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PATH DECOMPOSITION METHOD FOR $\delta-$POTENTIALS

A. Laissaoui, L. Chetouani[1]
Département de physique , Faculté des Sciences,
Université Mentouri, Constantine, Algeria.

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The propagators relating to $\delta-$potentials are exactly determined according to the method of path decomposition. The determination uses, after a suitable choice of decomposition surfaces, the images method and the spectral decomposition of the propagator at $x = 0$.

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We know all that thanks the development of the various formulations, moreover all equivalent, the physical phenomena explained by quantum mechanics are understood better. Among these formulations, we can quote most known because of its success: the formulation of Feynman [1] which uses the tool of the path integral.

Not a long time ago, obtaining solutions for certain problems of standard quantum mechanics by the formalism of the path integral was considered as an important step. Thanks to various techniques which were developed such as the canonical and space-time transformations, the solutions in question could be obtained via the spectral decomposition of the propagators or Green's functions [2]. The solutions in question having been able to be obtained, this approach currently knows an interest more and more growing and this in practically all the fields of physics thanks to the many applications which were found.

Moreover, we know all which it is thanks to quantum mechanics that important and well-known phenomena such as the effect of tunnel, could be well explained. Using either the approach of Feynman this effect is analyzed according to the semiclassical method WKB or the instantons method [2], which is richer.

However, a phenomena such as the tunnel effect was also explained according to another method: it acts of the method so-called path decomposition method [3] whose principle consists, by subdividing the configuration space in forbidden and allowed regions, to determine the dynamics of a system starting from information relating to the isolated regions.

We propose in this paper, to examine this path decomposition method where only one limited number of papers were devoted to it [4–7], by determining the propagators for interactions having a simple forms in order to see the possibilities of this method for other interactions. These interactions, which we choose as Dirac's functions δ are described by the following potentials

- $V(x) = \lambda\delta(x)$ which is static
- $V(x,t) = V(x - f(t))$ which is moving and as particular case: $V(x,t) = \lambda\delta(x - vt)$

[1]E-mail address: chetoua1@caramail.com

These potentials of form in δ are, a long time ago, taken as model in nuclear physics for nucleon-nucleon interaction.

Moreover, knowing that $\theta(x) = \int\limits_{-\infty}^{x} \delta(y)dy$ because $\frac{d\theta(x)}{dx} = \delta(x)$, i.e. that it also possible with a superposition of several potentials in δ, to form an another potential having a form of a Heaviside' s function θ, which also it is of interest in nuclear physics. It is thus obvious that the potential δ is important, not only for pedagogical level to illustrate the method of path decomposition, but also for the determination of the propagator which can be extended to other potentials by taking δ as an elementary interaction in the calculation of the propagator. With a null range, these potentials allow in addition, to reduce the calculation of the integrals and thus to simplify calculations.

Let us give the essential elements of the path decomposition method. In the nonrelativistic case, we know that the amplitude of probability or propagator is by definition the matrix element of the evolution operator. It becomes a sum over all possible paths in the path integral formalism, where which each path $x\,(t)$ is affected of a complex weight equal to $\exp(\frac{i}{\hbar}Action)$. Thus, in the D-dimensional configuration space, the propagator is

$$\begin{aligned} K\left(b,T\left|a\right.\right) &= \left\langle b\left|\exp\left(-\frac{i}{\hbar}HT\right)\right|a\right\rangle \\ &= \int Dx\,(t)\exp\left[\frac{i}{\hbar}\int_0^T dt\; S\left(x\left(t\right)\right)\right], \end{aligned} \tag{1}$$

where

$$S\left(x(t)\right) = \int_0^T dt\left(\frac{m}{2}\dot{x}^2 - V\left(x(t)\right)\right), \tag{2}$$

is the usual action, with obviously the following conditions

$$x\left(0\right) = a\ , \qquad x\left(T\right) = b. \tag{3}$$

Now, let us consider a $(D-1)$ dimensional surface $\sum$ separating space into two distinct regions. It is obvious that if "a" is in one region and "b" is in the other, $\sum$ is inevitably crossed by the paths $x(t)$ at distinct times. Let us indicate respectively by t and x_σ, the time and the position of the last or first crossing of $\sum$.

For these two choices, the path can be divided according to the time "t" and the position x_σ into two parts where one part goes directly from x_σ to the endpoint without ever crossing $\sum$.

To simplify, we consider the one-dimensional case where x_σ is consisting of a single point $\sum$ such a $a < x_\sigma < b$.

It is known, that the propagator K is better defined if imaginary time is used. In this case, the action S is replaced by iS_E, where

$$S_E\left(x\right) = \int_0^T dt\left[\frac{m}{2}\dot{x}^2 + V\left(x\right)\right], \tag{4}$$

is the action corresponding to the inverted potential.

