has a qualitative shape of a kink. Note also that the kink state $|\alpha\rangle$ is for $\Lambda \to \infty$ orthogonal to the vacuum state, $\langle v|\alpha\rangle \sim \exp\big(-\ln\Lambda\big)$. These states belong to the sectors with different topological charges:

$$\langle\alpha|Q|\alpha\rangle = v^{-1}\langle 0|W^{-1}(v)\phi(L)W(v)|0\rangle = 8i[\phi(L), \phi(0)] = \epsilon_\Lambda(L) = 1.$$

$$\langle v|Q|v\rangle = v^{-1}\langle 0|U^{-1}(v)\phi(L)U(v)|0\rangle = v^{-1}\langle 0|\phi(L)|0\rangle = 0. \tag{8.36}$$

Quantitative predictions of the properties of kink and antikink in quantum theory as obtained by LF Hamiltonian matrix diagonalizations using the DLCQ method [63, 64] will be presented in the second half of this chapter.

Finally, let us discuss the LF momentum of the coherent-state vacuum $U(v)|0\rangle$ and of the transformed Fock states $U(v)a^{\dagger}_{m_1}a^{\dagger}_{m_2}...|0\rangle$. The VEV of normal-unordered P^+ would be

$$\langle v|P^+|v\rangle = \frac{\pi}{L}\sum_{n=1/2}^{\Lambda}\big(n + \frac{32}{\pi}v^2\big). \tag{8.37}$$

The first term is removed by normal ordering. The second term, equal to $16v^2\delta_\Lambda(0)$ is a consequence of the fact that $\partial_-\epsilon_\Lambda(x^- - L) = 2\delta_\Lambda(x^- - L)$ which for $\Lambda \to \infty$ is singular just at the endpoints $x^- = \pm L$. This term is generated in the vacuum expectation value of P^+ (8.13) due to the shift (8.21). For finite Λ this constant C is a finite number. It is also present in the expectation values of the LF momentum of particle states:

$$\langle 0|a_l U^{-1}(v)P^+U(v)a^{\dagger}_l|0\rangle = p^+_l + C, \tag{8.38}$$

and similarly for many-particle states. Thus the LF momentum of the transformed states is shifted by the same constant value which is physically irrelevant since it cancels in the differences between any two levels. We shall therefore subtract this unphysical constant. Let us remark that the necessity to perform the (trivial) renormalization of the P^+ operator may seem a little unusual but actually it is natural and physically transparent: the shift of the scalar field due to $U(v)$ is equal to a constant in the whole box except for the endpoints. Hence the derivative of the shifted field is equal to the derivative of the unshifted field at the interval $-L < x^- < L$ and as a consequence the eigenvalue of the momentum operator will remain unchanged if we subtract contributions of the points $x^- = \pm L$.

Previous attempts to understand SSB in the LF theory were made either without imposing boundary conditions explicitly or by employing periodic ones [52], typically starting from the symmetric phase of the theory. Can one give a formulation of the broken phase using PBC? The problem is complicated because one has to solve the operator constraint for the dependent zero mode ϕ_0. At present, this appears possible only for small coupling, where one can use perturbation theory. Perturbative solution is however quite interesting because it corresponds to the semiclassical regime of the broken phase (small coupling implies a large value of the condensate v) and one can compare the results with the results of the previous section. The physical picture obtained by imposing antiperiodic boundary condition should be quite accurate far from the critical region, i.e. also for small value of the coupling constant. As already discussed, a derivation of a semiclassical vacuum state similar to the case of antiperiodic boundary conditions seems not to be possible for PBC. One may therefore expect that the physical vacuum state will coincide with the Fock vacuum and SSB will manifest itself by the presence of two Hamiltonians [50].

The field equation for the scalar field following from the Lagrangian (8.10) is

$$4\partial_+\partial_-\phi = \mu^2\phi + \lambda\phi^3. \tag{8.39}$$

The scalar field can be decomposed as $\phi(x) = \phi_0(x^+) + \varphi(x^+, x^-)$, with ϕ_0 being the x^--independent part carrying $p^+ = 0$. Projection of the field equation (8.39) on the zero-mode sector

$$\mu^2\phi_0 = -\lambda \int\limits_{-L}^{+L} \frac{\mathrm{d}x^-}{2} \big(\phi_0 + \varphi\big)^3 \tag{8.40}$$

shows that ϕ_0 is a dependent variable which has to be expressed in terms of all other (normal) modes [13]. The perturbative solution of the classical zero mode constraint to order λ was given by Robertson [65]. It has two physical branches:

$$\begin{aligned}
\phi_0^{(1)} &= \frac{\mu}{\sqrt{\lambda}} - \frac{3}{2}\frac{\sqrt{\lambda}}{\mu} \int\limits_{-L}^{+L} \frac{\mathrm{d}x^-}{2L}\varphi^2 - \frac{1}{2}\frac{\lambda}{\mu^2} \int\limits_{-L}^{+L} \frac{\mathrm{d}x^-}{2L}\varphi^3 \\
\phi_0^{(2)} &= -\frac{\mu}{\sqrt{\lambda}} + \frac{3}{2}\frac{\sqrt{\lambda}}{\mu} \int\limits_{-L}^{+L} \frac{\mathrm{d}x^-}{2L}\varphi^2 - \frac{1}{2}\frac{\lambda}{\mu^2} \int\limits_{-L}^{+L} \frac{\mathrm{d}x^-}{2L}\varphi^3.
\end{aligned} \tag{8.41}$$

To the given order it can be taken over to the quantum theory since there is no ordering ambiguity. Note that the solutions contain a constant piece and their structure differs completely from the perturbative solution in the symmetric phase because of the opposite sign of the μ^2-term in the field equation. Under $\varphi \to -\varphi$, we have $\phi_0^{(1)} \to -\phi_0^{(2)}$ and vice versa. When these two solutions are inserted into the PBC Hamiltonian, analogous to (8.12)

$$P^- = \frac{1}{2} \int\limits_{-L}^{+L} \frac{\mathrm{d}x^-}{2} \big[-\mu^2 \big(\phi_0 + \varphi\big)^2 + \frac{\lambda}{2} \big(\phi_0 + \varphi\big)^2 \big], \tag{8.42}$$

one indeed gets through $O(\lambda)$ two Hamiltonians

$$P^- = \frac{1}{2} \int\limits_{-L}^{+L} \frac{\mathrm{d}x^-}{2} \big[2\mu^2\varphi^2 + \frac{\lambda}{2}\varphi^4 \pm 2\mu\sqrt{\lambda}\varphi^3 - \frac{\mu^4}{2\lambda} - \frac{9}{2}\lambda\phi^2 \int\limits_{-L}^{+L} \frac{\mathrm{d}x^-}{2L}\varphi^2\big]. \tag{8.43}$$

Their structure is similar to the Hamiltonians P_v^- from the case of antiperiodic boundary conditions. Each Hamiltonian separately violates the symmetry under $\varphi \to -\varphi$ but the transformation connects them. Any of them can be chosen for calculating physical properties of the system. Their eigenstates will also be connected by the parity transformation.

As the next step, we could consider a two-dimensional theory of a self-interacting complex scalar field. The corresponding Hamiltonian has a continuous symmetry instead of the discrete one. Since the full treatment requires a discussion of the LF version of the Coleman theorem which prohibits SSB in one space dimension [47], we will instead study the $O(2)$ symmetric sigma model in two space dimensions. It is defined by the classical Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^\dagger\partial^\mu\phi + \frac{1}{2}\mu^2\phi^\dagger\phi - \frac{1}{4}\lambda(\phi^\dagger\phi)^2. \tag{8.44}$$

The system will be studied in a finite volume $V = 4LL_\perp, -L \le x^- \le L, \ -L_\perp \le x_\perp \le L_\perp$. Scalar fields are taken antiperiodic in both x^- and the transverse coordinate $x_\perp$. In terms of two real scalar fields introduced by $\phi(x) = \sigma(x) + i\pi(x)$, the corresponding LF Lagrangian density

$$\begin{aligned}\mathcal{L}_{lf} &= 2\partial_+\sigma\partial_-\sigma + 2\partial_+\pi\partial_-\pi - \frac{1}{2}(\partial_\perp\sigma)^2 - \frac{1}{2}(\partial_\perp\pi)^2 + \\ &+ \frac{\mu^2}{2}\big(\sigma^2+\pi^2\big) - \frac{\lambda}{4}(\sigma^2+\pi^2)^2 \end{aligned} \tag{8.45}$$

is invariant under $O(2)$ rotations

$$\begin{aligned}&\sigma(x) \to \sigma(x)\cos\alpha - \pi(x)\sin\alpha, \\ &\pi(x) \to \sigma(x)\sin\alpha + \pi(x)\cos\alpha. \end{aligned} \tag{8.46}$$

The associated conserved current is $j^\mu = \sigma\partial^\mu\pi - \partial^\mu\sigma\pi$. The field expansions at $x^+ = 0$ are

$$\sigma(\underline{x}) = \frac{1}{\sqrt{V}}\sum_{\underline{n}}\frac{1}{\sqrt{p_n^+}}\big[a(p_{\underline{n}})e^{-ip_{\underline{n}}\cdot\underline{x}} + a^\dagger(p_{\underline{n}})e^{ip_{\underline{n}}\cdot\underline{x}}\big], \tag{8.47}$$

$$\pi(\underline{x}) = \frac{1}{\sqrt{V}}\sum_{\underline{n}}\frac{1}{\sqrt{p_n^+}}\big[c(p_{\underline{n}})e^{-ip_{\underline{n}}\cdot\underline{x}} + c^\dagger(p_{\underline{n}})e^{ip_{\underline{n}}\cdot\underline{x}}\big]. \tag{8.48}$$

We use the notation $\underline{x} = (x^-, x_\perp), \underline{n} \equiv (n, n_\perp), p_{\underline{n}} = (p_n^+, p_{n_\perp}) = (\frac{2\pi}{L}n, \frac{\pi}{L_\perp}n_\perp)$ with $n, n_\perp = 1/2, 3/2, \dots\infty$. The conjugate momenta are $\Pi_\sigma = 2\partial_-\sigma, \Pi_\pi = 2\partial_-\pi$. The σ field operators satisfy the commutation relation

$$\big[\sigma(0,\underline{x}), \sigma(0,\underline{y})\big] = -\frac{i}{8}\epsilon_a(x^- - y^-)\delta_a(x_\perp - y_\perp). \tag{8.49}$$

The commutator of the π fields has the same form. The Hamiltonian is

$$P^- = \int_V d^2\underline{x}\big[(\partial_\perp\sigma)^2 + (\partial_\perp\pi)^2 + 2V(\sigma^2+\pi^2)\big],$$

$$V(\sigma^2+\pi^2) = -\frac{\mu^2}{2}\big(\sigma^2+\pi^2\big) + \frac{\lambda}{4}(\sigma^2+\pi^2)^2, \tag{8.50}$$

where $d^2\underline{x} = \frac{1}{2}dx^-dx_\perp$. In principle, both $\sigma(x)$ and $\pi(x)$ can be transformed by the unitary operators $U_\sigma(b)$ and $U_\pi(b)$ in analogy with Eq.(8.21). It is simpler however to start by shifting only one field which we choose in accord with the standard treatment to be $\sigma(x)$:

$$U_\sigma(b)\sigma(\underline{x})U_\sigma^\dagger(b) = \sigma(\underline{x}) - b, \tag{8.51}$$

(the $\epsilon_a(L - x^-)\epsilon_a(L - x^\perp)$ factor multiplying b is implicit here) with

$$U_\sigma(b) = \exp\big[-4ib\int_V d^2\underline{x}\Pi_\sigma(\underline{x})\big] = \exp\big[-8ib\int_{-L_\perp}^{+L_\perp} dx_\perp\sigma(L, x_\perp)\big]. \tag{8.52}$$

By minimization of $\langle b;0|P^-|b;0\rangle$, where $|b;0\rangle = U_\sigma(b)|0\rangle$, we again find that the physical vacuum $|v;0\rangle = U_\sigma(v)|0\rangle$ corresponds to the value $b^2 = \frac{\mu^2}{\lambda} \equiv v^2$ and

$$|v;0\rangle = \exp\Big\{ -v\sum_{\underline{n}} \tilde{c}(p_{\underline{n}})\big[a^\dagger(p_{\underline{n}}) - a(p_{\underline{n}})\big]\Big\}|0\rangle, \tag{8.53}$$

$$\tilde{c}(p_{\underline{n}}) = \frac{8}{\pi}\sqrt{\frac{L_\perp}{2\pi}}\frac{(-1)^{n+n_\perp}}{\sqrt{n}n_\perp}. \tag{8.54}$$

The rotations (8.46) are implemented by the unitary operators $V(\alpha) = e^{i\alpha Q}$, where $Q = \int\limits_V d^2\underline{x}j^+(\underline{x})$:

$$\sigma(x) \to V(\alpha)\sigma(x)V^\dagger(\alpha), \pi(x) \to V(\alpha)\pi(x)V^\dagger(\alpha), \tag{8.55}$$

$$V(\alpha) = \exp\big[\alpha\sum_{\underline{n}}\big(a^\dagger(p_{\underline{n}})c(p_{\underline{n}}) - c^\dagger(p_{\underline{n}})a(p_{\underline{n}})\big)\big]. \tag{8.56}$$

The operators $V(\alpha)$ extend the "primary" vacuum $|v;0\rangle$ to the infinite family $|v;\alpha\rangle = V(\alpha)|v\rangle$. Explicitly, we get

$$|\alpha;v\rangle = \exp\Big\{-v\sum_{\underline{n}}\tilde{c}(p_{\underline{n}})\Big[\big(a^\dagger(p_{\underline{n}})-a(p_{\underline{n}})\big)\cos\alpha+\big(c^\dagger(p_{\underline{n}})-c(p_{\underline{n}})\big)\sin\alpha\Big]\Big\}|0\rangle. \tag{8.57}$$

In spite of the presence of the box length $L_\perp$ in the coherent state (8.53), the orthogonality $\langle v;\alpha|v;\alpha'\rangle = \delta_{\alpha\alpha'}$ holds in the limit of infinite number of longitudinal modes n.

We can interpret the relation for vacuum and particle matrix elements of P^- (cf. Eq.(8.28)) as defining an effective Hamiltonian $P^-_v = U^\dagger_\sigma(v)P^-U_\sigma(v)$:

$$\begin{aligned} P^-_v &= \int\limits_V d^2\underline{x}\Big[(\partial_\perp\sigma)^2 + (\partial_\perp\pi)^2 + 2\mu^2\sigma^2 \\ &+ 2\sqrt{\lambda}\mu\sigma(\sigma^2+\pi^2) + \frac{\lambda}{2}(\sigma^2+\pi^2)^2\Big]. \end{aligned} \tag{8.58}$$

The form of the above Hamiltonian suggests that $\sigma(x)$ corresponds to a massive field because its mass term has a correct sign while the mass term is missing for $\pi(x)$ which became a Goldstone boson field. This tree-level result is more rigorously expressed by the Goldstone theorem.

In the usual proof of Goldstone theorem [46], one inserts a complete set of four-momentum operator eigenstates into the VEV of the commutator $[Q,\pi(x)] = \sigma(x)$ and then invokes translational invariance to show a singularity in the spectral function for $p^2 = 0$ [46, 66]. This means that there exists a massles state in the spectrum. We can proceed analogously because we have all the necessary components for the proof. A difference with respect to the usual theory is that here we have an explicit realization of the vacuum in the Fock representation, not just an abstract state with postulated properties. The states $|\alpha;v\rangle$ represent however only an approximative variational estimate of the true degenerate family of ground states. But its existence tells us that there must exist exact eigenstates of the LF Hamiltonian with energy lower than the energy of the Fock vacuum $|0\rangle$. This is sufficient for the usual proof of the Goldstone theorem. Some ingredients of the proof are actually valid also for the approximative $|\alpha; v\rangle$ states. Namely, the above commutator is a rigorous consequence of Eqs.(8.46) and (8.55). To show that, one only has to use the infinitesimal form of both transformation laws and compare the leading terms in the expansion. The vacuum expectation value of the commutator is v

$$\langle v;0|\big[Q,\pi(0)\big]|v;0\rangle = \langle 0|U_\sigma^{-1}(v)\sigma(0)U_\sigma(v)|0\rangle = v. \tag{8.59}$$

If we denote the set of exact vacuum state by $|\Omega_\alpha\rangle$, then we should also have

$$\langle\Omega_0|\big[Q,\pi(0)\big]|\Omega_0\rangle = \langle\Omega_0|\sigma(0)|\Omega_0\rangle = f_v, \tag{8.60}$$

where f_v is the (precisely unknown) expectation value of he σ field in the exact vacuum $|\Omega_\alpha\rangle$. Let $|n\rangle$ be the set of simultaneous eigenstates of the LF momentum and energy operators, $P^\mu|n\rangle = p^\mu|n\rangle$, where $p^\mu = (E_n^-, P_n^+, P_n^1)$. Inserting such a complete set into the relation (8.60) in the form of $\hat{1} = \sum_n |n\rangle\langle n|$, using the definition of the charge as a volume integral of $j^+(x)$ as well as translational invariance of the theory

$$j^+(x) = \exp\left(ix_\mu P^\mu\right)j^+(0)\exp\left(-ix_\mu P^\mu\right), \quad \exp\big(ix_\mu P^\mu\big)|\Omega_\alpha\rangle = |\Omega_\alpha\rangle, \tag{8.61}$$

we find

$$\begin{aligned}&\frac{2}{V}\sum_n \delta^2(\underline{p}_n)\exp\big(-\frac{i}{2}E_n^- x^+\big)\langle\Omega_0|j^+(0)|n\rangle\langle n|\pi(0)|\Omega_0\rangle - \\ &-\frac{2}{V}\sum_n \delta^2(\underline{p}_n)\exp\big(\frac{i}{2}E_n^- x^+\big)\langle\Omega_0|\pi_0|n\rangle\langle n|j^+(0)|\Omega_0\rangle = f_v.\end{aligned} \tag{8.62}$$

It follows from the VEV of the volume integral of the commutator $[\partial_\mu j^\mu, \pi(0)] = 0$ that f_v has to be x^+-independent:

$$\Big[\Big(\partial_+\int_{-L}^{+L}\frac{\mathrm{d}x^-}{2}j^+(x) + \int_{-L}^{+L}\frac{\mathrm{d}x^-}{2}\partial_- j^-(x)\Big), \pi(0)\Big] = \partial_+\big[Q,\pi(0)\big] = 0, \tag{8.63}$$

where the second term in the commutator vanishes due to the fact that the current always satisfies periodic boundary conditions. In the SL theory, this term vanishes due to locality. In order that the left-hand side of the equation (8.60) is also x^+-independent, there must exist an eigenstate $|G\rangle$ of P^μ which for $p^+ = 0,\ p^\perp = 0$ (so that the delta function is non-zero) has $E^- = 0$ (so that the x^+-depence vanishes), while $\langle\Omega_0|\pi(0)|G\rangle \neq 0,\ \langle\Omega_0|j^+(0)|G\rangle \neq 0$. Since $M^2 = E^-p^+ - p_\perp^2$, this state is massless. Note that the Nambu-Goldstone state is not simply $c^\dagger(\underline{k})|\Omega_0\rangle$ since the latter is not an eigenstate of P^-. The correct linear combination of Fock states representing the Goldstone boson can be (at least in principle) obtained by a Hamilton matrix diagonalization.

Thus, the approximate description of the not-trivial vacuum structure in terms of coherent states labeled by a continuous parameter enabled us to demonstrate the Nambu-Goldsone phenomenon in the light front scalar theory in a manner analogous to the usual formulation. Later in these notes we will extend this approach to the (abelian) gauge theory in three space dimensions and explain the spontaneous symmetry breaking of the gauge theory and the occurence of the Higgs phenomenon (mass generation) along similar lines.

9 Quantum solitons from numerical LF Hamiltonian diagonalizations

We have already mentioned the method of Discretized Light Cone Quantization (DLCQ). It is based on the finite-volume formulation of the given LF model accompanied by building of the many-particle Fock space. In contrast to the usual field theory, this is a well defined step due to the fact that the LF Fock vacuum is an eigenstate of the full LF Hamiltonian, not just its free part. The hamiltonian matrix H_{ij} is then calculated from the (rescaled) Hamiltonian operator in Fock representation $H = \frac{2\pi}{L}P^-$ as $\langle i|H|j\rangle$. The states symbolically denoted by $|j\rangle$ are composed from j particles with the momenta $p_1^+, p_2^+ \dots p_j^+$ ($p_1^+ \equiv \frac{2\pi}{L}k_1$, etc.) in such a way that $p_1^+ + p_2^+ + \dots = P^+$ where P^+ is the total LF momentum of the system. For the moment, we will again consider the two-dimensional scalar model with quartic self-interaction. Taking into account the fact that we are dealing with bosons so that each state can be multiply occupied (the indices j_i) , the general form of the Fock state normalized to unity is

$$|j\rangle = \frac{a_{l_1}^{\dagger j_1}}{\sqrt{j_1!}}\frac{a_{l_2}^{\dagger j_2}}{\sqrt{j_2!}}...\frac{a_{l_n}^{\dagger j_m}}{\sqrt{j_m!}}|0\rangle. \tag{9.1}$$

The indices satisfy the constraints $l_1 j_1 + l_2 j_2 + \dots + l_n j_m = K$, $j_1 + j_2 + \dots j_m = j$. K is the dimensionless (rescaled) LF momentum $K = \frac{L}{2\pi}P^+$ which is an integer. The number of Fock states grows rapidly with K and their generation, enumeration and storage becomes a task for a computer. In a similar way, computation of the hamiltonian matrix elements which involves commuting of many creation and annihilation operators, and the final step, the numerical diagonalization of the large but sparse matrix, is efficiently performed on a computer. This is the essence of the DLCQ approach. The entries of the diagonalized matrix are real numbers because one has to choose definite numerical values for all masses and coupling constants present in the Hamiltonian. The results of the diagonalization are not only a few lowest eigenvalues of the mass operator $\hat{M}^2 = P^+P^-$ but also the corresponding wave functions expressing the probability amplitude for the given Fock state to be present in the resulting bound state with given values of the discrete momenta of its constituents. Schematically, a generic bound state will have the structure

$$|BS\rangle = \psi_1(p_K^+)a_K^\dagger|0\rangle + \sum_n \psi_2(p_{K-n}^+, p_n^+)a_K^\dagger a_{K-n}^\dagger|0\rangle +$$

$$+\sum_{m,n}\psi_3(p_{K-m-n}^+, p_m^+, p_n^+)a_{K-m-n}^\dagger a_m^\dagger a_n^\dagger|0\rangle + \dots + \psi_K(p_1^+)a^{\dagger K}(p_1^+)|0\rangle. \tag{9.2}$$

The summation runs over all integers (half-integers in the case of antiperiodic boundary conditions) for which all momenta $K-n, K-m-n$ etc. are positive. Obviously, we can talk about two-particle, three-particle etc. sectors of the Fock states. The amplitudes $\psi_i(p_{j_1}^+, p_{j_2}^+, \dots)$ are the output of the numerical diagonalization of the hamiltonian matrix. They are complex numbers stored in an array whose first part is a set of positive integers j_i which represent the dimensionless momenta of the particles in the given Fock state. The amplitudes encode an important physical information and can be used to calculate additional observables. For more complicated theories, they depend on additional quantum numbers like spin or flavour. A necessary requirement in the DLCQ method is to work with sufficiently high values of the harmonic resolution K so that the results become stable. The stability indicates that one has in practice achieved the continuum limit. A more reliable option is to extrapolate the observables computed for (sufficiently large) finite values of K to infinite k. It is believed that the Fock expansion in the LF scheme converges fast so that states with higher particle numbers have negligible amplitudes and the given bound state is reasonably described by a few low-particle number states.

So far we have discussed the two-dimensional situation. The problem with higher dimensional models is not only that the dimension of the Fock space grows even more rapidly because of the presence of perpendicular components of the momenta labeled by two additional integers n_x, n_y. The main difficulty is a necessity to perform a non-perturbative renormalization in the Hamiltonian framework in order to deal with finite quantities. A consistent renormalization program of this kind has not been formulated so far.

There exists a different approach to the bound-state problem in the LF theory called the Tamm-Dancoff method. It is based on a projection of the fundamental relativistic eigenvalue equation (the relativistic version of the Schroedinger equation in the second-quantized form) $P^+P^-\psi(x) = M_i^2\psi(x)$ onto a few lowest Fock sectors (typically two- and three-particle states). Since one works in the continuum formulation, one arrives at a set of coupled integral equations. It is beyond the scope of the present work to give more details of the method which can be found in the review [16]. Recently, a systematic program of non-perturbative renormalization within the Tamm-Dancoff method has been formulated in the approach called covariant light front dynamics which substitutes quantization on a preferred surface $x^+ = 0$ by a general surface tangent to the light cone and parametrized by a light-like vector ω [67, 68], $\omega^2 = 0$.

The DLCQ method was proposed and applied in the work of Pauli and Brodsky [14, 15]. In the latter case, the two-dimensional Yukawa theory was analyzed omitting the scalar zero mode. In this section, we will discribe a different application of the method to a scalar theory in the phase of broken symmetry. Since the interaction is of the $\lambda\phi^4$ kind this discussion will be a quantitative extension of our analytical treatment of the broken phase described in the previous section. The chosen theory is interesting because as it is known from its classical version and from the semiclassical approximations, the spectrum of the model in the broken phase consists of collective excitations called solitons [60]. Since the DLCQ method is intrinsically non-perturbative (although one cannot completely avoid certain plausible approximations) one is in a good position to compute properties of genuine quantum solitons, called kinks in this particular model, and compare them with classical and semiclassical results. This permits us to test the reliability of the semiclassical methods, to calculate masses ab initio from the micsoscopic Hamiltonian and also to determine additional observables from Hamiltonian eigenfunctions (probability amplitudes).

In the variational approach, kinks can be well-approximated by coherent states. This appears to have two implications for the Fock space expansion in our discretized approach: one may need an infinite number of bosons to describe solitons, and, since the dimensionless total longitudinal momentum K automatically provides a cutoff on the number of bosons, convergence in K may be difficult to achieve for a kink-like state [55].

Here, we show how a nonperturbative evaluation of topological excitations and their observables is feasible in a finite Fock basis.

A popular nonperturbative numerical approach to field theory is the Euclidean lattice formulation. In the topologically non-trivial sector of the two-dimensional ϕ^4 theory, results available from lattice simulations are far limited to the determination of the kink mass [69]. The results for the configuration average of the kink profile are not smooth and are difficult to interpret, probably due to finite volume limitations.

These Euclidean lattice calculations are highly non-trivial and a brief overview displays the degree of effort needed to reveal topological observables. In one approach, one computes the kink mass from the decay of the correlation functions of an operator with nonvanishing projection on the topological sector under consideration, the dual field in the present case. On a finite lattice, the definition of such an operator is often ambiguous. Another approach involves integrating the difference between the expectation values of the lattice actions with antiperiodic and periodic boundary conditions. To obtain results for the continuum field theory, one has to work in the critical region of the lattice theory. Here, calculations are severely hampered by the phenomena of critical slowing down. Given these difficulties, it is understandable that Euclidean lattice calculations of the mass and other properties of the kink-antikink state have not been reported to date.

Let us mention for completeness that the present model was studied also from the point of view of constructive field theory. It was proven rigorously that in quantum theory a stable kink state is separated from the vacuum by a mass gap of the order λ^{-1} and from the rest of the spectrum by an upper gap [70]. More detailed nonperturbative information on the spectrum of the mass operator or on other observables from rigorous approaches is not available. This illustrates one of the problems of quantum field theory: it is very difficult to solve even relatively simple models if one insists on mathematical rigour.

In this situation, the DLCQ approach represents a powerful method capable to generate predictions for physical observables nonperturbatively and from first principles. We will present results of DLCQ computations for the case of both antiperiodic and periodic boundary conditions. As will become clear soon, these two situations correspond to the different regimes of the theory distinguished by different values of a specific quantum number, called topological charge. It stemms from the topological properties of the theory defined by the choice of boundary conditions.

Before diving to the DLCQ procedure, we shall briefly discuss the classical solitons in the usual [60] as well as LF version of the theory [71]. It is convenient to start with a general potential part $V(\phi)$ of the classical Lagrangian density which is assumed to possess multiple minima. Then the equation of motion is

$$\partial_0^2\phi(x) - \partial_1^2\phi(x) = -\frac{\delta V}{\delta\phi}(x). \tag{9.3}$$

The (classical) energy is given by

$$E(\phi) = \int_{-\infty}^{+\infty} \mathrm{d}x \Big[\frac{1}{2}\big(\partial_0\phi\big)^2 + \frac{1}{2}\big(\partial_1\phi\big)^2 + V(\phi)\Big]. \tag{9.4}$$

Let the minimum of $V(\phi)$, adjusted to zero, occurs in N values ϕ_0^i of the field $\phi(x)$. The minimum of the energy, equal to zero, is then achieved for these space-time independent configurations of the field. This is in agreement with the equation of motion, which takes the form $\delta U/\delta\phi = 0$. For the static solutions, the equation becomes

$$\partial_1^2\phi(x) = \frac{\delta V}{\delta\phi}(x). \tag{9.5}$$

The solitons are defined as the solution of the above static equation which have finite energy and are non-dissipative, having energy density that is spatially localized. It follows that for $x \to \pm\infty$ these solutions approach one of the minima ϕ_0^i. This provides boundary conditions for the problem. To solve the equation (9.5), one multiplies it by $\partial_1\phi(x)$ and integrates to get

$$\int\limits_{-\infty}^{+\infty} \mathrm{d}x\partial_1\phi\,\partial_1^2\phi = \int\limits_{-\infty}^{+\infty} \mathrm{d}x\frac{\delta V}{\delta\phi}\partial_1\phi \;\;\Rightarrow\;\; d\phi/dx = \pm\big[V(\phi)\big]^{1/2}. \tag{9.6}$$

The integration constant is zero because $\partial_1\phi$ and $V(\phi)$ vanish for $x \to -\infty$. Integrating the latter equation again, we find

$$x - x_0 = \pm\int_{\phi(x_0)}^{\phi(x)} \frac{\mathrm{d}\phi'}{\big[2V(\phi')\big]^{1/2}}. \tag{9.7}$$

Since by construction $V(\phi)$ approaches arbitrary two minima for $x \to \pm\infty$ which are equal to zero, it is positive between them. Choosing x_0 and $\phi(x_0)$, we may integrate Eq.(9.7) and invert the result to find the solution $\phi(x)$. For the quartic potential $V(\phi) = \frac{1}{4}\lambda(\phi^2 - \frac{\mu^2}{\lambda})^2$ the two minima are at $\phi_0^{\pm} = \pm\mu/\lambda$. There are two solutions interpolating between these two minima: one starting at ϕ_0^+ for $x \to \infty$ and ending at ϕ_0^- for $x \to -\infty$ and the second going in the opposite direction. For the static solution, we have

$$\partial_1^2\phi = \frac{\delta V}{\delta\phi} = \lambda\phi^3 - \mu^2\phi \tag{9.8}$$

leading to the equation (9.7) in the form

$$x - x_0 = \pm\int_{\phi(x_0)}^{\phi(x)} \frac{\mathrm{d}\phi'}{\sqrt{\lambda/2}\big[\phi'^2 - \mu^2/\lambda\big]}. \tag{9.9}$$

We can choose $\phi(x_0) = 0$ and use the simple integration formula

$$\int_0^a \frac{\mathrm{d}x}{x^2 - c^2} = \frac{1}{2c}\ln\frac{c-a}{c+a} \tag{9.10}$$

to calculate the right hand side. Exponentiating both sides, one finds

$$\phi_0(x) = \pm\frac{\mu}{\sqrt{\lambda}}\mathrm{th}\Big[\frac{\mu}{\sqrt{2}}(x - x_0)\Big]. \tag{9.11}$$

The two signs correspond to the two possibilities to connect two minima of the potential and are called a kink and an antikink. The translational invariance of the solution is seen from the fact that changing the point x_0 only moves the hyperbolic tangent along the x-axis. The energy density of the solution is

$$\varepsilon(x) = \frac{1}{2}\big(\partial_1\phi_0\big)^2 + V(\phi_0) = 2V(\phi_0) \tag{9.12}$$

and taking the derivative according to Eq.(9.6), we arrive at

$$\varepsilon(x) = \frac{\mu^4}{2\lambda}\mathrm{sech}^4\big[\frac{\mu}{\sqrt{2}}(x - x_0)\big], \tag{9.13}$$

where $\mathrm{sech}(x) = 1/\mathrm{ch}(x)$. The classical energy or mass of the kink is

$$M_{cl} = \int_{-\infty}^{+\infty} \mathrm{d}x\, \varepsilon(x) = \frac{2\sqrt{2}}{3}\frac{\mu^3}{\lambda}. \tag{9.14}$$

In obtaining this result, the integration formula

$$\int \mathrm{d}x \mathrm{ch}^{-4}(x) = \frac{\mathrm{sh}(x)}{3}\big[\mathrm{ch}^{-3}(x) + 2\mathrm{ch}^{-1}(x)\big] \tag{9.15}$$

was used.

Let us turn now to a calculation of the properties of a classical kink in the light front theory. The main difference is the structure of the equation of motion where the operator $\partial_\mu\partial^\mu = 4\partial_+\partial_-$ is not quadratic in the time and space derivatives and consequently one cannot look for static solutions simply by making the time derivative term vanish. One however has the mass-squared operator $M^2 = P^+P^-$ at the disposal:

$$M^2 = 4\int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}x^-}{2} : \left(\frac{\partial\phi(x^-)}{\partial x^-}\right)^2 :\ 2\int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}y^-}{2} : V\left(\phi(y^-)\right) : \tag{9.16}$$

whose classical minimum is given by

$$\frac{\delta M^2}{\delta\phi(x)} = 0 \tag{9.17}$$

leading to

$$-2\frac{\partial^2\phi_{cl}(x^-)}{\partial x^{-2}}\int\limits_{-\infty}^{+\infty}\frac{\mathrm{d}y^-}{2}V\left(\phi(y^-)\right) + V'(\phi_{cl}(x^-)\int\limits_{-\infty}^{+\infty}\frac{\mathrm{d}y^-}{2}\left(\frac{\partial\phi_{cl}(y^-)}{\partial y^-}\right)^2 = 0, \tag{9.18}$$

where $V'(\phi) \equiv \delta V(\phi)/\delta\phi$. We have also used partial integration in the first term and the relation $\delta\phi(x^-)/\delta\phi(y^-) = \delta(x^- - y^-)$. $\phi_{cl}(x)$ is the anticipated solution for which the integrals in the above relations are some constants (unknown at this stage):

$$A = \int\limits_{-\infty}^{+\infty}\frac{\mathrm{d}x^-}{2}V\left(\phi_{cl}(x^-)\right),\ \ B = \int\limits_{-\infty}^{+\infty}\frac{\mathrm{d}x^-}{2}\left(\frac{\partial\phi_{cl}(x^-)}{\partial x^-}\right)^2. \tag{9.19}$$

Then Eq.(9.18) takes the form

$$-2A\frac{\partial^2\phi_{cl}(x^-)}{\partial x^{-2}} + BV'\left(\phi_{cl}(x^-)\right) = 0. \tag{9.20}$$

After rescaling $y = -x^- \sqrt{\frac{B}{2A}}$ we finally find the classical equation

$$-\frac{\partial^2 \phi_{cl}(x^-)}{\partial y^2} + V'\left(\phi_{cl}(x^-)\right) = 0, \tag{9.21}$$

which has the same form as the equation in the SL theory. We can identify therefore

$$\phi_{cl}(x^-) = \phi_0(x). \tag{9.22}$$

This equation means that the shape of the LF solution in terms of the LF space variable x^- is identical to the shape of the solution (9.11) of the space-like theory.

Now we proceed to the DLCQ treatment of the $\lambda\phi^4$ theory to find the quantum counterparts of the above classical solutions and their properties. For completeness, let us recall that we start from the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4, \tag{9.23}$$

which leads to the LF Hamiltonian

$$P^- = \int_{-L}^{+L} \frac{\mathrm{d}x^-}{2}\Big[-\mu^2\phi^2 + \frac{2\lambda}{4!}\phi^4\Big] \equiv \frac{L}{2\pi}H \tag{9.24}$$

where L defines our compact domain $-L \le x^- \le +L$. The main goal in this section will be to compute the energy spectrum of H.

The LF momentum operator is

$$P^+ = 4\int_{-L}^{+L} \frac{\mathrm{d}x^-}{2}\partial_-\phi\partial_-\phi \equiv \frac{2\pi}{L}K \tag{9.25}$$

where K denotes the dimensionless momentum operator. The mass squared operator $M^2 = P^+P^- = KH$. In DLCQ with antiperiodic BC, the field expansion has the form

$$\phi(x^-) = \frac{1}{\sqrt{4\pi}}\sum_n \frac{1}{\sqrt{n}}\left[a_n e^{-i\frac{n\pi}{L}x^-} + a_n^\dagger e^{i\frac{n\pi}{L}x^-}\right]. \tag{9.26}$$

Here $n = \frac{1}{2}, \frac{3}{2}, \ldots$.

The normal ordered Hamiltonian is given by

$$\begin{aligned} H \quad = \quad & -\mu^2\sum_n \frac{1}{n}a_n^\dagger a_n + \frac{\lambda}{4\pi}\sum_{k\le l, m\le n} \frac{1}{N_{kl}^2}\frac{1}{\sqrt{klmn}}a_k^\dagger a_l^\dagger a_n a_m \delta_{k+l,m+n} + \\ & + \frac{\lambda}{4\pi}\sum_{k,l\le m\le n} \frac{1}{N_{lmn}^2}\frac{1}{\sqrt{klmn}}\left[a_k^\dagger a_l a_m a_n + a_n^\dagger a_m^\dagger a_l^\dagger a_k\right]\ \delta_{k,l+m+n} \end{aligned} \tag{9.27}$$

with

$$\begin{aligned} N_{lmn} &= 1,\ l \neq m \neq n, \\ &= \sqrt{2!},\ l = m \neq n,\ l \neq m = n, \\ &= \sqrt{3!},\ l = m = n, \end{aligned} \tag{9.28}$$

and

$$\begin{aligned} N_{kl} &= 1, k \neq l, \\ &= \sqrt{2!},\ k = l. \end{aligned} \tag{9.29}$$

The final result of the DLCQ numerical computations will be certain sets of numerical data that need to be analyzed and interpreted. It will be useful to have an analytical guidance. This is provided by the so-called constrained variational approach introduced in this context in [72]. The result of the constrained variational calculation, being semi-classical, is especially reliable in the weak coupling region and we can use its functional form to extract the kink mass from the numerical results of matrix diagonalization.

Let us first discuss the simpler unconstrained variational treatment. The main ingredients of the method are following. Choose as a trial state the coherent state

$$\mid \alpha \rangle = \mathcal{N} \exp\left(\sum_n \alpha_n a_n^\dagger \right) \mid 0 \rangle \tag{9.30}$$

where $\mathcal{N}$ is a normalization factor. This is a reasonable choice if the coupling constant is small because in this case we are close to the classical limit. The coherent states are an adequate approximation in this regime since they are by construction the "most classical" quantum states [73, 74].

With antiperiodic BC we have

$$\frac{\langle \alpha \mid \phi(x^-) \mid \alpha \rangle}{\langle \alpha \mid \alpha \rangle} = \frac{1}{\sqrt{4\pi}}\, f(x^-) \tag{9.31}$$

where

$$f(x^-) = \sum_{m=1}^{N} \frac{1}{\sqrt{m - \frac{1}{2}}} \left[\alpha_{m-\frac{1}{2}} e^{-i\frac{\pi}{L}(m-\frac{1}{2})x^-} + \alpha^*_{m-\frac{1}{2}} e^{i\frac{\pi}{L}(m-\frac{1}{2})x^-} \right], \tag{9.32}$$

where the defining property $a_n|\alpha\rangle = \alpha_n|\alpha\rangle$ has been used. The expectation value of the Hamiltonian (9.24) will have the same form as (9.24) with $\phi(x^-)$ replaced by $f(x^-)$. Minimizing this expectation value, we obtain

$$f_{min} = \pm\sqrt{\frac{24\pi\mu^2}{\lambda}} = \pm\sqrt{\frac{3}{g}}. \tag{9.33}$$

Let us choose $f(x^-) = \sqrt{3/g}$ for $0 < x^- < L$ and $f(x^-) = -\sqrt{3/g}$ for $-L < x^- < 0$.Then we get

$$\alpha_{m-\frac{1}{2}} = \sqrt{\frac{3}{g}}\, \frac{i}{\pi}\, \frac{1}{\sqrt{m - \frac{1}{2}}}\ ,\ \ m = 1, 2, 3, \ldots, \tag{9.34}$$

and

$$f(x^-) = \frac{2}{\pi}\sqrt{\frac{3}{g}}\sum_j \frac{1}{j}\ \sin\ \frac{j\pi x^-}{L} \tag{9.35}$$

where $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$, etc. The number density of bosons with momentum fraction $x = \frac{j}{K}$ is given by

$$\chi(x) = \frac{\langle \alpha \mid a_j^\dagger a_j \mid \alpha\rangle}{\langle \alpha \mid \alpha\rangle} = \alpha_j^2 \tag{9.36}$$

where $\alpha_j \sim \frac{1}{\sqrt{j}}$.

We also find

$$\frac{1}{\langle \alpha \mid \alpha\rangle}\frac{2\pi}{L}\int dx^- \langle \alpha \mid \phi^2(x^-) \mid \alpha\rangle = \frac{2}{\pi^2}\frac{3}{g}\sum_j \frac{1}{j^2}. \tag{9.37}$$

In the unconstrained variational calculation, the expectation value of the LF momentum operator is infinite for an infinite number of modes for the reason we discussed in the previous section: $f(x^-)$ is discontinous, here at $x^- = 0$, hence the space derivative in (9.25) is infinite. To cure this deficiency for the present purpose, it is convenient to switch to a constrained variational calculation. Its details are presented in the Appendix E, where it is shown that in the limit $\langle K\rangle \to \infty$ the expectation value of the Hamiltonian H in the generalized coherent states has the functional form

$$\frac{\langle \alpha \mid H \mid \alpha\rangle}{\langle \alpha \mid \alpha\rangle} = -\frac{6\pi\mu^4}{\lambda} + \frac{32\mu^6}{\lambda^2\langle K\rangle}\ . \tag{9.38}$$

Interpreting the state $|\alpha\rangle$ to be a kink state, we identify the first term as the vacuum energy density which is the classical vacuum energy density. The second term is identified as $\frac{M_{kink}^2}{\langle K\rangle}$. Then we get the classical kink mass $M_{kink} = \frac{4\sqrt{2}\mu^3}{\lambda}$.

An observable that yields considerable insight for the spatial structure of the topological object is the Fourier transform of its form factor. The form factor is defined as the matrix element of the field operator in a physical state. We compute the Fourier transform of the form factor of the lowest state which, according to Goldstone and Jackiw [75], in the weak coupling (static) limit, represents the kink profile, i.e. a semiclassical or quantum counterpart of the classical solution. Let $\mid K\rangle$ and $\mid K'\rangle$ denote this state with momenta K and K'. Then in the continuum LF theory, the starting formula reads

$$\int_{-\infty}^{+\infty} dq^+ \exp\{-\frac{i}{2}q^+ a\}\langle K' \mid \Phi(x^-) \mid K\rangle = \phi_c(x^- - a). \tag{9.39}$$

It has to be adapted to the application within the DLCQ method, where we diagonalize the Hamiltonian for a given $K = \frac{L}{2\pi}P^+$. For the computation of the form factor, we need the same state at different K values since $K' = K + q$. We can proceed as follows. We diagonalize the Hamiltonian, say, at $K = 41$ (even particle sector). Then we diagonalize the Hamiltonian at the neighbouring K values, $K = 40.5, 41.5, 39.5, 42.5, 38.5, 43.5, 37.5, 44.5, 36.5, 45.5$ (odd particle sectors). In this particular example, the dimensionless momentum transfer ranges from -4.5 to $+4.5$. If K is large enough to be near the continuum limit, then, in the spontaneous symmetry broken phase, with degenerate even and odd states, we can be confident that all these lowest states correspond to the *same* physical state observed at different longitudinal momenta. The test that the states are degenerate is that they have the same M^2, so the eigenvalues of H fall on a linear trajectory as a function of $\frac{1}{K}$, cf. Eq.(9.38).

As the next step, one computes the matrix element of the field operator between the lowest state at $K = 40$ and the other specified values of K and sum the amplitudes which corresponds to the choice of the shift parameter $a = 0$. The summation replaces integration in (9.39). In summing the amplitudes, we need to be careful about the phases. First we note that K is a conserved quantity, so eigenfunctions at different K values have an independent arbitrary complex phase factor. To fix the phases, we accept the guidance of the coherent state analysis. We set the overall sign of the lowest states for all K values such that the matrix elements $\langle K + n \mid a_n^\dagger \mid K\rangle$ is positive and $\langle K - n \mid a_n \mid K\rangle$ is negative. In addition, there is one overall complex phase that we apply to the profile function so that it is real at the boundaries. That the sum of all terms for the profile function produces the shape of a kink, with very small imaginary component, is nevertheless a non-trivial result. It is a further non-trivial result that the magnitude of the kink represents a physically sensible result.

Let us analyze the numerical results. With antiperiodic BC, for integer (half integer) values of K we have even (odd) number of particles. The dimensionality of the matrix in the even and odd sectors for different values of K is for example equal to $295,\ 61316,\ 813177$ for $K = 295,\ 39.5,\ 54.5$ and to $336,\ 67243,\ 880962$ for $K = 16,\ 40,\ 55$. All results presented here were obtained on small clusters of computers (≤ 15 processors) using the Many Fermion Dynamics (MFD) code adapted to bosons [76] with the Lanczos diagonalization method. Since the Hamiltonian exhibits the $\phi \to -\phi$ symmetry, the even and odd particle sectors of the theory are decoupled, i.e. matrix elements of the Hamiltonian between these two sectors vanish. Let us compare the situation between the two phases in a qualitative way. In the symmetric phase, where one has a positive μ^2, and at weak coupling, the lowest state in the odd particle sector is a single particle carrying all the momentum. In the even particle sector, the lowest state consists of two particles. Thus for massive particles, there is a distinct mass gap between odd and even particle sectors. In the broken phase we are studying μ^2 is negative and, at weak coupling, the situation is drastically different. Now, the lowest states in the odd and even particle sectors consist of the maximum number of particles carrying the lowest allowed momentum. Thus, in the continuum limit, the possibility arises that the states in the even and odd particle sectors become degenerate. A clear signal of SSB is the degeneracy of the spectrum in the even and odd particle sectors. Thus at finite K, we can compare the spectra for an integer K value (even particle sector) and its neighbouring half integer K value (odd particle sector) and look for degenerate states.

In Fig. 9.1 we show the lowest four energy eigenvalues in the broken symmetry phase for the even and odd particle sectors for λ=1.0 as a function of $\frac{1}{K}$. The points represent results at half integer increments in K from $K = 10$ to $K = 55$. The overall trend is towards smoother behavior at higher K. There is an apparent small oscillation superimposed on a generally linear trend for each state. The oscillations represent probably an artifact of discretization. These oscillations decrease with increasing K. The smooth curves in Fig. 9.1 are linear fits to the eigenvalues in the range from $K = 40$ to $K = 55$ constrained to have the same intercept.

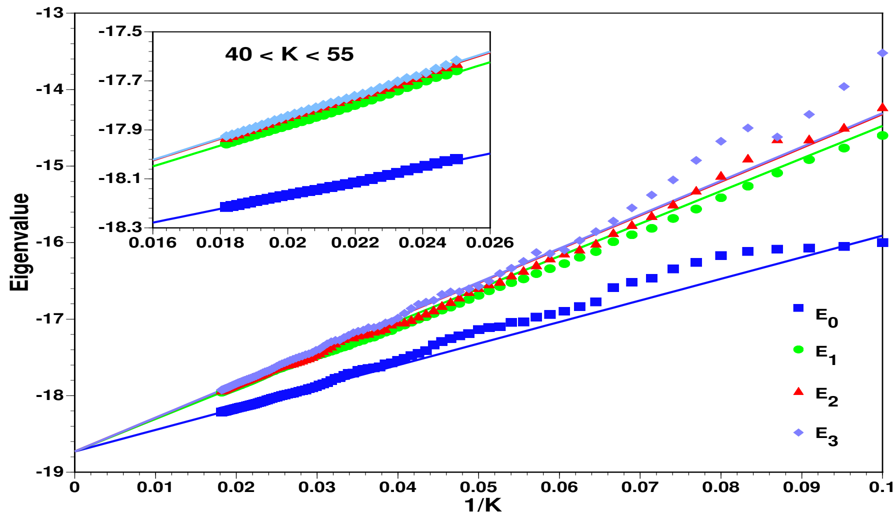

Fig. 9.1. Lowest four eigenvalues for even and odd sectors as a function of $\frac{1}{K}$ for λ=1.0. The inset shows the details over the range $40 \leq K \leq 55$. The discrete points are the DLCQ eigenvalues while the straight lines are the linear fits to the $40 \leq K \leq 55$ data constrained to have the same intercept.

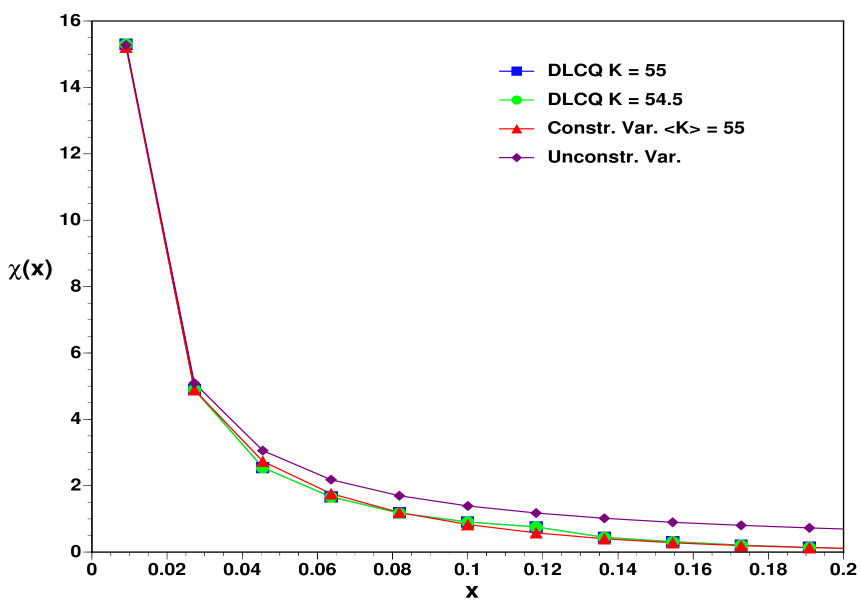

Fig. 9.2. The number density $\chi(x)$ for even ($K = 55$) and odd ($K = 54.5$) sectors for $\lambda = 1$ compared with unconstrained and constrained ($\langle K \rangle = 55$) variational results.

Tab. 9.1. Comparison of vacuum energy density and soliton mass extracted from the continuum limit of our DLCQ data, with classical results. For soliton mass, the semi-classical result [77] is also shown.

λ	vacuum energy		soliton mass		
	classical	DLCQ	classical	semi-classical	DLCQ
1.0	-18.85	-18.73 $\pm$0.05	5.66	5.19	5.3 $\pm$0.2

With guidance from the constrained variational calculation, see Eq. (9.38), we can extract the kink mass from the linear fit to the DLCQ data for the ground state eigenvalue. We fit the $\lambda = 1.0$ data in the range $40 \leq K \leq 55$ to a linear form (C_1+C_2/K). The reason for this choice of the K range is that the finite-vilume effects seem reasonably absent. We quote C_1 as the vacuum energy density and C_2 as the kink mass in Table 9.1. We obtain the uncertainties from the spread in these results arising from constrained fits to subsets of the data in this same range. For comparison, the corresponding classical values (classical vacuum energy density $\mathcal{E} = -6\pi\mu^4/\lambda$) are also presented. The agreement appears reasonable.

Next we examine the behaviour of the number density $\chi(x)$ for the kink state. In the broken phase, the ground states in the even and odd particle sectors are degenerate in the continuum limit. In Fig. 9.2 we show $\chi(x)$ for $K = 55$ and $K = 54.5$ for λ = 1. For this coupling the number densities for even and odd sectors are almost identical to each other indicative of degenerate states. In Fig. 9.2, we also compare the DLCQ number density with that predicted by the unconstrained and constrained variational calculations. At sufficiently large K and low λ, they appear to agree at a level which is reasonable for the comparison of a quantal result with a semi-classical result.

Following Goldstone and Jackiw, one can calculate the Fourier transform of the form factor of the kink state in DLCQ at weak coupling. In Fig. 9.3(a) we show the profile calculated in DLCQ for $\lambda = 1$ at three selected K values. It is clear that at $\lambda = 1$ the profile is that of a kink which appears reasonably converged with increasing K. In Fig. 9.3(b) the $K = 41$ DLCQ profile is compared with that of a constrained variational coherent state calculation of Eq. (F.14) with $\langle K \rangle = 41$. In the unconstrained variational calculation, this function is discontinuous at $x^- = 0$ and $\langle K \rangle$, the expectation value of the dimensionless LF momentum operator, is infinite. In the variational calculation where $\langle K \rangle$ is constrained to be finite, the kink profile is a smooth function of x^- as seen in Fig. 9.3(b). In the limit $\langle K \rangle \to \infty$, the kink profile from constrained variational calculation approaches that of the unconstrained case. For each K shown, we utilize 11 sets of DLCQ results to construct the profile function. Thus, for $K = 41$ we employ results at $K = 41$ and at $K = 36.5$ through 45.5 in unit steps.

In the light front literature it has been suggested that (see the review [16]) without a field mode carrying exactly zero momentum, it would be impossible to describe spontaneous symmetry breaking.

In this section, we shall show that it is possible to obtain the correct physics of the broken phase even without the explicit presence of the notorious zero-momentum mode as anticipated by Rozowsky and Thorn [72]. We will also compute the Fourier transform of the form factor of the lowest state (i.e., the " profile" of the kink-antikink configuration) and its parton content, results not yet available from other methods. In addition, at weak coupling, we extract the value of the vacuum condensate from two observables, namely the computed vacuum energy density

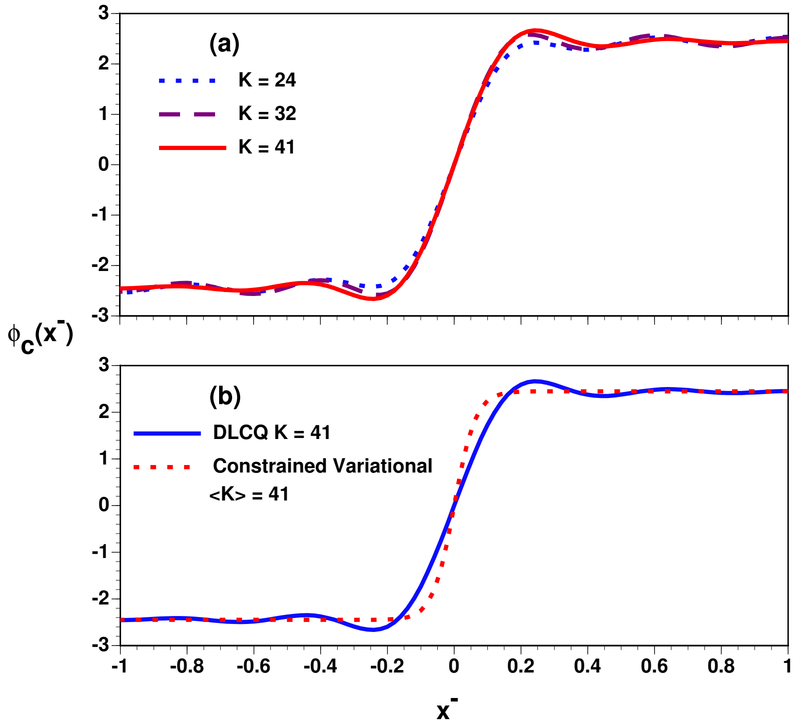

Fig. 9.3. Fourier Transform of the kink form factor at λ=1; (a) results for $K = 24, 32$, and 41 each obtained with DLCQ eigenstates from 11 values of K centered on the designated K value; (b) comparison of DLCQ profile at K=41 with constrained variational result with $\langle K\rangle = 41$.

and the asymptote of the profile function.

Let us highlight the main results of the DLCQ analysis of quantum kinks for the case of periodic boundary conditions. The scalar field can be decomposed in this case as $\phi(x^-) = \phi_0 + \Phi(x^-)$, where ϕ_0 is the zero mode operator. Since it will be deliberately omitted in our Hamiltonian, $\Phi(x^-)$ is the normal mode operator (9.26) where now the index n runs over integers instead of half-integers.

We again diagonalize this Hamiltonian in the basis of all many-boson configurations at a fixed K where K is the sum of the values of the dimensionless momenta of all bosons in the configuration. The Hamiltonian is symmetric under $\phi \to -\phi$ and thus, with PBC, the Hamiltonian matrix becomes block diagonal in even and odd particle number sectors. The dimensionality of the largest matrix we solve, $K = 60$, is equal to 483 338 (even sector) and 483 129 (odd sector).

Since we dropped the $P^+ = 0$ mode, degenerate vacuum states, characterized by a spatially uniform field expectation value, are not explicitly present in our formulation. However, one may expect degeneracy of the energy levels in the even and odd particle sectors at sufficiently high resolution, K. The argument is as follows. For small coupling, variational coherent states $|\alpha\rangle$ represent a good approximation of the lowest lying physical states [72]. Even and odd states are

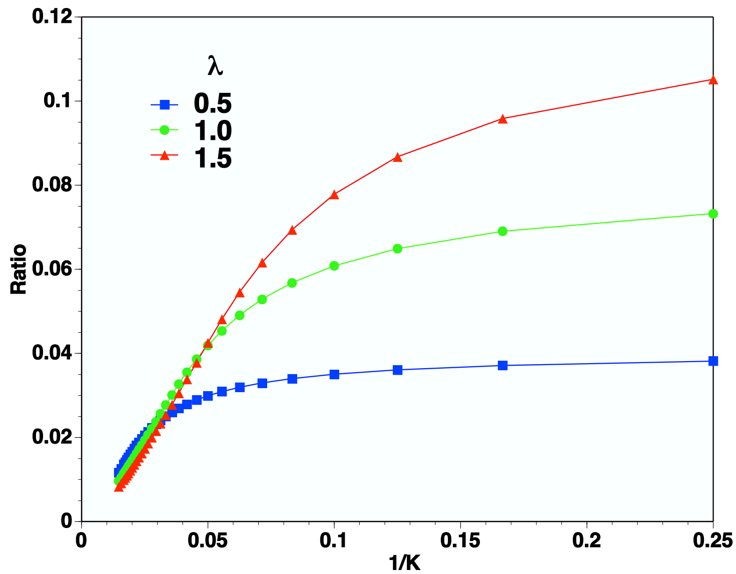

Fig. 9.4. Ratio of lowest state even-odd energy difference to classical energy density for $\lambda = 0.5, 1.0, 1.5$.

linear combinations of $|\alpha\rangle$ and $|-\alpha\rangle$ and for large enough K they have the same energy. We also expect that our lowest state will be an excitation above the vacuum state and we will show it corresponds to a configuration with properties of a kink-antikink pair. In Fig. 9.4 we present a ratio, the difference between the lowest eigenvalues of different parity divided by the classical vacuum energy density, as a function of the inverse resolution. Curves for $\lambda = 0.5, 1.0, 1.5$ all demonstrate the trend to degeneracy in the continuum limit ($K \to \infty$). That is, we obtain SSB at each coupling through degeneracy of even and odd parity states when we extrapolate to the continuum limit. At any finite K the lifting of the degeneracy is simply a reflection of the tunneling present in a finite system. As seen from Fig. 9.4, the tunneling is relatively strong for $K \leq 20$.

The obtained behaviour of the few lowest eigenvalues with K is quite similar to the case of antiperiodic B as they follow smooth curves that become more linear as K increases.

The lowest state is expected to be a kink-antikink configuration and should have a positive invariant mass (twice the mass of the single kink at weak coupling). One can extract this mass from finite K results analogously to the case of antiperiodic BC. For massive states, the light front energy E scales like $(1/K)$ so that it approaches zero in the infinite K limit. On the other hand, a coherent state variational calculation shows that in the infinite K limit, the energy of the lowest state approaches the classical ground state energy density $\mathcal{E} = -(6\pi\mu^4/\lambda)$ for small λ. As we show in what follows, our results are increasingly compatible with a coherent state as K increases towards the continuum limit. Thus, we fit our finite K results for the eigenvalues at small λ to the formula $C + M^2/K$, where C is the vacuum energy density and M is the kink-antikink mass.

Tab. 9.2. Comparison of vacuum energy and soliton mass from the continuum limit of our results, with classical results. Semi-classical results for the mass [77] are also shown. The estimated uncertainties in the last significant digit are quoted in parenthesis.

λ	vacuum energy		soliton mass		
	classical	this work	classical	semi-class.	this work
0.5	-37.70	-37.90(4)	11.31	10.84	11.26(4)
1.0	-18.85	-18.97(2)	5.657	5.186	5.563(7)

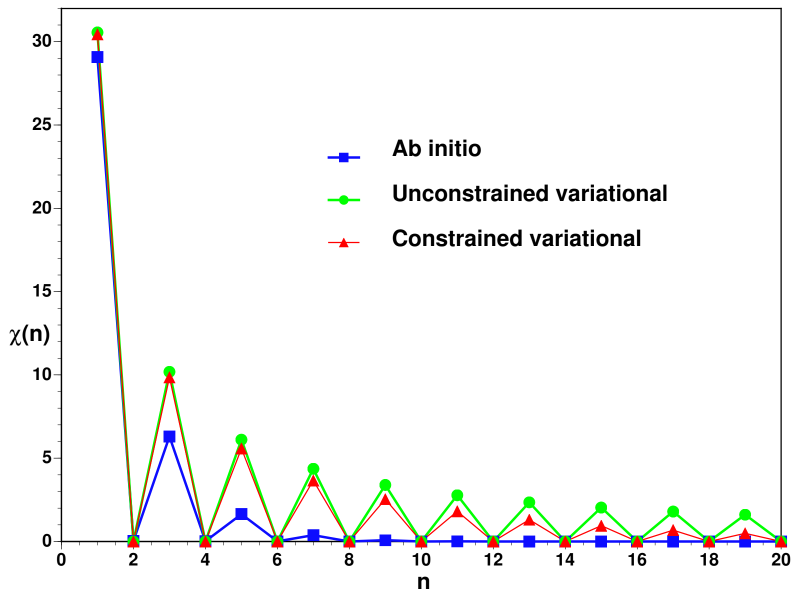

Fig. 9.5. Comparison of the number density $\chi(n)$ from our approach ("Ab initio") ($K = 60$) and the constrained and unconstrained coherent state variational calculation for $\lambda = 1.0$.

Extracted values of C and the kink or soliton mass $M/2$ are compared to their classical and one loop corrected ("semi-class.") counterparts in Table 9.2. The vacuum energy nearly coincides with the classical result probably as the result of dropping the zero mode. $M/2$ is also close to the classical value since our coupling constant is small and the kink-antikink interactions are weak. Our soliton mass and vacuum energy at $\lambda = 1$ are in reasonable agreement with results using antiperiodic boundary conditions. The fact that the mass of the quantum kink is larger than the semi-classical one is peculiar to the choice $\mu^2 = 1$ and does not occur for μ^2 away from 1 [69].

As an example of another observable, we again evaluate the occupation number density $\chi(n)$, the analog of the parton distribution function of more realistic theories. Note that in the unconstrained variational state [72], the shape of $\chi(n)$ is independent of the coupling λ which affects only its overall normalization. On the other hand, in the variational calculation, constrained to have a fixed value of $\langle K \rangle$, λ affects not only the overall normalization but also the shape of the distribution. In Fig. 9.5 our result at $K = 60$ is compared with that of the unconstrained and constrained ($\langle K\rangle = 60$) coherent state approximation for $\lambda = 1.0$. We find that the shape and normalization of our $\chi(n)$ depends on λ. Our results display the same sawtooth pattern as the constrained and unconstrained variational results. This is due to the PBC and and omission of the zero mode, the sawtooth pattern was not present in the number density in the case of antiperiodic BC. It is natural to expect sensitivity to the boundary conditions in topologically non-trivial sectors even in the infinite volume limit.

Next, we compute the Fourier transform of the formfactor for the lowest state, using the discrete version of the formula (9.39). The Fourier transform represents the classical kink-antikink profile in the weak coupling limit and thus yields information about the spatial structure of the low lying states. Let $\mid K\rangle$ and $\mid K'\rangle$ denote this state with momenta K and K'. Then we follow the technique discribed in the case of antiperiodic BC, namely we sum the matrix elements of the field operator for the same state but seen at different values of K.

The topology of a kink - antikink structure and other properties of the form factor thus rely on the detailed behavior of the amplitudes over a range of K values. The result for the lowest eigenstate for $\lambda = 1$ is presented in Fig. 9.6. This striking kink - antikink behavior is particular to the lowest state.

Taking the vacuum energy results of Table 9.2 together with the $\phi_c(x^-)$ results of Fig. 9.3, we have two independent but consistent methods for extracting $\langle\phi\rangle$ in the weak coupling limit. From our vacuum energy density for $\lambda = 1.0$, we obtain $\langle\phi\rangle = (\mathcal{E}/\pi\mu^2)^{1/2} = 2.457$. From the calculation of the profile function, shown in Fig. 9.3, we extract $\langle\phi\rangle$ as the asymptotic ($x^- = \pm 1$ in units of L) intercepts to be equal to 2.447. These results may be compared with the classical value of $\sqrt{6} = 2.449$, which agrees with the result from the variational coherent state.

One can summarize the results of the present chapter as follows. We have demonstrated the phenomenon of spontaneous symmetry breaking in a discretized light front approach without P^+ zero mode and calculated several nonperturbative physical quantities. The degeneracy of energy levels is both a signature of spontaneous symmetry breaking and essential for the existence of kinks. We find that a finite Fock space yields features of the lowest excitation that are similar to those of a variational coherent state ansatz. We have extracted the quantum kink mass and the vacuum energy density for small λ by extrapolating our lowest eigenvalue to the continuum limit. At weak coupling, the mass of the quantum kink is closer to the classical value than to the semi-classical mass. We have extracted the number density of elementary constituents of the lowest state and compared it with the coherent state prediction. We have also evaluated the Fourier transform of the lowest state form factor in a fully non-perturbative quantum approach and obtained a kink profile (antiperiodic BC) and a kink-antikink profile (periodic BC). These results can be interpreted as indicative of the viability of DLCQ for addressing non-trivial phenomena in quantum field theory.

10 Discretized LF quantization of the two-dimensional Yukawa model

This model describes interaction of the charged massive fermion field with neutral massive scalar field. Its two-dimensional version was studied in the Hamiltonian framework in ([78]) where spectrum of bound states was investigated. The same problem was later on analyzed in the light front formalism using the discretized approach (the DLCQ method) [14, 15]. A clear advantage of the DLCQ computations of bound-state properties was demonstrated. The main reason was

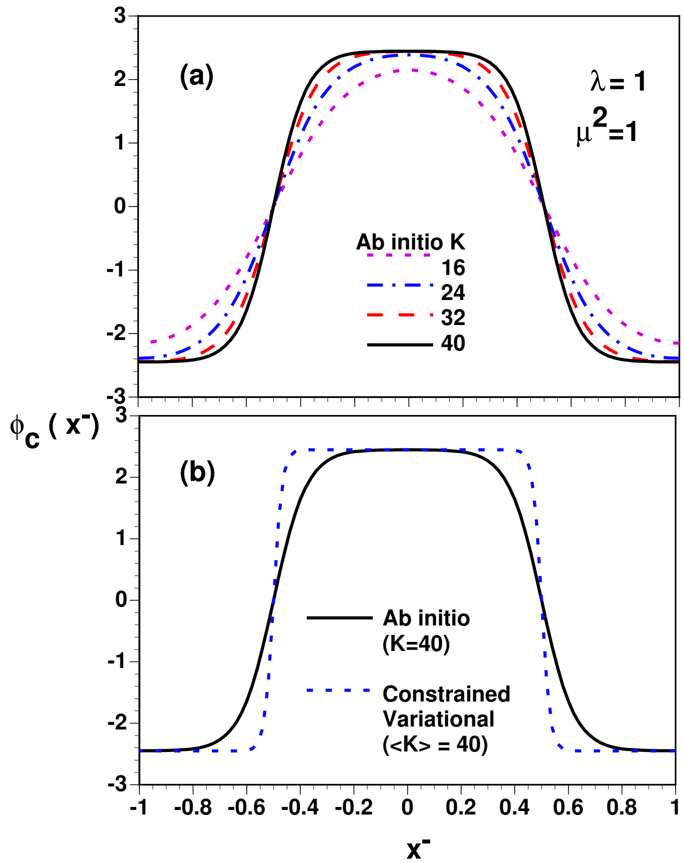

Fig. 9.6. Fourier Transform of the kink-antikink form factor at $\lambda = 1.0$. Results are plotted in units of L. (a) Convergence with K. (b) Comparison of our result (Ab initio) (K=40) with constrained variational calculation ($\langle K \rangle = 40$).

a consistent Fock expansion of composite states not available in the conventional form of the theory where disconnected vacuum fluctuations considerably complicate calculations.

We will briefly describe the LF quantization of D = $1+1$ Yukawa model in a finite volume focusing on the solution to the scalar zero mode constraint and on contributions of this zero mode to the Hamiltonian. The scalar ZM was neglected in the original treatment [14, 15].

Consider the covariant Lagrangian

$$\mathcal{L} = \frac{i}{2}\overline{\psi}\gamma^{\mu} \overleftrightarrow{\partial_{\mu}} \psi - m\overline{\psi}\psi + \frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi - \frac{1}{2}\mu^2\phi^2 - g\overline{\psi}\psi\phi \tag{10.1}$$

with the trilinear interaction between massive fermion and boson fields. Expressed in terms of the light front variables, the Lagrangian is

$$\begin{aligned}\mathcal{L}_{lf} &= i\psi_2^\dagger \overleftrightarrow{\partial_+}\, \psi_2 + i\psi_1^\dagger \overleftrightarrow{\partial_-}\, \psi_1 - m(\psi_1^\dagger\psi_2 + \psi_2^\dagger\psi_1) + \\ &+ 2\partial_+\phi\partial_-\phi - \frac{1}{2}\mu^2\phi^2 - g(\psi_1^\dagger\psi_2 + \psi_2^\dagger\psi_1)\phi \end{aligned} \tag{10.2}$$

leading to three field equations

$$2i\partial_+\psi_2 = m\psi_1 + g\psi_1\phi, \tag{10.3}$$

$$2i\partial_-\psi_1 = m\psi_2 + g\psi_2\phi, \tag{10.4}$$

$$4\partial_+\partial_-\phi = -\mu^2\phi - g(\psi_1^\dagger\psi_2 + \psi_2^\dagger\psi_1). \tag{10.5}$$

Since the scalar field is taken to be periodic in x^- coordinate, it can be decomposed into the zero mode and normal mode parts, $\phi(x) = \phi_0(x^+) + \varphi(x^+, x^-)$. As before, fermion field is antiperiodic, $\psi(-L) = -\psi(L)$. Projecting the equation (10.5) onto the ZM sector by integration over the "volume" $2L$, we get the constraint analogous to (8.40) from the $\lambda\phi^4$ theory,

$$\phi_0 = -\frac{g}{\mu^2}\int_{-L}^{+L}\frac{\mathrm{d}x^-}{2L}\big(\psi_1^\dagger\psi_2 + \psi_2^\dagger\psi_1\big). \tag{10.6}$$

The constraint equation (10.4) has a simple solution

$$\psi_1(x) = \frac{1}{4i}\int_{-L}^{+L}\frac{\mathrm{d}x^-}{2}\epsilon_a(x^- - y^-)\big[m + g\phi_0(x^+) + \varphi(y^-, x^+)\big]\psi_2(y^-, x^+), \tag{10.7}$$

which contains the ZM (10.6). Obviously, the two equations are coupled. Let us try to find the corresponding solution ϕ_0. Neglecting for simplicity the problem of ordering of operators, let us introduce the short-hand notation:

$$\psi_1(x) = mf_2(x) + g\phi_0 f_2(x) + gf_3(x), \tag{10.8}$$

where

$$\begin{aligned} f_2(x^-) &= \frac{1}{4i}\int_{-L}^{+L}\frac{\mathrm{d}x^-}{2}\epsilon_a(x^- - y^-)\psi_2(y^-, x^+) = \\ &= \frac{1}{\sqrt{2L}}\sum_n \frac{1}{p_n^+}\Big(b_n e^{-\frac{i}{2}p_n^+x^-} + d_n^\dagger e^{\frac{i}{2}p_n^+x^-}\Big), \end{aligned} \tag{10.9}$$

$$\begin{aligned} f_3(x^-) &= \frac{1}{4i}\int_{-L}^{+L}\frac{\mathrm{d}x^-}{2}\epsilon_a(x^- - y^-)\varphi(y^-, x^+)\psi_2(y^-, x^+) = \\ &= -\frac{1}{2L}\sum_{m,n}\frac{1}{\sqrt{p_m^+}}\Big[\frac{1}{p_m^+ + p_n^+}\Big(a_m^\dagger d_n^\dagger e^{\frac{i}{2}(p_m^+ + p_n^+)x^-} - a_m b_n e^{-\frac{i}{2}(p_m^+ + p_n^+)x^-}\Big) + \\ &+ \frac{2}{p_m^+ - p_n^+}\Big(a_m^\dagger b_n e^{\frac{i}{2}(p_m^+ - p_n^+)x^-} - d_n^\dagger a_m e^{-\frac{i}{2}(p_m^+ - p_n^+)x^-}\Big)\Big]. \end{aligned} \tag{10.10}$$

In the above sum, only terms with $m \neq n$ contribute. We have also displayed the Fock form of the operators f_2 and f_3 obtained by using the basic field expansion (3.35) valid at $x^+ = 0$

$$\psi_2(0, x^-) = \frac{1}{2L} \sum_n \Big(b_n e^{-\frac{i}{2} p_n^+ x^-} + d_n^\dagger e^{\frac{i}{2} p_n^+ x^-} \Big). \tag{10.11}$$

Inserting the expression (10.8) into the constraint (10.6), we find

$$\phi_0 = -\frac{g}{\mu^2} \int\limits_{-L}^{+L} \frac{\mathrm{d}x^-}{2L} \Big[(m + g\phi_0)\big(f_2^\dagger \psi_2 + \psi_2^\dagger f_2\big) + g\big(f_3^\dagger \psi_2 + \psi_2^\dagger f_3\big) \Big] \tag{10.12}$$

with the solution

$$\phi_0 = -\frac{g}{\mu^2} \frac{mF_2 + gF_3}{1 + \frac{g^2}{\mu^2} F_2}, \tag{10.13}$$

where we used the abbreviations

$$F_2 = \int\limits_{-L}^{+L} \frac{\mathrm{d}x^-}{2L} \big(f_2^\dagger \psi_2 + \psi_2^\dagger f_2\big), \quad F_3 = \int\limits_{-L}^{+L} \frac{\mathrm{d}x^-}{2L} \big(f_3^\dagger \psi_2 + \psi_2^\dagger f_3\big). \tag{10.14}$$

The LF Hamiltonian is obtained in the canonical way described in the case of the Federbush model. Here it reads

$$P^- = \int\limits_{-L}^{+L} \frac{\mathrm{d}x^-}{2} \big[\mu^2 \varphi^2 + \mu^2 \phi_0^2 + m\big(\psi_1^\dagger \psi_2 + \psi_2^\dagger \psi_1\big) + g\big(\phi_0 + \varphi\big)\big(\psi_1^\dagger \psi_2 + \psi_2^\dagger \psi_1\big) \Big]. \tag{10.15}$$

It can be simplified by inserting the solution of ϕ_0. In the original DLCQ treatment of the LF Yukawa model ([14, 15]), the zero-mode terms were neglected.

11 LF massive Schwinger model

Most of physically relevant models of quantum field theory nowadays are gauge theories. They are based on the gauge principle. The interaction of the fermionic or scalar matter fields with the massless vector fields is derived in these theories from the requirement that the corresponding Lagrangian be invariant under a group of local transformations, i.e. those with space-time dependent parameters. The change of phase of the matter fields is compensated by a suitable transformation of the massless gauge (vector) field so that the Lagrangian (or the action) remains unchanged. A simpler class of gauge theories are the models with abelian group of symmetry characterized by the property that two such transformations commute. If they are non-commuting, the theory is called non-abelian and is quite complicated from the technical point of view. The microscopic theory of strong interactions, quantum chromodynamics or QCD, is the prominent representative of the non-abelian gauge theories.

Although the gauge principle determines the structure of the interactions uniquely, there is a well known difficulty: a conflict between the relativistically covariant description of the vector field and the number of its physical degrees of freedom. For example, in four-dimensional quantum electrodynamics, QED$(3+1)$, the gauge field $A^\mu(x)$ has four components while the observable electric and magnetic fields are derived only from two transverse gauge potentials. Similarly, in two space-time dimensions, the gauge field has two components, $A^0(x)$ and $A^1(x)$ in the conventional space-like parametrization, or $A^+(x)$ and $A^-(x)$ in the LF description, while there are no physical gauge fields in two dimesions (the only exception is one space-independent component of the gauge field in the finite-volume formulation which is physical, i.e. gauge invariant). The standard solution of the problem is the gauge-fixing, which ammounts to eliminating the redundant degrees of freedom by imposing chosen conditions. Then it is not quite straightforward to compare the emerging physical picure in different gauges. In principle, it would be more satisfactory to formulate the theory in a gauge-independent manner.

So far, we did not study any gauge model in these notes. We rather tried to elucidate principal differences between the conventional and light front field theories using simple examples of massive scalar and fermion fields. In this chapter, we will analyze the simplest gauge theory, namely quantum electrodynamics with massive fermions in two dimensions, QED$(1+1)$, also called the massive Schwinger model [101, 102, 103], in the light front formulation. It is rather clear that the original massless model proposed by Schwinger [79] can be consistently formulated and its physical contents understood only as the massless limit of the massive theory in the LF approach. This follows from the fact that the mass has a different status in the LF field theory [7] in comparison with the SL theory (we have discussed unitary equivalence of the fields with different masses, the property not shared by the usual form of field theory). Two-dimensional models are particularly sensitive to this aspect due to the lack of transverse momenta. Non-vanishing m serves as an infrared regulator. Setting $m = 0$ from very beginning simply leads to the loss of important physical information contained for example in the fermionic constraint. Also the Pauli-Jordan function involving $\psi_1(x)$ and some correlation functions would vanish for massless theory while they are non-zero in the limiting sense. This attitude is supported also by a calculation of the axial anomaly in the Weyl-gauge version of the present model [104], where the correct (mass-independent) result emerged due to a cancellation of mass dependence in a ratio of two factors.

Although massive QED$(1+1)$ is an abelian gauge theory, it is sufficiently non-trivial and encodes in a baby-version many of essential aspects of more complicated gauge theories. Surprisingly enough, even its exactly soluble massless version is not understood uniquely neither in the space-like field theory nor in the LF approach. In other words, there is no "canonical" physical picture of this prototype gauge theory since various authors differ in their explanation of the essential physical aspects of the model (vacuum structure, number of the so-called vacuum angles, presence of chiral symmetry and a mechanism of its breaking, the explanation of the U(1) problem). The massive Schwinger model appears to be the ideal play ground for testing different ideas of the LF gauge theory and we will describe a few possible approaches to the gauge poblem within this model.

11.1 Formulation without a gauge condition

The classical Lagrangian density of the two-dimensional QED looks formally as that of the four-dimensional quantum electrodynamics:

$$\mathcal{L} = \frac{i}{2}\overline{\psi}(x)\gamma^\mu \overleftrightarrow{\partial_\mu} \psi(x) - m\overline{\psi}(x)\psi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - ej_\mu(x)A^\mu(x), \tag{11.1}$$

where m is the mass of the Fermi field $\psi(x)$ and e is the coupling constant, i.e. the electric charge. Of course, in addition to the fields depending on time and one space variable, the gauge potential $A^\mu(x)$ as well as the vector current $j^\mu(x) = \overline{\psi}(x)\gamma^\mu\psi(x)$ has only two components here. The electromagnetic tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The γ-matrices in the SL formalism are expressed as before in terms of Pauli matrices, $\gamma^0 = \sigma^1, \gamma^1 = i\sigma^2$. The conjugate Fermi field $\psi = \psi^\dagger\gamma^0$. Let us recall that the first term in the Lagrangian is invariant under a phase transformation with a global parameter, $\psi(x) \to e^{-i\Lambda}\psi(x)$. If the parameter is x-dependent, i.e. local, this term is changed by $\partial_\mu\Lambda(x)j^\mu(x)$, where $\Lambda(x)$ is called the gauge function. To maintain the invariance, an introduction of the vector field $A^\mu(x)$ transforming as $A^\mu(x) \to A^\mu(x) + \frac{1}{e}\partial^\mu\Lambda(x)$ is mandatory. Its kinetic term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is invariant (because $F_{\mu\nu}$ is), as well as the fermion mass term. The change of $A^\mu(x)$ induces a change of the last, interacting term in the Lagrangian, which precisely cancels the change of the fermion kinetic term, leading to the gauge invariance of the classical Lagrangian. We say that the theory is invariant under the abelian gauge group U(1) which consists of the infinite number of phase factors $U(\Lambda) = e^{-i\Lambda(x)}$. Under the action of this group, the fields transform as

$$\psi(x) \to U(\Lambda)\psi(x),\ \overline{\psi}(x) \to \overline{\psi}(x)U^{-1}(\Lambda),$$

$$A_\mu(x) \to U(\Lambda)\Big(A_\mu(x) - \frac{i}{e}\partial_\mu\Big)U^{-1}(\Lambda). \tag{11.2}$$

The most common way to eliminate the redundant gauge-field components is to impose a set of conditions on $A^\mu(x)$ at the classical level and to take the corresponding form of the theory over to the quantum level. For the usual SL formulation, the examples are the temporal gauge $A^0 = 0$ and the axial gauge $A^1 = 0$ which in one space dimension is equivalent to the Coulomb gauge $\partial_1 A^1 = 0$ in continuum theory. Such conditions are however not arbitrary but should be obtained from the transformation law for the gauge field by a specific and explicit choice of the gauge function $\Lambda(x)$. Here the boundary conditions may play a role.

In terms of LF space and time variables $x^\mu = x^\pm$, the Lagrangian (11.1) has the form

$$\begin{aligned}\mathcal{L}_{LF} &= i\psi_2^\dagger \overleftrightarrow{\partial_+} \psi_2 + i\psi_1^\dagger \overleftrightarrow{\partial_-} \psi_1 + \frac{1}{2}(\partial_+A^+ - \partial_-A^-)^2 - \\ &- m(\psi_2^\dagger\psi_1 + \psi_1^\dagger\psi_2) - \frac{e}{2}j^+A^- - \frac{e}{2}j^-A^+.\end{aligned} \tag{11.3}$$

We recall that the dynamical (ψ_+) and dependent (ψ_-) projections of the fermi field are defined as $\psi_\pm = \Lambda_\pm\psi$, where $\Lambda_\pm = \frac{1}{2}\gamma^0\gamma^\pm, \gamma^\pm = \gamma^0 \pm \gamma^1$. In the chosen representation of the γ-matrices, $\psi_+^\dagger = (0, \psi_2^\dagger), \psi_-^\dagger = (\psi_1^\dagger, 0)$ and the vector current $j^\pm = 2\psi_\pm^\dagger\psi_\pm$ has the components $j^+ = 2{\psi_2}^\dagger\psi_2,\ j^- = 2{\psi_1}^\dagger\psi_1$. The classical Euler-Lagrange equations read

$$2i\partial_+\psi_2 = m\psi_1 + eA^-\psi_2, \quad \partial_+(\partial_+A^+ - \partial_-A^-) = -\frac{e}{2}j^-, \tag{11.4}$$

$$2i\partial_-\psi_1 = m\psi_2 + eA^+\psi_1, \quad \partial_-(\partial_+A^+ - \partial_-A^-) = \frac{e}{2}j^+. \tag{11.5}$$

We will consider the theory on a finite interval $-L \le x^- \le L$ with (anti)periodic fields: $\psi(-L) = -\psi(L), A^\pm(-L) = A^\pm(L)$. This implies that the gauge field can be decomposed into the x^--independent zero-mode (ZM) part $A_0^\pm$ and the normal mode (NM) part $A_n^\pm(x^-)$. [9] The classical Lagrangian (11.3) is then invariant under the gauge transformations (GT)

$$\psi_2(x) \to e^{-i\Lambda(x^-)}\psi_2(x), \quad A^\pm(x) \to A^\pm(x) + \frac{2}{e}\partial_\mp\Lambda(x^-). \tag{11.6}$$

The fermion constraint (11.5) implies that $\psi_1(x)$ transforms in the same way as $\psi_2(x)$. A_0^+ is a physical variable invariant under "small" GT (see below).