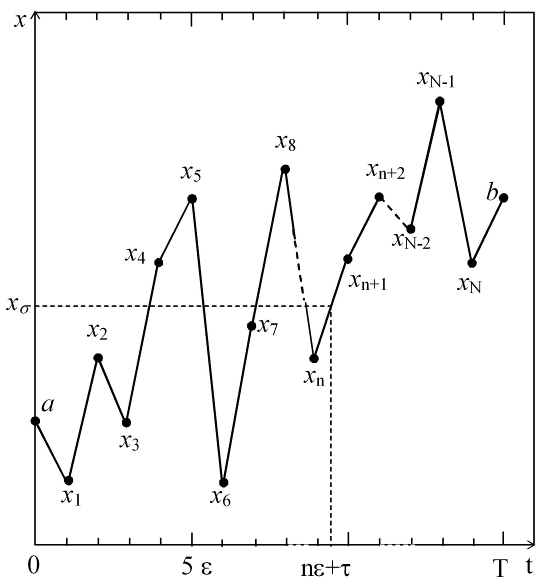

Fig. 1. Path connecting points a and b.

The interval of time $(0, T)$, we subdivide it into $(N+1)$ steps having each one a length equal to ε, with $T = (N+1)\varepsilon$ and the path $x(t)$ becomes a polygon and K is a limit

$$K(b, -iT\,|a) = \lim_{N\longrightarrow\infty} K_N(x_T, -iT\,|x_0), \quad x_T = x_{N+1}, \tag{5}$$

where

$$K_N(x_T, -iT\,|x_0) = \left(\frac{m}{2\pi\hbar\varepsilon}\right)^{\frac{N+1}{2}} \int_{-\infty}^{+\infty} dx_1 \ldots \int_{-\infty}^{+\infty} dx_N \, \exp\left[\frac{-S_E(0, N)}{\hbar}\right], \tag{6}$$

is the discrete form of the propagator and

$$S_E(n_1, n_2) = \sum_{j=n_1}^{n_2} \left[\frac{m(x_{j+1} - x_j)^2}{2\varepsilon} + V(x_j)\,\varepsilon\right], \tag{7}$$

the discretized action.

The path connecting x_0 to x_T is specified by the points x_j: $j = 1, \ldots N$ (Fig.1). The last crossing x_σ of $\sum$ is on the segment $[x_n, x_{n+1}]$ and we can define $(N+1)$ disjoint sets of polygonal lines P_n, $n = 0, \ldots N$, where P_n are the paths for which the greatest value of "j" such as $x_j \leqslant x_\sigma$ is $j = n$.

It is clear that

$$K_N(x_T, -iT\,|x_0) = \sum_{n=0}^{N} K_{N,n}(x_T, -iT\,|x_0), \tag{8}$$

where

$$K_{N,n} = \left(\frac{m}{2\pi\hbar\varepsilon}\right)^{\frac{N+1}{2}} \int_{-\infty}^{+\infty} dx_1 \cdots dx_{n-1} \exp\left[\frac{-S_E\left(0,n-1\right)}{\hbar}\right] \times \int_{-\infty}^{x_\sigma} dx_n \int_{x_\sigma}^{+\infty} dx_{n+1} \cdots dx_N \ \exp\left[\frac{-S_E\left(n,N\right)}{\hbar}\right], \quad (9)$$

is the sum over all paths of P_n.

We can notice that $K_{N,n}$ is not a product of two independent integrals, because the variable x_n is in $S_E\left(0,n-1\right)$ and in $S_E\left(n,N\right)$ at the same time.

The most important steps of the path decomposition method are as follows. They consist of two steps:

(1) to separates the variables x_{n+1} and x_n, by using the following identity

$$\left(\frac{m}{2\pi\hbar\varepsilon}\right)^{\frac{1}{2}} \exp\left[-\frac{m\left(x_{n+1}-x_n\right)^2}{2\hbar\varepsilon}\right] = \int_0^\varepsilon d\tau \ \exp\left[-\frac{m\left(x_\sigma - x_n\right)^2}{2\hbar\tau}\right] \times \left(\frac{m}{2\pi\hbar\tau}\right)^{\frac{1}{2}} \frac{\hbar}{m}\frac{\partial}{\partial x}\left[\left(\frac{m}{2\pi\hbar\left(\varepsilon-\tau\right)}\right)^{\frac{1}{2}} \exp\left[-\frac{m\left(x_{n+1}-x\right)^2}{2\hbar\left(\varepsilon-\tau\right)}\right]\right]\Bigg|_{x=x_\sigma}, \quad (10)$$

which can be verified . After having carried out the derivation and in order to use the formula [10]

$$\int_0^\infty x^{\nu-1}\exp\left(-\frac{\beta}{x}-\gamma x\right)dx = 2\left(\frac{\beta}{\gamma}\right)^{\frac{\nu}{2}} \ K_{\ \nu}\left(2\sqrt{\beta\gamma}\right) \qquad \left[\operatorname{Re}\beta>0,\ \ \operatorname{Re}\gamma>0\right],$$

we introduce the following change $\tau\in\left[0,\varepsilon\right]\to y\in\left[0,\infty\right]$ defined by