The conjugate momenta of the dynamical field components are calculated according to

$$\Pi_{\varphi_i} = \frac{\delta\mathcal{L}_{LF}}{\delta\partial_+\varphi_i}, \tag{11.7}$$

where φ_i stands for any of the dynamical fields. One easily finds

$$\Pi_{\psi_2} = i\psi_2^\dagger,\ \Pi^\dagger_{\psi_2} = -i\psi_2,\ \ \Pi_{\psi_1} = \Pi^\dagger_{\psi_1} = 0,$$

$$\Pi_{A_n^+} = \partial_+A_n^+ - \partial_-A_n^-,\ \ \Pi_{A_0^+} = \partial_+A_0^+,\ \ \Pi_{A_n^-} = \Pi_{A_0^-} = 0. \tag{11.8}$$

The conjugate momenta of the non-dynamic field components A_n^- and ψ_1 vanish because there are no time derivatives of these field in the Lagrangian. The LF Hamiltonian P^- and momentum P^+ are obtained from the energy-momentum tensor $T^{\mu\nu}$

$$T^{\mu\nu}(x) = \sum_i \frac{\delta\mathcal{L}_{LF}}{\delta\partial_\mu\varphi_i}\partial^\nu\varphi_i - g^{\mu\nu}\mathcal{L}_{LF} \tag{11.9}$$

which is analogous to the free-field tensor (3.41), as

$$P^- = \int_{-L}^{+L}\frac{\mathrm{d}x^-}{2}\left[\Pi^2_{A_n^+} + m(\psi_2^\dagger\psi_1 + \psi_1^\dagger\psi_2) - (2\partial_-\Pi_{A_n^+} - ej_n^+)A_n^-\right] + P^-_{ZM},$$

$$P^- + ZM = \Pi^2_{A_0^+} + Lej_0^+A_0^-, \tag{11.10}$$

$$P^+ = \int_{-L}^{+L}\frac{\mathrm{d}x^-}{2}\left[4i\psi_2^\dagger\partial_-\psi_2 + 2\Pi_{A_n^+}\partial_-A_n^+\right]. \tag{11.11}$$

The pair of "Maxwell" equations is easily decomposed into the normal-mode and zero-mode parts:

$$\partial_+(\partial_+A_n^+ - \partial_-A_n^-) = -\frac{e}{2}j_n^-,\ \ \partial_-\Pi_{A_n^+} = \frac{e}{2}j_n^+, \tag{11.12}$$

[9] Quantities defined in the normal-mode sector will be labeled by small n in this section.

$$\partial_+^2 A_0^+ - = -\frac{e}{2} j_0^-, \;\; j_0^+ = 0. \tag{11.13}$$

The second one is the Gauss' law valid in the normal-mode sector, the third one is the already discussed dynamical equation for the gauge zero mode and the last one is the Gauss' law in the zero-mode sector. It is equivalent to the relation $Q = 0$ (because $j_0^+ = Q/L$) which clearly cannot be satisfied as an operator equation on the quantum level. It is the first class constraint in the language of Dirac-Bergmann quantization and has to be imposed weakly as a condition on physical states: $Q|phys\rangle = 0$.

As a matter of fact, neither the normal-mode Gauss' law can be satisfied as an operator equation on the quantum level. Indeed, assuming the standard equal-LF time (anti)commutation relations

$$\begin{aligned} &\big\{\psi_2(0,x^-), {\psi_2}^\dagger(0,y^-)\big\} = \frac{1}{2}\delta_a(x^- - y^-), \\ &\big[A_n^+(0,x^-), \Pi_{A_n^+}(0,y^-)\big] = i\delta_n(x^- - y^-), \end{aligned} \tag{11.14}$$

one immediately sees that the Gauss' law as an operator statement is incompatible with the latter commutator as it leads to an operator relation

$$\big[A_n^+(x^-), e j_n^+(y^-)\big] = 2i\partial_-^y \delta_n(x^- - y^-) \tag{11.15}$$

which contradicts the rules of canonical quantization: A^+ and j^+ should commute since they are independent variables.

The operator $G_n(x) = 2\partial_- \Pi_{A_n^+}(x) - e j_n^+(x)$ of the Gauss' law, being non-zero, is in fact closely related to the generator of gauge transformations. The unitary operator $\Omega[\Lambda_n]$

$$\Omega[\Lambda_n] = \exp\big[-\frac{i}{e}\int_{-L}^{+L}\frac{\mathrm{d}y^-}{2} G_n(y^-)\Lambda_n(y^-)\big] \tag{11.16}$$

indeed implements the "small" gauge transformation (i.e. those with the periodic gauge function $\Lambda_n(x)$) quantum-mechanically:

$$\begin{aligned} &\Omega\big[\Lambda_n\big] A_n^+(x^-)\Omega^\dagger\big[\Lambda_n\big] = A_n^+(x^-) + \frac{2}{e}\partial_-\Lambda_n(x^-), \\ &\Omega\big[\Lambda_n\big]\psi_2(x^-)\Omega^\dagger\big[\Lambda_n\big] = e^{-i\Lambda_n(x^-)}\psi_2(x^-). \end{aligned} \tag{11.17}$$

These relation follow from the commutators

$$\begin{aligned} &\big[G_n(y^-), A_n^+(x^-)\big] = 2i\partial_-^x \delta_n(x^- - y^-), \\ &\big[G_n(y^-), \psi_2(x^-)\big] = e\delta_a(x^- - y^-)\psi_2(y^-) \end{aligned} \tag{11.18}$$

and the operator identities

$$\exp(A)B\exp(-A) = B + [A,B], \quad \exp(A)B\exp(-A) = \exp(\rho)B, \tag{11.19}$$

valid if $[A, B]$ is a c-number and $[A, B] = \rho B$, respectively.

A possible solution to the problem of incompatibility between the canonical commutator and the operator validity of the Gauss' law is the approach called the "minimal quantization" [105, 106]. This method consists in solving the Gauss' law explicitly on the classical level and inserting the solution back to the Lagrangian. The latter can be then rewritten in terms of gauge-invarint fields (this idea goes back to the original formulation due to Dirac [107],Dirgi2). In the present context, the solution of the Gauss' law can be written in the symbolic form as

$$A_n^-(x) = -\frac{e}{2}\frac{1}{\partial_-^2} j_n^+(x) + \frac{1}{\partial_-}\partial_+ A^+(x). \tag{11.20}$$

The inverse operators can be given a precise form in terms of Green's functions $\mathcal{G}_\infty(x^- - y^-)$ and $\mathcal{G}_\in(x^- - y^-)$:

$$\mathcal{G}_\infty(x^- - y^-) = 1/2\epsilon_n(x^- - y^-),\ \ \partial_-^2 \mathcal{G}_\in(x^- - y^-) = \delta_n(x^- - y^-). \tag{11.21}$$

In the following, we will study a few aspects of the massive Schwinger model in the gauge-fixed framework.

11.2 Vacuum structure in the light-cone gauge

Let us study vacuum properties of the LF massive Schwinger model choosing the most common gauge condition, namely the finite-volume version of the light cone gauge which in the continuum theory is defined by the condition $A^+(x) = 0$. In the case of periodic gauge field in a finite volume, he gauge transformation (11.6) tells us that the zero mode A_0^+ cannot be removed by any choice of the gauge function Λ and hence it is a physical, gauge-invariant degree of freedom. It obeys a dynamical equation (see below) and is therefore called the dynamical zero mode.

As has been already discussed, if the dynamical zero modes are absent or neglected in the given model, the physical ground state contains no quanta. This remarkable simplification of the vacuum aspects of dynamics causes however at the same problems with the understanding of vacuum degeneracy within the light-front theory. For example, it is not quite clear how one could reproduce non-zero fermion condensate in the present model [79,80]. An approach to these problems, which uses initialization of fields on both characteristic surfaces $x^\pm = 0$ has been proposed by some authors [81,82]. It leads however to a complicated theory with its own subtleties. Another proposed solution [83] uses the so-called light-cone representation but actually is not a light-front quantization.

In the approach developed here, the LF vacuum problem is studied strictly within the the framework of the Hamiltonian LF quantization using basic principles of quantum theory. In particular, it will be shown that despite the triviality of the LF vacuum in the sector of normal Fourier modes, the physical vacuum of this simple gauge theory quantized on $x^+ = 0$ surface can have a rich structure in terms of dynamical quanta. Moreover, a degenerate set of light-front vacuum states can emerge at the quantum level. Both properties are a direct consequence of a topologically non-trivial residual "large" gauge symmetry present in the formulation with compactified LF coordinate x^- [84, 85].

The general idea is of course not new. The key role of topology in vacuum aspects of gauge field theories is well established. Gauge transformations with non-trivial topological properties are known to be responsible for the vacuum degeneracy [86, 87, 88, 89, 90, 91, 92]. Among various topologically non-trivial gauge field configurations, studied usually within the path integral approach, instantons [93] are a well known example.

In the light front theory, a few attempts have been made to relate non-trivial vacuum structure to the dynamical zero modes (ZM) of the gauge field [85, 94, 95, 96, 97, 98]. They transform in a simple way under a finite-volume analogue of large gauge transformations [85] which are in continuum theory characterized by a gauge function tending to a non-zero constant at spatial infinity. The zero-mode dynamics is usually studied in terms of wavefunctions in coordinate representation [99] which are lowest-energy solutions of a Schrödinger equation with some "vacuum potential" (see for example [95, 96, 100]).

The present approach is based on the quantum-mechanical implementation of large gauge transformations by unitary operators, which leads in a natural way to the description of the physical vacuum in terms of coherent states of the dynamical gauge-field zero mode as well as of fermion bilinear Fock operators.

As we have already see, periodic boundary condition for the $A^\mu(x^+, x^-)$ field defined on the finite interval $-L \leq x^- \leq L$ imply a decomposition of the gauge field into the zero-mode (ZM) part A_0^μ and the part A_n^μ containing only normal Fourier modes. The ZM A_0^+ becomes a physical variable [84, 99, 109, 110, 85] since it cannot be gauged away. A natural gauge condition, which completely eliminates redundant gauge degrees of freedom and which we adopt here, is $A_n^+ = 0, A_0^- = 0$. In quantum theory, the gauge ZM satisfies the commutation relation

$$\Big[A_0^+(x^+), \Pi_{A_0^+}(x^+)\Big]_{x^+=0} = \frac{i}{L}, \tag{11.22}$$

where $\Pi_{A_0^+} = \partial_+ A_0^+$ is the operator of the spatially constant LF electric field. The fermi field satisfying at $x^+ = 0$ the familiar anticommutation relation

$$\Big\{\psi_2(x^-), {\psi_2}^\dagger(y^-)\Big\} = \frac{1}{2}\delta(x^- - y^-) \tag{11.23}$$

is expanded at $x^+ = 0$ as

$$\psi_2(x^-) = \frac{1}{\sqrt{2L}} \sum_{n=\frac{1}{2}}^{\infty} \Big(b_n e^{-\frac{i}{2}p_n^+ x^-} + d_n^\dagger e^{\frac{i}{2}p_n^+ x^-}\Big), \tag{11.24}$$

with $p_n^+ = \frac{2\pi}{L} n, n = \frac{1}{2}, \frac{3}{2}, \ldots \infty$ and with Fock operators satisfying

$$\{b_n, b_{n'}^\dagger\} = \{d_n, d_{n'}^\dagger\} = \delta_{n,n'}. \tag{11.25}$$

While the LF momentum operator P^+ is given in terms of ψ_2 quanta alone, the gauge invariant (see below) LF Hamiltonian of the model is expressed in terms of the both unconstrained variables ψ_2 and A_0^+. This light-cone gauge Hamiltonian is derived from the Lagrangian density (11.3) in the usual canonical way after setting $A_n^+ = 0,\ A_0^- = 0$ in it. With this choice, the fermionic constraint and the Gauss' law become

$$2i\partial_- \psi_1 = m\psi_2 + eA_0^+\psi_1, \quad -\partial_-^2 A_n^- = \frac{e}{2} j^+ \tag{11.26}$$

and can be inverted to find $\psi_1(x)$ and $A_n^-(x)$. Inserting these constrained field components to the Hamiltonian, we finally find

$$\begin{aligned} P^- &= L\Pi^2_{A_0^+} - \frac{e^2}{4}\int_{-L}^{+L}\frac{\mathrm{d}x^-}{2}\int_{-L}^{+L}\frac{\mathrm{d}y^-}{2} j^+(x^-)\mathcal{G}_2(x^- - y^-)j^+(y^-) + \\ &+ m_f^2\int_{-L}^{+L}\frac{\mathrm{d}x^-}{2}\int_{-L}^{+L}\frac{\mathrm{d}y^-}{2}\Big[{\psi_2}^\dagger(x^-)\mathcal{G}_a(x^- - y^-; A_0^+)\psi_2(y^-) + h.c.\Big]. \end{aligned} \tag{11.27}$$

$\mathcal{G}_2$ is the periodic Green's function $\sim (p+)^{-2}$ (mentioned at the end of the previous section) corresponding to the operator ∂_-^2 in the Gauss' law and the antiperiodic $\mathcal{G}_a$ is given by

$$\begin{aligned} \mathcal{G}_a(x^- - y^-; A_0^+) = \frac{1}{4i}\exp\Big(-\frac{ie}{2}(x^- - y^-)A_0^+\Big)\Big[\epsilon_a(x^- - y^-) + \\ + i\epsilon_a(L - y^-)\tan(\frac{eL}{2}A_0^+)\Big], \end{aligned} \tag{11.28}$$

where ϵ_a is the sign function, $\partial_-\epsilon_a(z^-) = 2\delta_a(z^-)$. The solution of the first equation in (11.26) is expressed as

$$\psi_1(x) = \int_{-L}^{+L}\frac{\mathrm{d}y^-}{2}\mathcal{G}_a(x^- - y^-; A_0^+)\psi_2(x^+, y^-). \tag{11.29}$$

It generalizes the solution of the free fermionic constraint.

As before, the Gauss' law in the ZM sector is equivalent to the condition of electric neutrality of the physical states, $Q|phys\rangle = 0$.

The LF Hamiltonian (11.27) exhibits a residual symmetry [99, 110, 85, 91, 111] which is not explicitly present in the continuum formulation. It corresponds to transformations with non-trivial topological properties. In the LF theory, the associated gauge function is linear in x^- (and hence non-vanishing at $x^- = \pm L$) with a coefficient, given by a specific combination of constants. These simple properties follow from the requirement to maintain boundary conditions for the gauge and matter fields, respectively.

For the considered $U(1)$ theory, the corresponding gauge function has the form $\Lambda_\nu = \frac{\pi}{L}\nu x^-$ and defines a winding number ν:

$$\Lambda_\nu(L) - \Lambda_\nu(-L) = 2\pi\nu, \;\; \nu \in Z. \tag{11.30}$$

Thus, the residual gauge symmetry of the Hamiltonian (11.27) is

$$A_0^+ \rightarrow A_0^+ + \frac{2\pi}{eL}\nu, \;\; \psi_+(x^-) \rightarrow e^{-i\frac{\pi}{L}\nu x^-}\psi_+(x^-). \tag{11.31}$$

Let us discuss the ZM part of the symmetry first. At the quantum level, it is convenient to work with the rescaled ZM operators $\hat{\zeta}$ and $\hat{\pi}_0$:

$$A_0^+ = \frac{2\pi}{eL}\hat{\zeta}, \;\; \Pi_{A_0^+} = \frac{e}{2\pi}\hat{\pi}_0, \;\; \Big[\hat{\zeta}, \hat{\pi}_0\Big] = i. \tag{11.32}$$

Note that the box length dropped out from the ZM commutator. The shift transformation of A_0^+ is for $\nu = 1$ implemented by the unitary operator $\hat{Z}_1$:

$$\hat{\zeta} \to \hat{Z}_1 \hat{\zeta} \hat{Z}_1^\dagger = \hat{\zeta} + 1, \quad \hat{Z}_1 = \exp(i\hat{\pi}_0). \tag{11.33}$$

The transformation of the ZM operator $\hat{\zeta}$ is accompanied by the corresponding transformation of the vacuum state. The latter is defined by $a_0 \Psi_0 = 0$, where $a_0(a_0^\dagger)$ is the usual annihilation (creation) operator of a boson quantum:

$$a_0 = \frac{1}{\sqrt{2}}\left(\hat{\zeta} + i\hat{\pi}_0\right), \quad a_0^\dagger = \frac{1}{\sqrt{2}}\left(\hat{\zeta} - i\hat{\pi}_0\right), \quad \left[a_0, a_0^\dagger\right] = 1. \tag{11.34}$$

$\hat{Z}_\nu = (\hat{Z}_1)^\nu$ is essentially the displacement operator defining the coherent states [73] $|\nu; z\rangle = \hat{Z}_\nu \Psi_0$:

$$\hat{Z}_\nu = \exp\left[\frac{-\nu}{\sqrt{2}}\left(a_0^\dagger - a_0\right)\right], \; a_0|\nu; z\rangle = \frac{-\nu}{\sqrt{2}}|\nu; z\rangle. \tag{11.35}$$

One can interpret the displaced vacuum expressed in terms of the harmonic oscillator Fock states $|n\rangle$ (C_n are the corresponding amplitudes) [73, 112]

$$|\nu; z\rangle = \sum_{n=0}^{\infty} C_n(\nu)|n\rangle \tag{11.36}$$

as the condensate of zero-mode gauge bosons.

Alternatively, working in the coordinate representation, one has $\hat{\pi}_0 = -i\frac{d}{d\zeta}$ and the vacuum wavefunction (wf) $\Psi_0(\zeta) = \pi^{-\frac{1}{4}} \exp(-\frac{1}{2}\zeta^2)$ transforms as

$$\Psi_0(\zeta) \to \Psi_\nu(\zeta) = \exp(\nu\frac{d}{d\zeta})\Psi_0(\zeta) = \pi^{-\frac{1}{4}} \exp(-\frac{1}{2}(\zeta + \nu)^2). \tag{11.37}$$

The ZM kinetic energy term of the LF Hamiltonian (11.27) is given by

$$P_0^- = -\frac{1}{2}\frac{e^2 L}{2\pi^2}\frac{d^2}{d\zeta^2}. \tag{11.38}$$

Usually, a Schrödinger equation with the ZM hamiltonian is invoked to find the lowest energy eigenfunctions subject to a gauge-fixing periodicity condition at the boundaries of the fundamental domain $0 \le \zeta \le 1$ [99, 95]. Here we are naturally led to consider the eigenstates of a_0 with a non-vanishing eigenvalue ν instead [98]. Since the operators $\hat{Z}_\nu$ commute with P_0^-, the infinite set of vacuum states $\Psi_\nu(\zeta), \nu \in Z$ is degenerate, i.e. the corresponding LF energy

$$E_0 = \int_{-\infty}^{+\infty} d\zeta \Psi_\nu(\zeta) P_0^- \Psi_\nu(\zeta) = \frac{e^2 L}{8\pi^2} \tag{11.39}$$

is independent of ν. In addition, the vacuum states are not invariant under $\hat{Z}_1$,

$$\hat{Z}_1 \Psi_\nu(\zeta) = \Psi_{\nu+1}(\zeta) \tag{11.40}$$

and those which differ by unity in the value of ν have a non-zero overlap. The latter property resembles tunnelling due to instantons in the usual formulation. Note however that in our picture one does not consider minima of the *classical* action. The lowest energy states have been obtained within the quantum mechanical treatment of the symmetry consisting of the c-number shifts of an operator. This residual symmetry within the light-cone gauge generates copies of the gauge field A_0^+ which are essentially Gribov copies [113].

Implementation of the large gauge transformations for the dynamical fermion field $\psi_2(x)$ is based on the commutator

$$\left[\psi_2(x^-), j^+(y^-)\right] = \psi_2(y^-)\delta_a(x^- - y^-) \tag{11.41}$$

which follows from the basic anticommutation relation (11.23). The unitary operators $\hat{F}_\nu = (\hat{F}_1)^\nu$ that implement the phase transformation (11.31) are

$$\psi_2(x^-) \to \hat{F}_\nu \psi_2(x^-)\hat{F}_\nu^\dagger, \quad \hat{F}_\nu = \exp\left[i\frac{\pi}{L}\nu \int\limits_{-L}^{+L} \frac{\mathrm{d}x^-}{2} x^- j^+(x^-)\right], \tag{11.42}$$

as can be shown using the operator identity (11.19) for the case $[A, B] = \rho A$.

The Hilbert space transforms correspondingly. But since physical states are states with zero total charge and the pairs of operators $b_k^\dagger d_l^\dagger$, which create these states, are invariant, it is only the vacuum state that changes to

$$|0\rangle \to \hat{F}_\nu|0\rangle = \exp\left[\nu \sum_{m=1}^{\infty} \frac{(-1)^m}{m}(A_m^\dagger - A_m)\right]|0\rangle \equiv |\nu; f\rangle. \tag{11.43}$$

The boson Fock operators $A_m, A_m^\dagger$ [114], where

$$A_m^\dagger = \sum_{k=\frac{1}{2}}^{m-\frac{1}{2}} b_k^\dagger d_{m-k}^\dagger + \sum_{k=\frac{1}{2}}^{\infty} \left[b_{m+k}^\dagger b_k - d_{m+k}^\dagger d_k\right], \tag{11.44}$$

satisfy $\left[A_m, A_{m'}^\dagger\right] = m\delta_{m,m'}$. They emerge naturally after taking a Fourier transform of the $j^+(x^-)$ current expressed in terms of fermion modes. This yields

$$j^+(x^-) = \frac{1}{L}\sum_{m=1}^{\infty}\left[A_m e^{-\frac{i}{2}p_m^+ x^-} + A_m^\dagger e^{\frac{i}{2}p_m^+ x^-}\right] \tag{11.45}$$

as well as the exponential operator in Eq.(11.43). The states $|\nu; f\rangle$ are not invariant under $\hat{F}_1$: $|\nu; f\rangle \to |\nu + 1; f\rangle$, in analogy with Eq.(11.40).

To construct the physical vacuum state of the massive Schwinger model, one first defines the operator of the full large gauge transformations $\hat{T}_1 = \hat{Z}_1\hat{T}_1$ as a product of the commuting operators $\hat{Z}_1$ and $\hat{F}_1$. The requirement of gauge invariance of the physical ground state [22] then leads to the θ vacuum, which is obtained by diagonalization, i.e. by summing the degenerate vacuum states $|\nu\rangle = (\hat{T}_1)^\nu|0\rangle = |\nu; z\rangle|\nu; f\rangle$ with the appropriate phase factor:

$$|\theta\rangle = \sum_{\nu=-\infty}^{+\infty} e^{-i\nu\theta}\left(\hat{T}_1\right)^\nu|0\rangle,\ \ \hat{T}_1|\theta\rangle = e^{i\theta}|\theta\rangle. \tag{11.46}$$

($|0\rangle$ here denotes both the fermion and gauge boson Fock vacuum). Thus $|\theta\rangle$ is an eigenstate of $\hat{T}_1$ with the eigenvalue $\exp(i\theta)$. In other words, it is invariant up to a phase, which is the usual result [80, 22].

The physical meaning of the vacuum angle θ as the constant background electric field [103] can be found by a straightforward calculation:

$$\langle\theta|\Pi_{A_0^+}|\theta\rangle = \frac{e\theta}{2\pi}, \tag{11.47}$$

where the infinite normalization factor $\langle\theta|\theta\rangle$ has been divided out.

11.3 Formulation in the LF Weyl gauge

Let us turn now to the analysis of the massive Schwinger model in the light front temporal or Weyl gauge $A^-(x) = 0$. The solution of the fermionic constraint for can be again given in a closed form as

$$\psi_1(x^+, x^-) = m\int_{-L}^{+L}\frac{\mathrm{d}y^-}{2}G_a(x^- - y^-; A^+)\psi_2(x^+, y^-), \tag{11.48}$$

$$G_a(x^- - y^-; A^+) = \frac{1}{4i}\left[\epsilon_a(x^- - y^-) + i\mathrm{tg}\alpha\right] e^{\frac{ie}{2}(y^- - x^-)A_0^+} e^{-\frac{ie}{2}[\vartheta(x^-) - \vartheta(y^-)]}.$$

The quantities ϑ and α are equal to

$$\vartheta(x^-) = \frac{1}{2}\int_{-L}^{+L}\frac{\mathrm{d}y^-}{2}\epsilon_n(x^- - y^-)A_n^+(y^-),\ \ \alpha = \frac{eL}{2}A_0^+. \tag{11.49}$$

A rigorous way to quantize the model in the Weyl gauge is to perform the Dirac-Bergmann analysis and use $A^- = 0$ as a supplementary condition in that procedure. Here we simply prescribe the standard (anti)commutation relations between the independent fields ψ_2, A_0^+ and A_n^+ and their conjugate momenta at $x^+ = 0$ and obtain all other from the solution (11.48):

$$\begin{aligned}
&\left\{\psi_2(x^-), \psi_2^\dagger(y^-)\right\} = \frac{1}{2}\delta_a(x^- - y^-),\ \ \left[A_n^+(x^-), \Pi_{A_n^+}(y^-)\right] = i\delta_n(x^- - y^-),\\
&\left\{\psi_2(x^-), \psi_1^\dagger(y^-)\right\} = \left\{\psi_1(x^-), \psi_2^\dagger(y^-)\right\} = \frac{m}{2}G_a(x^- - y^-; A^+),\\
&\left[A_0^+, \Pi_{A_0^+}\right] = \frac{i}{L}.
\end{aligned} \tag{11.50}$$

Gauss' law is not an equation of motion in the Weyl gauge. One has to impose it as a condition on states which selects the physical subspace. The ZM condition is again simply $Q|phys\rangle = 0$ while the NM part reads

$$\langle phys|G_n(x^-)|phys\rangle = 0, \quad G_n(x^-) = 2\partial_-\Pi_{A_n^+}(x^-) - ej^+(x^-). \tag{11.51}$$

Furthermore, the Gauss' law is closely related to the generator of residual time-independent gauge transformations. Using the above field algebra and the operator identitie (11.19), we obtain

$$A_n^+(x^-) \to \Omega[\Lambda]A_n^+(x^-)\Omega^\dagger[\Lambda] = A_n^+(x^-) + \frac{2}{e}\partial_-\Lambda_n(x^-),$$

$$\psi_2(x^-) \to \Omega[\Lambda]\psi_2(x^-)\Omega^\dagger[\Lambda] = e^{-i\Lambda(x^-)}\psi_2(x^-), \tag{11.52}$$

where the unitary operator, implementing the residual symmetry, is

$$\Omega[\Lambda] = \exp\left[-\frac{i}{e}\int\limits_{-L}^{+L}\frac{\mathrm{d}y^-}{2}G_n(y^-)\Lambda_n(y^-)\right]\exp\left[ieQ\Lambda_0\right]\hat{T}_\nu, \quad \hat{T}_\nu = \hat{Z}_\nu\hat{F}_\nu,$$

$$\hat{Z}_\nu = \exp(i\nu\hat{\pi}_0), \; \hat{F}_\nu = \exp\left[i\frac{\pi}{L}\nu\int\limits_{-L}^{+L}\frac{\mathrm{d}y^-}{2}y^-j^+(y^-)\right], \; \hat{\pi}_0 = \frac{2\pi}{e}\Pi_{A_0^+}. \tag{11.53}$$

The operator $\hat{T}_\nu$ implements the residual large gauge transformations

$$\psi_2(x^-) \to \hat{T}_\nu\psi_2\hat{T}_\nu^{-1} = e^{-ie\Lambda_\nu}\psi_2(x^-), \; A_0^+ \to \hat{T}_\nu A_0^+\hat{T}_\nu^{-1} = A_0^+ + 2\partial_-\Lambda_\nu$$

which are manifestly present in the finite volume with (anti)periodic fields. The gauge function $\Lambda_\nu = \frac{\pi}{L}x^-\nu$ satisfies $\Lambda_\nu(-L) - \Lambda_\nu(L) = 2\pi\nu, \; \nu \equiv Z$. It is a part of the most general decomposition of the full gauge function $\Lambda(x^-) = \Lambda_n(x^-) + \Lambda_0 + \Lambda_\nu$ compatible with our boundary conditions.

Next, the independent fields can be expanded at $x^+ = 0$ as ($p_n^+ = \frac{2\pi}{L}n$)

$$\psi_2(x^-) = \frac{1}{\sqrt{2L}}\sum_{n=\frac{1}{2}}^{\infty}\left[b_n e^{-\frac{i}{2}p_n^+x^-} + d_n^\dagger e^{\frac{i}{2}p^+x_n^-}\right],$$

$$A_n^+(x^-) = \frac{1}{\sqrt{2L}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{p_n^+}}\left[g_n e^{-\frac{i}{2}p_n^+x^-} + g_n^* e^{\frac{i}{2}p_n^+x^-}\right],$$

$$\Pi_{A_n^+}(x^-) = -\frac{2i}{\sqrt{2L}}\sum_{n=1}^{\infty}\sqrt{p_n^+}\left[h_n e^{-\frac{i}{2}p_n^+x^-} - h_n^* e^{\frac{i}{2}p_n^+x^-}\right]. \tag{11.54}$$

The Fock operators then satisfy $\{b_m, b_n^\dagger\} = \{d_m, d_n^\dagger\} = [g_n, h_m^*] = \delta_{mn}$. Note the unusual commutator corresponding in fact to an indefinite-metric space [115]. The associated quanta are unphysical (there are no physical gauge degrees of freedom except for a zero mode in a compactified theory) and have to carry zero energy and momentum. The chosen form of the above commutator indeed leads for example to the vanishing of the norm of the one-particle ghost state $|g\rangle \equiv g_m^*|0\rangle$ as well as to $\langle g|P^+|g\rangle$=0.

The full gauge invariance of our theory requires also the vacuum state to be invariant. The Fock LF vacuum satisfies $Q|0\rangle = 0$ and $\langle 0|G_n(x^-)|0\rangle = 0$, but not $\hat{T}_\nu|0\rangle = |0\rangle$. In a complete analogy to the case of the light cone gauge, one has to form a superposition $|\theta\rangle$

$$|\theta\rangle = \sum_{\nu=\infty}^{\infty} e^{-i\nu\theta}\hat{T}_\nu|0\rangle \tag{11.55}$$

which is invariant up to a phase: $\hat{T}_\nu|\theta\rangle = \exp(i\theta)|\theta\rangle$. It is easy to show that the NM Gauss' law is satisfied by the theta vacuum, $\langle\theta|G_n(x^-)|\theta\rangle = 0$. The Fock representation of the individual vacua $|\nu\rangle = \hat{T}_\nu|0\rangle$ is given by $\hat{Z}_\nu = \exp\big[-\nu/\sqrt{2}\big(a_0^\dagger - a_0\big)\big]$, where $a_0 = 1/\sqrt{2}(\hat{\zeta} + i\hat{\pi}_0), \hat{\zeta} = 2\pi A_0^+/eL$ and $[a_0, a_0^\dagger] = 1$. Also, inserting the Fourier transform of the j^+ current

$$j^+(x^-) = \frac{1}{L}\left[A_0 + \sum_{m=1}^{\infty}\left(A_m e^{-\frac{i}{2}p_m^+ x^-} + A_m^\dagger e^{\frac{i}{2}p_m^+ x^-}\right)\right],$$

$$A_m = \sum_{k=\frac{1}{2}}^{\infty}\left[b_k^\dagger b_{m+k} - d_k^\dagger d_{m+k}\right] + \sum_{k=\frac{1}{2}}^{m-\frac{1}{2}} d_{m-k}b_k \tag{11.56}$$

to the expression for $\hat{F}_\nu$ we get the ame ermion Fock vacuum structure as in the light cone gauge,

$$\hat{F}_\nu|0\rangle = \exp\big[\nu\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\big(A_m^\dagger - A_m\big)\big]|0\rangle. \tag{11.57}$$

By a direct calculation one finds $[A_m, A_n^\dagger] = m\delta_{mn}$ which is equivalent to the commutator in x-representation with the Schwinger term:

$$\big[j^+(x^-), j^+(y^-)\big] = \frac{i}{\pi}\partial_-^x\delta(x^- - y^-). \tag{11.58}$$

It is known that the Schwinger model has interesting chiral properties [80, 22] such as spontaneous chiral symmetry breaking (SCSB), axial anomaly and the $U(1)$ problem. What is the chiral structure of the massive model in the LF version of the theory ? The answer is rather surprising. Since the $\psi_-(x^-)$ fermi field component satisfies the constraint (11.5), its chiral transformations are determined by those of the independent componet $\psi_+(x^-)$. Defining $\psi_+(x^-) \to \exp(-i\beta\gamma_5)\psi_+(x^-)$ classically, one finds

$$\psi_2(x^-) \to e^{-i\beta}\psi_2(x^-), \;\; \psi_1(x^-) \to e^{-i\beta}\psi_1(x^-), \tag{11.59}$$

i.e. both components rotate with the same sign. There is simply not enough independent spinor degrees of freedom to have true chiral transformations (this is no longer true in four dimensions where two components of $\psi_+(x)$ will transform with opposite sign). To summarize, there is no chiral symmetry on the classical level associated with two-dimensional LF fermi fields. Hence, there is nothing to be implemented on the quantum level, no possibility of spontaneous chiral symmetry breaking, no puzzle of the non-existent Goldstone boson known as the $U(1)$ problem. The present picture provides the simplest conceivable solution to the $U(1)$ problem in the Schwinger model. We want to emphasize that the above conclusion does not mean that LF theory has missed something: it simply clarifies the physical picture as much as possible (but not more).

In the usual picture with spontaneous chiral symmetry breaking, the vacuum expectation value of the fermi bilinear serves as an order parameter. The known result is $\langle\theta|\overline{\psi}\psi|\theta\rangle = \frac{e}{2\sqrt{\pi^3}}e^{\gamma_E}\cos(\theta)$ with γ_E being the Euler constant. Although lacking order-parameter interpretation, this VEV will still be non-zero in our case due to the fermionic structure of the theta vacuum following from topological properties. We have all ingredients (explicit Fock representation of the theta vacuum and exact solution of the fermionic constraint) for a calulation of $\langle\theta|\overline{\psi}\psi|\theta\rangle$ which however is rather complicated and lies beyond the scope of the present review.

On the other hand, we can still define the axial-vector current j_5^μ because we have an independent γ_5 matrix in the LF theory. Naively, j_5^+ appears to be identical to the vector current component j^+. However, in quantum theory j^μ has to be taken as normal ordered product of fermi fields. This is equivalent to having C-parity odd j^μ which is dictated by opposite charges of fermion and antifermion. There is no such a requirement for the axial current. Then, using the point-splitting regularization of j_5^μ as well as the solution (11.48) to the fermionic constraint, one finds [104]

$$\partial_\mu j_5^\mu(x) = 2im\overline{\psi}(x)\gamma_5\psi(x) - \frac{e}{\pi}\partial_+ A^+ \tag{11.60}$$

which agrees for $m = 0$ with the space-like result $-\frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}$. The derivation performed in the continuum theory relied crucially on the existence of a $1/x^+$ singularity in the "bad" component j_5^-. This singularity is a consequence of the fact that the contraction (i.e., vacuum expectation value or a correlation function) $\langle j_5^-\rangle$ of $j_5^- \sim m^2$ is given by the Bessel function $K_1(m\sqrt{-x^+x^-})$. In the present finite volume treatment the same mechanism works: although the contraction $\langle\psi_1^\dagger(x+\epsilon)\psi_1(x)\rangle$ is now expressed as an infinite series instead of an integral, this series contains the continuum result [35] in addition to (here irrelevant) L-dependent terms.

Another ingredient of our calculation was a *gauge invariant* version of the anticommutator (11.50). Without this correction, the lhs and rhs of the anticommutator transform differently under GT leading to inconsistent Fock calculations. The problem is not present in gauge-fixed schemes such as the usual light cone gauge. An alternative treatment of this difficulty, related to the "minimal quantization" method, is to define a gauge invariant fermi field $\hat{\psi}_2(x^-) = \exp(\frac{ie}{2}\vartheta(x^-))\psi_2(x^-)$ and require $\{\hat{\psi}_2(x^-), \hat{\psi}_2^\dagger(y^-)\} = i/2\delta_a(x^- - y^-)$. Some operators will change their form when expressed in terms of this field. For example, we find

$$P^+ = \int_{-L}^{+L}\frac{\mathrm{d}x^-}{2}\left[4i\hat{\psi}_2^\dagger\partial_-\hat{\psi}_2 + G_n(x^-)A_n^+\right], \tag{11.61}$$

where the second term vanishes on physical states. Hence $\langle phys|P^+|phys\rangle$ is gauge invariant. On the other hand, P^- being manifestly gauge invariant will keep its form when expressed in terms of $\hat{\psi}_2$ and $\hat{\psi}_1$. But it contains at the same time redundant gauge variables. To clarify the physical picture, it is useful to transform the theory to a different representation [115, 111].

Let us consider a unitary operator ($\vartheta(x^-;A_n^+)$ is defined above Eq.(11.49))

$$U[\vartheta] = \exp\Big[-\frac{ie}{2}\int_{-L}^{+L}\frac{\mathrm{d}x^-}{2}j^+(x^-)\vartheta(x^-;A_n^+)\Big]. \tag{11.62}$$

Using the field algebra (11.50) and the BCH formulae, we find that A_0^+,$\Pi_{A_0^+}$ and $A_n^+(x^-)$ do not change under $U[\vartheta]$ while

$$U[\vartheta]\psi_{2(1)}(x^-)U^\dagger[\vartheta] = e^{-i\frac{e}{2}\vartheta(x^-)}\psi_{2(1)}(x^-), \tag{11.63}$$

$$U[\vartheta]\Pi_{A_n^+}U^\dagger[\vartheta] = \Pi_{A_n^+} + \frac{e}{2}\int_{-L}^{+L}\frac{\mathrm{d}y^-}{2}\frac{1}{2}\epsilon(x^- - y^-)j^+(y^-). \tag{11.64}$$

The Hamiltonian changes its structure: $P^- \to U[\vartheta]P^-U^\dagger[\vartheta] = P^-_{lc} + P^-_{gh}$,

$$\begin{aligned}
P^-_{lc} &= \int_{-L}^{+L}\frac{\mathrm{d}x^-}{2}\Big[\Pi^2_{A_0^+} - \frac{e^2}{4}\int_{-L}^{+L}\frac{\mathrm{d}y^-}{2}j^+(x^-)G_2(x^- - y^-)j^+(y^-) + \\
&+ m\Big(\psi_2^\dagger(x^-)\psi_1(x^-) + \psi_1^\dagger(x^-)\psi_2(x^-)\Big)\Big], \quad \partial_-^2 G_2(x^-) = \delta_n(x^-), \\
P^-_{gh} &= \int_{-L}^{+L}\frac{\mathrm{d}x^-}{2}\big[\Pi^2_{A_n^+}(x^-) + \frac{e}{2}\int_{-L}^{+L}\frac{\mathrm{d}y^-}{2}\Pi_{A_n^+}(x^-)\epsilon(x^- - y^-)j^+(y^-)\big].
\end{aligned} \tag{11.65}$$

P^-_{lc} is the Hamiltonian of the usual light-cone (LC) gauge $A_n^+(x^-) = 0$ while unphysical P^-_{gh} has vanishing matrix elements on physical states defined by the transformed Gauss' operator $\langle phys|\hat{G}_n|phys\rangle = 0$. $\hat{G}_n = 2\partial_-\Pi_{A_n^+}$ does not generate any GT of $\psi_{2(1)}$. They are the fermi fields in the LC gauge,

$$\psi_1(x^-) = \frac{m}{4i}\int_{-L}^{+L}\frac{\mathrm{d}y^-}{2}\Big[\epsilon(x^- - y^-) + i\mathrm{tg}\alpha\Big]e^{-\frac{e}{2}(x^- - y^-)A_0^+}\psi_2(y^-). \tag{11.66}$$

One of the advantages of the transformation to the LC gauge representation is that now it is simple to derive the Fock representation of $\psi_1(x^-)$:

$$\psi_1(x^-) = \frac{m}{\sqrt{2L}}\sum_{n=\frac{1}{2}}^{\infty}\Big[\frac{b_n}{p_n^+ - eA_0^+}\,e^{-\frac{i}{2}p_n^+x^-} - \frac{d_n^\dagger}{p_n^+ + eA_0^+}\,e^{\frac{i}{2}p_n^+x^-}\Big] \tag{11.67}$$

The theta vacuum becomes $U[\vartheta]|\theta\rangle \equiv |\hat{\theta}\rangle$ and contains the unphysical ghost quanta,

$$|\hat{\theta}\rangle = \sum_{\nu=-\infty}^{\infty}e^{i\nu(\hat{\pi}_0-\theta)}\exp\sum_{n=1}^{\infty}(-1)^n\Big[\frac{\nu}{\sqrt{2L}}\frac{e}{p_n^{+3/2}}\big(g_n^* - g_n\big) + \frac{\nu}{n}\big(A_n^\dagger - A_n\big)\Big]|0\rangle.$$

12 Light front bosonization

It is a remarkable property of two-dimensional field theory in the conventional space-like formulation and fermionic field variables. Thus, there exists a kind of "duality" in the description of corresponding dynamics, as has been explicitly found by Coleman [43] and Mandelstam [116] for the case of sine-Gordon and Thirring models, and implicitly by Klaiber [117] and others. These results have later on been made more rigorous by Schroer and Truong [118] and Lehmann and Stehr [119]. For the case of the massive Schwinger model, bosonization was demonstrated by Coleman, Jackiw, Susskind and Kogut [102, 103, 101].