$$\tau = \frac{\varepsilon}{1+y}.$$

For $\nu=-\frac{1}{2}$, $\beta=\frac{m(x_{n+1}-x_\sigma)^2}{2\hbar\varepsilon}$ and $\gamma=\frac{m(x_\sigma-x_n)^2}{2\hbar\varepsilon}$. The second member of (10) is equal to

$$\frac{m}{\pi\hbar\varepsilon}\left(x_{n+1}-x_\sigma\right)\left(\frac{\beta}{\gamma}\right)^{\frac{\nu}{2}} \exp\left\{-\beta-\gamma\right\}K_{-1/2}(2\sqrt{\beta\gamma}),$$

and with the expression of $K_{\pm\frac{1}{2}}\left(z\right)=\sqrt{\frac{\pi}{2z}}e^{-z}$ [10], we obtain the first member of (10).

This identity can still be explained [11] by considering the composition law of the free propagator

$$K_0\left(x_{n+1},-i\varepsilon\left|x_n\right.\right) = \int_{-\infty}^{+\infty} dx_\sigma K_0\left(x_{n+1},-i\left(\varepsilon-\tau\right)\left|x_\sigma\right.\right)K_0\left(x_\sigma,-i\tau\left|x_n\right.\right).$$

Let us introduce the change of variables $x_\sigma\to\tau$, $\int_{-\infty}^{+\infty}dx_\sigma\to\int_0^\varepsilon Jd\tau$, where J is the Jacobian $J=\frac{\delta x_\sigma}{\delta\tau}=\frac{x_{n+1}-x_\sigma}{\varepsilon-\tau}=v$ which express the velocity of $x\left(\tau\right)=x_\sigma$ at $\sum$. This Jacobian disappear if we consider the propagator's derivative

$$\frac{x_{n+1-}x_\sigma}{\varepsilon-\tau}K_0\left(x_{n+1},-i\left(\varepsilon-\tau\right)\left|x_\sigma\right.\right) = \frac{\hbar}{m}\frac{\partial}{\partial x}\left.K_0\left(x_{n+1},-i\left(\varepsilon-\tau\right)\left|x\right.\right)\right|_{x=x_\sigma},$$

where $x_n < x_\sigma < x_{n+1}$, and
(2) using the development of the potential

$$V(x_n)\,\varepsilon = [V(x_n)\,\tau + V(x_\sigma)(\varepsilon-\tau)] - \frac{\partial}{\partial x_\sigma} V(x_n)(x_\sigma - x_n)(\varepsilon-\tau) + \cdots, \quad (11)$$

to have (with the limit $N \to \infty$) a separable form for the integrals. Thus we obtain a factorized expression of the propagator

$$K(b,-iT\,|a) = \int_0^T dt\,\frac{\hbar}{2m}\frac{\partial}{\partial x}\,K^r(b,-i(T-t)\,|x)|_{x=x_\sigma}\,K(x_\sigma,-it\,|a), \quad (12)$$

which is an integral over the last crossing time t of a product of two propagators

- one, $K(x_\sigma,-it\,|a)$,
- and the other, K^r defined by

$$K^r(b,-i(T-t)\,|x) = \int\limits_{x(0)=x}^{x(T-t)=b} D^r x\left(t^{'}\right)\exp\left[\frac{-S_E\left(x\left(t^{'}\right)\right)}{\hbar}\right], \quad (13)$$

  with $x > x_\sigma$, which is called the restricted propagator. All the paths of K^r which never cross $\sum$ and this restriction is expressed by the measure $D^r(x)$.

For the case $b < x_\sigma < a$, the same steps leads to an expression similar to that of (12) with only a minus sign (-) to add moreover. If the points "a", "b" are on the same side of $\sum$, a class of paths not crossing $\sum$ (Fig. 2) can be found and it is necessary to take this into account in the propagator expression.

By introducing, the function of Heaviside θ for the various cases of positions a and b, we may write the generalized path decomposition expression of the propagator in a single formula according to the time of crossing:

- when t represents the last crossing time of x_σ on $\sum$ it is expressed as

$$K(b,-iT\,|a) = [\theta(b-x_\sigma) - \theta(x_\sigma-a)]^2\,K^r(b,-iT\,|a)$$
$$+[\theta(b-x_\sigma)-\theta(x_\sigma-b)]\int_0^T dt\frac{\hbar}{2m}\frac{\partial}{\partial x}\,K^r(b,-i(T-t)\,|x)|_{x=x_\sigma}\,K(x_\sigma,-it\,|a), \quad (14)$$

- and when t is the first crossing time, it becomes

$$K(b,-iT\,|a) = [\theta(a-x_\sigma) - \theta(x_\sigma-b)]^2\,K^r(b,-iT\,|a)$$
$$+[\theta(a-x_\sigma)-\theta(x_\sigma-a)]\int_0^T dt\,K(b,-i(T-t)\,|x_\sigma)\,\frac{\hbar}{2m}\frac{\partial}{\partial x}\,K^r(x,-it\,|a)|_{x=x_\sigma}. \quad (15)$$

To illustrate the method, let us make two applications by considering the two following potentials

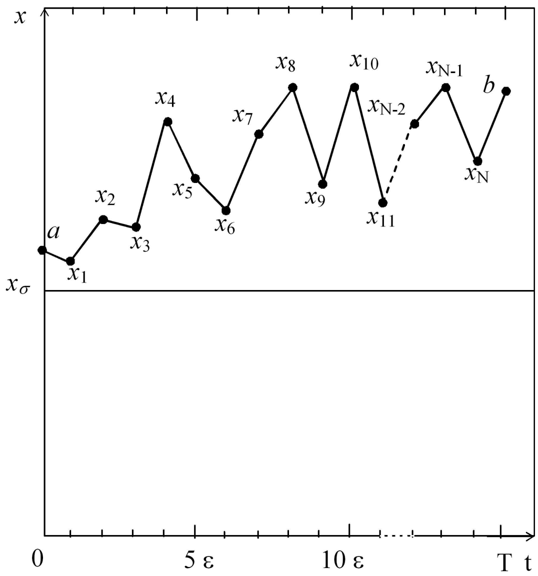

Fig. 2. Path of the class P_0.

- the first, $V(x) = \lambda\delta(x)$ is static (with $\lambda > 0$)
- and the second, $V(x,t) = \lambda\delta(x - vt)$ is moving.

For the static potential, it is adequate to choose the point $x_\sigma = 0$ for the position of surface $\sum$. Thus, the path decomposition expressions of the propagator according to the last and the first crossing time, are

$$\begin{aligned} K\left(b, -iT\left|a\right.\right) &= \left[\theta\left(b\right) - \theta\left(-a\right)\right]^2 K^r\left(b, -iT\left|a\right.\right) + \left[\theta\left(b\right) - \theta\left(-b\right)\right] \\ &\times \int_0^T dt\, \frac{\hbar}{2m}\frac{\partial}{\partial x}\, K^r\left(b, -i\left(T-t\right)\left|x\right.\right)\Big|_{x=0} K\left(0, -it\left|a\right.\right), \end{aligned} \tag{16}$$

$$\begin{aligned} K\left(b, -iT\left|a\right.\right) &= \left[\theta\left(a\right) - \theta\left(-b\right)\right]^2 K^r\left(b, -iT\left|a\right.\right) + \left[\theta\left(a\right) - \theta\left(-a\right)\right] \\ &\times \int_0^T dt\, K\left(b, -i\left(T-t\right)\left|0\right.\right)\frac{\hbar}{2m}\frac{\partial}{\partial x}\, K^r\left(x, -it\left|a\right.\right)\Big|_{x=0}, \end{aligned} \tag{17}$$

where the restricted propagators are defined as follows

$$K^r\left(b, -i\left(T-t\right)\left|x\right.\right) = \int D_1^r x\left(t'\right)\; \exp\left[-\frac{1}{\hbar}\int_0^{T-t} dt'\left(\frac{1}{2}m\dot{x}^2 + \lambda\,\delta\left(x\right)\right)\right], \tag{18}$$

$$K^r\left(x, -it\left|a\right.\right) = \int D_2^r x\left(t'\right)\; \exp\left[-\frac{1}{\hbar}\int_0^{t} dt'\left(\frac{1}{2}m\dot{x}^2 + \lambda\,\delta\left(x\right)\right)\right], \tag{19}$$

and $D_1^r x\left(t^{'}\right)$, $D_2^r x\left(t^{'}\right)$ are the measures which restrict the integrations to the intervals $]0, \infty[$ and $]-\infty, 0[$ respectively.

Our potential $\lambda\delta(x)$ being localized at the origin, it is obvious that it does not have any effect on the calculation of the restricted propagator K^r. It is consequently the same as that relative to an infinite barrier. According to the method images [4], its expression is as follows

$$K^r\left(y,-iT\left|z\right.\right)=\sqrt{\frac{m}{2\pi\hbar T}}\left\{\exp\left[-\frac{m\left(y-z\right)^2}{2\hbar T}\right]-\exp\left[-\frac{m\left(y+z\right)^2}{2\hbar T}\right]\right\}. \tag{20}$$

In the case where "a", "b" are not on the same side of $\sum$, we have

$$\begin{aligned}K\left(b,-iT\left|a\right.\right) &= \left[\theta\left(b\right)-\theta\left(-b\right)\right]\int_0^T dt\,\frac{\hbar}{2m}\frac{\partial}{\partial x}\,K^r\left(b,-i\left(T-t\right)\left|x\right.\right)\Big|_{x=0}\\ &\quad\times K\left(0,-it\left|a\right.\right).\end{aligned} \tag{21}$$