The bosonization corresponding rules are summarized by

$$j^\mu(x) = \frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\phi(x),$$

$$j_5^\mu(x) = \frac{1}{\sqrt{\pi}}\partial_\nu\phi(x),$$

$$\overline{\psi}(x)\psi(x) = c : \cos 2\sqrt{\pi}\phi(x) :,$$

$$\overline{\psi}(x)\gamma^5\psi(x) = c : \sin 2\sqrt{\pi}\phi(x) :,$$

$$\psi_{L(R)}(x) = \sqrt{\frac{c\mu}{2\pi}} : \exp\big[-i\sqrt{\pi}\Big(\int_{-\infty}^{x} d\xi\dot{\phi}(\xi) \pm \phi(x)\Big)\big] : \tag{12.1}$$

Some elements of bosonization have been treated also in the LF formulation [114] but no systematic study was done.

Let us start with the case of free LF fermions in the continuum formulation. All necessary ingredients are paralel to those discussed in a previous section. The goal here will be to find an equivalent representation of the fermion field in terms of a bosonic field which obeys canonical bosonic commutation relation.

For this purpose, let us consider the commutation relation between two dynamical current components

$$j^+(x) = 2 : {\psi_2}^\dagger(x)\psi_2(x) : . \tag{12.2}$$

It is convenient again first to Fourier transform the current,

$$j^+(x^-) = \int_0^\infty \frac{dp^+}{4\pi\sqrt{\pi}}[A(p^+)e^{-\frac{i}{2}p^+x^-} + A^\dagger(p^+)e^{\frac{i}{2}p^+x^-}], \tag{12.3}$$

where the composite operator $A(k^+)$ is given by

$$\begin{aligned} A(k^+) &= 2\int\limits_{-\infty}^{+\infty}\frac{dx^-}{2}j^+(x^-)e^{\frac{i}{2}k^+x^-} = \\ &= \int\limits_0^{\infty}\frac{dp^+}{2\pi\sqrt{p^+(p^++k^+)}}\left[b^\dagger(p^+)b(p^++k^+)-d^\dagger(p^+)d(p^++k^+)\right]+ \\ &+ \int\limits_0^{k^+}\frac{dp^+}{2\pi\sqrt{p^+(k^+-p^+)}}d(p^+)b(k^+-p^+). \end{aligned} \tag{12.4}$$

This is the continuum version of the current bosonization that was described in the previous section for the finite-volume massive Schwinger model. By a direct calculation one finds

$$[A(p^+),A^\dagger(p'^+)] = 2\pi p^+\delta(p'^+-p^+),\ \ [A(p^+),\ A(p'^+)] = 0 \tag{12.5}$$

and as a consequence also

$$[j^+(x^-),j^+(y^-)] = \frac{i}{\pi}\partial_-^x\delta(x^--y^-). \tag{12.6}$$

The latter relation is the basis for deriving the bosonized version of the fermi field operator ψ_2. Indeed, consider the exponential operator

$$\Phi(x^-) = c_1e^{-ic_0\phi(x^-)} = c_1c_2:e^{-ic_0\phi(x^-)}: \ = c_1c_2e^{-ic_0\phi^{(+)}(x^-)}e^{-ic_0\phi^{(-)}(x^-)}, \tag{12.7}$$

where $\phi^{(\pm)}$ are the positive and negative frequency parts of $\phi(x)$ (parts with creation and annihilation operators) and the constant $c_2 = \exp\left\{1/2c_0^2\left[\phi^{(+)}(x^-),\phi^{(-)}(x^-)\right]\right\}$. This factorized form is the result of using the first opoerator identity from (11.19). The scalar field ϕ given by

$$\phi(x^-) = \sqrt{\pi}\int\limits_{-\infty}^{+\infty}\frac{dy^-}{2}\frac{1}{2}\epsilon(x^--y^-)j^+(y^-). \tag{12.8}$$

The constants c_0 and c_1 are left as free parameters for the moment. The operator ϕ satisfies the commutation relation for the canonical scalar LF field at $x^+=y^+=0$:

$$\begin{aligned} [\phi(x^-),\phi(y^-)] &= \frac{\pi}{4}\int\limits_{-\infty}^{+\infty}\frac{du^-}{2}\int\limits_{-\infty}^{+\infty}\frac{dv^-}{2}\epsilon(x^--u^-)\epsilon(y^--v^-)\left[j^+(u^-),j^+(v^-)\right] \\ &= -\frac{i}{8}\epsilon(x^--y^-), \end{aligned} \tag{12.9}$$

The crucial point for deriving the above commutation relation is the Schwinger term in the current-current commutator. Inserting the momentum representation of the antisymmetric sign function

$$\epsilon(x^--y^-) = \frac{2i}{\pi}\int\limits_0^{\infty}\frac{dp^+}{2}\frac{1}{p^+}\left[e^{-\frac{i}{2}p^+(x^--y^--i\epsilon)}-e^{\frac{i}{2}p^+(x^--y^-+i\epsilon)}\right] \tag{12.10}$$

into Eq.(12.8) one finds

$$\phi(x^-) = \frac{i}{4\pi}\int\limits_0^\infty dq^+ \frac{1}{q^+}\big[A(q^+)e^{-\frac{i}{2}q^+x^-} - A^\dagger(q^+)e^{\frac{i}{2}q^+x^-}\big] \tag{12.11}$$

and

$$\Phi(x^-) = c_1 c_2 \exp[-\hat{A}^\dagger(x^-)]\exp[\hat{A}(x^-)] \tag{12.12}$$

with

$$\hat{A}(x^-) = \frac{c_0}{4\pi}\int\limits_0^\infty dq^+ \frac{A(q^+)}{q^+}e^{-\frac{i}{2}q^+x^-}. \tag{12.13}$$

Can the operator $\Phi(x^-)$ satisfy anticommutation relation ? It turns out that the constant c_0 may be chosen in such a way that it can. Indeed, applying the operator relation following from (11.19) $\exp(A)\exp(B) = \exp[A,B]\exp(B)\exp(A)$ in the product $\Phi(x^-)\Phi(y^-)$ to interchange order of the two operators, one gets

$$\begin{aligned}
\Phi(x^-)\Phi(y^-) &= c_1^2 c_2^2 e^{-\hat{A}^\dagger(x^-)}e^{\hat{A}(x^-)}e^{-\hat{A}^\dagger(y^-)}e^{\hat{A}(y^-)} = \\
&= c_1^2 c_2^2 \exp\big\{-[\hat{A}(x^-),\hat{A}^\dagger(y^-)]\big\}\exp\big\{-[\hat{A}^\dagger(x^-),\hat{A}(y^-)]\big\} \times \\
&\times\ e^{-\hat{A}^\dagger(y^-)}e^{\hat{A}(y^-)}e^{-\hat{A}^\dagger(x^-)}e^{\hat{A}(x^-)} = \\
&= \exp[-\frac{c_0^2}{4\pi}\int\limits_0^\infty \frac{dq^+}{q^+}e^{-\frac{i}{2}q^+(x^- - y^-)}] \times \\
&\times\ \exp[\frac{c_0^2}{4\pi}\int\limits_0^\infty \frac{dq^+}{q^+}e^{\frac{i}{2}q^+(x^- - y^-)}]\Phi(y^-)\Phi(x^-).
\end{aligned} \tag{12.14}$$

Taking into account the definition of the sign function (12.10) and choosing $c_0 = 2\sqrt{2\pi}$, the two exponentials in the last line of the above equations combine to $\exp[i\pi\epsilon(x^- - y^-)] = -1$ yielding the desired anticommutation property.

The situation is slightly more complicated for the anticommutor between $\Phi(x)$ and $\Phi^\dagger(y)$ which we should find to be proportional to $\delta(x^- - y^-)$. We remind here the form of the two-point function of the massive LF scalar field $\tilde{\phi}(x)$ of a mass μ that will be needed. The field expansion is

$$\tilde{\phi}(x^-) = \int\limits_0^\infty \frac{dq^+}{4\pi q^+}\big[a(q^+)e^{-\frac{i}{2}q^+x^- - \frac{i}{2}\frac{\mu^2}{q^+}x^+} + a^\dagger(q^+)e^{\frac{i}{2}q^+x^- + \frac{i}{2}\frac{\mu^2}{q^+}x^+}\big], \tag{12.15}$$

where small imaginary parts for $x^\pm$ are understood. Using the Fock commutation relation $[a(p^+), a^\dagger(p'^+)] = 2\pi p^+\delta(p^+ - p'^+)$, one straightforwardly gets for the two-point function

$$D(x) = \langle 0|\tilde{\phi}(0,0)\tilde{\phi}(x^+,x^-)|0\rangle = \int\limits_0^\infty \frac{dp^+}{8\pi p^+}e^{\frac{i}{2}p^+(x^- + i\epsilon) + \frac{i}{2}\frac{\mu^2}{p^+}(x^+ + i\delta)}. \tag{12.16}$$

We already found that this integral is equal to

$$\begin{aligned} D(x) &= -\frac{i}{16}|\mathrm{sgn}(x^+) + \mathrm{sgn}(x^-)|\Big(N_0(\mu\sqrt{x^2}) - i\mathrm{sgn}(x^+)J_0(\mu\sqrt{x^2})\Big) + \\ &+ \frac{1}{8\pi}|\mathrm{sgn}(x^+) - \mathrm{sgn}(x^-)|\tilde{K}(x^+, x^-), \end{aligned} \tag{12.17}$$

where $\tilde{K}$ is equal to

$$\tilde{K}(x^+, x^-) = K_0(\mu\sqrt{(x^+ - i\delta)(x^- + i\epsilon)}, \ \text{ if } \ x^+ > 0, x^- < 0 \tag{12.18}$$

or to a complex conjugate expression if $x^+ < 0, x^- > 0$. $J_0(z)$ $(K_0(z))$ is the (modified) Bessel function of the order 0. This expression is useful because the 2-point correlation function of the composite field $\phi(x)$ (12.11) is the same as the one of the elementary scalar field $\tilde{\phi}(x)$:

$$\langle 0|\phi(0,0)\phi(x^+, x^-)|0\rangle = D(x). \tag{12.19}$$

The calculation of the anticommutator in the bosonized form requires an explicit presence of the small imaginary part of the LF time in the two operators Φ. As in the previous calculation, we get, using the familiar operator identities,

$$\Phi(x^+, x^-)\Phi^\dagger(0,0) = \exp\,[\hat{A}(x^+, x^-), \hat{A}^\dagger(0,0)] : \Phi(x^+, x^-)\Phi^\dagger(0,0) :,$$

$$\Phi^\dagger(0,0)\Phi(x^+, x^-) = \exp\,[\hat{A}(0,0), \hat{A}^\dagger(x^+, x^-)] : \Phi^\dagger(0,0)\Phi(x^+, x^-) : . \tag{12.20}$$

It follows that

$$\Phi(x^+, x^-)\Phi^\dagger(0,0) + \Phi^\dagger(0,0)\Phi(x^+, x^-) =: \Phi(x^+, x^-)\Phi^\dagger(0,0) : \times$$

$$\Big[\exp\Big(\int_0^\infty \frac{dk^+}{k^+} e^{-\frac{i}{2}k^+x^- - \frac{i}{2}\frac{\mu^2}{k^+}x^+}\Big) + \exp\Big(\int_0^\infty \frac{dk^+}{k^+} e^{\frac{i}{2}k^+x^- + \frac{i}{2}\frac{\mu^2}{k^+}x^+}\Big)\Big]. \tag{12.21}$$

From Eqs.(5.10) and (12.18) we see that the integrals $\hat{E}_1, \hat{E}_1^*$ in the exponentials of the latter expression are related to the function $D(x)$. For $x^2 < 0$ we have

$$\hat{E}_1 = 2K_0\left(\mu\sqrt{(x^+ - i\delta)(x^- + i\epsilon)}\right). \tag{12.22}$$

Taking into account the form of the Bessel function $K_0(z)$ for small z, $K_0(z) \approx -\gamma_E - ln(\frac{z}{2}) + O(z^2)$, where γ_E is the Euler's constant, we find

$$\big\{\Phi(x^+, x^-), \Phi^\dagger(0,0)\big\} = e^{-2\gamma_E}\frac{4}{\mu^2} : \Phi(x^+, x^-)\Phi^\dagger(0,0) : \times$$

$$\times\left[\frac{1}{(x^+ - i\delta)(x^- + i\epsilon)} + \frac{1}{(x^+ + i\delta)(x^- - i\epsilon)}\right] \tag{12.23}$$

which for $x^+ = 0$ reduces to the equal-time anticommutator

$$\left\{\Phi(0,x^-),\Phi^\dagger(0,0)\right\} = c_2^2 e^{-2\gamma_E}\frac{8\pi}{\epsilon\mu^2}\delta(x^-), \tag{12.24}$$

because $:\Phi(0,0)\Phi^\dagger(0,0):=1$ and we have used the relation $(x^- + i\epsilon)^{-1} - (x^- - i\epsilon)^{-1} = -2i\pi\delta(x^-)$. The normalization constant c_1 can now be determined from the requirement that the anticommutator (12.24) is equal to $1/2\delta(x^-)$. One finds $c_1 = \mu\sqrt{\epsilon}\frac{e^{\gamma_E}}{4\sqrt{\pi}}$. Since the constant c_2 was

$$\exp\Big(-\frac{1}{2}\big[\hat{A}(x),\hat{A}^\dagger(x)\big]\Big) = \exp\Big(-\frac{1}{2}K_0(\mu\epsilon)\Big) = \sqrt{\frac{\mu\epsilon}{2}} \tag{12.25}$$

the final expression for the bosonized dynamical Fermi field component is

$$\Phi(x) = C : e^{-i2\sqrt{\pi}\phi(x)} :$$

$$C = c_1 c_2 = e^{\gamma_E}\sqrt{\frac{\mu}{8\pi}}. \tag{12.26}$$

It is interesting to note that a small imaginary part of the time argument of the bosonized fermion field was used also in the original space-like treatment by Mandelstam:

$$\psi_L(x) = (c\mu/2\pi)^{1/2}e^{\mu/8\epsilon} : \exp\big[-2i\pi\int_{-\infty}^{x} d\xi\dot{\phi}(\xi) - \frac{i}{2}\beta\phi(x)\big] : \tag{12.27}$$

has been found in the original Mandelstam's paper [116] starting from the commutation relation

$$\left[\phi^+(x,t+dt),\phi^-(y,t)\right] = \Delta_+\big((x-y)^2 - (dt+i\epsilon)^2\big), \tag{12.28}$$

where

$$\Delta_+ = -\frac{1}{4\pi}\ln\big\{c^2\mu^2\big[x^2 - (dt+i\epsilon)^2\big]\big\} + O(x^2). \tag{12.29}$$

μ is a mass parameter and β defines the interacting term $\cos\big(\beta\phi(x)\big)$ of the sine-Gordon model.

The last two properties we have to understand is if one really has $j^- = -\partial_+\phi$ and if one can write the mass term as something proportional to $\cos(\phi)$.

From Eq.(12.8) we easily get, using partial integration

$$j^+(x) = \frac{2}{\sqrt{\pi}}\partial_-\phi(x). \tag{12.30}$$

In a similar way, assuming the vector-current conservation $\partial_+ j^+(x) + \partial_- j^-(x) = 0$, we find

$$\partial_+\phi(x) = -\sqrt{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}y^-}{2}\frac{1}{2}\epsilon(x^- - y^-)\partial_- j^-(x^+,y^-) = -\frac{\sqrt{\pi}}{2}j^-(x^-). \tag{12.31}$$

The Eqs.(12.30,12.31) are summarized in the familiar statement

$$j^{\mu}(x) = \frac{\epsilon^{\mu\nu}}{\sqrt{\pi}}\partial_{\nu}\phi(x). \tag{12.32}$$

To derive the bosonized form of the mass term, one has to take into account the fact that the second component of the Fermi field satisfies an equation of constraint in the LF formulation. This implies

$$\chi(x) = \frac{m}{4i}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}y^-}{2}\epsilon(x^- - y^-)\Phi(x^+, y^-) \tag{12.33}$$

The free fermionic Hamiltonian becomes

$$P^- = \frac{m^2}{4i}\int\limits_{-\infty}^{+\infty}\frac{\mathrm{d}x^-}{2}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}y^-}{2}\epsilon(x^- - y^-)\Big[\Phi^\dagger(x)\Phi(x^+, y^-) - \Phi^\dagger(x^+, y^-)\Phi(x)\Big] \tag{12.34}$$

or

$$P^- = \frac{m^2}{4i}C^2\int\limits_{-\infty}^{+\infty}\frac{\mathrm{d}x^-}{2}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}y^-}{2}\epsilon(x^- - y^-)\Big[\, : e^{2i\sqrt{\pi}\phi(x^-)} : \; : e^{-2i\sqrt{\pi}\phi(y^-)} : -$$

$$: e^{2i\sqrt{\pi}\phi(y^-)} : \; : e^{-2i\sqrt{\pi}\phi(x^-)} : \Big] \tag{12.35}$$

It is instructive to compare this result with the derivation of the bosonized form of the mass in the conventional field theory. In one approach ([120]) one uses the commutators between pseudoscalar density J_5 and the axial current density j_5^0:

$$J_5(x) = \psi_L^\dagger\psi_R - \psi_R^\dagger\psi_L = F_+\big(\phi(x)\big) - F_-\big(\phi(x)\big),$$

$$j_0^5 = \psi_L^\dagger\psi_L - \psi_R^\dagger\psi_R,$$

$$\Big[j_5^0(x), F_-\big(\phi(y)\big)\Big] = -2\delta(x-y)F_-\big(\phi(y)\big). \tag{12.36}$$

The Fermi field $\psi(x)$ has the upper and lower components ψ_R and ψ_L and satisfies the canonical anticommutation relation. Since from the current bosonization one has $j_5^0(x) = \frac{1}{\sqrt{\pi}}\partial_0\phi(x) = \frac{1}{\sqrt{\pi}}\Pi_\phi(x)$, the two commutators can be rewritten as

$$\frac{1}{\sqrt{\pi}}\Big[\Pi_\phi(x), F_\pm\Big(\phi(y)\Big)\Big] = \pm 2\delta(x-y)F_\pm\big(\phi(y)\big). \tag{12.37}$$

The corresponding solution is

$$F_\pm\big(\phi(x)\big) = c\exp\big\{\pm 2i\sqrt{\pi}\phi(x)\big\} \tag{12.38}$$

so that $J(x) = c\cos 2\sqrt{\pi}\phi(x)$ (c is a constant). Note that this result is based on the canonical commutator of the scalar field

$$\Big[\phi(x), \Pi_\phi(y)\Big] = i\delta(x-y) \tag{12.39}$$

and cannot be used in the LF theory where $\Pi \neq \partial_+\phi$. Thus one can expect a modification of the bosonized mass term in the LF formulation.

Bosonic represention of the two-dimensional massive LF fermion field can be derived also in the discretized formulation. The starting expressions are the Fock expansions of the dynamical fermion field and current

$$\psi_2(0,x^-) = \frac{1}{\sqrt{2L}}\sum_{n=\frac{1}{2}}^{\infty}\Big[b_n e^{-\frac{i}{2}p_n^+x^- - \frac{i}{2}\frac{m^2}{p^+}x^+} + d_n^\dagger e^{\frac{i}{2}p_n^+x^- + \frac{i}{2}\frac{m^2}{p^+}x^+}\Big],$$

$$j^+(x) = \frac{1}{L}\Big[A_0 + \sum_{m=1}^{\infty}\Big(A_m e^{-\frac{i}{2}p_m^+x^-} - A_m^\dagger e^{\frac{i}{2}p_m^+x^-}\Big)\Big],$$

where $A_0 = Q$. We already had the relation

$$\Big[A_m, A_n^\dagger\Big] = m\delta_{m,n} \tag{12.40}$$

and the current-current commutator with the Schwinger term

$$\Big[j^+(x^-), j^+(y^-)\Big] = \frac{1}{\pi}\partial_-^x\delta_n(x^- - y^-).$$

From the DLCQ definition of the scalar field $\phi(x)$

$$\phi(x) = \sqrt{\pi}\int_{-L}^{+L}\frac{\mathrm{d}x^-}{2}\epsilon_N(x^- - y^-)j_N^+(y^-) \tag{12.41}$$

we find

$$\phi(x) = \sum_{m=1}^{\infty}\frac{1}{p_m^+}\Big[A_m e^{-\frac{i}{2}p_m^+x^-} - A_m^\dagger e^{\frac{i}{2}p_m^+x^-}\Big],$$

$$A_m = \sum_{k=\frac{1}{2}}^{\infty}\Big(b_k^\dagger b_{k+m} - d_k^\dagger d_{k+m}\Big) + \sum_{k=\frac{1}{2}}^{m-\frac{1}{2}} d_{m-k}b_k. \tag{12.42}$$

To show that the dynamical Fermi field component $\psi_2(x)$ has a representation in terms of a bosonic field, one can proceed as follows. First, calculate the commutator

$$\big[A_m, \psi_2(x)\big] = -e^{\frac{i}{2}p_m^+x^- + \frac{i}{2}\frac{m^2}{p^+}x^+}\psi_2(x), \tag{12.43}$$

where a few nontrivial cancellations between different terms occured. Since $A_m|0\rangle = 0$, we immediately get that the state $\psi_2(x)|0\rangle$ is an eigenstate of the annihilation operator A_m:

$$A_m\psi_2(x)|0\rangle = -e^{\frac{i}{2}p_m^+x^- + \frac{i}{2}\frac{m^2}{p^+}x^+}\psi_2(x)|0\rangle \tag{12.44}$$

and hence it is a boson coherent state. Its form has to be

$$\psi_2(x)|0\rangle = \mathcal{N}\exp\Big\{-\sum_{m=1}^{\infty}\frac{A_m^\dagger}{m}e^{\frac{i}{2}p_m^+x^-}\Big\}|0\rangle. \tag{12.45}$$

or

$$\psi_2(x)|0\rangle = c'\exp\Big\{-\sum_{n=1}^{\infty}\frac{1}{n}\Big(A_n^\dagger e^{\frac{i}{2}p_n^+x^- + \frac{i}{2}\frac{m^2}{p_n^+}x^+} - A_n e^{-\frac{i}{2}p_n^+x^- - \frac{i}{2}\frac{m^2}{p_n^+}x^+}\Big)\Big\}|0\rangle. \tag{12.46}$$

To check the validity of the representation (12.45) it is sufficient to act on it with A_m and to use the first operator relation (11.19) in the form $B\exp(-A) = \exp(-A)B + [A,B]\exp(-A)$. In the next step, one can show that the same relation holds for an arbitrary Fock state generated by $A_n^\dagger$, i.e. in the whole space of states. Hence the bosonization correspondence holds as an operator relationship.

Let us try to generalize the above considerations to the case of interacting models. First, consider the light-front massive Schwinger model in the continuum formulation and in the LC gauge $A^+ = 0$. The corresponding Lagrangian is

$$\mathcal{L}_{lf} = i{\psi_2}^\dagger \overleftrightarrow{\partial_+}\psi_2 + i{\psi_1}^\dagger \overleftrightarrow{\partial_-}\psi_1 - m({\psi_2}^\dagger\psi_1 + {\psi_1}^\dagger\psi_2) + \frac{1}{2}(\partial_-A^-)^2 - \frac{e}{2}j^+A^-. \tag{12.47}$$

Using the constraints

$$2i\partial_-\psi_1(x) = m\psi_2(x), \quad \partial_-^2A^-(x) = -\frac{e}{2}j^+(x) \tag{12.48}$$

one gets the Hamiltonian in terms of dynamical variable $\psi_2(x)$ only which is analogous to the finite-volume Hamiltonian (11.27):

$$\begin{aligned} P^- &= \int_{-\infty}^{+\infty}\frac{\mathrm{d}x^-}{2}\Big[-\frac{e^2}{4}j^+\frac{1}{\partial_-^2}j^+ + \\ &+ \frac{m^2}{2i}\int_{-\infty}^{\infty}\frac{\mathrm{d}y^-}{2}\epsilon(x^- - y^-)\Big({\psi_2}^\dagger(x)\psi_2(y) - {\psi_2}^\dagger(y)\psi_2(x)\Big)\Big]. \end{aligned} \tag{12.49}$$

Expressing now the field ψ_2 and the current j^+ by their bosonic forms, we find

$$\begin{aligned} P^- &= \int_{-\infty}^{+\infty}\frac{\mathrm{d}x^-}{2}\Big[\frac{e^2}{\pi}\phi^2 + \frac{m^2}{2i}C^2\int_{-\infty}^{\infty}\frac{\mathrm{d}y^-}{2}\epsilon(x^- - y^-)\times \\ &\times \Big(:e^{2i\sqrt{\pi}\phi(x^-)}::e^{-2i\sqrt{\pi}\phi(y^-)}: - :e^{2i\sqrt{\pi}\phi(y^-)}::e^{-2i\sqrt{\pi}\phi(x^-)}:\Big)\Big]. \end{aligned} \tag{12.50}$$

Another possibility is to consider the massive Schwinger model in the Weyl gauge $A^- = 0$. The corresponding LF Hamiltonian in the continuum theory is

$$P^- = \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}x^-}{2} \left[\Pi^2_{A^+} + m\left({\psi_2}^\dagger \psi_1 + {\psi_1}^\dagger \psi_2\right)\right]. \tag{12.51}$$

The dependent component ψ_1 in (12.51) satisfies the constraint

$$2i\partial_- \psi_1(x) = m\psi_2(x) + e\psi_1 A^+(x), \tag{12.52}$$

which has a similar stucture than the fermionic constraint in the Thirring model (see below). In a full analogy with the finite-volume treatment, it can be inverted by means of the Green's function $G(z^-; A^+)$:

$$\psi_1(x) = m \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}y^-}{2} \mathcal{G}(x^- - y^-; A^+)\psi_2(x^+, y^-),$$

$$\mathcal{G}(x^- - y^-; A^+) = \frac{1}{2i}\epsilon_a(x^- - y^-)e^{-ie\vartheta(x^-) + ie\vartheta(y^-)},$$

$$\vartheta(x^-) = \frac{1}{2}\int\limits_{-\infty}^{\infty} \frac{1}{2}\epsilon(x^- - z^-)A^+(x^+, z^-). \tag{12.53}$$

It is interesting to observe that an exponential form of the Green's function $\mathcal{G}(x^- - y^-)$ emerges also in the LF treatent of the massive Thirring model. In the conventional field theory, massive Thirring model was found to be equivalent to the sine-Gordon model by Coleman, Mandelstam, Schroer and others. It is defined by the Lagrangian density

$$\mathcal{L} = \frac{i}{2}\overline{\psi}\gamma^\mu \overleftrightarrow{\partial_\mu} \psi - m\overline{\psi}\psi - \frac{1}{2}g j_\mu j^\mu \tag{12.54}$$

where $j^\mu =: \overline{\psi}\gamma^\mu\psi:$. The corresponding LF expressions are

$$\mathcal{L}_{lf} = i{\psi_2}^\dagger \overleftrightarrow{\partial_+} \psi_2 + i{\psi_1}^\dagger \overleftrightarrow{\partial_-} \psi_1 - m\big({\psi_2}^\dagger\psi_1 + {\psi_1}^\dagger\psi_2\big) - \frac{1}{2}g j^+ j^-,$$

$$j^+ = 2 : {\psi_2}^\dagger\psi_2 :, \;\; j^- = 2 : {\psi_1}^\dagger\psi_1 : . \tag{12.55}$$

The Euler-Lagrange equations read

$$\begin{aligned} 2i\partial_+\psi_2 &= m\psi_1 + g j^-\psi_2, \\ 2i\partial_-\psi_1 &= m\psi_2 + g j^+\psi_1. \end{aligned} \tag{12.56}$$

The latter equation is a constraint. It can be used to bring the LF Hamiltonian to the form

$$P^- = m\int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}x^-}{2}\left[{\psi_2}^\dagger\psi_1 + {\psi_1}^\dagger\psi_2\right] \tag{12.57}$$

Derivation of the Hamiltonian as well as its form is identical to the case of the Federbush model. The difference is that one can write down solution of the dynamical equation in the latter case and this leads to a simplification of the bilinear fermionic structure because the exponential factors cancel. The Hamiltonian of the massive Thirring model is more compex due to the presence of exponential factors. Note that the interaction term $j_\mu j^\mu$ disappeared from P^- and the interaction is contained solely in the $\psi_1(x)$ which is the solution of the constraint:

$$\psi_1(x) = \frac{m}{2i} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}y^-}{2} \epsilon(x^- - y^-) \exp\Big\{ \frac{ig}{2} \int\limits_{x^-}^{y^-} dz^- j^+(z^-) \Big\} \psi_2(x^+, y^-). \tag{12.58}$$

Defining the Green's function G

$$G(x^- - y^-) = \frac{1}{2i} \epsilon(x^- - y^-) \exp\Big\{ - ig\phi(x^-) + ig\phi(y^-) \Big\},$$

$$\phi(x^-) = \sqrt{\pi} \int\limits_{-\infty}^{+\infty} \frac{dz^-}{2} \frac{1}{2} \epsilon(x^- - z^-) j^+(z^-), \tag{12.59}$$

the dependent Fermi field component can be written as

$$\psi_1(x) = m \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}y^-}{2} G(x^- - y^-) \psi_2(x^+, y^-). \tag{12.60}$$

Inserting the bosonic representation of $\psi_2(x)$ into P^-, we get

$$P^- = \frac{m^2}{2i} C^2 \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}x^-}{2} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}y^-}{2} \epsilon(x^- - y^-)$$

$$\Big(: e^{2i\sqrt{\pi}\phi(x^-)} : e^{-\frac{ig}{2}\phi(x^-) + \frac{ig}{2}\phi(y^-)} : e^{-2i\sqrt{\pi}\phi(y^-)} : - h.c. \Big) \tag{12.61}$$

This LF Hamiltonian can be further treated by using operator identities. The detailed analysis is a bit technical and will not be discussed here.

13 Higgs mechanism in a LF formulation

The Higgs mechanism in the LF formalism was studied on the tree level in the continuum formulation [53]. It was assumed that a scalar field contains a c-number piece which gave a justification for performing a usual shift in the Lagrangian leading to the generation of the mass term for the gauge field. A support for the above assumption comes from the fact that the solution of the zero-mode constraint of the real scalar field in the DLCQ analysis contains such a constant non-operator part [65, 56].

In the present work, we study the SSB of an abelian symmetry in the Higgs model. Our approach is based on the discrete light-cone quantization method (DLCQ) considered as a hamiltonian analytical framework with large but finite number of Fourier modes to approximate quantum field theory with its infinite number of degrees of freedom. A (regularized) unitary operator that shifts the scalar field by a constant will be used to transform the Fock space. The motivation for this step is a natural physical requirement to find ground states in the broken phase which would correspond to a lower LF energy than the usual Fock vacuum. This is suggested already by considering minima of the classical LF potential energy. A procedure, equivalent to transforming the states, is to work with a transformed Hamiltonian and calculate its matrix elements between the usual Fock states. In this way one naturally arrives at the effective type of the Hamiltonian that incorporates the usual pattern of the Higgs mechanism.

The Lagrangian density of the abelian Higgs model that we wish to analyze has the form

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu\phi)^\dagger D^\mu\phi + \frac{1}{2}\mu^2\phi^\dagger\phi - \frac{\lambda}{4}(\phi^\dagger\phi)^2, \tag{13.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $D^\mu\phi = \partial^\mu\phi + ieA^\mu\phi, \mu^2 > 0$. The Lagrangian is invariant under two groups of transformations: the global rotations of the complex scalar field $\phi(x) \to \exp\big(-i\beta\big)\phi(x)$ and the local gauge transformations

$$\phi(x) \to \exp\big(-i\omega(x)\big)\phi(x), \;\; A^\mu(x) \to A^\mu(x) + \partial^\mu\omega(x)/e. \tag{13.2}$$

In terms of the LF variables, the Lagrangian (13.1) is

$$\begin{aligned}\mathcal{L}_{lf} &= \frac{1}{2}\big(\partial_+A^+ - \partial_-A^-\big)^2 + \big(\partial_+A^i + \frac{1}{2}\partial_iA^-\big)\big(2\partial_-A^i + \partial_iA^+\big) - \\ &-\frac{1}{2}\big(\partial_1A_2 - \partial_2A_1\big)^2 + \partial_+\phi^\dagger\partial_-\phi + \partial_-\phi^\dagger\partial_+\phi - \frac{1}{2}\partial_i\phi^\dagger\partial_i\phi - \frac{ie}{2}\phi^\dagger \overleftrightarrow{\partial_+}\phi A^+ - \\ &-\frac{ie}{2}\phi^\dagger\overleftrightarrow{\partial_-}\phi A^- - \frac{ie}{2}\phi^\dagger\overleftrightarrow{\partial_i}\phi A^i + \frac{e^2}{2}\big(A^+A^- - A^iA^i\big)\phi^\dagger\phi + \frac{\mu^2}{2}\phi^\dagger\phi - \frac{\lambda}{4}\big(\phi^\dagger\phi\big)^2.\end{aligned} \tag{13.3}$$

Writing $\phi = \sigma + i\pi$, the conserved current corresponding to the global symmetry is $J^\mu(x) = -i\phi^\dagger(x)\overleftrightarrow{\partial^\mu}\phi(x) = 2\sigma(x)\overleftrightarrow{\partial^\mu}\pi(x)$.

We will work in a finite volume $V = 8L^3$ with space coordinates restricted to $-L \le x^-, x^1, x^2 \le L$. Our notation is $x^\mu = (x^+, \underline{x}), \underline{x} = (x^-, x^1, x^2),\ p.x = \frac{1}{2}p^-x^+ + \frac{1}{2}p^+x^- - p^1x^1 - p^2x^2$. The gauge field will be chosen periodic in all three directions, while the scalar field will be antiperiodic: $A^\mu(x^+, x^- = -L, x, y) = A^\mu(x^+, x^- = L, x, y), \sigma(x^+, x^- = -L, x, y) = -\sigma(x^+, x^- = L, x, y)$, and similarly in the perpendicular directions x^1, x^2).[10] The boundary conditions imply discrete values of the three-momentum labeled by an integer or a half-integer and also lead to the presence of global and proper zero modes of the gauge field [96]. The proper ZM are constrained variables that can modify the LF Hamiltonian. For small coupling the corrections may be evaluated perturbatively [97]. We shall however neglect the gauge-field

[10] The alternative periodic BC for scalar fields imply presence of the constrained zero mode which in some cases lead to a non-vanishing vacuum expectation value (VEV) of a field [50, 51, 27]. We believe however that the overall physical picture of the symmetry breakdown is considerably clearer with the choice of antiperiodic BC.

ZM in the present discussion because they are not crucial for the phenomenon under study. The fields below refer then to the sector of normal Fourier modes.

The LF Hamiltonian, obtained in the canonical way from the Lagrangian (13.3), reads

$$\begin{aligned} P^- &= \int\limits_{-V} \mathrm{d}^3\underline{x}\Big\{F_{12}^2 + \Pi_{A^+}^2 + 2\Pi_{A^+}\partial_- A^- - \Pi_{A^i}\partial_i A^- - 2e\sigma \overleftrightarrow{\partial_-} \pi A^- \\ &- \quad 2e\sigma \overleftrightarrow{\partial_i} \pi A^i - e^2 A^2\big(\sigma^2+\pi^2\big) + (\partial_i\sigma)^2 + (\partial_i\pi)^2 - \\ &- \quad \mu^2\big(\sigma^2+\pi^2\big) + \frac{\lambda}{2}\big(\sigma^2+\pi^2\big)^2\Big\}. \end{aligned} \tag{13.4}$$

Here $A^2 = A^+A^- - A^iA^i, i = 1,2$ and the canonical momenta are

$$\begin{aligned} \Pi_{A^+} &= \partial_+ A^+ - \partial_- A^-,\ \Pi_{A^i} = 2\partial_- A^i + \partial_i A^+ \\ \Pi_{A^-} &= 0,\ \Pi_\sigma = 2\partial_-\sigma - e\pi A^+, \Pi_\pi = 2\partial_-\pi + e\sigma A^+. \end{aligned} \tag{13.5}$$

At $x^+ = 0$, we assume the usual LF commutation rules

$$\begin{aligned} &\big[\sigma(x^+,\underline{x}),\sigma(x^+,\underline{y})\big] = \frac{-i}{8}\epsilon(x^- - y^-)\delta^2(x_\perp - y_\perp), \\ &\big[\pi(x^+,\underline{x}),\pi(x^+,\underline{y})\big] = \frac{-i}{8}\epsilon(x^- - y^-)\delta^2(x_\perp - y_\perp), \\ &\big[A^+(x^+,\underline{x}),\Pi_{A^+}(x^+,\underline{y})\big] = \frac{i}{2}\delta^3(\underline{x}-\underline{y}) \\ &\big[A^i(x^+,\underline{x}),\Pi_{A^j}(x^+,\underline{y})\big] = \frac{i}{2}\delta^{ij}\delta^3(\underline{x}-\underline{y}). \end{aligned} \tag{13.6}$$

The mode expansions of the scalar fields are

$$\begin{aligned} \sigma(0,\underline{x}) &= \frac{1}{\sqrt{V}}\sum_{\underline{n}}\frac{1}{\sqrt{p_n^+}}\big[a(p_{\underline{n}})e^{-ip_{\underline{n}}\cdot\underline{x}} + a^\dagger(p_{\underline{n}})e^{ip_{\underline{n}}\cdot\underline{x}}\big], \\ \pi(0,\underline{x}) &= \frac{1}{\sqrt{V}}\sum_{\underline{n}}\frac{1}{\sqrt{p_n^+}}\big[c(p_{\underline{n}})e^{-ip_{\underline{n}}\cdot\underline{x}} + c^\dagger(p_{\underline{n}})e^{ip_{\underline{n}}\cdot\underline{x}}\big], \end{aligned} \tag{13.7}$$

where $p_{\underline{n}} = (p_n^+, p_{n_1}, p_{n_2}), p_n^+ = \frac{2\pi}{L}n, n = 1/2, 3/2, \ldots$, and similarly for the perpendicular components. The global rotations are implemented by the unitary operators $V(\beta)$ in terms of the charge $Q = \int\limits_{-V} \mathrm{d}^3\underline{x}J^+(x)$:

$$\begin{aligned} \sigma(x) &\to V(\beta)\sigma(x)V^\dagger(\beta) = \sigma(x)\cos\beta - \pi(x)\sin\beta, \\ \pi(x) &\to V(\beta)\pi(x)V^\dagger(\beta) = \sigma(x)\sin\beta + \pi(x)\cos\beta. \end{aligned} \tag{13.8}$$

Here

$$V(\beta) = e^{i\beta Q} \tag{13.9}$$

with

$$V(\beta) = \exp\Big\{ \sum_{\underline{n}}^{\Lambda} \Big(a^\dagger(p_{\underline{n}}) c(p_{\underline{n}}) - c^\dagger(p_{\underline{n}}) a(p_{\underline{n}}) \Big) \Big\}. \tag{13.10}$$

The Hamiltonian (13.4) is invariant under x^+-independent gauge transformations. They are implemented by the unitary operator

$$U[\omega(\underline{x})] = \exp\Big\{ i \int_{-V} \mathrm{d}^3\underline{x} \big[2\Pi_{A^+}\partial_- - \Pi_{A^i}\partial_i + eJ^+ \big] \omega(\underline{x}) \Big\} \tag{13.11}$$

Indeed, we easily find

$$\begin{aligned} U[\omega(\underline{x})]\phi(x)U^\dagger[\omega(\underline{x})] &= \exp\big(-i\omega(\underline{x})\big)\phi(x), \\ U[\omega(\underline{x})]A^\mu(x)U^\dagger[\omega(\underline{x})] &= A^\mu(x) + e^{-1}\partial^\mu\omega(\underline{x}). \end{aligned} \tag{13.12}$$

Consider now the unitary operators

$$\begin{aligned} U_\sigma(b) &= \exp\Big\{ -2ib \int_{-V} \mathrm{d}^3\underline{x}\Pi_\sigma(x) \Big\} \\ U_\pi(b) &= \exp\Big\{ -2ib \int_{-V} \mathrm{d}^3\underline{x}\Pi_\pi(x) \Big\}. \end{aligned} \tag{13.13}$$

They shift the corresponding scalar field by a constant. To follow the usual treatment, we will perform only shifts in the σ-direction:

$$\begin{aligned} U_\sigma(b)\sigma(x)U_\sigma^{-1}(b) &= \sigma(x) - 2ib \int_{-V} \mathrm{d}^3\underline{y} \big[\Pi_\sigma(y), \sigma(x) \big] \\ &= \sigma(x) - b\epsilon_\Lambda(L - x^-)\epsilon_\Lambda(L - x^1)\epsilon_\Lambda(L - x^2). \end{aligned} \tag{13.14}$$

The subscript Λ attached to the sign function $\epsilon(\underline{x})$ indicates that their Fourier series is truncated at Λ:

$$\epsilon_\Lambda(x^-) = \frac{4i}{L} \sum_{n=\frac{1}{2}}^{\Lambda} \frac{1}{p_n^+} \Big(e^{-ip_n^+ x^-} - e^{ip_n^+ x^-} \Big) \tag{13.15}$$

and analogously for the perpendicular components. The point is that one has to take a large but finite number of field modes in all three space directions in order to have a well-defined operator $U_\sigma(b)$. In practice, for $\Lambda \approx 10^3$ the sign functions are equal to unity to a very good approximation everywhere on the finite interval $-L < x^-, x^1, x^2 < L$ except for a very small neighborhood of the end-points. Therefore we will not write these sign functions explicitly henceforth.

By means of the shift operator $U_\sigma(b)$, we can define a set of states $|b\rangle = U_\sigma(b)|0\rangle$ ($|0\rangle$ is the Fock vacuum). Minimizing the expectation value of the energy density $V^{-1}\langle b|P^-|b\rangle$, we easily find that the minimum of the LF energy, equal to $-\frac{\mu^4}{2\lambda}$ is achieved for $b = \frac{\mu}{\sqrt{\lambda}} \equiv v$. It is lower than the usual (vanishing) value of the LF energy in the "trivial" vacuum $|0\rangle$. From Eq.(13.14) we also have the property that the vacuum expectation value of the σ-field is non-zero which is the indication of broken symmetry:

$$\langle v|\sigma(x)|v\rangle = \langle 0|U_\sigma^{-1}(v)\sigma(x)U_\sigma(v)|0\rangle = v. \tag{13.16}$$

Here, the sign functions multiplying the value v are understood as in Eq.(13.14). The accompanying vacuum degeneracy is easily obtained by rotating our "trial" vacuum (chosen in the σ-direction):

$$V(\beta)|v\rangle = V(\beta)U_\sigma(v)|0\rangle \equiv |v;\beta\rangle. \tag{13.17}$$

Thus we have an infinite set of vacuum states corresponding to the above minimum of the LF energy and labeled by the angle β.