By using the relation (20) and the expression of $K\left(0,-it\left|a\right.\right)$ which is given in Appendix A, and after certain arrangements, the propagator (21) take the form

$$\begin{aligned}K\left(b,-iT\left|a\right.\right) &= \sqrt{\frac{m}{2\pi\hbar T}}\Big\{\exp\left[-\frac{m\left(b-a\right)^2}{2\hbar T}\right]\\ &\quad+\frac{\gamma}{2}\int_0^\infty dz\exp\left(\frac{\gamma z}{2}\right)\exp\left[\frac{-m\left(|b|+|a|+z\right)^2}{2\hbar T}\right]\Big\}.\end{aligned} \tag{22}$$

For the cases where "a" and "b" are in the same side of $\sum$, and taking account the restricted part of (16), we obtain also

$$\begin{aligned}K\left(b,-iT\left|a\right.\right) &= K^r\left(b,-iT\left|a\right.\right)+\sqrt{\frac{m}{2\pi\hbar T}}\Big\{\exp\left[-\frac{m\left(|b|+|a|\right)^2}{2\hbar T}\right]\\ &\quad+\frac{\gamma}{2}\int_0^\infty dz\exp\left(\frac{\gamma z}{2}\right)\exp\left[\frac{-m\left(|b|+|a|+z\right)^2}{2\hbar T}\right]\Big\},\end{aligned} \tag{23}$$

which, after having replaced K^r (20) by its expression, we obtain

$$K\left(b,-iT\left|a\right.\right)=K_0\left(b,-iT\left|a\right.\right)+\frac{\gamma}{2}\int_0^\infty dz\exp\left(\frac{\gamma z}{2}\right)\left(K_0\left|b\right|+\left|a\right|+z,-iT\left|0\right.\right), \tag{24}$$

the same expression as that of the preceding case (22). This expression obtained by the method of path decomposition is exactly the same as that found by other methods [9] [12].

Now, let us consider to the moving potential $V(x,t)=V(x-f(t))$. By choosing the decomposition surface $\sum$ at the moving point $x=f(t)$, it is easy to show that for this form of potential the propagator admits the following decomposition (see Appendix B)

$$K\left(B,-iT\left|A\right.\right) = \exp\left[-\frac{m}{\hbar}\left(B\dot{f}_b - A\dot{f}_a\right)\right]\exp\left[-\frac{m}{\hbar}\int_0^T dt\frac{1}{2}\dot{f}^{\,2}\left(t\right)\right] \times\int_0^T dt\,\frac{\hbar}{2m}\frac{\partial}{\partial X}\,K^r\left(B,-i\left(T-t\right)\left|X\right.\right)\big|_{X=0}K\left(0,-it\left|A\right.\right), \tag{25}$$

with $B = b - f(T)$, $A = a - f(0)$, $X(t) = x(t) - f(t)$, and

$$K^r\left(B,-i\left(T-t\right)\left|X\right.\right) = \int D^rX\left(t^{'}\right)\exp\left[-\frac{1}{\hbar}\int_0^{T-t}dt^{'}mX\ddot{f}\right] \times\exp\left[-\frac{1}{\hbar}\int_0^{T-t}dt^{'}\left(\frac{1}{2}m\dot{X}^2+V\left(X\right)\right)\right], \tag{26}$$

is the restricted propagator. Thus, we see that it appears a factor of phase which is independent of the potential and for the restricted propagator an additional term $mX\ddot{f}$ in the action, also independent of the potential.

For the particular case $V = \lambda\delta(x - vt)$, where the potential is centered at the moving point $x = vt$ having a constant speed v, $f = vt$ $(\ddot{f} = 0)$. The potential is reduced to $V\left(X\right) = \lambda\delta\left(X\right)$.

In the restricted propagator, $\delta\left(X\right)$ has no effect since it is added to an infinite barrier. So, its expression is

$$K^r\left(B,-i\left(T-t\right)\left|X\right.\right) = \sqrt{\frac{m}{2\pi\hbar\left(T-t\right)}}\left\{\exp\left[-\frac{m\left(B-X\right)^2}{2\hbar\left(T-t\right)}\right] -\exp\left[-\frac{m\left(B+X\right)^2}{2\hbar\left(T-t\right)}\right]\right\}, \tag{27}$$

and by returning to the previous variables (38), the propagator (25) takes the following form,

$$K\left(b,-iT\left|a\right.\right) = \exp\left[-\frac{m}{\hbar}v\left[b-a-vT\right]\right]\exp\left[-\frac{m}{\hbar}\frac{T}{2}v^2\right] \sqrt{\frac{m}{2\pi\hbar T}}\left\{\exp\left[-\frac{m\left[b-a-vT\right]^2}{2\hbar T}\right]+\frac{\gamma}{2}\int_0^\infty dz\exp\left(\frac{\gamma z}{2}\right) \exp\left[-\frac{m\left(\left|b-vT\right|+\left|a\right|+z\right)^2}{2\hbar T}\right]\right\}. \tag{28}$$

This expression is identical to that obtained by [13] changing T into $+iT$.

In conclusion, we can note that, for the potential $V(x) = \lambda\delta(x)$, the determination of propagator require only local knowledge of the wave at the point $x = 0$ with in addition the method of images, by choosing a surface $\sum$ centered at the point $x = 0$ in the path decomposition method.

For the moving potential $V(x-f(t)$, the path decomposition method is also applicable via a simple change of variable, and the surface of decomposition $\sum$ being centred on the moving point $x=f(t)$.

The propagators for $\lambda\delta(x)$ and $\lambda\delta(x-vt)$ obtained by using the method of path decomposition are similar to those given by [9, 12, 13].

Let us mention finally that the method can be extended to the sum of several δ (or of the rectangular barrier). Calculations are in progress and the results can be found elsewhere.

Appendix A

In order to calculate $K\left(b,-iT\left|a\right.\right)$ from the path decomposition expression according to the last crossing time of $\sum$ (16) it is necessary to have $K\left(0,-iT\left|a\right.\right)$. With the same method which uses the first crossing time of $\sum$ (17), we can find $K\left(0,-iT\left|a\right.\right)$, but this requires a prior the knowledge of $K\left(0,-iT\left|0\right.\right)$ at the origin. For that, let us solve the Schrödinger' s equation

$$\left[-\frac{\partial^2}{\partial x^2}+\gamma\,\delta\left(x\right)\right]\psi_k\left(x\right)=k^2\,\psi_k\left(x\right), \tag{29}$$

where $E=\frac{\hbar^2k^2}{2m}$, $\lambda=-\frac{\hbar^2\gamma}{2m}>0$ and the boundary conditions are

$$\left\{\begin{array}{ll}\psi_k\left(0^+\right) & =\psi_k\left(0^-\right),\\ \psi_k^{'}\left(0^+\right)-\psi_k^{'}\left(0^-\right) & =-\gamma\psi_k\left(0\right).\end{array}\right.$$

For a wave propagating towards the right side

$$\psi_{\underrightarrow{k}}\left(x\right)=\left\{\begin{array}{ll}Ae^{ikx} & \text{for } x>0\\ Be^{ikx}+Ce^{-ikx} & \text{for } x<0,\end{array}\right. \tag{30}$$

we have by using the boundary conditions

$$\left\{\begin{array}{l}A=B\left(\frac{2k}{2k-i\gamma}\right),\\ C=B\left(\frac{i\gamma}{2k-i\gamma}\right).\end{array}\right.$$

The wave function take then the form

$$\psi_{\underrightarrow{k}}\left(x\right)=B\left\{\begin{array}{ll}\left(\frac{2k}{2k-i\gamma}\right)e^{ikx} & \text{for } x>0\\ e^{ikx}+\left(\frac{i\gamma}{2k-i\gamma}\right)e^{-ikx} & \text{for } x<0.\end{array}\right.$$

and with the normalization's condition of $\psi_{\underrightarrow{k}}\left(x\right)$

$$\int_{-\infty}^{+\infty}dx\psi^*_{\underrightarrow{k}}\left(x\right)\psi_{\underrightarrow{k'}}\left(x\right)=\delta(k-k'),$$

we fix $B=\dfrac{1}{\sqrt{2\pi}}$. We can obtain in the same way, the wave propagating towards the left side. At the origin $x=0$, the waves are

$$\begin{cases} \psi_{\underset{\longrightarrow}{k}}(0)=\dfrac{1}{\sqrt{2\pi}}\,\dfrac{2k}{2k-i\gamma}, \\ \psi_{\underset{\longleftarrow}{k}}(0)=\dfrac{1}{\sqrt{2\pi}}\,\dfrac{2k}{2k+i\gamma}, \end{cases} \tag{31}$$

and by using the spectral decomposition we have for the propagator, the following expression

$$K(0,-iT\,|0\,)=\int_{-\infty}^{+\infty}dk\exp\left(\frac{-\hbar T\,k^2}{2m}\right)\left[\begin{array}{l}\psi_{\underset{\longrightarrow}{k}}(0)\,\psi^{\star}_{\underset{\longrightarrow}{k}}(0)\;\theta(k)+\\ \psi_{\underset{\longleftarrow}{k}}(0)\,\psi^{\star}_{\underset{\longleftarrow}{k}}(0)\;\theta(-k)\end{array}\right]$$

$$=\frac{1}{2\pi}\left\{\int_{-\infty}^{+\infty}dk\exp\left(\frac{-\hbar T\,k^2}{2m}\right)+\int_{-\infty}^{+\infty}dk\,\frac{\gamma}{2ik-\gamma}\exp\left(\frac{-\hbar T\,k^2}{2m}\right)\right\}. \tag{32}$$

Another form of (32) can be obtained by using

$$\frac{\gamma}{2ik-\gamma}=\frac{\gamma}{2}\int_0^{\infty}dz\exp\left[-\left(ik-\frac{\gamma}{2}\right)z\right],$$

which after integration on k, can be rewritten as

$$K(0,-iT\,|0\,)=\sqrt{\frac{m}{2\pi\hbar T}}\left[1+\frac{\gamma}{2}\int_0^{\infty}dz\exp\left(\frac{\gamma z}{2}-\frac{mz^2}{2\hbar T}\right)\right]. \tag{33}$$