The next step in the Hamiltonian formalism is to construct the space of states. A natural possibility would be to apply a string of creation operators of all fields to the new vacuum, chosen to be $|v;0\rangle$, and calculate the corresponding matrix elements of P^-. A simpler option is to build a usual set of Fock states from the Fock vacuum $|0\rangle$ and transform all of them by $U_\sigma(v)$. This type of states is known as displaced number states in quantum optics [62]. In either case one can easily see that instead of the original Hamiltonian (13.4) one actually works with the new "effective" LF Hamiltonian

$$\tilde{P}^- = U_\sigma^{-1}(v)P^-U_\sigma(v) \tag{13.18}$$

in which the σ-field is shifted by the value v. This of course leads to the structure known from the lagrangian formalism in the conventional field theory [121]: the mass term of the gauge field of the form $e^2v^2A^2$ is generated, the pion field becomes massless and the σ-field acquires mass equal to $\sqrt{2}\mu$. The change in the Hamiltonian density shows this explicitely:

$$\begin{aligned}\delta P^- &= -\frac{\mu^4}{2\lambda} + 3\mu^2\sigma^2 + \mu^2\pi^2 - e^2v^2A^2 - 2e^2v\sigma A^2 + \\ &+ 2\sqrt{\lambda}\mu\sigma\Big(\sigma^2+\pi^2\Big) - 2ev\Big(\partial_-\pi A^- + \partial_i\pi A^i\Big).\end{aligned} \tag{13.19}$$

The latter non-diagonal term and the kinetic term $(\partial_i\pi)^2$ can be removed by introducing the new field B:

$$B^+(x) = A^+ + \frac{2}{ev}\partial_-\pi,\ B^i(x) = A^i(x) - \frac{1}{ev}\partial_i\pi(x), \tag{13.20}$$

while $B^- = A^-$. In this way, the π field disappeared from the quadratic part of the Hamiltonian but it is still present in the interacting part. One may suspect that it is actually a redundant degree of freedom because the gauge freedom has not been removed.

In a full analogy with the space-like treatment, a clearer physical picture is obtained in the unitary gauge. Introducing the radial and angular field variables:

$$\phi(x) = \rho(x)e^{i\Theta(x)/v}, \tag{13.21}$$

the LF Hamiltonian will take the form

$$P_r^- = \int\limits_{-V} \mathrm{d}^3\underline{x} \Big\{ \Pi_{A^+}^2 + 2\Pi_{A^+}\partial_- A^- - \Pi_{A^i}\partial_i A^- + F_{12}^2 + (\partial_i\rho)^2 +$$

$$+ \rho^2(\partial_i\Theta/v)^2 - 2e\rho^2\partial_- A^-\Theta/v - 2e\rho^2 A^i\partial_i\Theta/v - e^2\rho^2 A^2 - \mu^2\rho^2 + \frac{\lambda}{2}\rho^4\Big\}. \quad (13.22)$$

To fix the gauge at the classical Lagrangian level, one observes that the gauge transformations simply shift the angular field variable $\Theta(x)$ by the gauge function $\omega(x)$. Choosing $\omega(x) = -\Theta(x)/v$, one has

$$\phi(x) \to \rho(x),\ A^\mu(x) \to B^\mu(x) = A^\mu(x) - \frac{1}{ev}\partial^\mu\Theta(x) \quad (13.23)$$

with the corresponding Lagrangian

$$\mathcal{L}_u = -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}|\partial_\mu\rho - ieB_\mu\rho|^2 + \frac{1}{2}\mu^2\rho^2 - \frac{\lambda}{4}\rho^4. \quad (13.24)$$

Taking this gauge fixing over to the quantum theory, defined by the commutation relation at $x^+ = 0$

$$\Big[\rho(x^+,\underline{x}), \rho(x^+,\underline{y})\Big] = -\frac{i}{8}\epsilon(x^- - y^-)\delta^2(x_\perp - y_\perp), \quad (13.25)$$

we find the quantum LF Hamiltonian P_u^- in the unitary gauge. It coincides with the Hamiltonian (13.22) except for the missing Θ-terms and the B^μ replacing the A^μ field. The equal-LF time algebra (13.25) enables us to introduce the shift operator ($\Pi_\rho = 2\partial_-\rho$)

$$U_\rho(v) = \exp\Big\{ -2iv \int\limits_{-V} \mathrm{d}^3\underline{x}\Pi_\rho(x)\Big\} \quad (13.26)$$

which defines the "effective" LF Hamiltonian $\tilde{P}_u^- = U_\rho^{-1}(v)P_u^- U_\rho(v)$ corresponding to the unitary gauge:

$$\tilde{P}_u^- = \int\limits_{-V} \mathrm{d}^3\underline{x} \Big\{ \Pi_{B^+}^2 + 2\Pi_{B^+}\partial_- B^- - \Pi_{B^i}\partial_i B^- + G_{12}^2 +$$

$$+ (\partial_i\rho)^2 - e^2(\rho+v)^2B^2 - \mu^2(\rho+v)^2 + \frac{\lambda}{2}(\rho+v)^4\Big\}. \quad (13.27)$$

From its form it is easy to find that it describes one massive scalar field ρ and a vector field with the mass e^2v^2. The massive vector field emerged as a combination of the the massless gauge field A^μ and the scalar Θ field.

Another possibility is to analyze the symmetry breaking in the light-cone gauge. This means that we set $A^+ = 0$ in the normal-mode sector. The starting Hamiltonian and conjugate momenta are then the expressions (13.4),(13.5) without the terms containing A^+. One proceeds as in the case without the gauge fixing, namely defines the shift operator $U_\sigma(v)$ and constructs the infinite set of degenerate (approximative) vacuum states by applying the unitary operator $V(\beta)$ (Eq.(13.10)) to the coherent-state vacuum $|v\rangle$. The corresponding effective LF Hamiltonian is obtained by the transformation (13.18). One observes an important difference as compared with the unitary-gauge treatment. It is related to the fact that the choice $A^+ = 0$ eliminates the A^+A^- part of the vector field mass term generated by shifting the σ field in the $-e^2A^2(\sigma^2+\pi^2)$ term in the Hamiltonian (13.4). Thus the massive vector field seems to have only two components and this is not correct. The resolution of this difficulty comes from the observation [53] that in the light-cone gauge the Gauss' law becomes a constrained equation for the A^- component of the gauge field:

$$\partial_-^2 A^-(x) + \partial_-\partial_i A^i(x) = e\sigma(x)\overleftrightarrow{\partial_i}\,\pi(x). \tag{13.28}$$

The shift of the σ field by means of the operator $U_\sigma(v)$ generates an additional term of the form $ev\partial_-\pi(x)$ on the righ-hand side of this equation. Upon inserting the shifted constraint to the Hamiltonian, the latter piece leads to the new term $e^2v^2\pi^2$ ($i = 1,2$):

$$\tilde{P}^-_{lc} = \int\limits_{-V} \mathrm{d}^3\underline{x}\Big[F_{12}^2 + (\partial_i A^i)^2 + (\partial_i\pi)^2 + e^2v^2\big(\pi^2 + A_i^2\big) + \dots\Big]. \tag{13.29}$$

To see that this Hamiltonian corresponds to a free massive vector meson field, it is useful to consider the gauge-invariant form of the scalar electrodynamics with a massive vector field [122]. It differs from the Lagrangian of the massless scalar QED by the term $\frac{1}{2}(mA^\mu - \partial^\mu B)^2$ which makes the vector field massive. B is a scalar field and m a mass parameter. The usual formulation with the condition $\partial_\mu A^\mu = 0$ corresponds to the gauge $B = 0$. In the $A^+ = 0$ gauge we obtain

$$P^- = \int\limits_{-V} \mathrm{d}^3\underline{x}\Big[F_{12}^2 + \big(\partial_i A^i\big)^2 + (\partial_i B)^2 + m^2A_i^2 + m^2B^2\Big], \tag{13.30}$$

plus the interaction terms. Comparing the two Hamiltonians, one can see that also in the light-cone gauge picture of the LF Higgs mechanism the gauge field became massive possessing three components (π, A^1, A^2) with the mass $m = ev$. The mass term of the σ field is generated as in the previous case.

In summary, we gave three versions of the Higgs phenomenon in the light front abelian Higgs model for different gauge choices. Our light front formulation was based on the finite-volume quantization with antiperiodic boundary conditions for the scalar fields. Minimization of the LF energy led to the semiquantum description of the degenerate vacuum states. In this way, the concept of the trivial LF vacuum containing no quanta was generalized to a more complex vacuum state with the non-trivial structure. The overall picture of the spontaneous breaking of the (abelian) gauge symmetry was thus found to be quite analogous to the conventional theory quantized on the space-like hypersurface, namely one scalar field and the gauge field become massive (the tree-level masses e^2v^2 and $\sqrt{2}\mu$, respectively) and there is no massless Goldstone boson in the particle spectrum.

14 Two-dimensional perturbative LF scattering matrix in DLCQ

Perturbative amplitudes of self-energy and scattering processes have been studied in the continuum formulation of light front quantization since its inception [123]. Chang and Yan [8] showed the formal equivalence of S matrix elements in light front quantization and the more familiar space-like quantization.

Most of the applications of the DLCQ method following the work of Pauli and Brodsky [14, 15] have been to bound state spectra. It is however an interesting question to develop perturbation theory in a finite volume and to study the continuum limit of the corresponding amplitudes. In particular, it is important to understand the role of the constrained zero modes which are explicitly present in the DLCQ approach for obtaining answers that agree with the usual covariant results. There exist claims about iconsistencies of the LF perturbative results [124]. In this section we will show that the Hamiltonian LF perturbation theory in a finite volume with (anti)periodic boundary conditions yields correct results provided it is applied consistently and carefully. This agreement is non-trivial taking into account the fact that the Hamiltonian scheme is not manifestly covariant and has different structure. One uses energy denominators instead of propagators and calculates individual time-ordered diagrams, not the covariant (Feynman) ones. Also, the zero-mode contributions in a finite volume have to be added and extrapolated to the infinite-volume limit. This issues will be studied in this subsection for a simple case of one loop scattering amplitude in $\frac{\lambda}{4!}\phi^4$ theory in the continuum and discretized formulation including also a few numerical results.

Let us first review the calculation of the scattering amplitude in the forward limit (scattering angle equal to zero) and in the continuum formulation. For simplicity we will again study two-dimensional theory.

Consider the scattering amplitude at one loop level in ϕ^4 theory. p_1, p_2 are the initial momenta and p_3, p_4 are the final momenta. Let us denote $s = (p_1+p_2)^2 = (p_1+p_2)^+(p_1+p_2)^- = (p_1^+ + p_2^+)^2 m^2/p_1^+ p_3^+$ and $t = (p_1 - p_3)^2 = (p_1 - p_3)^+(p_1 - p_3)^- = -(p_1^+ - p_3^+)^2 m^2/p_1^+ p_3^+$. According to the rules of (Hamiltonian) light front perturbation theory, listed in the Appendix, we have to consider two cases separately. For $p_1^+ > p_3^+$, which defines one possible x^+ ordering, the amplitude (Fig. 14.1a) is

$$
\begin{aligned}
M_{fi} &= \frac{1}{2}\frac{\lambda^2}{4\pi}\,\theta(p_1^+ - p_3^+)\int_0^{p_1^+ - p_3^+} dq_1^+\,\frac{1}{q_1^+}\,\frac{1}{p_1^+ - p_3^+ - q_1^+}\,\times \\
&\times\ \frac{1}{p_1^- + p_2^- - p_3^- - p_2^- - q_1^- - (p_1 - p_3 - q_1)^-} \\
&= \frac{1}{2}\frac{\lambda^2}{4\pi m^2}\frac{p_1^+ p_3^+}{p_1^+ + p_3^+}\frac{\theta(p_1^+ - p_3^+)}{p_1^+ - p_3^+}\int_0^{p_1^+ - p_3^+} dq_1^+\Big[\frac{1}{q_1^+ - p_1^+} - \frac{1}{q_1^+ + p_3^+}\Big], \qquad (14.1)
\end{aligned}
$$

where the step function $\theta(p_1^+ - p_3^+)$ incorporates the chosen condition on the momenta or equivalently the given time ordering of the two vertices in Fig. 14.1a. The energy denominator corresponds to the difference of the LF energies between the incoming state and the intermediate state. The latter contains four particles as can be visualized by drawing a vertical line through the loop in the diagram. The second form of the integrand has been obtained by using the dispersion relation for a free quantum $k^- = m^2/k^+$ for all energies in the denominator (recall that in the

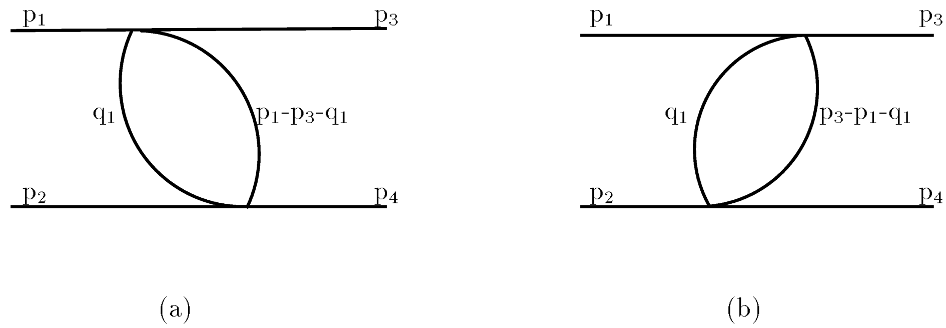

Fig. 14.1. ϕ^4 scattering diagrams in time-ordered LF perturbation theory

quanta are off energy shell but on mass shell in the Hamiltonian form of the perturbation theory) and performing some simple algebraic manipulations to simplify the expression. For $p_1^+ < p_3^+$, the scattering amplitude (Fig. 14.1b) is equal

$$\begin{aligned} M_{fi} &= \frac{1}{2}\frac{\lambda^2}{4\pi}\,\theta(p_3^+ - p_1^+)\int_0^{p_3^+ - p_1^+} dq_1^+\,\frac{1}{q_1^+}\,\frac{1}{p_3^+ - p_1^+ - q_1^+}\times \\ &\times\quad \frac{1}{p_3^- + p_2^- - p_1^- - p_2^- - q_1^- - (p_3 - p_1 - q_1)^-} \\ &= \frac{1}{2}\frac{\lambda^2}{4\pi m^2}\frac{p_1^+ p_3^+}{p_1^+ + p_3^+}\frac{\theta(p_3^+ - p_1^+)}{p_3^+ - p_1^+}\int_0^{p_3^+ - p_1^+} dq_1^+\Big[\frac{1}{q_1^+ - p_3^+} - \frac{1}{q_1^+ + p_1^+}\Big]. \end{aligned} \tag{14.2}$$

We have used overall energy conservation $p_1^- + p_2^- = p_3^- + p_4^-$ and hence $p_2^- - p_4^- = p_3^- - p_1^-$.

We are interested in the forward scattering amplitude, i.e., in $\mid p_1^+ - p_3^+ \mid \to 0$ limit. In this limit q_1^+ is very small compared to both p_1^+ and p_3^+ and it is legitimate to expand the integrands. We get,

$$\begin{aligned} \frac{1}{q_1^+ - p_1^+} - \frac{1}{q_1^+ + p_3^+} &\approx -\frac{p_1^+ + p_3^+}{p_1^+ p_3^+}, \\ \frac{1}{q_1^+ - p_3^+} - \frac{1}{q_1^+ + p_1^+} &\approx -\frac{p_1^+ + p_3^+}{p_1^+ p_3^+}. \end{aligned} \tag{14.3}$$

Thus, in the forward scattering limit, we get,

$$M_{fi} = -\frac{1}{2}\frac{\lambda^2}{4\pi m^2}, \tag{14.4}$$

because the integrand does not depend on the integration variable, the integration contributes only the factor coming from the upper limit and the momentum-dependent factors cancel. Alternatively, we can use the first form of the integrand, change the variables according to $P^+ =$

$p_3^+ - p_1^+, q_1^+ = yP^+$ to find

$$M_{fi} = \frac{1}{2}\frac{\lambda^2}{4\pi}\int_0^1 dy \frac{1}{y(1-y)t - m^2 + i\epsilon} \tag{14.5}$$

After decomposing the denominator into two fractions and performing an elementary integration, the explicit result is

$$M_{fi}(t) = -\frac{1}{2}\frac{\lambda^2}{4\pi}\frac{1}{t\sqrt{\frac{1}{4}-\frac{m^2}{t}}}\log\left(\frac{2\sqrt{\frac{1}{4}-\frac{m^2}{t}}-1}{2\sqrt{\frac{1}{4}-\frac{m^2}{t}}+1}\right). \tag{14.6}$$

In the forward scattering limit $t \to 0$, one again finds the result (14.4).

In order to calculate the one-loop scattering amplitude in DLCQ perturbation theory for the $\lambda/(4!)^{-1}\phi^4$ (1+1) model with periodic boundary conditions, we need to derive the light front Hamiltonian with $O(\lambda^2)$ zero-mode effective interactions. This will be done in the subsequent section where we will also show that contributions from zero modes vanish in the continuum limit. We shall therefore ignore these contributions here. Recall that the mode expansion for the normal mode field $\phi_n(x^-)$ is

$$\phi_n(x^-) = \frac{1}{\sqrt{2L}}\sum_{k_n^+>0}\frac{1}{\sqrt{k_n^+}}\left[a_n e^{-ikx} + a_n^\dagger e^{ikx}\right]. \tag{14.7}$$

The notation is $kx \equiv \frac{1}{2}k_n^+ x^-$ and $k_n^+ = \frac{2\pi}{L}n, n = 1, 2, \ldots \infty$.

The scattering amplitude can be calculated by the old fashioned perturbation theory formula (see the Appendix)

$$T_{fi} = \sum_j \frac{\langle p'|H_I|j\rangle\langle j|H_I|p\rangle}{p^- - p_j^-}, \tag{14.8}$$

where H_I denotes the interacting Hamiltonian. Using the formula (14.8) with $|p\rangle \to |p_1^+, p_2^+\rangle$, $|p'\rangle \to |p_3^+, p_4^+\rangle$ and with four-particle intermediate states, one finds the following expression for the second-order normal mode scattering amplitude

$$\begin{aligned} T_{fi} \quad &= \quad \frac{\delta_{p_4^+ + p_3^+, p_2^+ + p_1^+}\,\theta(p_3^+ - p_1^+)}{(2L)^2\sqrt{p_4^+ p_3^+ p_2^+ p_1^+}}\frac{\lambda^2}{4}\sum_{q_1^+}\frac{1}{q_1^+(p_3^+ - p_1^+ - q_1^+)} \times \\ &\times \quad \frac{1}{p_3^- - p_1^- - q_1^- - (p_3 - p_1 - q_1)^-} \end{aligned} \tag{14.9}$$

plus another term with $1 \leftrightarrow 3$. The above equation must be treated with care. Due to the presence of the θ-function, p_1^+ may approach p_3^+ to an arbitrary precision but not to the exact value. In DLCQ, we have,

$$t = (p_1^+ - p_3^+)(p_1^- - p_3^-) = -m^2\frac{(p_1^+ - p_3^+)^2}{p_1^+ p_3^+} = -m^2\frac{(n_1 - n_3)^2}{n_1 n_3}, \tag{14.10}$$

Tab. 14.1. $M(\tilde{t})$ versus $\tilde{t}$ in DLCQ. For the definition of $\tilde{t}$, see the text. The correct answer is -1 in our choice of units.

n_1	$-\tilde{t}$	$M(\tilde{t})$
6	$.166667 \times 10^{0}$	-.980000
8	$.833333 \times 10^{-1}$	-.989796
10	$.500000 \times 10^{-1}$	-.993827
20	$.111111 \times 10^{-1}$	-.998615
30	$.476190 \times 10^{-2}$	-.999405
100	$.408163 \times 10^{-3}$	-.999949
500	$.160643 \times 10^{-4}$	-.999998
1000	$.400802 \times 10^{-5}$	-.999999

independent of L. For convenience, we set $m^2 = 1.0$ and without loss of generality take $p_1^+ > p_3^+$. The scattering amplitude (up to the irrelevant factor $\frac{\lambda^2}{8\pi}$) is

$$M(t) = \frac{n_1 n_3}{n_1 + n_3} \frac{1}{n_1 - n_3} \sum_{n=1}^{n_1 - n_3} \Big[\frac{1}{n - n_1} - \frac{1}{n + n_3}\Big]. \tag{14.11}$$

With antiperiodic boundary condition, the mode expansion for the field is

$$\phi(x^-) = \frac{1}{\sqrt{2\pi}} \sum_{1,3,\ldots} \frac{1}{\sqrt{n}} \left[a_m e^{-\frac{i}{2}\frac{\pi}{L} m x^-} + a_m^\dagger e^{-\frac{i}{2}\frac{\pi}{L} m x^-}\right]. \tag{14.12}$$

The scattering amplitude in the t-channel in this case reads

$$M(t) = 2\frac{n_1 n_3}{n_1 + n_3} \frac{1}{n_1 - n_3} \sum_{n=1}^{n_1 - n_3 - 1} \Big[\frac{1}{n - n_1} - \frac{1}{n + n_3}\Big]. \tag{14.13}$$

Let us evaluate the scattering amplitude given in Eq. (14.11) in DLCQ. Note that the minimum allowed value for n_1, n_3 is 1. Thus we start from $n_1 = 2$. In this case $n_3 = 1$ and DLCQ gives the answer -1 for the scattering amplitude for $t = -1/2$ which is obviously wrong. It is easy to check that for each n_1, since the maximum n_3 is $n_1 - 1$, the corresponding minimum t is - $\frac{1}{n_1(n_1-1)}$ and for this particular t DLCQ always gives the answer -1 for the scattering amplitude which is wrong for finite n_1 but is correct for $n_1 \to \infty$.

The next maximum value of n_3 is $n_1 - 2$ and we denote the corresponding t by $\tilde{t} = -\frac{4}{n_1(n_1-2)}$. In Table 14.1. we present the behavior of $M(\tilde{t})$ with n_1 as $\tilde{t} \to 0$. It is clear from Fig. 14.1. that DLCQ produces the correct answer which is -1 in our units, for the limit of forward scattering. Again, the limit may be approached to an arbitrary numerical precision. For a given n_1, we increase n_3 by steps of 2 and study the behavior of $M(t)$ as a function of t for small values of t. The result is plotted in Fig. 14.2. Recall that for $n_1 = 2$, $n_3 = 1$,

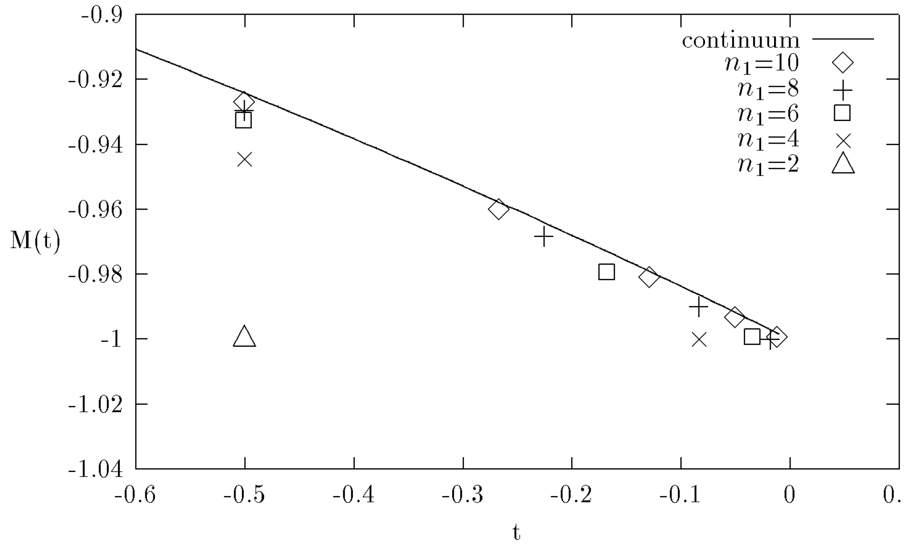

Fig. 14.2. The behaviour of the amplitude $M(t)$ as a function of t for $n_1 = 2, 4, 6, 8, 10$ for small values of t.

$t = -1/2$ and $M(t) = -1$. For $n_1 = 4$, $n_3 = 2$, $t = -1/2$ and $M(t) = -0.94$ which is close to the continuum limit (-0.92). Thus, for very small n_1, with periodic boundary condition, the convergence is from below. We can see that the results for very small n_1 are affected by discretization but reliable results emerge already for $n_1 = 10$. This is further confirmed by Fig. 14.3. where we present the results for $n_1 = 10$, 20 and 30. The continuum result given in Eq. (14.6) is also plotted for comparison. In Fig. 14.4. we present the result for $n_1 = 2000$ and the continuum result. It is evident that DLCQ reproduces the continuum answer for the entire range of t including the forward scattering limit $t = 0$. One can evaluate the scattering amplitude given in Eq. (14.13) for antiperiodic boundary condition in DLCQ. For the minimum value of $n_1 = 3$, $n_3 = 1$, $t = -4/3$, $M(t) = -3/4$ which is away from the continuum limit. For $n_1 = 9$, $n_3 = 3$, $t = -4/3$, $M(t) = -0.81$ which is closer to the continuum limit (-0.82). Thus for very small n_1, with antiperiodic boundary condition, the convergence is from above. We can see that results for very small n_1 are affected by discretization but reliable results emerge already for $n_1 = 9$. The behavior of $M(t)$ as a function of t for small values of n_1 is plotted in Fig. 14.5. In Fig. 14.6. we present the result for $n_1 = 2001$ and the continuum result. It is evident that DLCQ reproduces the continuum answer for the entire range of t including the forward scattering limit $t = 0$ also for antiperiodic boundary condition.

The question whether DLCQ can produce the correct continuum limit is nontrivial in 3+1 dimensions due to divergences and the need to renormalize the calculated perturbative amplitudes. Two dimensional scalar field theory allowed us to unambiguously answer this question. The conclusion is that the continuum limit of DLCQ produces the correct covariant limit for the one-loop scattering amplitude including processes with $p^+ = 0$ exchange in the t-channel. In the next sec-

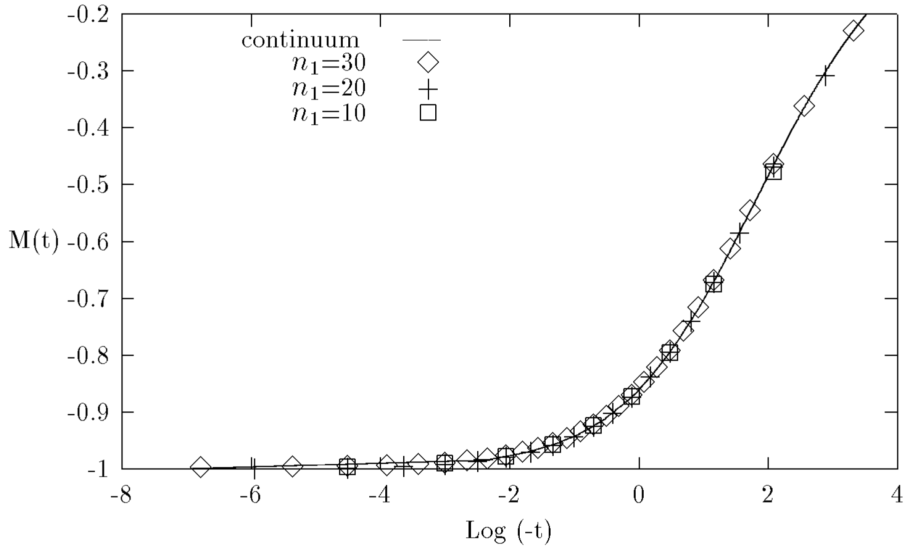

Fig. 14.3. The amplitude $M(t)$ plotted as a function of $log(-t)$ in DLCQ for $n_1 = 10, 20, 30$ and compared with the continuum result.

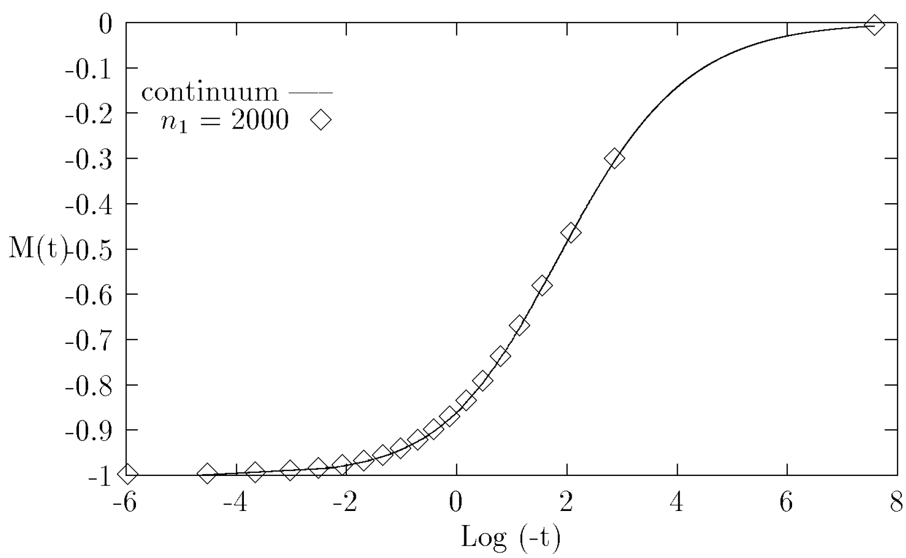

Fig. 14.4. The amplitude $M(t)$ plotted as a function of $log(-t)$ in DLCQ for $n_1 = 2000$ and compared with the continuum result.

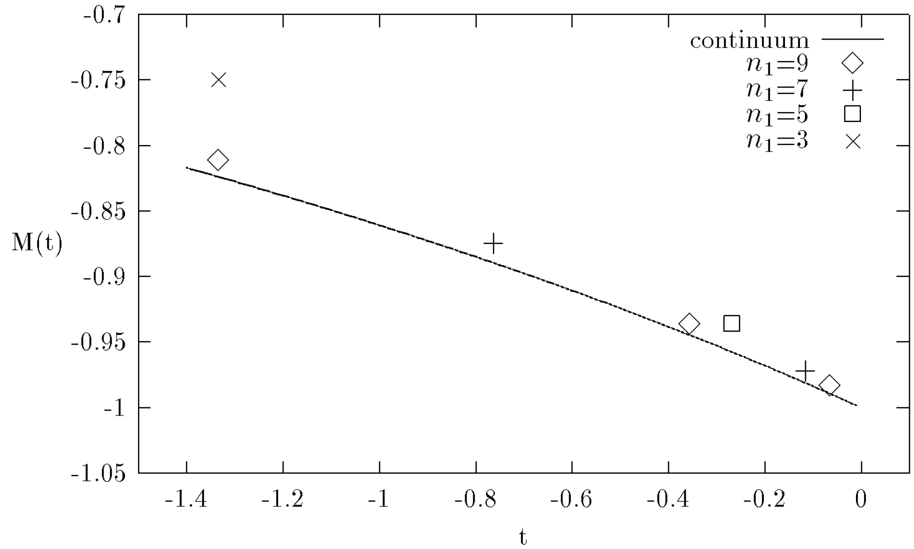

Fig. 14.5. The amplitude $M(t)$ plotted as a function of t in DLCQ for $n_1 = 3, 5, 7, 9$ for small values of t and antiperiodic boundary condition.

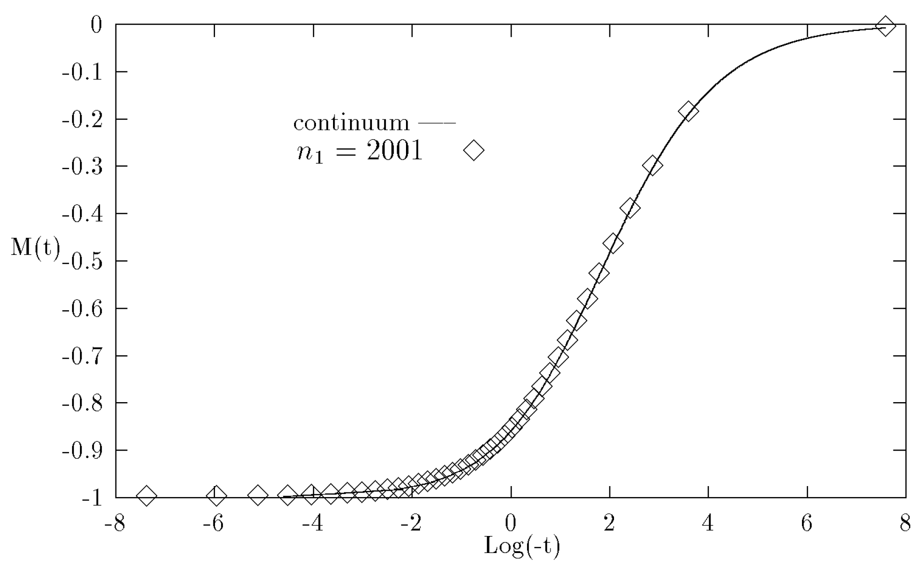

Fig. 14.6. The amplitude $M(t)$ plotted as a function of $log(-t)$ in DLCQ for $n_1 = 2001$ and compared with the continuum result for antiperiodic boundary condition.

tion, we will extend this analysis to the self-energy and scattering processes calculated in three seemingly equivalent schemes, namely the genuine LF theory, the so called infinite-momentum frame approach and the ”near-LC” formalism.

15 LF, infinite-momentum frame and near-light cone perturbative amplitudes

Problems pertaining to compactification near and on the light front have become interesting also in the context of string theory [125]. Compactification means that the space variable of a quantum field has topology of a circle, so that in fact one is using a finite domain with periodic boundary conditions. Perturbative scalar field theory has been taken as a testing ground for the zero-mode dynamics encountered also in the string theory context [125]. In the formalism of compactification near the light front certain divergences were found in the one loop scattering amplitude in scalar field theory *at finite box length* as one tried to approach the light front. These divergences were presumed to be caused by the longitudinal zero momentum modes in the light front theory.

Zero modes in the light front formalism have been studied for a long time [13, 84, 126]. We have already discussed the role of the dynamical ZM in the vacuum structure of the massive Schwinger model and the constrained ZM in the two-dimensional Yukawa model. In the latter case as well as in the DLCQ treatment of scalar field theory with periodic boundary conditions, the zero modes are dependent and they have to be determined in terms of the non-zero modes by solving a nonlinear operator equation. Thus the zero mode in scalar light front theory is quite different from the zero mode in equal time theory where it is a dynamical mode just as any non-zero mode. It is important to keep this distinction in mind.

Since zero modes pose a major challenge in the nonperturbative context, attempts have been made to perform the quantization on a space like surface [127] close to the light front (a parameter η characterizes the “closeness”, see the Appendix). By taking $\eta \to 0$ one is supposed to reach the light front surface. However, this limiting procedure need not be smooth since a light front surface cannot be reached from a space-like surface by a *finite* Lorentz transformation. On the other hand, S-matrix elements should be independent of η for *any* value of η since this parameterization simply labels different space-like surfaces. Thus any η dependence in an S-matrix element signals breakdown of Lorentz invariance as in the results of Ref. [125].

Let us recall the major differences between the discretized versions of near light front theory and light front theory. We shall restrict the ”longitudinal” coordinate x^- to a finite interval while keeping two transverse coordinates unbounded. To avoid confusion, we shall denote the light front box length by L and the near light front box length by L_{et}. In order to check Lorentz invariance one has to perform the continuum limit of DLCQ. Let us consider the mass operator $M^2 = P^+P^- - (P^\perp)^2$, where P^+, P^- are the light-front momentum and energy operators and $P^\perp \equiv (P^1, P^2)$. In DLCQ one often introduces $P^+ = \frac{2\pi}{L}K$ and $P^- = \frac{L}{2\pi}H$. The semi-positive definite operator K, the harmonic resolution, is dimensionless momentum and H, the Hamiltonian, has the dimension of M^2. In DLCQ the mass operator is given by $M^2 = KH - (P^\perp)^2$. The box length L has disappeared from the operator. Eigenvalues of K represent the total momentum of the system. The continuum limit is given by $K \to \infty$. This is to be contrasted with the near light front discretization where the box length does not disappear from the mass operator. Also the momentum operator is not semi-positive definite. Nevertheless for

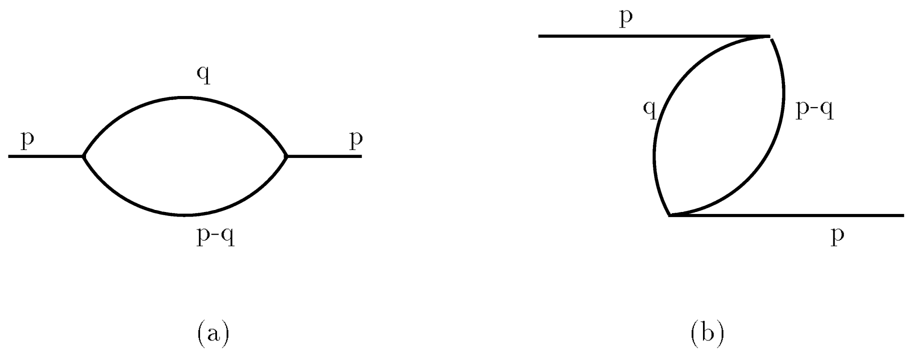

Fig. 15.1. Two time-ordered one-loop diagrams of the boson self-energy in the Hamiltonian perturbation theory for the $\lambda\phi^3$ model. Only the first diagram contributes in the LF case.

the ease of comparisons, let us denote the total dimensionless momentum in the near light front case by K. The longitudinal momentum P in this case can take both positive and negative values and we can put only $\mid P \mid = \frac{2\pi}{L_{et}} K$.

The infinite momentum frame [128, 2] is a concept that allows one to simulate perturbative light front theory calculations in an equal time framework by taking the external total longitudinal momentum to infinity. In scalar field theory, in the discretized version, one can ask whether one can simulate DLCQ perturbation theory by considering the infinite momentum frame starting from the equal time formulation. Obviously this cannot be achieved by taking K very large since that should correspond to the continuum limit of DLCQ. One choice is to take $L_{et} \to 0$ since this can simulate infinite momentum for non-zero modes. Then one can ask the question whether L_{et} drops out of scattering amplitudes and if they in turn approach DLCQ scattering amplitudes. Of course by taking $L_{et} \to 0$ we have moved as far away from the continuum limit as possible and if we find Lorentz non-invariant answers we should not be surprised. Another choice is to discretize the near light front theory, let $\eta \to 0$ and see whether L_{et} dependence drops out (characteristic of the DLCQ formalism). At finite η, $L \to \infty$ readily reproduces covariant answers, but at finite L, $\eta \to 0$ produces divergent answers. From this one cannot conclude anything about DLCQ since Lorentz invariance is broken. Note that in the discretized near light front formulation, where modes are specified by integers n, the expression $\frac{n}{\eta}$, encountered in [125], presents for the zero mode ($n = 0$) a $\frac{0}{0}$ problem for $\eta \to 0$ which means that the limit is undefined.

To clarify this issues, we perform and compare perturbative calculations for scalar field theory in the continuum and discretized versions of three formulations, namely, light front quantization, infinite momentum limit of equal time quantization and space-like quantization parameterized by η. For simplicity, we consider here mainly the self-energy diagram in ϕ^3 theory. The same overall picture emerges also in the case of scattering diagram in ϕ^4 theory.

First we compare results of the light front perturbation theory with those of the covariant perturbation theory, both in continuum formulation.

Consider the one loop self-energy diagram in ϕ^3 theory. Note that in this case there is only one time ordered diagram (the first diagram of Fig. (15.1)) in the light front case. Using the

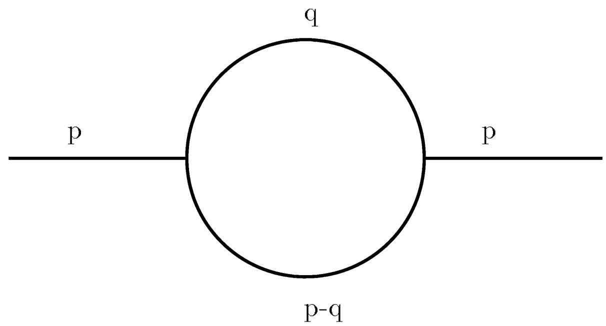

Fig. 15.2. Feynman diagram of the one-loop boson self-energy in the standard perturbation theory for the $\lambda\phi^3$ model.

rules of light front old fashioned perturbation theory [6] (see also the Appendix), we have

$$\Sigma(p^2) = \frac{1}{2}\lambda^2 \int_0^{p^+} \frac{dq^+ d^2q^\perp}{2(2\pi)^3} \frac{1}{q^+(p^+ - q^+)} \frac{1}{p^- - \frac{(q^\perp)^2+m^2}{q^+} - \frac{(p^\perp - q^\perp)^2+m^2}{p^+-q^+} + i\epsilon} \tag{15.1}$$

The factor $1/2$ is a symmetry factor. Introducing $y = q^+/p^+$, we get

$$\Sigma(p^2) = \frac{1}{2}\frac{\lambda^2}{2(2\pi)^3} \int_0^1 dy d^2q^\perp \frac{1}{y(1-y)p^2 - (q^\perp)^2 - m^2 + i\epsilon} \tag{15.2}$$

with $p^2 = p^+p^- - (p^\perp)^2$.

Note that the integrand is nonsingular at $y = 0$.