Now, we apply the equation (17)

$$\begin{aligned} K(0,-iT\,|a\,) &= [\theta(a)-\theta(-a)]\int_0^{T}dt\,K(0,-i(T-t)\,|0\,) \\ &\quad\times\frac{\hbar}{2m}\frac{\partial}{\partial x}\,K^{r}(x,-it\,|a\,)|\,, \end{aligned} \tag{34}$$

and by the substitution of (33) , and (20) in the above equation, we obtain

$$\begin{aligned} K(0,-iT\,|a\,) &= \frac{ma}{2\pi\hbar}[\theta(a)-\theta(-a)]\left\{\int_0^{T}dt\,\frac{\exp\left[-ma^2/2\hbar t\right]}{t^{\frac{3}{2}}\sqrt{T-t}}\right. \\ &\quad\left.\times\left[1+\frac{\gamma}{2}\int_0^{\infty}dz\exp\left[\frac{\gamma z}{2}-\frac{mz^2}{2\hbar(T-t)}\right]\right]\right\}. \end{aligned} \tag{35}$$

After a simple calculation, using the transformation of Laplace and its inverse, we finally obtain

$$\begin{aligned} K(0,-iT\,|a\,) &= \sqrt{\frac{m}{2\pi\hbar T}}\left\{\exp\left(-\frac{ma^2}{2\hbar T}\right)\right. \\ &+ \left.\frac{\gamma}{2}\int_0^{\infty}dz\exp\left(\frac{\gamma z}{2}\right)\exp\left[-\frac{m(|a|+z)^2}{2\hbar T}\right]\right\}. \end{aligned} \tag{36}$$

Appendix B

Let us show how to apply the method of path decomposition for the potentials of the form $V(x,t) = V(x - f(t))$, centered at the moving point $x = f(t)$. The discrete form of the propagator K being

$$K_{N,n} = \left(\frac{m}{2\pi\hbar\varepsilon}\right)^{\frac{N+1}{2}} \int_{-\infty}^{+\infty} dx_1 \cdots dx_{n-1} \int_{-\infty}^{x_\sigma} dx_n \int_{x_\sigma}^{+\infty} dx_{n+1} \cdots dx_N$$

$$\times \exp\left[-\frac{1}{\hbar}\sum_{j=0}^{n-1}\left[\frac{m\left(x_{j+1}-x_j\right)^2}{2\varepsilon} + V\left(x_j - f_j\right)\varepsilon\right]\right]$$
$$\times \exp\left[-\frac{1}{\hbar}\left(\frac{m\left(x_{n+1}-x_n\right)^2}{2\varepsilon} + V\left(x_n - f_n\right)\varepsilon\right)\right]$$
$$\times \exp\left[-\frac{1}{\hbar}\sum_{j=n+1}^{N}\left[\frac{m\left(x_{j+1}-x_j\right)^2}{2\varepsilon} + V\left(x_j - f_j\right)\varepsilon\right]\right], \tag{37}$$

where $f_j = f\left(j\varepsilon\right)$ and $x_\sigma = f\left(n\varepsilon + \tau\right)$, let us carry out the following change: $x \longrightarrow X = x - f$

$$\begin{cases} x_j \longrightarrow X_j = x_j - f_j \quad \text{and} \quad x_\sigma \longrightarrow X_\sigma = x_\sigma - f_\sigma = 0 \\ Dx \Leftrightarrow DX \end{cases} . \tag{38}$$

Then

$$K_{N,n} = \left(\frac{m}{2\pi\hbar\varepsilon}\right)^{\frac{N+1}{2}} \int_{-\infty}^{+\infty} dX_1 \cdots dX_{n-1} \int_{-\infty}^{X_\sigma} dX_n \int_{X_\sigma}^{+\infty} dX_{n+1} \cdots dX_N$$
$$\times \exp\left[-\frac{1}{\hbar}\sum_{j=0}^{N}\frac{m(f_{j+1}-f_j)^2}{2\varepsilon}\right] \exp\left[-\frac{1}{\hbar}\sum_{j=0}^{n-1}\left(\frac{m(X_{j+1}-X_j)^2}{2\varepsilon} + V\left(X_j\right)\varepsilon\right)\right]$$
$$\times \exp\left[-\frac{m}{\hbar}X_j\frac{(f_{j+1}-f_j)}{\varepsilon}\right]\Big|_{j=0}^{j=n} \exp\left[\frac{1}{\hbar}\sum_{j=0}^{n-1}\frac{mX_{j+1}(f_{j+2}-2f_{j+1}+f_j)}{\varepsilon}\right]$$
$$\times \exp\left[-\frac{1}{\hbar}\left(\frac{m(X_{n+1}-X_n)^2}{2\varepsilon} + V\left(X_n\right)\varepsilon\right)\right]$$
$$\times \exp\left[-m\frac{X_{n+1}(f_{n+2}-f_{n+1})-X_n(f_{n+1}-f_n)-[X_{n+1}(f_{n+2}-2f_{n+1}+f_n)]}{\varepsilon\hbar}\right]$$
$$\times \exp\left[-\frac{m}{\hbar}X_j\frac{(f_{j+1}-f_j)}{\varepsilon}\right]\Big|_{j=n+1}^{j=N+1} \exp\left[\frac{1}{\hbar}\sum_{j=n+1}^{N}\frac{mX_{j+1}(f_{j+2}-2f_{j+1}+f_j)}{\varepsilon}\right]$$
$$\times \exp\left[-\frac{1}{\hbar}\sum_{j=n+1}^{N}\left(\frac{m(X_{j+1}-X_j)^2}{2\varepsilon} + V\left(X_j\right)\varepsilon\right)\right]. \tag{39}$$