Next we derive this result starting from the Feynman diagram. The corresponding amplitude Fig.(15.2) is

$$-i\Sigma(p^2) = \frac{1}{2}\frac{(-i\lambda)^2}{(2\pi)^4} \int d^4k \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(p+k)^2 - m^2 + i\epsilon}. \tag{15.3}$$

Using $d^4k = \frac{1}{2}dk^+dk^-d^2k^\perp$, we have

$$\begin{aligned}\Sigma(p^2) \quad &= -\frac{i}{2}\frac{(-i\lambda)^2}{(2\pi)^4}\frac{1}{2}\int_{-\infty}^{+\infty} dk^+ \int d^2k^\perp \int_{-\infty}^{+\infty} dk^- \frac{1}{k^+(p^+ + k^+)} \times \\ &\times \frac{1}{k^- - \frac{(k^\perp)^2+m^2}{k^+} + i\frac{\epsilon}{k^+}} \frac{1}{p^- + k^- - \frac{(p^\perp + k^\perp)^2+m^2}{p^+ + k^+} + i\frac{\epsilon}{p^+ + k^+}}.\end{aligned} \tag{15.4}$$

Let us now perform the k^- integration. Let $p^+ > 0$. For $k^+ > 0$, $p^+ + k^+ > 0$, both poles are in the lower half of the complex k^- plane. We can close the contour in the upper half plane and the integral is zero. For $k^+ < 0$, if $p^+ < -k^+$, $p^+ + k^+ < 0$, both poles are in the upper half plane. We can close the contour in the lower half plane and the integral again is zero. For $p^+ > 0$, we get a non-vanishing contribution when $k^+ < 0$ and $k^+ > -p^+$. Then, closing the contour in the upper half plane, we get

$$\begin{aligned}\Sigma(p^2) \quad &= \frac{1}{2}\frac{(-i\lambda)^2}{2(2\pi)^3}\int_{-p^+}^{0}\frac{dk^+d^2k^\perp}{k^+(p^++k^+)}\times \\ &\times\frac{1}{p^- + \frac{(k^\perp)^2+m^2}{k^+} - \frac{(p^\perp-k^\perp)^2+m^2}{p^++k^+} - i\frac{\epsilon}{k^+} + i\frac{\epsilon}{p^++k^+}}\end{aligned} \tag{15.5}$$

or

$$\Sigma(p^2) = \frac{1}{2}\lambda^2\int_0^{p^+}\frac{dq^+d^2q^\perp}{2(2\pi)^3}\,\frac{1}{q^+(p^+-q^+)}\,\frac{1}{p^- - \frac{(q^\perp)^2+m^2}{q^+} - \frac{(p^\perp-q^\perp)^2+m^2}{p^+-q^+} + i\epsilon}. \tag{15.6}$$

We recover the expression (Eq. (15.1)) from old fashioned perturbation theory with energy denominator and integration over three momentum.

The same amplitudes can be calculated in the DLCQ framework. As we already pointed out, light front quantization in a finite volume with periodic fields has some conceptual advantages. First of all, it allows one to work explicitly with Fourier modes of quantum fields, carrying vanishing light front momentum p^+ – the zero modes. While in the case of gauge fields some ZM are dynamically independent, ZM of scalar fields are always dependent (constrained) variables, as follows from the structure of the equations of motion, containing $\partial_\mu\partial^\mu = 4\partial_+\partial_- - \partial_\perp^2, \partial_\perp^2 \equiv \partial_i\partial_i, i = 1, 2$. Analogously to the case of two dimensions, due to periodic boundary conditions in x^- and $x^\perp \equiv (x^1, x^2)$ $(-L \leq x^- \leq L, -L_\perp \leq x^\perp \leq L_\perp)$, the full scalar field can be decomposed as $\phi(x) = \phi_0(x^+, x^\perp) + \phi_n(x^+, \underline{x})$, where $\underline{x} \equiv (x^-, x^\perp)$. The mode expansion for the normal-mode field $\phi_n(\underline{x})$ is

$$\phi_n(\underline{x}) = \frac{1}{\sqrt{V}}\sum_{\underline{k}}\frac{1}{\sqrt{k^+}}\left[a_{\underline{k}}e^{-i\underline{kx}} + a^\dagger_{\underline{k}}e^{i\underline{kx}}\right]. \tag{15.7}$$

Here we have used the notation $\underline{kx} \equiv \frac{1}{2}k^+x^- - k^\perp x^\perp$ and $k^+ = \frac{2\pi}{L}n, n = 1, 2, \dots N,\ k^\perp = \frac{2\pi}{L_\perp}n^\perp, n^\perp = 0, \pm1, \pm2, \dots \pm N_\perp$. In the following, the integration over the 3-dimensional volume V will be denoted by $\int_V d^3\underline{x} \equiv \frac{1}{2}\int_{-L}^{L}dx^- \int_{-L_\perp}^{L_\perp} d^2x^\perp$.

Let us calculate now the self-energy loop in the ϕ^3 theory. The corresponding DLCQ Hamiltonian, obtained in the canonical way from the energy-momentum tensor, is

$$P^- = \int_V d^3\underline{x}\left[m^2\phi^2 + (\partial_\perp\phi)^2 + \frac{\lambda}{3}\phi^3\right]. \tag{15.8}$$

It contains ZM terms, which have to be expressed by means of the normal-mode field $\phi_n(\underline{x})$. To do so we need to obtain the lowest-order solution of the ZM constraint. As we have already seen in the two-dimensional version of the theory, the latter is simply the ZM projection of the equation of motion

$$(4\partial_+\partial_- - \partial_\perp^2)\phi = -m^2\phi - \frac{\lambda}{2}\phi^2 \tag{15.9}$$

and reads

$$(m^2 - \partial_\perp^2)\phi_0 = -\frac{\lambda}{2}\int_{-L}^{L}\frac{dx^-}{2L}(\phi_0^2 + \phi_n^2). \tag{15.10}$$

It can be solved iteratively and to the lowest order in λ one has

$$\phi_0 = -\frac{\lambda}{2}\frac{1}{m^2-\partial_\perp^2}\int_{-L}^{L}\frac{dx^-}{2L}\phi_n^2. \tag{15.11}$$

The symbolic inverse operator $(m^2-\partial_\perp^2)^{-1}$ is defined in momentum representation by replacing $\partial_\perp^2$ by the minus square of the perpendicular momentum of the composite operator in the integrand. In the Fock representation, one finds

$$\phi_0(x^\perp) = -\frac{\lambda}{V}\sum_{\underline{k}_1,\underline{k}_2}\frac{\delta_{k_1^+,k_2^+}}{\sqrt{k_1^+k_2^+}}\frac{e^{-i(k_1^\perp-k_2^\perp)x^\perp}}{m^2+(k_1^\perp-k_2^\perp)^2}a^\dagger_{\underline{k}_1}a_{\underline{k}_2} - \frac{\lambda}{2m^2}\frac{1}{V}\sum_{\underline{k}_1}\frac{1}{k_1^+}, \tag{15.12}$$

where the second term comes from the normal ordering. This term will be neglected henceforth because it generates divergent terms in the Hamiltonian, which are presumably a manifestation of the well known pathology of the $\lambda\phi^3$ theory (no lower bound of the energy). Indeed, in the case of the $\lambda\phi^4$ interaction, the constrained zero mode is expressed automatically as a normal-ordered product of creation and annihilation operators without a c-number piece.

The interacting Hamiltonian P^-_{int} contains a term, corresponding to the usual one of the continuum formulation, plus the ZM term, calculated to $O(\lambda^2)$:

$$P^-_{int} = P_{NM} + P^{-(2)}_{ZM}, \quad P_{NM} = \frac{\lambda}{3}\int_V d^3\underline{x}\phi_n^3, \tag{15.13}$$

$$P^{-(2)}_{ZM} = \int_V d^3\underline{x}\left[\phi_0(m^2-\partial_\perp^2)\phi_0 + \frac{\lambda}{3}(\phi_0\phi_n^2+\phi_n\phi_0\phi_n+\phi_n^2\phi_0)\right] \tag{15.14}$$

with ϕ_0 given by Eq.(15.11). The symmetric operator ordering has been used in the last term. The $O(\lambda^2)$ self-energy amplitude, corresponding to the first term in (15.13), can be calculated by the Hamiltonian (time-ordered) perturbation theory formula

$$T_{fi} = \sum_n \frac{\langle \underline{p}'|P_{NM}|n\rangle\langle n|P_{NM}|\underline{p}\rangle}{p^- - p_n^-}, \tag{15.15}$$

where H_I denotes the interacting Hamiltonian, $|\underline{p}\rangle \equiv a^\dagger_{\underline{p}}|0\rangle$ and the summation runs over the two-particle intermediate states $|n\rangle \equiv 2^{-\frac{1}{2}}a^\dagger_{\underline{i}_1}a^\dagger_{\underline{i}_2}|0\rangle$. After inserting the field expansion (15.7) and performing the operator commutations, we arrive at

$$T_{fi} = \frac{\delta_{\underline{p},\underline{p}'}}{\sqrt{p^+Vp'^+V}}\frac{\lambda^2}{4}\sum_{\underline{q}}\frac{1}{q^+(p^+-q^+)}\frac{1}{\frac{(p^\perp)^2+m^2}{p^+}-\frac{(q^\perp)^2+m^2}{q^+}-\frac{(p^\perp-q^\perp)^2+m^2}{p^+-q^+}}, \tag{15.16}$$

where $q^+ < p^+ = 2\pi K L^{-1}, |q^\perp| < 2\pi\Lambda_\perp L_\perp^{-1}$ and $K, \Lambda_\perp$ are integers. From this expression, the continuum answer for the self-energy $\Sigma(p^2)$ (15.1) or (15.2) can be extracted in the infinite volume limit $K, L, \Lambda_\perp, L_\perp \to \infty$ (p^+ kept fixed) with $\frac{1}{V}\Sigma_{\underline{q}} \to \frac{1}{(2\pi)^3}\int\frac{dq^+}{2}d^2q_\perp$, $\frac{V}{2}\delta_{\underline{p},\underline{k}} \to (2\pi)^3\delta(\underline{p}-\underline{q})$. We recall that Σ corresponds to the invariant amplitude M_{fi} which differs by $(2\pi)^3$ times the kinematical factor (first term in (15.16)) from T_{fi}.

As we have anticipated in the previous discussion, the ZM Hamiltonian does not contribute in the continuum limit. Indeed, the first term in (15.14) to $O(\lambda^2)$ is

$$\begin{aligned} P^{-(2)}_{ZM_1} &= \frac{1}{2}\frac{\lambda^2}{V}\Big\{\sum_{\underline{k}_1\ldots\underline{k}_4}\frac{\delta_{k_1^+,k_2^+}\delta_{k_3^+,k_4^+}}{\sqrt{k_1^+k_2^+k_3^+k_4^+}}\delta^2_{k_1^\perp+k_3^\perp,k_2^\perp+k_4^\perp}\frac{a^\dagger_{\underline{k}_1}a^\dagger_{\underline{k}_3}a_{\underline{k}_2}a_{\underline{k}_4}}{m^2+(k_1^\perp-k_2^\perp)^2} \\ &+ \sum_{\underline{k}_1,k_2^\perp}\frac{1}{k_1^{+2}}\frac{a^\dagger_{\underline{k}_1}a_{\underline{k}_1}}{m^2+(k_1^\perp-k_2^\perp)^2}\Big\}. \end{aligned} \tag{15.17}$$

The second term of (15.14) has the same structure with the individual coefficients -1 and $-\frac{2}{3}$ instead of the overall $\frac{1}{2}$ and thus the full $O(\lambda^2)$ ZM Hamiltonian is equal to

$$\begin{aligned} P^{-(2)}_{ZM} &= -\frac{1}{2}\frac{\lambda^2}{V}\Big\{\sum_{\underline{k}_1\ldots\underline{k}_4}\frac{\delta_{k_1^+,k_2^+}\delta_{k_3^+,k_4^+}}{\sqrt{k_1^+k_2^+k_3^+k_4^+}}\delta^2_{k_1^\perp+k_3^\perp,k_2^\perp+k_4^\perp}\frac{a^\dagger_{\underline{k}_1}a^\dagger_{\underline{k}_3}a_{\underline{k}_2}a_{\underline{k}_4}}{m^2+(k_1^\perp-k_2^\perp)^2} \\ &+ \frac{1}{3}\sum_{\underline{k},q^\perp}\frac{1}{k^{+2}}\frac{a^\dagger_{\underline{k}}a_{\underline{k}}}{m^2+(k^\perp-q^\perp)^2}\Big\}. \end{aligned} \tag{15.18}$$

Its contribution to the boson self-energy in the first order perturbation theory is

$$\tilde{T}_{fi} = -\frac{1}{6}\frac{\delta_{\underline{p}',\underline{p}}}{\sqrt{p^+Vp'^+V}}\frac{\lambda^2}{p^+}\sum_{q^\perp}\frac{1}{m^2+(p^\perp-q^\perp)^2}. \tag{15.19}$$

The corresponding M-amplitude vanishes in the continuum limit due to the extra L^{-1} factor (a similar result in the case of $\frac{\lambda}{4!}\phi^4(1+1)$ has been obtained in Ref. [51]:

$$\tilde{\Sigma}(p^+,p^\perp) = -\frac{1}{6}\frac{\lambda^2}{(2\pi)^2}\frac{1}{L}\frac{1}{p^+}\int d^2q^\perp\frac{1}{m^2+(p^\perp-q^\perp)^2}. \tag{15.20}$$

In this way, DLCQ calculation yields the correct covariant result for the one-loop self-energy in $\lambda\phi^3$ theory in the infinite-volume limit.

In order to calculate the one-loop scattering amplitude in DLCQ perturbation theory for $(4!)^{-1}\phi^4$ (3+1) model, we again need to derive the light front Hamiltonian with $O(\lambda^2)$ ZM effective interactions. Following the same steps as in the previous subsection with $(3!)^{-1}\lambda\phi^3$ interaction replaced by $(4!)^{-1}\lambda\phi^4$, we find

$$P^-_{int} = \frac{2\lambda}{4!}\int_V d^3\underline{x}\ \phi_n^4(\underline{x}) \quad + \quad P^{-(2)}_{ZM}, \tag{15.21}$$

where the second-order ZM Hamiltonian is

$$P^{-(2)}_{ZM} = \int_V d^3\underline{x}\left[\phi_0(m^2-\partial_\perp^2)\phi_0 + \frac{2\lambda}{4!}\ 4\phi_0\phi_n^3\right]. \tag{15.22}$$

In the last term, the symmetric operator ordering between the lowest-order solution of the ZM constraint

$$\phi_0 = -\frac{\lambda}{3!}\frac{1}{m^2-\partial_\perp^2}\int_{-L}^{L}\frac{dx^-}{2L}\phi_n^3 \tag{15.23}$$

and ϕ_n^3 is assumed. In the Fock representation, one obtains

$$\begin{aligned}\phi_0(x^\perp) &= -\frac{\lambda}{2}\frac{1}{V^{\frac{3}{2}}}\sum_{\underline{k}_1,\underline{k}_2,\underline{k}_3}\frac{1}{\sqrt{k_1^+k_2^+k_3^+}}\frac{\delta_{k_1^+,k_2^++k_3^+}}{m^2+(k_1^\perp-k_2^\perp-k_3^\perp)^2}\times \\ &\times \left[a^\dagger_{\underline{k}_3}a^\dagger_{\underline{k}_2}a_{\underline{k}_1}e^{-i(k_3^\perp+k_2^\perp-k_1^\perp)x^\perp}+h.c.\right]. \qquad (15.24)\end{aligned}$$

Using the formula (15.15) with $|\underline{p}\rangle \to |\underline{p}_1,\underline{p}_2\rangle$, $|\underline{p}'\rangle \to |\underline{p}_3,\underline{p}_4\rangle$ and with four-particle intermediate states, one finds after a lot of algebra for the second-order NM scattering amplitude the expression

$$\begin{aligned}T_{fi} &= \frac{\delta_{\underline{p}_4+\underline{p}_3,\underline{p}_2+\underline{p}_1}}{\sqrt{p_4^+Vp_3^+Vp_2^+Vp_1^+V}}\frac{\lambda^2}{4}\sum_{\underline{q}}\frac{1}{q^+(p_3^+-p_1^+-q^+)}\times \\ &\times \frac{1}{p_3^- - p_1^- - (p_3-p_1-q)^-}+(1\leftrightarrow 3) \qquad (15.25)\end{aligned}$$

The continuum-limit invariant scattering amplitude M_{fi}, extracted from (15.25), coincides with the covariant answer. It follows that for consistency the ZM contribution has to vanish in the continuum limit. That this is indeed the case can be checked in the first-order perturbation theory. In the Fock representation, part of the ZM Hamiltonian relevant for $2 \to 2$ scattering, takes the form

$$P^{-(2)}_{ZM_4} = -\frac{\lambda^2}{4}\frac{1}{V^2}\sum_{q^\perp}\sum_{\underline{k}_4,\underline{k}_3,\underline{k}_2,\underline{k}_1}\frac{\delta_{\underline{k}_4+\underline{k}_3,\underline{k}_2+\underline{k}_1}}{\sqrt{k_4^+k_3^+k_2^+k_1^+}}\frac{1}{k_1^+-k_2^+}\frac{a^\dagger_{\underline{k}_3}a^\dagger_{\underline{k}_1}a_{\underline{k}_4}a_{\underline{k}_2}}{m^2+(q^\perp+k_2^\perp-k_1^\perp)^2}.$$

The corresponding scattering amplitude is

$$\begin{aligned}\tilde{T}_{fi} &= -\frac{\delta_{\underline{p}_4+\underline{p}_3,\underline{p}_2+\underline{p}_1}}{\sqrt{p_4^+Vp_3^+Vp_2^+Vp_1^+V}}\frac{\lambda^2}{8}\frac{1}{p_3^+-p_1^+}\sum_{q^\perp}\frac{1}{m^2+(q^\perp+p_1^\perp-p_3^\perp)^2}+ \\ &+ (1\leftrightarrow 3) \qquad (15.26)\end{aligned}$$

and the invariant amplitude $\tilde{M}$ indeed vanishes for $L \to \infty$:

$$\begin{aligned}&\tilde{M}_{fi}(p_3^+-p_1^+,p_3^\perp-p_1^\perp) = \\ &-\frac{\lambda^2}{8(2\pi)^3}\frac{1}{p_3^+-p_1^+}\frac{1}{L}\int d^2q^\perp\frac{1}{m^2+(q^\perp+p_1^\perp-p_3^\perp)^2}++(1\leftrightarrow 3). \qquad (15.27)\end{aligned}$$

15.1 Near light front approach

Let us focus now on the one-loop self-energy diagram calculated within the near light front time-ordered perturbation theory. In the continuum version, for the ϕ^3 self-energy, we have, using the formula (H.26) from the Appendix

$$\begin{aligned}
\Sigma(p^2) &= \frac{1}{2}\lambda^2 \int_{-\infty}^{+\infty} \frac{dq_- d^2q^\perp}{(2\pi)^3} \Big(\frac{1}{\frac{1}{\eta^2}(E_{on}(p) - E_{on}(q) - E_{on}(p-q)) + i\epsilon} \\
&- \frac{1}{\frac{1}{\eta^2}(E_{on}(p) + E_{on}(q) + E_{on}(p-q)) - i\epsilon} \Big) \qquad (15.28) \\
&= \Sigma_I(p^2) + \Sigma_{II}(p^2). \qquad (15.29)
\end{aligned}$$

The two contributions correspond to two different time orderings (Figs. 15.1a,b) in old fashioned perturbation theory.

Let us now take the $\eta \to 0$ limit of these expressions. First consider $\Sigma_I(p^2)$. We have, $\lim_{\eta\to 0} E_{on}(q) =\mid q \mid + \frac{\eta^2(m^2+(q^\perp)^2)}{2|q|} + ...$ Without loss of generality, we shall set $p_- > 0$. Then we get

$$\begin{aligned}
\Sigma_I(p^2) &= \frac{1}{2}\frac{\lambda^2}{(2\pi)^3} \int_{-\infty}^{+\infty} dq_- d^2q^\perp \, \frac{1}{2 \mid q_- \mid} \, \frac{1}{2 \mid p_- - q_- \mid} \times \\
&\times \frac{1}{\frac{p_-}{\eta^2} - \frac{|q_-|}{\eta^2} - \frac{|p_- - q_-|}{\eta^2} + \frac{m^2+(p^\perp)^2}{2p_-} - \frac{m^2+(q^\perp)^2}{2|q_-|} - \frac{m^2+(p^\perp - q^\perp)^2}{2|p_- - q_-|}}. \qquad (15.30)
\end{aligned}$$

Now we have to distinguish various regions. For $q_- > 0$, $p_- - q_- > 0$, we get

$$\begin{aligned}
\Sigma_I(p^2) &= \frac{1}{2}\frac{\lambda^2}{(2\pi)^3} \int_0^{p_-} dq_- \int_{-\infty}^{+\infty} d^2q^\perp \, \frac{1}{q_-} \, \frac{1}{p_- - q_-} \times \\
&\times \frac{1}{\frac{m^2+(p^\perp)^2}{2p_-} - \frac{m^2+(q^\perp)^2}{2(q_-)} - \frac{m^2+(p^\perp - q^\perp)^2}{2(p_- - q_-)}} + \mathcal{O}(\eta^2)
\end{aligned} \qquad (15.31)$$

which agrees with the light front answer. For $q_- > 0$, $p_- - q_- < 0$, the amplitude scales as η^2 which vanishes as $\eta \to 0$. For $q_- < 0$, $p_- - q_- > 0$ the amplitude again scales as η^2 and thus vanishes also.

Next we consider $\Sigma_{II}(p^2)$. In the limit $\eta \to 0$, we get

$$\Sigma_{II}(p^2) = -\frac{1}{2}\lambda^2 \int_{-\infty}^{+\infty} \frac{dq_- d^2q^\perp}{(2\pi)^3} \, \frac{1}{2 \mid q_- \mid} \, \frac{1}{2 \mid p_- - q_- \mid} \times$$

$$\times \frac{1}{\frac{1}{\eta^2}\Big(p_- + \mid q_- \mid + \mid p_- - q_- \mid\Big) + \frac{m^2+(p^\perp)^2}{2p_-} + \frac{m^2+(q^\perp)^2}{2|q_-|} + \frac{m^2+(p^\perp - q^\perp)^2}{2|p_- - q_-|}}. \qquad (15.32)$$

For the three cases namely, (a) $q_- > 0$, $p_- - q_- > 0$, (b) $q_- > 0$, $p_- - q_- > 0$, and (c) $q_- < 0, p_- - q_- > 0$, we find that $\Sigma_{II}(p^2)$ scales as η^2 which vanishes in the limit.

Thus we observe that for ϕ^3 self-energy, for finite η there are two time ordered diagrams. As $\eta \to 0$ the "backward moving" diagram vanishes as η^2 and we get the light front perturbation theory answer. It is important to note that for any value of η, the sum of the two contributions should be independent of η as dictated by Lorentz invariance. However, it is sufficient for our purposes to show η independence in the limit $\eta \to 0$.

Let us now consider the first term of ϕ^3 self energy diagram (Fig. 15.1a) in the discretized version. Restricting the longitudinal coordinate to a finite interval, we obtain

$$\begin{aligned} \Sigma_I(p^2) &= \frac{1}{2}\lambda^2 \frac{1}{2L}\sum_n \int \frac{d^2q^\perp}{(2\pi)^2} \frac{1}{2\sqrt{(\frac{n\pi}{L})^2 + \eta^2((q^\perp)^2+m^2)}} \times \\ &\times \frac{1}{2\sqrt{(\frac{(j-n)\pi}{L})^2 + \eta^2((p^\perp - q^\perp)^2+m^2)}} \frac{1}{\frac{1}{\eta^2}(E_i - E_I) + i\epsilon} \end{aligned} \tag{15.33}$$

where the energy of the initial (i) and intermediate (I) state is given by

$$\begin{aligned} E_i &= \sqrt{(\frac{j\pi}{L})^2 + \eta^2((p^\perp)^2 + m^2)}, \\ E_I &= \sqrt{(\frac{n\pi}{L})^2 + \eta^2((q^\perp)^2 + m^2)} + \sqrt{(\frac{(j-n)\pi}{L})^2 + \eta^2((p^\perp - q^\perp)^2 + m^2)} \end{aligned} \tag{15.34}$$

and the discretized longitudinal momenta are

$$q_- = \frac{n\pi}{L}, \;\; p_- = \frac{j\pi}{L}, \;\; n, j = 0, \pm 1, \pm 2, \ldots \tag{15.35}$$

For $j, n \neq 0$, as $\eta \to 0$, we get the result independent of η and L.

For $n > j$, the amplitude vanishes as $\eta^2 L^2$ for fixed L. For $n = j = 0$, the amplitude diverges as $\frac{1}{\eta L}$.

For Fig. 15.1b, we have

$$\begin{aligned} \Sigma_{II}(p^2) &= -\frac{1}{2}\lambda^2 \frac{1}{2L}\sum_n \int \frac{d^2q^\perp}{(2\pi)^2} \frac{1}{2\sqrt{(\frac{n\pi}{L})^2 + \eta^2((q^\perp)^2+m^2)}} \times \\ &\times \frac{1}{2\sqrt{(\frac{(j-n)\pi}{L})^2 + \eta^2((p^\perp - q^\perp)^2+m^2)}} \frac{1}{\frac{1}{\eta^2}(E_i + E_I) - i\epsilon}. \end{aligned} \tag{15.36}$$

For $j, n \neq 0$, as $\eta \to 0$, the amplitude vanishes as $\eta^2 L^2$. For $n = j = 0$, the amplitude diverges as $\frac{1}{L\eta}$. It is not difficult to understand the origin of this divergence. We have already seen that there is no dynamical scalar zero mode on the light front and thus the sum over intermediate states cannot include this mode. On the other hand, for arbitrarily small but non-zero η (space-like quantization) there is a dynamical zero mode in the sum over intermediate states. By requiring this state to exist in the limit we are not approaching the light front theory but some peculiar (divergent) regime of the space-like theory. The light-front theory has its own mechanisms (constraints for zero modes) to replace this "missing" dynamical mode.

15.2 Infinite momentum frame approach

Let us calculate the same self-energy diagram of the ϕ^3 theory in the infinite momentum formulation. Using the rules of old fashioned perturbation theory, we obtain

$$\begin{aligned}\Sigma(p^2) &= \frac{1}{2}\lambda^2 \int_{-\infty}^{+\infty} \frac{d^3q}{(2\pi)^3} \frac{1}{2E_q} \frac{1}{2E_{p-q}} \Big(\frac{1}{E_p - E_q - E_{p-q} + i\epsilon} \\ &- \frac{1}{E_p + E_q + E_{p-q} - i\epsilon} \Big). \qquad (15.37)\\ &= \Sigma_I(p^2) + \Sigma_{II}(p^2). \qquad (15.38)\end{aligned}$$

Here $E_p = \sqrt{p^2 + (p^\perp)^2 + m^2}$. For ease of notation we have denoted the third component of the three-vector $\mathbf{p}$ as p. The two contributions correspond to two different time orderings in old fashioned perturbation theory. To facilitate the infinite momentum limit, we parametrize the internal momenta as follows: $\mathbf{q} = (xp, q^\perp)$, $\mathbf{p} - \mathbf{q} = ((1-x)p, p^\perp - q^\perp)$. It is important to note that the range of x is $-\infty < x < +\infty$. Now $\frac{d^3q}{2E_q} = \frac{pdxd^2q^\perp}{2E_q}$. Let us now take the infinite momentum, $p \to \infty$ limit of these expressions. It follows that $\frac{pdx}{2E_q} \to \frac{1}{2}\frac{dx}{|x|}$, $E_q \to \mid x \mid p + \frac{m^2+(q^\perp)^2}{2|x|p}$.

First consider $\Sigma_I(p^2)$. We get

$$\begin{aligned}\Sigma_I(p^2) &= \frac{1}{2}\frac{\lambda^2}{(2\pi)^3} \int_{-\infty}^{+\infty} dxd^2q^\perp \frac{1}{2x} \frac{1}{2 \mid 1-x \mid p} \times \\ &\times \frac{1}{p(1- \mid x \mid - \mid 1-x \mid) + \frac{m^2+(p^\perp)^2}{2p} - \frac{m^2+(q^\perp)^2}{2|x|p} - \frac{m^2+(p^\perp-q^\perp)^2}{2|1-x|p}}. \qquad (15.39)\end{aligned}$$

Now we have to distinguish various regions. For $x \geq 0$, $1-x \geq 0$, we get

$$\Sigma_I(p^2) = \frac{1}{2}\frac{\lambda^2}{(2\pi)^3} \int_0^1 dx \frac{1}{\frac{m^2+(p^\perp)^2}{2} - \frac{m^2+(q^\perp)^2}{2x} - \frac{m^2+(p^\perp-q^\perp)^2}{2(1-x)}} \qquad (15.40)$$

which agrees with the light front answer. For $x > 0, 1-x < 0$, the amplitude scales as $\frac{1}{p^2}$ which vanishes as $p \to \infty$. For $x < 0$, $1-x > 0$ the amplitude again scales as $\frac{1}{p^2}$ and vanishes in the limit.

Next we consider $\Sigma_{II}(p^2)$. In the limit $p \to \infty$, we get

$$\begin{aligned}\Sigma_{II}(p^2) &= -\frac{1}{2}\frac{\lambda^2}{(2\pi)^3} \int_{-\infty}^{+\infty} dx \frac{1}{2 \mid x \mid} \frac{1}{2 \mid 1-x \mid p} \times \qquad (15.41)\\ &\times \frac{1}{\Big(p(1+ \mid x \mid + \mid 1-x \mid) + \frac{m^2+(p^\perp)^2}{2p} + \frac{m^2+(q^\perp)^2}{2|x|p} + \frac{m^2+(p^\perp-q^\perp)^2}{2|1-x|p}\Big)}.\end{aligned}$$

For the three cases namely, (a) $x > 0, 1-x > 0$, (b) $x > 0, 1-x < 0$, and (c) $x < 0, 1-x > 0$, we find that $\Sigma_{II}(p^2)$ scale as $\frac{1}{p^2}$ which vanishes in the limit.

Thus in old fashioned perturbation theory in the infinite momentum limit ($p \to \infty$), the "backward going diagram" vanishes as $\frac{1}{p^2}$ in accordance with Weinberg's results [2].

In the finite-volume or discretized calculations, we again restrict the longitudinal coordinate to a finite interval. Specifically, we set $-L < x^3 < +L$. The longitudinal momenta $q^3 \to q^3_n = \frac{\pi}{L}n$, $n = 0, \pm 1, \pm 2, \ldots$. The field operator at $t = 0$ becomes

$$\phi(x) = \frac{1}{\sqrt{2L}} \sum_n \int d^2q^\perp \frac{1}{\sqrt{2\omega_n}} \left[a_n(q^\perp) e^{i\frac{n\pi x^3}{L} + iq^\perp \cdot x^\perp} + a^\dagger_n(q^\perp) e^{-i\frac{n\pi x^3}{L} - iq^\perp \cdot x^\perp} \right]. \quad (15.42)$$

Let us consider the ϕ^3 self energy. For the external momentum we set $p = (\frac{j\pi}{L}, p^\perp)$. We get

$$\Sigma(p^2) = \frac{1}{2}\lambda^2 \frac{1}{2L} \sum_n \int \frac{d^2q^\perp}{(2\pi)^2} \frac{1}{2\sqrt{(\frac{n\pi}{L})^2 + M(q^\perp)}} \frac{1}{2\sqrt{(\frac{(j-n)\pi}{L})^2 + M(p^\perp - q^\perp)}}$$

$$\Big(\frac{1}{\sqrt{(\frac{j\pi}{L})^2 + M(p^\perp)} - \sqrt{(\frac{n\pi}{L})^2 + M(q^\perp)} - \sqrt{(\frac{(j-n)\pi}{L})^2 + M(p^\perp - q^\perp)}}$$

$$- \frac{1}{\sqrt{(\frac{j\pi}{L})^2 + M(p^\perp)} + \sqrt{(\frac{n\pi}{L})^2 + M(q^\perp)} + \sqrt{(\frac{(j-n)\pi}{L})^2 + M(p^\perp - q^\perp)}} \Big), \quad (15.43)$$

where we used the abbreviation $M(p^\perp) \equiv (p^\perp)^2 + m^2$, etc. If we take the continuum limit, then $\frac{1}{2L}\sum_n \to \frac{dq}{2\pi}$ and we obtain the result of the previous subsection. Then, taking the infinite momentum limit, the second contribution drops out and we get the light front answer from the first contribution alone.

Suppose one takes the limit $L \to 0$, which is the opposite of the continuum limit $L \to \infty$. This is an attempt to simulate DLCQ results in a space like box. We do not expect the result to agree with the continuum limit of DLCQ which agrees with covariant perturbation theory results.

For $n \neq 0, j \neq 0, n < j$, in the limit $L \to 0$, the amplitude becomes independent of L. For $j = n = 0$, the amplitude diverges like $\frac{1}{L}$. For $n > j$ the amplitude vanishes like L^2. But none of these results have anything to do with either continuum or DLCQ results.

We should summarize the main conclusions of these a bit lengthy but instructive calculations. The first of them is a demonstration of one important feature of the covariant perturbation theory, namely that when Feynman amplitudes are rewritten in terms of the light front variables and the contour integration in the light-front energy complex plane is performed, the Feynman amplitudes reduce back to the continuum light front answers [3]. Also, as stressed already by Weinberg in 1966 [2], the light front perturbation theory (old-fashioned perturbation theory in a reference frame with "infinite-momentum" at that time) is more economical in the sense that one does not need to introduce Feynman parameters to combine propagators in the corresponding integrals, and the four-dimensional Euclidean integration is replaced by a two-dimensional one. Feynman parameters appear in the light front formulation naturally as light front longitudinal momentum fractions.

More specifically, after demonstrating that the continuum light front perturbation theory has no problem with zero modes and its results agree with the covariant results, we have analyzed the continuum limit of the light-front perturbation theory formulated in a finite volume with periodic fields (DLCQ method). This investigation was motivated by claims [125] that DLCQ is ill-defined since it is divergent when formulated as a limit of the space-like quantization on a hypersurface close to the light front [129]. In this connection, we have first shown that the DLCQ perturbation theory is consistent, because parts of the perturbative amplitudes due to the effective interactions induced by the constrained zero mode vanish in the infinite-volume limit and the covariant results are reproduced. Second, when one considers the light front limit ($\eta \to 0$) of the near light front discretized amplitudes, the zero-mode contribution indeed diverges for fixed box length. But this disagrees with the light front answer and actually cannot tell anything about the light front zero modes. The point is that the light front zero mode is not dynamical in the scalar theory and thus it is not present in the complete set of intermediate states. By letting $\eta \to 0$ one is forcing the dynamical space-like zero mode to exist on the light front which leads to an incorrect, diverging amplitude.

In other words, the $\eta \to 0$ limit does not lead us to the light front theory but to a peculiar non-covariant regime of the discretized space-like theory.

On the other hand, the continuum version of the near light front old fashioned perturbation theory reproduces the light front answers (which agree with the covariant ones). But since this formulation has no particular advantages there is no real reason to use it in practical calculations.

The conclusion relevant for the light front theory is that the "light-like compactification" is feasible and DLCQ is a consistent scheme. The discretized (compactified) formulation of the quantum theory on the light-like surface does exist as a straightforward light front field theory, but *not* as a limit of a space-like compactification.

16 Concluding remarks

Our intention in the present review was to give a pedagogical introduction to quantum field theory formulated in terms of light front space-time and field variables. We have tried to ellucidate principal differences between this approach and the usual field theory starting with the example of free massive scalar and fermion fields. We compared derivation of the vacuum state in both forms of field theory and reminded a few additional general aspects of the operator formalism not so frequently discussed in textbooks nowadays like causality and unitary non-equivalence of the fields with different masses. A large part of the review was based on the views and results of the present author (and in many cases also of his collaborators). The underlying general attitude is that although the light front field theory is a very promising theoretical scheme, one has to go beyond the conventional schemes to uncover its full potential. In particular, an opinion was emphasized that the concept of the non-perturbative LF vacuum as a state without particles is sound but not sufficient. We believe that it requires additional mechanisms to make it more complex. Details of such mechanisms depend on the dynamics and/or symmetries of the given model. It is possible that simplifications of the LF formulation could make an explicit construction of a complex non-perturbative vacuum state possible in case of simpler models like for example massive Schwinger model. Approximate description of degenerate sets of vacuum states was discussed also for the case of spontaneous breakdown of global symmetries as well as for abelian gauge symmetry. A conceptual basis for most of our analyses is formed by the quantization of LF fields in a finite volume with (anti)periodic boundary conditions in space coordinates, on the corresponding canonical operator formalism and on implementation of symmetries at quantum level in terms of (regularized) unitary operators. A necessity of paying attention to mathematical subtleties especially in the continuum formulation was pointed out and illustrated with the example of vanishing of surface terms in the commutators of the Poincaré algebra.

One of our aims was also to hightlight advantages of the light front formulation as compared to the conventional field theory. We discussed a few cases where the LF simplifications are quite striking. A simple example is the structure and solutions of the Dirac equation in two dimensions. One could compare also Hamiltonians of the LF and conventional formulation of the explicitly soluble Federbush model. The Hamiltonian LF diagonalization was shown to be very efficient in obtaining non-perturbative information about detailed properties of quantum solitons not available from lattice calculations. A detailed comparison of the conventional ”covariant” perturbation theory and the Hamiltonian LF perturbation theory showed that inspite of larger number of diagrams the latter is a very efficient tool and actually captures just the non-zero part of the usual perturbative amplitudes. A related observation is that the LF field theory cannot be obtained as a ”light-like” limit of the conventional theory quantized on a space-like hypersurface because this limit is singular. The two formulations of field theory have different structure of field degrees of freedom and hence transition from one representation to the other cannot be smooth.

We would like to conclude our discussion with two observations. First, one has to admit that despite its potential, the structure of the LF field theory is still not understood sufficiently well and its novel predictions have been so far rather rare. Our goal here was to at least partially explain why the LF formulation is considered by many workers to be conceptually as well as computationaly very well suited as a natural ”language” for relativistic elementary particle physics. It is probable that some solutions proposed by the present author are far from being perfect and will require further improvements. We believe however that they point out to the correct direction. We apologize to many colleagues in the LF community whose work did not find a proper space in this review. One reason is a certain heterogenity of the LF conceptual basis that is also reflected in the LF literature. Our intention was to make a modest attempt to formulate and interpret the LF theory, its ideas and predictions from a unified point of view.

Acknowledgement: This work was supported by the grant No. APVT 51-005704 of the Slovak Research and Development Agency, by the French NATO fellowship and by the IN2P3-CNRS. Hospitality of the Department of Physics and Astronomy of the Iowa State University and of the LPTA Laboratory at the Montpellier University where parts of this work have been done is also gratefully ackowledged.

Appendices

A Postulates of quantum field theory

Over its development, quantum field theory has been formulated on different levels of mathematical rigour. Nowadays it exists on the one hand in the form of a pragmatic computational scheme, based on the use of Feynman integrals for calculation of observable processes perturbatively, on the other hand in the form of highly abstract algebraic scheme, called local quantum physics. At present, we are far from the situation of the non-relativistic quantum mechanics, where the conceptual and mathematical foundations are in a reasonable symbiosis with the calculational strength. One important branch of a more rigorous approach to QFT was due to Wightman [130] who (among others) tried to define a minimal set of mathematical assumptions derived from observable phenomena. A modern formulation of general non-perturbative aspects of QFT can be found for example in lectures by F. Strocchi [22]. We include here a list of this postulates in a "popular" form for completeness since we referred to some of them in our disussion of the vacuum states in the main text. This list is probably not a unique one, in particular, the discovery of spontaneous symmetry breaking and gauge theories lead to relaxation of some postulates (uniqueness of the vacuum state, e.g.). The axioms can be divided into three main groups:

1. Postulates of quantum theory and covariance (structure of the Hilbert space, representations of the Lorentz group, definition of a quantum field).

2. Postulates of microcausality (or locality): Commutator of two operators that represent observable quantities vanish for space-like separations of their space-time arguments. This incorporates the fact that due to the finite speed of light such two points are causally independent.

3. Spectral postulates

   (a) in a general form: the spectrum of energy-momentum is contained in the closed forward light-cone V_+, which means that $a_\mu P^\mu$ is a positive operator for every $a^\mu \subset V_+$

   (b) in a sharper form, the following requirements are added
   i) the existence of a vacuum state, that is of a normalizable Lorentz invariant state $|0\rangle$
   ii) the existence of an eigenstate spectrum of the mass operator $\hat{M}^2 = \hat{P}_\mu \hat{P}^\mu$ and the identification of the particles described by quantum field theory with these eigenstates.

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B Quantization of massive scalar and fermion fields

B.1 Fock representation, operators of energy and momentum, ground state

In this Appendix, a few details of the calculation of the energy and momentum operators for the free massive scalar and Fermi fields are given. The formulae we need are

$$\phi(x) = \sum_p \frac{1}{\sqrt{4L\omega(p)}} \left[a(p)e^{-ip.x} + a^\dagger(p)e^{ip.x}\right], \quad [a(p), a^\dagger(q)] = \delta_{pq}. \tag{B.1}$$

For P^1 and at $t=0$, we have

$$\begin{aligned}
P^1 &= -\int\limits_{-L}^{+L} \mathrm{d}x \partial_0\phi\partial_1\phi = -\int_{-L}^{L} dx \Big\{ \sum_p (-i)\sqrt{\frac{\omega(p)}{4L}}\big[a(p)e^{ipx} - a^\dagger(p)e^{-ipx}\big] \\
&\times \sum_q \frac{iq}{\sqrt{4L\omega(q)}}\big[a(q)e^{iqx} - a^\dagger(q)e^{-iqx}\big]\Big\} = \\
&= -\frac{1}{4L}\sum_{p,q}\sqrt{\frac{\omega(p)}{\omega(q)}}q\Big[a(p)a(q)\int\limits_{-L}^{+L}\mathrm{d}x e^{i(p+q)x} - a(p)a^\dagger(q)\int\limits_{-L}^{+L}\mathrm{d}x e^{i(p-q)x} - \\
&- a^\dagger(p)a(q)\int\limits_{-L}^{+L}\mathrm{d}x e^{-i(p-q)} + a^\dagger(p)a^\dagger(q)\int\limits_{-L}^{+L}\mathrm{d}x e^{-i(p+q)}\Big] = \\
&= -\frac{1}{2}\sum_p p\big[a(-p)a(p) - a(p)a^\dagger(p) - a^\dagger(p)a(p) + a^\dagger(-p)a^\dagger(p)\big], \qquad \text{(B.2)}
\end{aligned}$$

where we have used the relation

$$\int\limits_{-L}^{+L}\mathrm{d}x e^{\pm i(p-q)x} = 2L\delta_{pq} \qquad \text{(B.3)}$$

and one summation was performed in the last step using this identity. Now, the first and the last terms in (B.2) vanish since the summation runs over positive and negative values of $p = 2\pi n/L$ and the change $p \to -p$ leads to the relation of the type $A = -A$ for these two terms. The second term can be rewritten with the help of the commutation relation (B.1) as $a(k)a^\dagger(k) = a^\dagger(k)a(k) + 1$ and we finally get

$$P^1 = \sum_p p a^\dagger(p)a(p) + \frac{1}{2}\sum_p p. \qquad \text{(B.4)}$$

The last term is a divergent constant which is physically irrelevant since it cancels in the physically measurable differences of the momenta of the given states. Formally, this subtraction is accomplished by prescribing the normal-ordered definition for P^1.

The calculation of the Fock representation of the Hamiltonian proceeds in a similar way. Evaluating the three terms in

$$H = \int\limits_{-L}^{+L}\mathrm{d}x\Big[\frac{1}{2}\big(\partial_0\phi^2\big) + \frac{1}{2}\big(\partial_1\phi\big)^2 + \frac{1}{2}\mu^2\phi^2\Big]. \qquad \text{(B.5)}$$

separately, using the formula (B.3), performing one summation by means of the Kronecker symbols and grouping the coefficients of the four operator structures, we find

$$\begin{aligned} H &= -\frac{1}{4}\sum_k \Big[\Big(\omega(p) - \frac{p+\mu^2}{\omega(p)}\Big)a(-p)a(p) - \Big(\omega(p) + \frac{p+\mu^2}{\omega(p)}\Big)a(p)a^\dagger(p) - \\ &- \Big(\omega(p) + \frac{p+\mu^2}{\omega(p)}\Big)a^\dagger(p)a(p) + \Big(\omega(k) - \frac{p+\mu^2}{\omega(p)}\Big)a^\dagger(-p)a^\dagger(p)\Big]. \end{aligned} \quad \text{(B.6)}$$

In this case the trick with $p \to -p$ does not work because there is no term linear in p and $\omega(-p) = \omega(p)$. The first and the last terms vanish simply because their coefficients are equal to zero while the coefficients of the second and the third terms are equal to $2\omega(p)$. In this way we get

$$\begin{aligned} H &= \frac{1}{2}\sum_p \omega(p)\big[a(p)a^\dagger(p) + a^\dagger(p)a(p)\big], \\ H &= \sum_p \sqrt{p^2+\mu^2}\; a^\dagger(p)a(p), \end{aligned} \quad \text{(B.7)}$$

where the infinite constant $1/2\sum_p \omega(p)$ was subtracted by the reasons mentioned above.

Let us sketch the similar calculations for the massive Fermi field. We will start from the old-fashioned ”pre-hole” formulation and show in detail how the Dirac’s reinterpretation of the negative-energy solutions of the Dirac equation leads to the particle-antiparticle picture. In two dimensions, the free Dirac equation looks formally like the usual four-dimensional one:

$$\big(i\gamma^\mu\partial_\mu - m\big)\psi(x) = 0. \quad \text{(B.8)}$$

The simplest realization of the three γ-matrices is given in terms of a set of 2×2 matrices which we choose for definiteness as

$$\beta = \gamma^0 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \;\; \gamma^1 = \beta\alpha^1 = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\gamma^5 = \gamma^0\gamma^1 = \alpha^1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad \text{(B.9)}$$

$\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices. Transition to a different representation is possible by using a suitable unitary matrix.

Let us look for the solution of the Eq.(B.8) in the form

$$\psi(x) = u(p)e^{-ip.x}. \quad \text{(B.10)}$$

The general solution will be a superposition of the plane waves (B.10) with some coefficients. It is clear that the quantity $u(p)$ is a two-dimensional object because (B.8) is a 2×2 matrix equation

$$\begin{pmatrix} i\partial_0 - m & i\partial_1 \\ -i\partial_1 & -i\partial_0 - m \end{pmatrix} u(p)e^{-ip^0x^0+ip^1x^1} = 0. \quad \text{(B.11)}$$

Writing $u(p) = \left(\begin{array}{c} a \\ b \end{array} \right)$ and performing the derivatives, we get a simple system of two linear equations

$$(p^0 - m)a - p^1 b = 0, \;\; p^1 a - (p^0 + m)b = 0, \tag{B.12}$$

which has a solution if the determinant of the matrix in p-representation, $m^2 - (p_0^2 - p_1^2)$, is equal to zero. Hence we have two types of solutions: one with positive and one with negative energy, $p^0 = \pm\sqrt{p_1^2 + m^2}$. Let us concentrate on the former one first. Eliminating for example a from the first equation and inserting it to the second one leads to an identity, i.e. the equation is satisfied for arbitrary a. Choosing the simplest option, $a = 1$, we get $b = (E(p) - m)/p$ and

$$u(p) = N \left(\begin{array}{c} 1 \\ \frac{E(p)-m}{p} \end{array} \right) = N \left(\begin{array}{c} 1 \\ \frac{p}{E(p)+m} \end{array} \right), \tag{B.13}$$

Here and in the following, $E(p) = +\sqrt{p^2 + m^2}$. N is the normalization factor. Also, we are using the notation p instead of p^1 if there will be no danger of a confusion. We will apply the covariant normalization condition, $u^\dagger(p)u(p) = \frac{E(p)}{m}$, which leads to

$$u_+(p) = \sqrt{\frac{E(p)+m}{2m}} \left(\begin{array}{c} 1 \\ \frac{p}{E(p)+m} \end{array} \right), \;\; u_-(-p) = \sqrt{\frac{E(p)+m}{2m}} \left(\begin{array}{c} \frac{p}{E(p)+m} \\ 1 \end{array} \right). \tag{B.14}$$

We have also displayed the second solution, $u_-(-p)$, which corresponds to the equation $(p_\mu \gamma^\mu + m)u(p) = 0$. The latter is obtained from the original one by changing $p^\mu \to -p^\mu$, i.e. $u_-(-p)$ is a negative-energy solution with $-p^1$. As a result, we have two independent solutions of the Dirac equation, namely

$$w_1(p) = u_+(p), \;\; w_2(p) = u_-(-p). \tag{B.15}$$

The first one corresponds to the positive energy, the second one to the negative energy. The full solution is a superposition of the plane waves with momentum-dependent coefficients and reads

$$\psi(x) = \frac{1}{\sqrt{2L}} \sum_p \sqrt{\frac{m}{E(p)}} \Big[b_1(p) w_1(p) e^{-ip.x} + b_2(p) w_2(p) e^{ip.x} \Big]. \tag{B.16}$$

The hermite conjugate field is defined as

$$\psi^\dagger(x) = \frac{1}{\sqrt{2L}} \sum_p \sqrt{\frac{m}{E(p)}} \Big[b_1^\dagger(p) w_1^\dagger(p) e^{ip.x} + b_2^\dagger(p) w_2^\dagger(p) e^{-ip.x} \Big]. \tag{B.17}$$

In the classical theory, the coefficients b are simply c-number amplitudes and we have the complex conjugate $b^*(p)$ instead of $b^\dagger(p)$. In quantum theory, these coefficients are operators that incorporate the property of creating and annihilating a particle with a mass m and momentum p. Choosing a commutation relation for these Fock operators would result in violation of the Pauli exclusion principle which is valid experimentally for real fermions and which is assumed also in the "toy" two-dimensional models. The anticommutators

$$\big\{ b_r(p), b_s^\dagger(q) \big\} = \big\{ b_r(p) b_s^\dagger(q) + b_s^\dagger(q) b_r(p) \big\} = \delta_{rs}\delta(p - q), \; r, s = 1, 2,$$

$$\left\{b_r(p), b_s(q)\right\} = \left\{b_r^\dagger(p), b_s^\dagger(q)\right\} = 0, \tag{B.18}$$

incorporate correctly the Pauli principle. For example, we get for the number operator $N_1(p)$ (see below)

$$N_+^2(p) = b_1^\dagger(p)b_1(p)b_1^\dagger(p)b_1(p) = b_1^\dagger(p)\big(1 - b_1^\dagger(p)b_1(p)\big)b_1(p) =$$

$$= b_1^\dagger(p)b_1(p) = N_+(p), \tag{B.19}$$

where the property $b_1^\dagger(p)b_1^\dagger(p) = b_1(p)b_1(p) = 0$ implied by the above anticommutation relations has been used. It follows that the occupation number for a state with given mass and momentum p can only be zero or one.

Let us calculate the Hamiltonian in terms of the $b(p), b^\dagger(p)$ operators, that is in the Fock representation. Inserting this expansion into the Hamiltonian

$$H = \int_{-L}^{+L} \mathrm{d}x \big[-\frac{i}{2}\psi^\dagger\alpha^1 \overleftrightarrow{\partial_1}\, \psi + m\psi^\dagger\beta\psi\big] \tag{B.20}$$

and evaluating all terms using the formula of the Kronecker symbol (B.3), the "spinor" identities (B.32) below as well as the relations

$$w_1^\dagger(-p)\alpha^1 w_2(p) = w_2^\dagger(-p)\alpha^1 w_1(p) = 1,$$

$$w_1^\dagger(-p)\beta w_2(p) = -w_2^\dagger(-p)\beta w_1(p) = -p/m, \tag{B.21}$$

we get

$$\begin{aligned} H \quad &= \quad \sum_p \frac{mp}{E(p)}\Big[\frac{p}{m}\Big(b_1^\dagger(p)b_1(p) - b_2^\dagger(-p)b_2(-p)\Big) \\ &- \quad b_1^\dagger(-p)b_2(-p) + b_2^\dagger(p)b_1(p)\Big] + \\ &+ \quad \sum_p \frac{m^2}{E(p)}\Big[b_1^\dagger(p)b_1(p) - b_2^\dagger(-p)b_2(-p) \\ &+ \quad \frac{p}{m}\Big(b_1^\dagger(-p)b_2(-p) - b_2^\dagger(p)b_1(p)\Big)\Big]. \end{aligned} \tag{B.22}$$

The terms off-diagonal in Fock operators cancel and using $p^2 + m^2 = E^2(p)$ we find

$$H = \sum_p E(p)\big[N_+(p) - N_-(p)\big], \;\; N_+(p) = b_1^\dagger(p)b_1(p), \;\; N_-(p) = b_2^\dagger(p)b_2(p). \tag{B.23}$$

We have written $b_2^\dagger(p)b_2(p)$ instead of $b_2^\dagger(-p)b_2(-p)$ in the above expression because $E(p)$ is quadratic in p and the change of variables $p \to -p$ leaves it unchanged. One notices immediately that the modes corresponding to negative-energy solutions contribute to the Hamiltonian with a negative sign and hence the energy operator is not positive definite. Dirac has overcome this difficulty by assuming that the physical vacuum satisfies the conditions

$$N_-(p)|0\rangle = |0\rangle, \;\; N_+(p)|0\rangle = 0, \tag{B.24}$$

for all momenta p. This means that the vacuum state $|0\rangle$ has all negative-energy states filled (due to the exclusion principle, there is only one fermion for given p) and the positive-energy states are empty. Then the vacuum energy has an infinite negative value $E_0 = -\sum_p E(p)$:

$$H|0\rangle = -\sum_p E(p)|0\rangle \tag{B.25}$$

and the same is true for the vacuum charge $Q_0 = -e\sum_p 1$, where

$$Q = -e\int_{-L}^{+L} \mathrm{d}x \psi^\dagger(x)\psi(x) = -e\sum_p \big[N_+(p) + N_-(p)\big]. \tag{B.26}$$

However, since only differences, not absolute values of these quantities are measurable, we can replace

$$\begin{aligned} H &\;\;\rightarrow\;\; H - E_0 = \sum_p E(p)\big[N_+(p) + \big(1 - N_-(p)\big)\big], \\ Q &\;\;\rightarrow\;\; Q - Q_0 = -e\sum_p \big[N_+(p) - \big(1 - N_-(p)\big)\big]. \end{aligned} \tag{B.27}$$

A negative-energy state contributes only if the value of $N_-(p)$ on this state is zero, i.e. when this state is empty, and the corresponding contribution to energy and charge is positive. Thus such an empty state is a state of an antiparticle and this can be formally incorporated to the formalism by writing

$$b_2(-p) = d^\dagger(p), \; b_2^\dagger(-p) = d(p). \tag{B.28}$$

Sometimes this reinterpretation is expressed in a form of a Bogoliubov-type of transformation

$$b(p) = \theta(p^0)b_1(p) + \theta(-p^0)b_2^\dagger(-p), \;\; d(p) = \theta(p^0)b_2(p) + \theta(-p^0)b_1^\dagger(-p), \tag{B.29}$$

where the step function $\theta(x) = 1$ for $x > 0$ and vanishes for $x \leq 0$. The second term in the expressions for modified H and Q (B.27) now reads

$$1 - N_-(p) = 1 - b_2^\dagger(-p)b_2(-p) = b_2(-p)b_2^\dagger(-p) = d^\dagger(p)d(p). \tag{B.30}$$

We then get for the field expansion and the Fock anticommutators the commonly used formulae

$$\begin{aligned} &\psi(x) = \frac{1}{\sqrt{2L}}\sum_p \sqrt{\frac{m}{E(p)}}\Big[b(p)u(p)e^{-ip.x} + d^\dagger(p)v(p)e^{ip.x}\Big], \\ &\big\{b(p), b^\dagger(q)\big\} = \big\{d(p), d^\dagger(q)\big\} = \delta_{pq}, \end{aligned} \tag{B.31}$$

where $u(p) \equiv w_1(p),\ v(p) \equiv w_2(p)$. The "spinor" identities that will be useful in the course of the calculations are

$$\overline{u}(p)u(p) = -\overline{v}(p)v(p) = 1,\ \ \overline{u}(p)v(p) = \overline{v}(p)u(p) = 0,$$

$$u^\dagger(p)u(p) = v^\dagger(p)v(p) = \frac{E(p)}{m},$$

$$u^\dagger(p)v(-p) = v^\dagger(p)u(-p) = 0,$$

$$u^\dagger(p)\alpha^1 u(p) = v^\dagger(p)\alpha^1 v(p) = \frac{p}{m}$$

$$u^\dagger(p)\alpha^1 v(-p) = v^\dagger(p)\alpha^1 u(-p) = 1,$$

$$u^\dagger(p)\gamma^0 u(p) = -v^\dagger(p)\gamma^0 v(p) = 1,$$

$$u^\dagger(p)\gamma^0 v(-p) = -v^\dagger(p)\gamma^0 u(-p) = -\frac{p}{m}. \tag{B.32}$$

In the case of the Federbush model, we also used the identities for different momenta

$$\begin{aligned}
u^\dagger(p)u(q) &= v^\dagger(p)v(q) \\
&= \sqrt{\frac{E(p)+m}{2m}}\sqrt{\frac{E(q)+m}{2m}}\Big(1 + \frac{pq}{\big(E(p)+m\big)\big(E(q)+m\big)}\Big) \\
&\equiv f_1(p,q), \\
u^\dagger(p)v(q) &= v^\dagger(p)u(q) \\
&= \sqrt{\frac{E(p)+m}{2m}}\sqrt{\frac{E(q)+m}{2m}}\Big(\frac{p}{E(p)+m} + \frac{q}{E()+m}\Big) \\
&\equiv f_2(p,q), \\
u^\dagger(p)\alpha^1 u(q) &= -v^\dagger(p)\alpha^1 v(q) \\
&= -\sqrt{\frac{E(p)+m}{2m}}\sqrt{\frac{E(q)+m}{2m}}\Big(1 - \frac{pq}{\big(E(p)+m\big)\big(E(q)+m\big)}\Big) \\
&\equiv f_3(p,q), \\
u^\dagger(p)\alpha^1 v(q) &= -v^\dagger(p)\alpha^1 u(q) \\
&= \sqrt{\frac{E(p)+m}{2m}}\sqrt{\frac{E(q)+m}{2m}}\Big(\frac{p}{E(p)+m} - \frac{q}{E(q)+m}\Big) \\
&\equiv f_4(p,q).
\end{aligned} \tag{B.33}$$

Using the relation (B.32), we have

$$
\begin{aligned}
P &= -i\int_{-L}^{+L} \mathrm{d}x \psi^\dagger \partial_1 \psi = \\
&- i\int_{-L}^{+L} \mathrm{d}x \Big\{ \sum_p \sqrt{\frac{m}{2LE(p)}} \Big[b^\dagger(p) u^\dagger(p) e^{-ipx} + d(p) v^\dagger(p) e^{ipx} \Big] \times \\
&\times \sum_q \sqrt{\frac{m}{2LE(q)}} iq \Big[b^(q) u(q) e^{iqx} - d^\dagger(q) v(q) e^{-iqx} \Big] \Big\} = \\
&= m \sum_p \frac{p}{E(p)} \Big[u^\dagger(p) u(p) b^\dagger(p) b(p) - u^\dagger(p) v(-p) b^\dagger(p) d^\dagger(-p) \\
&+ v^\dagger(p) u(-p) d(p) b(-p) - v^\dagger(p) v(p) d(p) d^\dagger(p) \Big].
\end{aligned} \tag{B.34}
$$

In the last step, one summation was performed by means of the Kronecker symbols $\delta_{p,\pm q}$. With the help of appropriate spinor identities from the list (B.32), we can see that the second and the third terms in (B.34) vanish. Using the anticommutation relation $d(p)d^\dagger(p) = 1 - d^\dagger(p)d(p)$ in the last term and subtrating the infinite irrelevant constant $-\sum_p p$, we finally obtain

$$
P^1 = \sum_p p \Big[b^\dagger(p) b(p) + d^\dagger(p) d(p) \Big]. \tag{B.35}
$$

The calculation of the Hamiltonian is only slightly more complicated because it contains two terms. We can proceed as for the momentum operator: insert the field expansions, integrate over x and perform summation over one of the momenta by means of the Kronecker symbols, or simply combine our erlier resuls (B.27) and (B.30) to obtain

$$
H = \int_{-L}^{+L} \mathrm{d}x \big[-i\psi^\dagger \alpha^1 \partial_1 \psi + m\psi^\dagger \gamma^0 \psi \big] = \sum_p E(p) \big(b^\dagger(p) b(p) + d^\dagger(p) d(p) \big). \tag{B.36}
$$

B.2 Negative-energy solutions for the LF fermion field

It is instructive to analyze the problem of negative-energy solutions of the Dirac equation in the light front theory. One immediately notices the striking simplicity of the LF description. As mentioned also in the main text, two projections of the LF Dirac equations

$$
2i\partial_+ \psi_2(x) = m\psi_1(x), \;\; 2i\partial_- \psi_1(x) = m\psi_2(x) \tag{B.37}
$$

can easily be combined into one equation for the dynamical component,

$$
\big(4\partial_+ \partial_- - m \big) \psi_2(x) = 0 \tag{B.38}
$$

which replaces the matrix equation (B.8) of the space-like theory. As in the latter, we are looking for the positive-energy one-particle solutions

$$
\psi_2(x) = e^{-ip_\mu x^\mu} = e^{-\frac{i}{2} p_n^+ x^- - \frac{i}{2} p_n^- x^+}, \;\; p_n^+ = 2\pi n/L. \tag{B.39}
$$

For the sake of this discussion, we have imposed periodic boundary conditions in x^- for the Fermi field. Notice that there is no "spinor" part multiplying the plane wave because the equation is free from any matrix structure. Inserting (B.39) into Eq.(B.38) yields

$$p_n^+ p_n^- \psi_2(x) = m^2 \psi_2(x). \tag{B.40}$$

The negative-energy solution $\exp\left(ip_\mu x^\mu\right)$ leads to the same relation which simply reproduces the dispersion law for a free quantum, $p^- = m^2/p^+$. For positive (negative) p^+, p^- is positive (negative). The general solution will be the superposition

$$\begin{aligned} \psi_2(x) &= \frac{1}{\sqrt{2L}} \sum_{n=-\infty}^{+\infty} a(p_n^+) e^{-ip_\mu x^\mu} = \\ &= \frac{1}{\sqrt{2L}} \sum_{n=1}^{+\infty} \left[b(p_n^+) e^{-\frac{i}{2}p^+x^- - \frac{i}{2}\hat{p}_n^- x^+} + b(-p_n^+) e^{\frac{i}{2}p^+x^- + \frac{i}{2}\hat{p}_n^- x^+} \right]. \end{aligned} \tag{B.41}$$

It is in principle sufficient now simply to redefine $b(-p_n^+) = d^\dagger(p_n^+)$. To paralel the discussion of the space-like fermion field, let us show in details the introduction of the Dirac sea. The solution of the constraint (the second equation in (B.37)) is given by

$$\psi_1(x) = \frac{m}{\sqrt{2L}} \sum_{n=1}^{\infty} \frac{1}{p_n^+} \left[b(p_n^+) e^{-\frac{i}{2}p^+x^- - \frac{i}{2}\hat{p}_n^- x^+} - b(-p_n^+) e^{\frac{i}{2}p^+x^- + \frac{i}{2}\hat{p}_n^- x^+} \right]. \tag{B.42}$$

This can be checked by inserting the latter into the original constraint or derived formally by means of the Green function (see the main text). If we insert these two components expressed in Fock representation into the expression of the LF Hamiltonian, also derived in the main text, we get

$$\begin{aligned} P^- &= m \int_{-L}^{+L} \frac{\mathrm{d}x^-}{2} \left[\psi_1^\dagger \psi_2 + \psi_2^\dagger \psi_1 \right] = \sum_{n=1}^{\infty} \frac{m^2}{p_n^+} \left[N_+(p_n^+) - N_-(-p_n^+) \right], \\ N_+(p_n^+) &= b^\dagger(p_n^+) b(p_n^+), \;\; N_-(-p_n^+) = b^\dagger(-p_n^+) b(-p_n^+). \end{aligned} \tag{B.43}$$

The mixed terms vanished because the x^--integration gives for them $L\delta_{p_m^+, -p_n^+} = L\delta_{m,-n} = 0$ for positive m, n.

Obviously, the above Hamiltonian is not bounded from below and this disease can be cured like in the case of the space-like field, assuming $N_+(p_n^+)|0\rangle = 0, \; N_-(-p_n^+)|0\rangle = |0\rangle$.

B.3 Orthogonality of Fock spaces for different masses – more details

Let us study two real scalar fields $\phi_1(x), \phi_2(x)$ (2.3) with masses μ_1 and μ_2. Let us assume that the Fock operator algebra (2.5) is satisfied separately for two species of the creation and annihilation operators which are independent, i.e. $[a_1(p), a_2(q)] = [a_1(p), a_2^\dagger(q)] = 0$, etc. Since the field equation (2.1) is second order in the time derivative, let us choose the boundary conditions at the initial time $t = 0$ as

$$\phi_1(0, x) = \phi_2(0, x), \partial_0 \phi_1(0, x) = \partial_0 \phi_2(0, x). \tag{B.44}$$

Inserting the field expansions (2.3) with the same momentum p into the two relations (B.44), changing $p \to -p$ in the second (creation-operator) parts and comparing the coefficients at the same plane waves, we find two algebraic equations

$$\frac{1}{\sqrt{\omega_1}}a_1(p) + \frac{1}{\sqrt{\omega_1}}a_1^\dagger(-p) = \frac{1}{\sqrt{\omega_2}}a_2(p) + \frac{1}{\sqrt{\omega_2}}a_2^\dagger(-p),$$

$$\sqrt{\omega_1}a_1(p) - \sqrt{\omega_1}a_1^\dagger(-p) = \sqrt{\omega_2}a_2(p) - \sqrt{\omega_2}a_2^\dagger(-p) \tag{B.45}$$

for four operators $a_1(p), a_1^\dagger(p), a_2(p), a_2^\dagger(p)$ ($\omega_1(p) \equiv \omega_1 = +\sqrt{p^2+\mu_1^2}, \omega_2(p) \equiv \omega_2 = +\sqrt{p^2+\mu_2^2}$). The solution expresses the annihilation and creation operators of the first scalar field as a linear combination of the Fock operators of the second field:

$$\begin{aligned} a_1(p) &= \frac{1}{\sqrt{4\omega_1\omega_2}}\Big[(\omega_1+\omega_2)a_2(p) + (\omega_1-\omega_2)a_2^\dagger(-p)\Big] \\ &\equiv c_1(p)a_2(p) + c_2(p)a_2^\dagger(-p). \end{aligned} \tag{B.46}$$

This relation permits us to express the vacuum of the first scalar field, defined as $a_1(p)|0_1\rangle = 0$, in terms of the vacuum of the second scalar field, defined as $a_2(p)|0_2\rangle = 0$ (the vacua are normalized: $\langle 0_1|0_1\rangle = \langle 0_2|0_2\rangle = 1$) in the following form:

$$|0_1\rangle = K\exp\Big(\sum_p c_3(p)a_2^\dagger(p)a_2^\dagger(-p)\Big)|0_2\rangle \equiv \hat{A}|0_2\rangle,$$

$$c_3(p) = \frac{\omega_2-\omega_1}{2(\omega_2+\omega_1)}, \tag{B.47}$$

where K is the normalization factor

$$K = \exp\Big\{\frac{1}{4}\sum_p \ln\Big(1 - \frac{(\omega_2-\omega_1)^2}{(\omega_2+\omega_1)^2}\Big)\Big\}. \tag{B.48}$$

Indeed, using the operator identity $B\exp(A) = \exp(A)B - [A,B]\exp(A)$, we get

$$\begin{aligned} a_2(p)\exp\Big(\sum_q c_3(q)a_2^\dagger(q)a_2^\dagger(-q)\Big)|0_2\rangle &= \exp\Big(\sum_q c_3(q)a_2^\dagger(q)a_2^\dagger(-q)\Big)a_2(p)|0_2\rangle \\ -\sum_q c_3(q)\big[a_2^\dagger(q)a_2^\dagger(-q), a_2(p)\big]&\exp\Big(\sum_q c_3(q)a_2^\dagger(q)a_2^\dagger(-q)\Big)|0_2\rangle \\ = \big(c_3(p)a_2^\dagger(-p) + c_3(-p)a_2^\dagger(-p)\big)&\exp\Big(\sum_q c_3(q)a_2^\dagger(q)a_2^\dagger(-q)\Big)|0_2\rangle, \end{aligned}$$

where we have used the vacuum definition $a_2(p)|0\rangle = 0$ as well as the commutator

$$\big[a_2(p), a_2^\dagger(q)a_2^\dagger(-q)\big] = a_2^\dagger(-q)\delta_{pq} + a_2^\dagger(q)\delta_{p,-q}. \tag{B.49}$$

With this result, we get by means of (B.46)

$$\begin{aligned} a_1(p)|0_1\rangle &= K\big(c_1(p)a_2(p) + c_2(p)a_2^\dagger(-p)\big)\exp\Big(\sum_q c_3(q)a_2^\dagger(q)a_2^\dagger(-q)\Big)|0_2\rangle = \\ &= K\Big(2c_1(p)c_3(p) + c_2(p)\Big)a_2^\dagger(-p)\hat{A}|0_2\rangle = 0, \end{aligned} \tag{B.50}$$

because the terms in the bracket add to zero (we have also made use of $c_3(-p) = c_3(p)$). Thus we can see that the complicated exponential state (B.47) is indeed the vacuum state for the annihilation operator $a_1(p)$. Now we are ready for a very important statement: the vacua $|0_1\rangle$ and $|0_2\rangle$ are orthogonal in the continuum limit. Indeed, we find

$$\langle 0_2|0_1\rangle = \langle 0_2|\hat{A}|0_2\rangle = K, \tag{B.51}$$

because the creation operators in the exponent of $\hat{A}$ annihilate $|0\rangle$ when acting to the right so that only the factor 1 yields a non-zero contribution leading to $\langle 0_2|0_2\rangle = 1$. Now, using the relation $\sum_p \to \frac{2L}{2\pi}\int \mathrm{d}p$ ($2L$ is the "volume" of our one-dimensional space) for a transition from the finite to infinite volume, we get

$$K = \exp\Big\{\frac{L}{4\pi}\int \mathrm{d}p \ln\Big(1 - \frac{(\omega_2-\omega_1)^2}{(\omega_2+\omega_1)^2}\Big)\Big\}. \tag{B.52}$$

The integral will be a negative number because its integrand is negative for $-\infty \le p \le \infty$ since it is equal to $-\ln\Big(\frac{1}{2} + \frac{1}{4}\big(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1}\big)\Big)$. Thus K will approach zero as $\exp(-cL)$, where c is a positive number. In other words, in the infinite-volume limit $L \to \infty$, the overlap between the two vacua as well as between arbitrary Fock states vanishes, i.e. the two Fock spaces become orthogonal. This means that there is no unitarity operator connecting these two spaces and one says that they are unitarily inequivalent.

B.4 Explicit form of the Pauli-Jordan function in two dimensions

It is instructive to compare the LF calculation of the Pauli-Jordan commutator function with the conventional approach. In the latter case, the definition of the PJ function of the scalar field yields after inserting the field expansion gives

$$\begin{aligned} i\Delta(x) &= \big[\phi(x),\phi(0)\big] = \frac{1}{(4\pi)^2}\int \frac{\mathrm{d}p^1\mathrm{d}q^1}{\omega(p^1)\omega(q^1)}\Big\{4\pi\omega(q^1)\delta(p^1-q^1)\times \\ &\times\ e^{-i\omega(p^1)x^0+ip^1x^1} - 4\pi\omega(q^1)\delta(p^1-q^1)e^{i\omega(p^1)x^0-ip^1x^1}\Big\} = \\ &= \frac{1}{4\pi}\int\frac{\mathrm{d}p^1}{\omega(p^1)}\Big\{e^{-i\omega(p^1)x^0+ip^1x^1} - e^{i\omega(p^1)x^0-ip^1x^1}\Big\}. \end{aligned} \tag{B.53}$$

The latter expression can be rewritten in a manifestly covariant form as

$$\Delta(x) = -\frac{1}{2\pi}\int \mathrm{d}^2p\ \epsilon(p^0)\delta(p^2-\mu^2)e^{-ip.x}. \tag{B.54}$$

Similarly, one has for the two-point correlation function

$$
\begin{aligned}
D(x-y) &= \langle 0|\phi(x)\phi(y)|0\rangle = \frac{1}{4\pi}\int \frac{\mathrm{d}p^1}{\omega(p^1)}\left\{e^{-i\omega(p^1)(x^0-y^0)+ip^1(x^1-y^1)}\right\} = \\
&= \int \frac{\mathrm{d}^2p}{2\pi}\delta(p^2-\mu^2)e^{-ip.x}. \qquad \text{(B.55)}
\end{aligned}
$$

Calling $x^1 = x$, $p^1 = p$, $\omega(p) = +\sqrt{p_1^2+\mu^2} \equiv p^0$ for simplicity and making the substitution $p^0 = m\sinh\varphi$, $p = \mu\cosh\varphi$, we have to calculate the integral

$$
f(x) = \frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}p}{p^0}e^{-i(p^0x^0-px)} = \frac{1}{4\pi}\int_{-\infty}^{+\infty}\mathrm{d}\varphi e^{-i\mu(x^0\cosh\varphi - x\sinh\varphi)}. \qquad \text{(B.56)}
$$

Now it is necessary to distinguish four cases:

$$
\begin{aligned}
&1.\ x^0>0,\ x^0>x, \qquad 2.\ x^0>0,\ x^0<x, \\
&3.\ x^0<0,\ |x^0|>x, \quad 4.\ x^0<0,\ |x^0|<x, \qquad \text{(B.57)}
\end{aligned}
$$

because the integrand has different structure for these four cases and hence the results will differ.

For the first case, set $x^0 = \sqrt{\lambda}\cosh\varphi_0,\ x = \sqrt{\lambda}\sinh\varphi_0$, where $\lambda = x^2 = x_0^2 - x_1^2$. This choice respects positivity of both x^0 and x as well as the relation of their magnitudes. Using the formula $\cosh(\varphi-\varphi_0) = \cosh\varphi\cosh\varphi_0 - \sinh\varphi\sinh\varphi_0$, we find

$$
f(x) = \frac{1}{4\pi}\int_{-\infty}^{+\infty}\mathrm{d}\varphi e^{-i\mu\sqrt{\lambda}\cosh(\varphi-\varphi_0)}. \qquad \text{(B.58)}
$$

The table of integrals [25] tells us that the latter integral is equal to

$$
f(x) = -\frac{i}{4}J_0(\mu\sqrt{\lambda}) - \frac{1}{4}N_0(\mu\sqrt{\lambda}). \qquad \text{(B.59)}
$$

J_0 and N_0 are the standard Bessel functions [25]. One can proceed similarly for the other three cases with the overall result for the two-point correlation function

$$
D(x) = -\frac{1}{4}\theta(\lambda)\Big[N_0(\mu\sqrt{\lambda}) + i\mathrm{sgn}x^0 J_0(\mu\sqrt{\lambda})\Big] + \frac{1}{2\pi}\theta(-\lambda)K_0(\mu\sqrt{\lambda}). \qquad \text{(B.60)}
$$

Note that in the LF calculation we did not have to perform the sophisticated change of variables and four cases of the integrand were available immediately.

C Details of calculations of discrete LF correlation functions

Let us describe the main elements of the calculation of the Pauli-Jordan function $\hat{D}(x)$ in the finite volume. As shown in the main text, there are a few contributions to the integral representation of the discrete PJ function corresponding to the segments of the rectangular integration contour. On the segment C_1 we have $u = iv$, where v is positive real. One immediately finds that this contribution to $\hat{D}(x)$ may be written as

$$\frac{1}{8\pi Q} P \int_0^{\pi Q+\xi} dv J_0(2\sqrt{v}) \cot\big((v-\xi)/(2Q)\big) - \frac{i}{8\pi Q} \int_0^{\pi Q+\xi} dv J_0(2\sqrt{v}), \tag{C.1}$$

where P denotes principal value. Each of the two integrals is real and finite. The second term can be evaluated in closed form and the result is

$$-\frac{i}{8\pi} \frac{\sqrt{\pi Q+\xi}}{Q} J_1(2\sqrt{\pi Q+\xi}), \tag{C.2}$$

and in particular it vanishes in the large Q limit.

In order to obtain the $L \to \infty$ limiting behavior of the first term of (C.1), one can replace the cotangent function by the inverse of its argument, so that after the change of variable $v = w^2/4$ one obtains

$$P \frac{1}{4\pi} \int_0^{\infty} dv \frac{J_0(2\sqrt{v})}{v-\xi} = P \frac{1}{2\pi} \int_0^{\infty} dw \frac{w J_0(w)}{w^2 - 4\xi} = -\frac{1}{4} N_0(2\sqrt{\xi}). \tag{C.3}$$

On the infinitesimal semicircle C_2 defined by $u - i\xi = \epsilon e^{i\theta}, -\pi/2 \le \theta \le \pi/2$, with $\epsilon \to 0^+$, we may use the approximation

$$\frac{1}{\exp\Big(\frac{1}{Q}(u-i\xi)\Big) - 1} \approx \frac{Q}{\epsilon e^{i\theta}}. \tag{C.4}$$

Replacing further the function I_0 by its value at u_0 and using the relation [25] $I_0(2i\sqrt{\xi}) = J_0(2\sqrt{\xi})$, we find that this contribution to $\hat{D}$ is equal to $\frac{i}{16\pi} J_0(2\sqrt{\xi})$. Note that this quantity is independent of L and thus survives in the large-Q limit.

On the horizontal semi-infinite line C_3 we may write $u = i\xi + Q(i\pi + v)$, where v is real, positive. The contribution to $\hat{D}(x)$ is given by

$$-\frac{1}{4\pi} \int_0^{\alpha Q} dv \frac{1}{e^v + 1} J_0\big(2\sqrt{\xi + \pi Q - iQv}\big). \tag{C.5}$$

The asymptotic analysis of Eq.(C.5) in the limit $Q \to +\infty$ is very lengthy and involves substitution of an integral representation of J_0 combined with the method of steepest descent. The final result is that the leading behavior of the expression (C.5) is given by a term proportional to $\exp(iQ)$. This vanishes in the limit $L \to \infty$ as long as for any finite L the quantity x^+ includes a small positive imaginary part such that $L \times \mathrm{Im}(x^+) \to \infty$. This requirement is satisfied for example by the choice $\mathrm{Im}(x^+) = O(L^{-1/2})$.

Finally, points on the line segment C_4 are described by $u = R + iv$, where v is real, positive. The contribution to $\hat{D}(x)$ from C_4 is then found to be dominated by $\exp\Big[-\big(\alpha - \sqrt{2\alpha}\big)Q\Big]$ and thus vanishes in the large-Q limit since we choose $\alpha > 2$.

D The Dirac-Bergmann quantization for constrained systems

In this Appendix, we first very briefly describe the Dirac-Bergmann algorithm for quantization of theories with constraints and then apply it to the light front $\lambda\phi^4(1+1)$ model. A detailed derivation of the method can be found in the Dirac's lectures [61] and its more recent applications in the monographs [131, 132]. The method (or its alternative [133]) is suitable and sometimes even inevitable for Lagrangians which are linear in velocities, e.a. in time derivatives of the fields. Here we just summarize the main steps of the procedure in a form of a recipe. It consists of the following steps.

1. Identify *primary constraints* φ_i, e. a. calculate conjugate momenta Π_i and construct

$$\varphi_i = \Pi_i - \frac{\delta\mathcal{L}}{\delta\partial_0\phi_i} \approx 0 \tag{D.1}$$

for those Π_i which are not time derivatives of fields. The $\approx$ sign indicates the "weak equality" which means that it should be implemented only after all Poisson brackets have been calculated. The Poisson bracket is defined in field theory as

$$\{A, B\} = \int dz \left[\frac{\delta A}{\delta\phi_i(z)}\frac{\delta B}{\delta\Pi_i(z)} - \frac{\delta B}{\delta\phi_i(z)}\frac{\delta A}{\delta\Pi_i(z)}\right] \quad . \tag{D.2}$$

2. Prescribe the *fundamental* Poisson brackets between pairs of fields and their conjugate momenta (even in the case if a field is non-dynamical, i.e. its conjugate momentum vanishes).

3. Construct the *primary Hamiltonian* from the canonical Hamiltonian H_c, the primary constraints and the Lagrange multiplier functions u_j:

$$H_p = H_c + \sum_i \int d^3x u_i(x)\varphi_i(x) \quad . \tag{D.3}$$

The multipliers u_i are determined from the self-consistency condition that the primary constraints do not depend on time. This means that they have vanishing Poisson brackets with the primary Hamiltonian:

$$\partial_0\varphi_i(x) = \{\varphi_i(x), H_p\} + \sum_j \int d^3y u_j(y)\{\varphi_i(x), \varphi_j(y)\} \approx 0 \quad . \tag{D.4}$$

One has to require the self-consistency for the secondary, terciary, etc. constraints (which may arise in the course of solving the above equation) until no new constraints appear.

4. The residual gauge freedom is indicated by a presence of the *first-class constraints* φ_k satisfying

$$\{\varphi_k(x), \varphi_i(y)\} = 0 \tag{D.5}$$

for all i. In this case a *subsidiary condition(s)* $\chi_k(x)$ has to be imposed to uniquely find $u_i(x)$. After this step, all constraints $\theta_\alpha(x)$ which include primary, secondary, ... constraints and the gauge-fixing (subsidiary) conditions are *second class*. A constraint θ_β is called second class if it has non-vanishing Poisson bracket with at least one of the other constraints θ_α:

$$\{\theta_\alpha(x), \theta_\beta(y)\} \neq 0 \tag{D.6}$$

at least for one β.

5. The second-class constraints change the canonical quantization procedure: the matrix $C_{\alpha\beta} = \{\theta_\alpha, \theta_\beta\}$ of the Poisson brackets between all second-class constraints is used to construct the *Dirac bracket* $\{A, B\}^*$

$$\{A, B\}^* = \{A, B\} - \{A, \theta_\alpha\} C^{-1}_{\alpha\beta} \{\theta_\beta, B\} \ . \tag{D.7}$$

6. To obtain a quantum theory, Dirac brackets are replaced by commutators according to the rule

$$\{A, B\}^* \to -i[A, B] \ . \tag{D.8}$$

Now the second-class constraints can be taken strongly and implemented in the Hamiltonian H_p.

Let us illustrate the method on a concrete light front example of the two-dimensional self-interacting real scalar field with the Lagrangian density

$$\mathcal{L}_{lf} = 2\partial_+\phi\partial_-\phi - \frac{1}{2}\mu^2\phi^2 - \frac{\phi^4}{4!}, \tag{D.9}$$

where the quadratic term is the true mass term, i.e. the Lagrangian corresponds to the theory in the symmetric phase. We consider the model on the finite interval $-L \le x^- \le L$ and impose periodic boundary conditions leading to the decomposition $\phi(x) = \phi_0 + \varphi(x)$. ϕ_0 is the x^--independent part, the zero mode, and φ is the sum of all Fourier modes with $p_n^+ \neq 0$. The Dirac-Bergmann procedure then has to include also the zero-mode sector. As we have seen in the main text, the canonical Hamiltonian of the model is

$$P^- = \int\limits_{-L}^{+L} \frac{\mathrm{d}x^-}{2} \Big[\mu^2\phi^2 + 2\frac{\lambda}{4!}\phi^4\Big] = \int\limits_{-L}^{+L} \frac{\mathrm{d}x^-}{2} \Big[\mu^2\phi_0^2 + \mu^2\varphi^2 + 2\frac{\lambda}{4!}\big(\phi_0 + \varphi\big)^4\Big]. \tag{D.10}$$

The canonical momentum $\Pi_\phi = 2\partial_-\phi$ leads to two primary constraints

$$\theta_1 = \Pi_\varphi - 2\partial_-\varphi \approx 0, \ \ \theta_2 = \Pi_0 \approx 0 \tag{D.11}$$

The primary Hamiltonian is

$$P_p^- = P_c^- + \int\limits_{-L}^{+L} \frac{\mathrm{d}y^-}{2} u_1(y^-)\theta_1(y^-) + Lu_2\theta_2. \tag{D.12}$$

The last term is y^- - independent and the integration yielded just the factor L. The fundamental Poisson brackets are

$$\big\{\varphi(x^-), \Pi_\varphi(y^-)\big\} = \delta_N(x^- - y^-), \ \ \big\{\phi_0, \Pi_{\phi_0}\big\} = \frac{1}{L}, \tag{D.13}$$

where $1/L$ is the zero-mode part of the full periodic delta function $\delta_P(z^-)$. The Poisson bracket of the primary constraints are easily calculated as

$$\big\{\theta_1(x^-), \theta_1(y^-)\big\} = -4\partial_-^x\delta_N(x^- - y^-), \ \ \big\{\theta_2, \theta_2\big\} = 0, \ \ \big\{\theta_1(x^-), \theta_2\big\} = 0. \tag{D.14}$$

Next we have to require that the primary constraints are conserved (the consistency condition):

$$\begin{aligned}\{\varphi_1(x^-), P_p^-\} &= \int\limits_{-L}^{+L} \frac{\mathrm{d}y^-}{2}\Big[-2\mu^2\varphi(y^-) - \frac{2\lambda}{3!}\big(3\phi_0^2\varphi(y^-) + 3\phi_0\varphi^2(y^-) + \\ &+ \varphi^3(y^-)\big) - 4u_1(y^-)\partial_-^x\Big]\delta_N(x^- - y^-) \approx 0. \qquad \text{(D.15)}\end{aligned}$$

Using partial integration and integrating with the help of the delta function, we find that this equation determines the Lagrange multiplier u_1:

$$\partial_- u_1(x^-) \approx -\frac{1}{2}\mu^2\varphi(x^-) - \frac{\lambda}{2.3!}\big[3\phi_0^2\varphi(x^-) + 3\phi_0\varphi^2(x^-) + \varphi^3(x^-)\big]. \qquad \text{(D.16)}$$

Note that without prescribing boundary condition, the inversion of this equation would not be unique since an arbitrary function of x^+ could be added to u_1. With periodic boundary condition, the inversion of Eq.(D.16) is unique. The solution is expressed in terms of the Green's function $G_1(x^- - y^-) = 1/2\epsilon_N(x^- - y^-)$ which is the inverse of the operator ∂_-: $\partial_-^x G_1(x^- - y^-) = \delta_N(x^- - y^-)$.

The consistency condition for the constraint θ_2 yields

$$\begin{aligned}\{\theta_2, P_p^-\} &= \Big\{\Pi_0, \int\limits_{-L}^{+L} \frac{\mathrm{d}x^-}{2}\Big[\mu^2\phi_0^2 + \\ &+ \frac{2\lambda}{4!}\Big(\phi_0^4 + 4\phi_0^3\varphi(x^-) + 6\phi_0^2\varphi^2(x^-) + 4\phi_0\varphi^3(x^-) + \varphi^4(x^-)\Big)\Big]\Big\} = \\ &= -2\mu^2\phi_0 - \frac{\lambda}{3}\int\limits_{-L}^{+L} \frac{\mathrm{d}x^-}{2L}\Big[\phi_0^3 + 3\phi_0\varphi^2(x^-) + \varphi^3(x^-)\Big] \approx 0. \qquad \text{(D.17)}\end{aligned}$$

We have to impose a consistency condition on the secondary constraint θ_3

$$\theta_3 = \mu^2\phi_0 + \frac{\lambda}{3!}\int\limits_{-L}^{+L} \frac{\mathrm{d}x^-}{2L}\Big(\phi_0 + \varphi\Big)^3 \qquad \text{(D.18)}$$

because no u multiplier appears in it:

$$\{\theta_3, P_p^-\} = \Big\{\mu^2\phi_0 + \frac{\lambda}{3!}\int\limits_{-L}^{+L} \frac{\mathrm{d}x^-}{2L}\Big(\phi_0^3 + 3\phi_0\varphi^2(x^-) + \varphi^3(x^-)\Big),$$

$$\int\limits_{-L}^{+L} \frac{\mathrm{d}y^-}{2} u_1(y^-)\big(\Pi_\varphi(y^-) - 2\partial_-\varphi(y^-)\big) + Lu_2\Pi_0\Big\}. \qquad \text{(D.19)}$$

Evaluation of individual terms yields a weak equation for the multiplier u_2:

$$\big(\mu^2 + \frac{\lambda}{2}\phi_0^2\big)u_2 + \frac{\lambda}{2}\int_{-L}^{+L}\frac{\mathrm{d}x^-}{2L}u_1(x^-)\big[2\phi_0\varphi(x^-) + \varphi^2(x^-)\big] \approx 0. \tag{D.20}$$

At this point, both multipliers are determined and the procedure terminates. The non-vanishing Poisson brackets between the final set of three constraints

$$\theta_1 = \Pi_\varphi - 2\partial_-\varphi,\ \ \theta_2 = \Pi_0,\ \ \theta_3 = \mu^2\phi_0 + \frac{\lambda}{3!}\int_{-L}^{+L}\frac{\mathrm{d}x^-}{2L}\big(\phi_0 + \varphi\big)^3 \tag{D.21}$$

are

$$\begin{aligned}
&\big\{\theta_1(x^-), \theta_1(y^-)\big\} = -4\delta_N(x^- - y^-),\\
&\big\{\theta_1(x^-), \theta_3\big\} = -\frac{\lambda}{2L}\varphi(x^-)\big[2\phi_0 + \varphi(x^-)\big],\\
&\{\theta_2, \theta_3\} = -\frac{\mu^2}{L} - \frac{\lambda}{2L}\big[\phi_0^2 + \int_{-L}^{+L}\frac{\mathrm{d}x^-}{2L}\varphi^2(x^-)\big].
\end{aligned} \tag{D.22}$$

They are second class because each has non-vanishing Poisson bracket with at least one of them. Denoting $\Delta \equiv \mu^2 + \frac{\lambda}{2}\big[\phi_0^2 + \int_{-L}^{+L}\frac{\mathrm{d}x^-}{2L}\varphi^3(x^-)\big]$ and $f(x^-) = \varphi(x^-)\big[2\phi_0 + \varphi(x^-)\big]$, these Poisson brackets determine the matrix C as:

$$C(x^-, z^-) = \begin{pmatrix} -4\partial_-^x\delta_N(x^- - z^-) & 0 & -\frac{\lambda}{2L}f(x^-) \\ 0 & 0 & -\Delta \\ \frac{\lambda}{2L}f(z^-) & \Delta & 0 \end{pmatrix}. \tag{D.23}$$

Its inverse can be found for example with Mathematica:

$$C^{-1}(z^-, y^-) = \begin{pmatrix} -\frac{1}{4}G_1(y^- - y^-) & \frac{\lambda}{8}\frac{G_1(z^- - y^-)f(y^-)}{L\Delta} & 0 \\ \frac{\lambda}{8}\frac{G_1(z^- - y^-)f(z^-)}{L\Delta} & -\frac{\lambda^2}{16}\frac{G_1(z^- - y^-)f(y^-)f(z^-)}{L^2\Delta^2} & \frac{1}{L^2\Delta} \\ 0 & -\frac{1}{L^2\Delta} & 0 \end{pmatrix}. \tag{D.24}$$

One can check that the matrices indeed satisfy the required property

$$\int_{-L}^{+L}\frac{\mathrm{d}z^-}{2}C_{\alpha\gamma}(x^- - z^-)C^{-1}_{\gamma\beta}(z^- - y^-) = \delta_{\alpha\beta}\delta(x^- - y^-), \tag{D.25}$$

which in the presence of zero modes reads

$$\int_{-L}^{+L}\frac{\mathrm{d}z^-}{2}C(x^- - z^-)C^{-1}(z^- - y^-) = \mathrm{diag}\Big(\delta_N(x^- - y^-), 1/L, 1/L\Big). \tag{D.26}$$

The last two entries correspond to the zero-mode sector.

Now we can calculate the Dirac brackets according to the formula (D.7), which here has a detailed form

$$\Big\{A(x^-),B(y^-)\Big\}^* = \Big\{A(x^-),B(y^-)\Big\} -$$

$$-\int_{-L}^{+L}\frac{\mathrm{d}u^-}{2}\int_{-L}^{+L}\frac{\mathrm{d}v^-}{2}\Big\{A(x^-),\theta_\alpha(u^-)\Big\}C^{-1}_{\alpha\beta}(u^-,v^-)\Big\{\theta_\beta(v^-),B(y^-)\Big\}, \tag{D.27}$$

and using the Poisson brackets between the fields and the three constraints:

$$\begin{aligned}
&\big\{\varphi(x^-),\theta_1(y^-)\big\} = \delta(x^- - y^-),\\
&\big\{\Pi_\varphi(x^-),\theta_1(y^-)\big\} = -2\partial^x_-\delta(x^- - y^-),\\
&\big\{\Pi_\varphi(x^-),\theta_3(y^-)\big\} = -\frac{\lambda}{2L}f(x^-)\\
&\{\phi_0,\theta_2\} = \frac{1}{L}, \quad \{\Pi_0,\theta_3\} = -\frac{\Delta}{L}.
\end{aligned} \tag{D.28}$$

The calculation is straightforward albeit a bit tedious. As the result, with the correspondence (D.8) and the strong relations

$$\Pi_\varphi = 2\partial_-\varphi,\ \ \Pi_0 = 0,\ \ \phi_0 = -\frac{1}{3!}\frac{\lambda}{\mu^2}\int_{-L}^{+L}\frac{\mathrm{d}x^-}{2L}\Big(\phi_0+\varphi\Big)^3, \tag{D.29}$$

the following set of equal-LF time quantum commutators is found:

$$\begin{aligned}
&\Big[\varphi(x^-),\varphi(y^-)\Big] = -\frac{i}{8}\epsilon_N(x^- - y^-),\\
&\Big[\varphi(x^-),\Pi_\varphi(y^-)\Big] = \frac{i}{2}\delta_N(x^- - y^-),\\
&\Big[\Pi_\varphi(x^-),\Pi_\varphi(y^-)\Big] = i\partial^x_-\delta_N(x^- - y^-),\\
&\Big[\phi_0,\Pi_0\Big] = 0,\\
&\Big[\varphi(x^-),\phi_0\Big] = i\frac{\lambda}{16\Delta}\int_{-L}^{+L}\frac{\mathrm{d}y^-}{2L}\epsilon(x^- - y^-)f(y^-),
\end{aligned}$$

$$\Big[\Pi_\varphi(x^-), \phi_0\Big] = i\frac{\lambda}{4L\Delta}f(x^-), \tag{D.30}$$

all the rest being equal to zero. The first three commutators correspond to the normal mode sector and are also found in the continuum theory. The mixed normal mode – zero mode sector commutators are specific to the finite-volume treatment. The assumed canonical commutator $\big[\phi_0, \Pi_0\big] = \frac{i}{L}$ has been changed to a vanishing one in accordance with the strong relation $\Pi_0 = 0$. Note the highly non-linear nature of the latter two commutators. It is not clear how one could use them outside an iterative treatment. This of course is tightly related to the lack of non-perturbative techniques of solving the fundamental ZM constraint (D.29). At the quantum level, also an operator-ordering problem arises in the canonical LF Hamiltonian due to the non-vanishing commutator $\big[\phi_0, \varphi\big]$. These are still the open questions.

E Regularized special functions in the finite volume

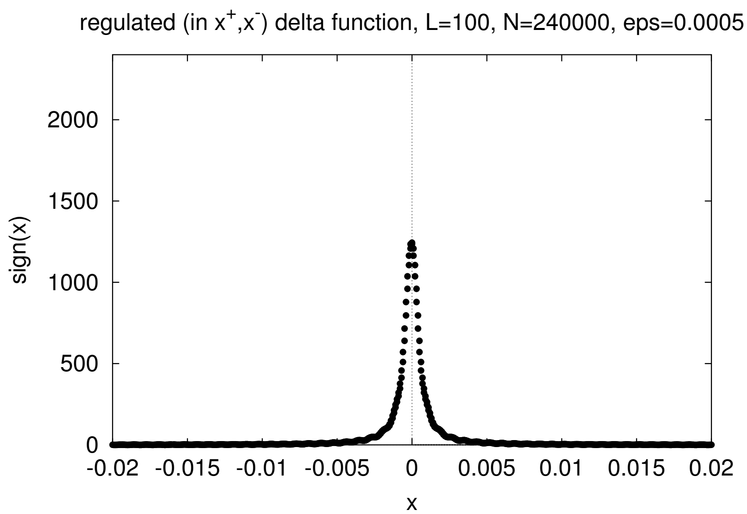

Fig. E.1. Regularized delta function. The number of terms $N = 24 \times 10^4$, the box length $L = 100$ and the regulator $\epsilon = 5 \times 10^{-4}$.

We will display detailed behaviour of the regularized sign function and Dirac delta functions in this Appendix. As defined in the main text, the regularization is twofold: a cutoff on number of modes and a convergence factor governed by a small parameter ϵ. Then the corresponding formulae read

$$\delta_\Lambda(x^- - y^-) = \frac{1}{2L}\sum_{n=\frac{1}{2}}^{\Lambda}\Big(e^{-\frac{i}{2}p_n^+(x^- - y^- - i\epsilon)} + e^{\frac{i}{2}p_n^+(x^- - y^- + i\epsilon)}\Big) \tag{E.1}$$

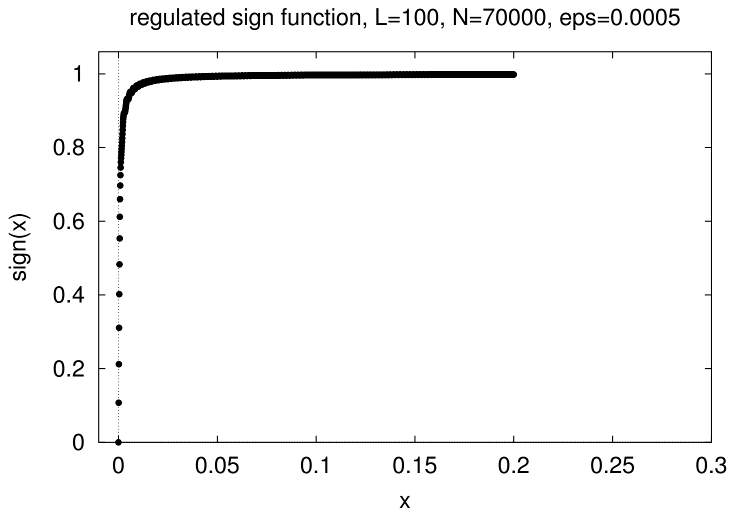

Fig. E.2. Regularized sign function $\epsilon_\Lambda(x^-)$ for $N = 7 \times 10^4$ and $L = 100$ in the neighborhood of $x^- = 0$.

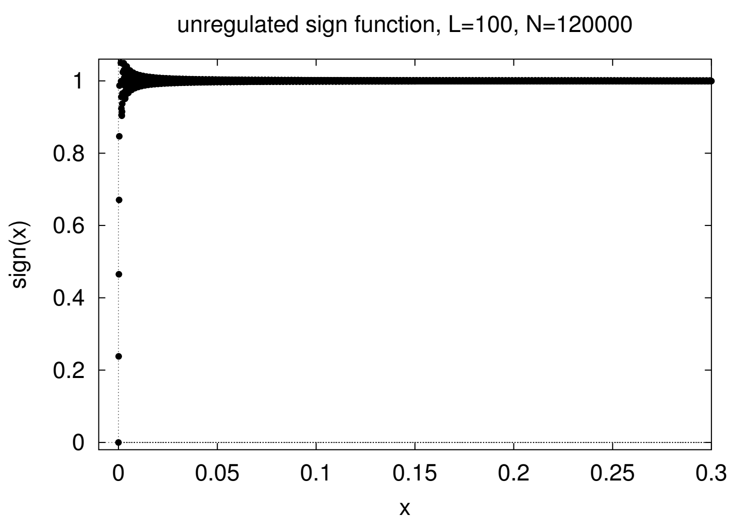

Fig. E.3. Detailed behaviour of the unregularized sign function $\epsilon_\Lambda(x^-)$ around the endpoint $x^- = 0.0$. The rapid oscillations are evident.

and

$$\epsilon_\Lambda(x^- - y^-) = \frac{4i}{L} \sum_{n=\frac{1}{2}}^{\Lambda} \frac{1}{p_n^+} \Big(e^{-\frac{i}{2} p_n^+ (x^- - y^- - i\epsilon)} - e^{\frac{i}{2} p_n^+ (x^- - y^- + i\epsilon)} \Big). \tag{E.2}$$

F Constrained variational method in the broken phase of $\lambda\phi^4$ theory

In this Appendix, we present details of the mathematical procedure to obtain the generalization of the unconstrained variational method which uses the coherent states and infinite value of the dimensionless momentum K, to the finite value of K. The idea is to add a chosen value of K via a Lagrange multiplier to the Hamiltonian and determine the coefficients of the generalized coherent states from the minimization of this constrained problem. The case of the antiperiodic boundary condition will be discussed.

With $\langle K \rangle = \frac{L}{2\pi} \frac{\langle \alpha | P^+ | \alpha \rangle}{\langle \alpha | \alpha \rangle}$, and $f' = \frac{\partial f(x^-)}{\partial x^-}$ we have

$$K = \frac{L}{4\pi^2} \int_{-L}^{+L} dx^- (f')^2 \, . \tag{F.1}$$

Minimizing

$$\frac{1}{\mu^2} \frac{\langle \alpha \mid H_\beta \mid \alpha \rangle}{\langle \alpha \mid \alpha \rangle} = \frac{1}{L} \int_{-L}^{+L} dx^- \Big[\beta \Big\{ \frac{L^2}{4\pi^2} (f')^2 - \langle K \rangle L \Big\} - \frac{1}{4} f^2 + \frac{\lambda}{192\mu^2} f^4 \Big] \tag{F.2}$$

we obtain

$$-2\beta \frac{L^2}{4\pi^2} \frac{\partial^2 f}{\partial (x^-)^2} - \frac{1}{2} f + \frac{\lambda}{48\pi\mu^2} f^3 = 0. \tag{F.3}$$

Putting $f(x^-) = f_0 F(u)$ where the variable $u = \frac{2x^- + L}{L} \overline{K}$ with

$$\overline{K} = \overline{K}(k) = \int_0^1 dt (1 - t^2)^{-\frac{1}{2}} (1 - k^2 t^2)^{-\frac{1}{2}}, \tag{F.4}$$

we have,

$$\frac{\partial^2 F}{\partial u^2} = -\frac{1}{4\overline{K}^2 \beta} F + \frac{\lambda f_0^2}{96 \overline{K}^2 \beta \pi \mu^2} F^3. \tag{F.5}$$

Comparing with the differential equation satisfied by the Jacobi Elliptic Function $sn(u, k)$, namely,

$$\frac{\partial^2 sn(u,k)}{\partial u^2} = -(1 + k^2) \, sn(u,k) + 2k^2 \, sn^3(u,k), \tag{F.6}$$

we get

$$f(x^-) = f_0 \, sn \left(\frac{x^-}{L} \overline{K}, k \right) \tag{F.7}$$

with

$$\beta = \frac{1}{4\overline{K}^2(1+k^2)} \;\text{ and }\; f_0^2 = \frac{48k^2\pi\mu^2}{\lambda(1+k^2)}. \tag{F.8}$$

Note that we have imposed APBC on the solution. By explicit calculation we get

$$\langle K\rangle = \frac{8\mu^2}{\pi\lambda}\overline{K}\Big[E(k) - \frac{1-k^2}{1+k^2}\overline{K}(k)\Big] \tag{F.9}$$

with

$$E(k) = \int_0^1 dt\, \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}} \tag{F.10}$$

and

$$\frac{\langle\alpha \mid H \mid \alpha\rangle}{\langle\alpha \mid \alpha\rangle} = -\frac{24k^2\pi\mu^4}{\lambda(1+k^2)^2} + \frac{64\mu^6}{\lambda^2(1+k^2)\langle K\rangle}\Big[E(k) - \frac{1-k^2}{1+k^2}\overline{K}(k)\Big]^2. \tag{F.11}$$

In the $\langle K\rangle \to \infty$ limit, $k \to 1$ and we get

$$\frac{\langle\alpha \mid H \mid \alpha\rangle}{\langle\alpha \mid \alpha\rangle} = -\frac{6\pi\mu^4}{\lambda} + \frac{32\mu^6}{\lambda^2\langle K\rangle}\;. \tag{F.12}$$

Interpreting the state $\mid \alpha\rangle$ to be a kink state, we identify the first term as the vacuum energy density which is the classical vacuum energy density. The second term is identified as $\frac{M_{kink}^2}{\langle K\rangle}$. Then we get the classical kink mass $M_{kink} = \frac{4\sqrt{2}\mu^3}{\lambda}$.

Using the Fourier expansion [134]

$$sn(u,k) = \frac{1}{\overline{K}}\frac{2\pi}{\sqrt{k^2}}\sum_{m=1}^{\infty}\frac{q^{m-\frac{1}{2}}}{1-q^{2m-1}}\;\sin\frac{(2m-1)\pi u}{2\overline{K}} \tag{F.13}$$

where $q = exp\left(-\pi\frac{\overline{K}(1-k^2)}{\overline{K}(k^2)}\right)$ we have

$$f(x^-) = \frac{2\pi}{\overline{K}}\sqrt{\frac{48\pi\mu^2}{\lambda(1+k^2)}}\sum_j \frac{q^j}{1-q^{2j}}\;\sin\frac{j\pi x^-}{L}\;. \tag{F.14}$$

In the limit $k^2 \to 1$, using $q \to limit_{k^2\to 1}\left(1 - \pi\,\frac{\overline{K}(k^2-1)}{\overline{K}(k^2)}\right)$ so that $(1-q^{2m-1})\overline{K} \to (2m-1)\frac{\pi^2}{2}$ since $\overline{K}(0) = \frac{\pi}{2}$, it is readily verified that in the limit $k^2 \to 1$, the expression for $f(x^-)$ in the constrained variational calculation given by Eq. (F.14) goes over to that in the unconstrained variational calculation given by Eq. (9.35).

G Light front perturbation theory

Perturbative calculations in the LF field theory are most often performed in the ("old-fashioned") Hamiltonian form. This perturbation theory is also called time-ordered formalism because one considers all possible orderings of vertices along the time arrow. The main reason for the Hamiltonian formulation (which is not manifestly covariant) is smaller number of time-ordered diagrams in comparison with their number in usual space-like perturbation theory where they can be obtained from covariant Feynman diagrams by integration over k^0. In literature, one can also find LF calculations [135, 136] starting from the usual Feynman amplitudes which are expressed in terms of covariant integrals.[11] The LF time-ordered diagrams are then generated by performing integration over the LF energy variable k^- (recall that $\mathrm{d}^4k = \frac{1}{2}\mathrm{d}k^-\mathrm{d}k^+\mathrm{d}k^1\mathrm{d}k^2$ in the LF parametrization.) Mathematically, this is a delicate step because a typical element of a Feynman diagram – the propagator of a scalar boson $i(k^2-m^2)^{-1}$ – is equal to $i(k^+k^- - k_\perp^2 - m^2)^{-1}$ in the LF formalism while it is more convergent in the space-like parametrization $i(k_0^2-\vec{k}^2-m^2)^{-1}$. As a consequence, in the contour integration method, there are additional contributions in the LF perturbative computations coming from e.g. the arch in the complex k^- plane [135] and one has to be very careful to take all of them correctly into account. Thus it seems more appropriate to base the LF perturbative calculations on a genuine (and more straightforward) approach which consists of the following steps.

As in the usual manifestly covariant perturbation theory, one works in the Dirac or interaction representation of field operators and states, in which the state vector Φ satisfies the equation

$$i\frac{\partial\Phi(t)}{\partial t} = H_I(t)\Phi(t). \tag{G.1}$$

H_I is the interaction part of the Hamiltonian constructed from the fields in interaction representation. Suppose that the state Φ can be expressed at arbitrary time t as $\Phi(t) = S(t,t_0)\Phi(t_0)$. Then one obtains from the Eq.(G.1)

$$i\frac{\partial S(t,t_0)}{\partial t} = H_I(t)S(t,t_0). \tag{G.2}$$

Let us try to find an iterative solution of this equation in the form $S(t,t_0) = \sum_n^\infty g^n S_n(t,t_0)$, where g is the coupling constant of the given model. Inserting this series into (G.2) and comparing terms of the same order in g, we obtain a sequence of relations

$$i\frac{\partial S_0(t,t_0)}{\partial t} = 0, \quad i\frac{\partial S_1(t,t_0)}{\partial t} = H_I(t)S_0, \ldots \quad i\frac{\partial S_n(t,t_0)}{\partial t} = H_I(t)S_{n-1}, \tag{G.3}$$

where we also used that $H_I \sim g$. The first relation tells us that $S(t,t_0) = \hat{1}$ since it has to be time-independent and obviously $S(t_0,t_0) = \hat{1}$. Then the second relation in (G.3) can be integrated yielding

$$S_1(t_1,t_0) = -i\int_{t_0}^{t_1} \mathrm{d}t_2 H_I(t_2). \tag{G.4}$$

[11] It is to be noted however that they are obtained from Feynman rules based on conventional space-like version of the theory.

Similarly, integrating the third relation in (G.3) for $n = 2$, one finds

$$S_2(t,t_0) = -i\int_{t_0}^{t} \mathrm{d}t_1 H_I(t_1) S_1(t_1,t_0) = (-i)^2 \int_{t_0}^{t} \mathrm{d}t_1 H_I(t_1) \int_{t_0}^{t_1} \mathrm{d}t_2 H_I(t_2), \tag{G.5}$$

and then also

$$\begin{aligned} S_3(t,t_0) &= -i\int_{t_0}^{t} \mathrm{d}t_1 H_I(t_1) S_2(t_1,t_0) = \\ &= (-i)^3 \int_{t_0}^{t} \mathrm{d}t_1 H_I(t_1) \int_{t_0}^{t_1} \mathrm{d}t_2 H_I(t_2) \int_{t_0}^{t_2} \mathrm{d}t_3 H_I(t_3), \end{aligned} \tag{G.6}$$

etc. For the scattering problems, one usually chooses $t_0 = -\infty, t = +\infty$. The above considerations were quite general as they were not based on any specific feature of the conventional space-like field theory, so they are equally valid if we choose the LF time x_n^+ instead of t_n and the LF Hamiltonian P_I^- instead of H_I. Then the first few terms in the perturbative series for the LF scattering matrix are given by

$$\begin{aligned} S &= \hat{1} - \frac{i}{2}\int_{-\infty}^{+\infty} \mathrm{d}x^+ P_I^-(x^+) + \left(-\frac{i}{2}\right)^2 \int_{-\infty}^{+\infty} \mathrm{d}x_1^+ P_I^-(x_1^+) \int_{-\infty}^{x_1^+} \mathrm{d}x_2^+ P_I^-(x_2^+) + \\ &+ \left(-\frac{i}{2}\right)^3 \int_{-\infty}^{+\infty} \mathrm{d}x_1^+ P_I^-(x_1^+) \int_{-\infty}^{x_1^+} \mathrm{d}x_2^+ P_I^-(x_2^+) \int_{-\infty}^{x_2^+} \mathrm{d}x_3^+ P_I^-(x_3^+) + \ldots \end{aligned} \tag{G.7}$$

In the next step, one calculates the matrix elements of the above operator. Assume that $|\Phi_i\rangle$ and $|\Phi_f\rangle$ are eigenstates of the free LF Hamiltonian P_0^- at $x^+ = -\infty$ and $x^+ = +\infty$, respectively. Then we have from the expression (G.7)

$$\begin{aligned} S_{fi} &\equiv \langle\Phi_f|S|\Phi_i\rangle = \delta_{fi} - \frac{i}{2}\int_{-\infty}^{+\infty} \mathrm{d}x^+ \langle\Phi_f|P_I^-(x^+)|\Phi_i\rangle - \\ &- \frac{1}{4}\int_{-\infty}^{+\infty} \mathrm{d}x_1^+ \langle\Phi_f|P_I^-(x_1^+) \int_{-\infty}^{x_1^+} \mathrm{d}x_2^+ P_I^-(x_2^+)|\Phi_i\rangle + \\ &+ \frac{i}{8}\int_{-\infty}^{+\infty} \mathrm{d}x_1^+ \langle\Phi_f|P_I^-(x_1^+) \int_{-\infty}^{x_1^+} \mathrm{d}x_2^+ P_I^-(x_2^+) \int_{-\infty}^{x_2^+} \mathrm{d}x_3^+ P_I^-(x_3^+)|\Phi_i\rangle + \ldots \end{aligned} \tag{G.8}$$

We can insert the complete set of states $\hat{1} = \sum_n |\Phi_n\rangle\langle\Phi_n|$ between any two $P_I^-(x_n^+)$ operators and use the translation operator to obtain all P_I^- at the same time $x^+ = 0$:

$$P_I^-(x^+) = e^{\frac{i}{2}P_0^- x^+} V(0) P e^{-\frac{i}{2}P_0^- x^+}, \tag{G.9}$$

where we have denoted the interacting Hamiltonian at $x^+ = 0$ as $V(0) \equiv V$ for simplicity. Since all states $|\Phi_n\rangle$ including $|\Phi_f\rangle$ and $|\Phi_i\rangle$ are eigenstates of P_0^- corresponding to the LF energy E_n^-, action of the translational operator $e^{\frac{i}{2}P_0^- x^+}$ on the state $|\Phi_n\rangle$ generates the factors $e^{\frac{i}{2}E_n^- x^+}$ which can be easily integrated leading to energy dependent denominator factors. Thus for example the first non-trivial term is

$$S_{fi}^{(1)} = -\frac{i}{2}\int_{-\infty}^{+\infty} \mathrm{d}x^+ e^{\frac{i}{2}(E_f^- - E_i^-)x^+} \langle\Phi_f|V|\Phi_i\rangle = -2\pi i\delta(E_f^- - E_i^-)\langle\Phi_f|V|\Phi_i\rangle. \tag{G.10}$$

The next correction is

$$
\begin{aligned}
S_{fi}^{(2)} &= -\frac{1}{4}\sum_n \int_{-\infty}^{+\infty} \mathrm{d}x_1^+ e^{\frac{i}{2}E_f^- x_1^+}\langle\Phi_f|Ve^{-\frac{i}{2}P_0^- x_1^+}|\Phi_n\rangle \times \\
&\times \int_{-\infty}^{x_1^+} \mathrm{d}x_2^+ \langle\Phi_n|e^{\frac{i}{2}P_0^- x_2^+}V|\Phi_i\rangle e^{-\frac{i}{2}E_i^- x_2^+} = \\
&= -2\pi i\delta(E_f^- - E_i^-)\sum_n \frac{\langle f|V|n\rangle\langle n|V|i\rangle}{E_i^- - E_n^-}. \qquad \text{(G.11)}
\end{aligned}
$$

Here the simplified notation is $|f\rangle \equiv |\Phi_f\rangle$, E_n^- is the energy of the intermediate state $|n\rangle$. Presence of appropriate convergence factors is understood, so that the energy in the exponents is actually $E_n^- \pm i\epsilon$ and $\pm i\epsilon$ terms are also present in the energy denominators. The convergence factors eliminate integral contributions at $x^+ = \pm\infty$. The summation runs over all many-particle intermediate Fock states generating elementary time-ordered processes (or diagrams in the graphical language). Only those bra-states contribute in which all creation (annihilation) operators are contracted with annihilation (creation) operators from the interaction Hamiltonian or from a ket-state. The latter situation corresponds to spectator particles not affected by interaction. Note that the energy is not conserved at vertices in this form of perturbation theory. We say that particles (or more properly particle states) are off energy shell instead of being off mass shell as in the manifestly covariant form of the perturbative expansion. In the latter approach, the space-like version of the representation (G.7) is not the final step. One introduces the time ordering operation T to set all upper integration limits to $+\infty$. In this way one is left with integrals of the form $\int_{-\infty}^{+\infty} \mathrm{d}^4x\mathcal{H}$, where $\mathcal{H}$ is the Hamiltonian density, which by inserting the field expansions are converted to four-dimensional integrals in the momentum space with the measure proportional to d^4k. Contrary to this, in the time-ordered perturbation theory summation over intermediate states includes summation over momenta not fixed by momentum conservation. In the continuum form of the theory, one thus performs only three-dimensional momentum integration (or one-dimensional integration in the case of two-dimensional theories). No k^- integration is necessary avoiding the subtleties mentioned earlier.

Finally, let us present also a formula for the fourth order correction to the scattering matrix. It reads

$$
S_{fi}^{(4)} = -2\pi i\delta(E_f^- - E_i^-)\sum_{l,m,n} \frac{\langle f|V|l\rangle\langle l|V|m\rangle\langle m|Vn\rangle\langle n|V|i\rangle}{(E_i^- - E_l^-)(E_i^- - E_m^-)(E_i^- - E_n^-)} \qquad \text{(G.12)}
$$

and can be applied for example to calculations of the so-called box diagrams, i.e. those with four internal lines.

One can formulate a set of rules analogous to Feynman rules which simplify the task of writing down perturbative amplitudes of elementary processes of given quantum fields with a prescribed interaction Hamiltonian [11]. The rules in the case of $(3+1)$ dimensional theory are:

1. Draw all time-ordered diagrams, i.e. all diagrams which correspond to a different ordering of the vertices along the x^+ arrow and which are topologically nonequivalent.

2. For each vertex, insert the factor $2(2\pi)^3\delta^3(p_f - p_i)$ which guarantees the four-momentum conservation, and an elementary matrix element of the given interaction Hamiltonian.

These matrix elements have to be worked out separately for each theory using the Fock representation.

3. For each internal line, sum over momentum and helicity. In the continuum theory, the former means integration $\int \frac{\mathrm{d}p^+\mathrm{d}^2p_\perp}{2(2\pi)^3 p}$.

4. Insert the energy denominator $(p_i^- - p_I^- + i\epsilon)^{-1}$ for each intermediate state I. Here $p_I^- = \sum_j p_j^-$ is the LF energy of the intermediate state equal to the sum of on-mass shell energies (since one is working in the interaction picture) $p_j^- = m^2/p_j^+$ of particles present in the intermediate state.

5. In theories with bosons, a symmetry factor has to be included for each boson loop since it is necessary to use symmetrized boson states.

H Field theory in near light front coordinates

Consider the set of coordinates

$$\begin{aligned}
x^+ &= \frac{1}{\sqrt{2}}(x^0 + x^3) + \frac{1}{2}\eta^2\frac{1}{\sqrt{2}}(x^0 - x^3) \\
x^- &= \frac{1}{\sqrt{2}}(x^0 - x^3) \\
x^\perp &= (x^1, x^2).
\end{aligned} \tag{H.1}$$

They generalize our LF time variable to an arbitrary "mixture" of usual t and x^3 coordinates. Then x^- is the third space (" longitudinal") coordinate. The metric tensor in the new coordinate system is given by

$$\begin{aligned}
\tilde{g}_{\mu\nu} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -\eta^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \\
\tilde{g}^{\mu\nu} &= \begin{bmatrix} \eta^2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.
\end{aligned} \tag{H.2}$$

Thus we have,

$$x^2 = \tilde{g}_{\mu\nu}x^\mu x^\nu =$$

$$2x^+x^- - \eta^2(x^-)^2 - (x^\perp)^2 = \tilde{g}^{\mu\nu}x_\mu x_\nu = \eta^2(x_+)^2 + 2x_+x_- - (x_\perp)^2. \tag{H.3}$$

Furthermore,

$$x_+ = x^-,\ x_- = x^+ - \eta^2 x^-. \tag{H.4}$$

The scalar product is $k \cdot x = k^+ x^- + k^- x^+ - \eta^2 k^- x^- - k^\perp \cdot x^\perp = k_+ x^+ + k_- x^- - k_\perp \cdot x_\perp$. Thus k_+ which is conjugate to x^+ is the energy and k_- which is conjugate to x^- is the longitudinal momentum. It is important to keep in mind that $-\infty < k_- < +\infty$.

For an on mass-shell particle of mass m, $k^2 = m^2$ yields

$$\eta^2 (k_+)^2 + 2k_+ k_- - (k^\perp)^2 - m^2 = 0 \tag{H.5}$$

which leads to the dispersion relation

$$k_+ = \frac{-k_- \pm \sqrt{(k_-)^2 + \eta^2 (m^2 + (k^\perp)^2)}}{\eta^2}. \tag{H.6}$$

For an on mass-shell particle, since $k^0 > k^3$, $k^0 > 0$ implies $k_+ > 0$ and hence the Lorentz invariant phase space factor

$$\frac{d^4 k}{(2\pi)^4} 2\pi \delta(k^2 - m^2)\theta(k_+) =$$

$$= \frac{dk_+ dk_- d^2 k^\perp}{(2\pi)^3} \delta(\eta^2 (k_+)^2 + 2k_+ k_- - (k^\perp)^2 - m^2)\theta(k_+) = \frac{dk_- d^2 k^\perp}{(2\pi)^3 2E_{on}}, \tag{H.7}$$

where $E_{on}(k) = \sqrt{(k_-)^2 + \eta^2 (m^2 + (k^\perp)^2)}$.

Consider now the free massive scalar theory with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 = \frac{1}{2}\eta^2 \partial_+ \phi \partial_+ \phi + \partial_+ \phi \partial_- \phi - \frac{1}{2}\partial^\perp \phi \cdot \partial^\perp \phi - \frac{1}{2} m^2 \phi^2. \tag{H.8}$$

The equation of motion

$$(\partial_\mu \partial^\mu + m^2)\phi = 0 \tag{H.9}$$

becomes

$$\Big(\eta^2 \partial_+ \partial_+ + 2\partial_+ \partial_- - (\partial^\perp)^2 + m^2\Big)\phi = 0. \tag{H.10}$$

The general solution is

$$\phi(x) = \int \frac{d^4 k}{(2\pi)^4} f(k) 2\pi \delta(k^2 - m^2) e^{ik\cdot x}, \tag{H.11}$$

or

$$\begin{aligned} \phi(x) &= \int \frac{dk_- d^2 k^\perp}{(2\pi)^3 2E_{on}(k)} [a(k) e^{i(k_{+on} x^+ + k_- x^- - k^\perp \cdot x^\perp)} + \\ &+ a^*(k) e^{-i(k_{+on} x^+ + k_- x^- - k^\perp \cdot x^\perp)}], \end{aligned} \tag{H.12}$$

where

$$k_{+(on)} = \frac{-k_- + \sqrt{(k_-)^2 + \eta^2 (m^2 + (k^\perp)^2)}}{\eta^2}. \tag{H.13}$$

In the quantum theory we have

$$\begin{aligned}\phi(x) &= \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\frac{dk_-d^2k^\perp}{2E_{on}(k)}\big[a(k)e^{i(k_{+on}x^++k_-x^--k^\perp\cdot x^\perp)}+ \\ &+ a^\dagger(k)e^{-i(k_{+on}x^++k_-x^--k^\perp\cdot x^\perp)}\big]. \end{aligned} \tag{H.14}$$

The conjugate momentum is

$$\pi(x)=\frac{\partial\mathcal{L}}{\partial\partial_+\phi}=\eta^2\partial_+\phi+\partial_-\phi, \tag{H.15}$$

$$\begin{aligned}\pi(x) &= -i\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\frac{dk_-d^2k^\perp}{2E_{on}(k)}E_{on}(k)\big[a(k)e^{i(k_{+on}x^++k_-x^--k^\perp\cdot x^\perp)} \\ &- a^\dagger(k)e^{-i(k_{+on}x^++k_-x^--k^\perp\cdot x^\perp)}\big]. \end{aligned} \tag{H.16}$$

We have,

$$[\phi(x),\pi(y)]_{x^+=y^+}=i\delta(x^--y^-)\delta^2(x^\perp-y^\perp), \tag{H.17}$$

provided

$$\begin{aligned}&\big[a(k),a^\dagger(k')\big]=(2\pi)^3 2E_{on}(k)\delta(k_--k'_-)\delta^2(k^\perp-k'^\perp), \\ &[a(k),a(k')]=0,\big[a^\dagger(k),a^\dagger(k')\big]=0. \end{aligned} \tag{H.18}$$

The Hamiltonian density is

$$\mathcal{H} \quad =\pi\partial_+\phi-\mathcal{L}=\frac{1}{2}\frac{(\pi-\partial_-\phi)^2}{\eta^2}+\frac{1}{2}\partial^\perp\phi\cdot\partial^\perp\phi+\frac{1}{2}m^2\phi^2 \tag{H.19}$$

and the Hamiltonian in the Fock representation takes the form

$$\begin{aligned} H &= \int dx^-d^2x^\perp\mathcal{H}= \\ &= \int\frac{dk_-d^2k^\perp}{(2\pi)^3 2E_{on}(k)}\frac{-k_-+\sqrt{(k_-)^2+(m^2+(k^\perp)^2)\eta^2}}{\eta^2}a^\dagger(k)a(k). \end{aligned} \tag{H.20}$$

The propagator is given by

$$\begin{aligned} iS_B(x) &= \langle 0\mid T(\phi(x)\phi(0))\mid 0\rangle= \\ &= \theta(x^+)\langle 0\mid\phi(x)\phi(0)\mid 0\rangle+\theta(-x^+)\langle 0\mid\phi(0)\phi(x)\mid 0\rangle \\ &= \frac{1}{(2\pi)^3}\int\frac{dk_-d^2k^\perp}{2E_{on}(k)}\Big[\theta(x^+)e^{-i(k_+x^++k_-x^--k^\perp\cdot x^\perp)} \\ &+ \theta(-x^+)e^{i(k_+x^++k_-x^--k^\perp\cdot x^\perp)}\Big]. \end{aligned} \tag{H.21}$$

Using

$$\theta(x) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} dy e^{iyx}\frac{1}{y-i\epsilon} \tag{H.22}$$

and changing integration variables, we get

$$\begin{aligned} iS_B(x) &= \frac{1}{(2\pi)^4}\int dk_+ dk_- d^2k^\perp e^{i(k_+x^+ + k_-x^- - k^\perp\cdot x^\perp)} \times \\ &\times \frac{i}{\eta^2(k_+)^2 + 2k_+k_- - (k^\perp)^2 - m^2 + i\epsilon}, \\ iS_B(x) &= \frac{1}{(2\pi)^4}\int d^4k e^{ik.x}\frac{i}{k^2 - m^2 + i\epsilon}. \end{aligned} \tag{H.23}$$

Finally, let us present the perturbative formula for the S matrix in the near light front version:

$$S_{fi} = \delta_{fi} - 2\pi i\delta(p_{+(on)f} - p_{+(on)i})T_{fi}, \tag{H.24}$$

$$T_{fi} = \langle f \mid V_S \mid i\rangle + \sum_n \frac{\langle f \mid V_S \mid n\rangle\langle n \mid V_S \mid i\rangle}{p_{+(on)i} - p_{+(on)n} + i\epsilon} + \dots \tag{H.25}$$

Since the longitudinal momentum p_- is conserved at the vertex, we get

$$T_{fi} = \langle f \mid V_S \mid i\rangle + \sum_n \frac{\langle f \mid V_S \mid n\rangle\langle n \mid V_S \mid i\rangle}{\frac{1}{\eta^2}(E_{(on)i} - E_{(on)n}) + i\epsilon} + \dots \tag{H.26}$$

The sum over intermediate states $\sum_n \to \int \frac{dk_- d^2k^\perp}{(2\pi)^3 2E_{on}}$.

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