This expression according to (10), (11) , can be rewritten as following

$$K_{N,n} = \exp\left[-\frac{m}{\hbar}X_j\tfrac{(f_{j+1}-f_j)}{\varepsilon}\right]\Big|_{j=0}^{j=N+1}\exp\left[-\frac{1}{\hbar}\sum_{j=0}^{N}\tfrac{m(f_{j+1}-f_j)^2}{2\varepsilon}\right]$$

$$\begin{aligned}
&\times\left(\frac{m}{2\pi\hbar\varepsilon}\right)^{\frac{N}{2}}\int_0^\varepsilon d\tau\int_{-\infty}^{+\infty}dX_1\cdots dX_{n-1}\int_{-\infty}^{X_\sigma}dX_n\int_{X_\sigma}^{+\infty}dX_{n+1}\cdots dX_N\\
&\times\exp\left[\frac{1}{\hbar}\sum_{j=0}^{N}\tfrac{mX_{j+1}(f_{j+2}-2f_{j+1}+f_j)}{\varepsilon}\right]\exp\left[-\frac{1}{\hbar}\sum_{j=0}^{n-1}\left(\tfrac{m(X_{j+1}-X_j)^2}{2\varepsilon}+V\left(X_j\right)\varepsilon\right)\right]\\
&\times\left(\frac{m}{2\pi\hbar\tau}\right)^{\frac{1}{2}}\exp\left[-\frac{1}{\hbar}\left(\tfrac{m(X_\sigma-X_n)^2}{2\tau}+V\left(X_j\right)\tau\right)\right]\left(\frac{m}{2\pi\hbar\left(\varepsilon-\tau\right)}\right)^{\frac{1}{2}}\\
&\times\frac{\hbar}{m}\frac{\partial}{\partial X}\exp\left[-\frac{1}{\hbar}\left(m\tfrac{(X_{n+1}-X)^2}{2(\varepsilon-\tau)}+V\left(X\right)\left(\varepsilon-\tau\right)\right)\right]\Big|_{X=0}\\
&\times\exp\left[-\frac{1}{\hbar}\sum_{j=n+1}^{N}\left(m\tfrac{(X_{j+1}-X_j)^2}{2\varepsilon}+V\left(X_j\right)\varepsilon\right)\right]. \qquad (40)
\end{aligned}$$

The factorized form of $K_{N,n}$ is then

$$\begin{aligned}
K_{N,n}\left(X_T,-iT\left|X_0\right.\right) &= \exp\left[-\frac{m}{\hbar}X_j\tfrac{(f_{j+1}-f_j)}{\varepsilon}\right]\Big|_{j=0}^{j=N+1}\exp\left[-\frac{1}{\hbar}\sum_{j=0}^{N}\tfrac{m(f_{j+1}-f_j)^2}{2\varepsilon}\right]\\
&\quad\times\int_0^\varepsilon d\tau\,\frac{\hbar}{2m}\frac{\partial}{\partial X}\,K^r_{N-n}\left(X_T,-i\left(T-n\varepsilon-\tau\right)\left|X\right.\right)\big|_{X=0}\\
&\quad\times K_n\left(0,-i\left(n\varepsilon+\tau\right)\left|X_0\right.\right), \qquad (41)
\end{aligned}$$

and at the limit $N\to\infty$, the path decomposition expression of the propagator is

$$\begin{aligned}
K\left(B,-iT\left|A\right.\right) &= \exp\left[-\frac{m}{\hbar}\left(B\dot{f}_b-A\dot{f}_a\right)\right]\exp\left[-\frac{m}{\hbar}\int_0^T dt\frac{1}{2}\dot{f}^2\left(t\right)\right] \qquad (42)\\
&\quad\times\int_0^T dt\,\frac{\hbar}{2m}\frac{\partial}{\partial X}\,K^r\left(B,-i\left(T-t\right)\left|X\right.\right)\big|_{X=0}K\left(0,-it\left|A\right.\right),
\end{aligned}$$

where

$$\begin{aligned}
K^r\left(B,-iT\left|X\right.\right) &= \int D^rX\left(t\right)\exp\left[-\frac{1}{\hbar}\int_0^T dt\left(\frac{1}{2}m\dot{X}^2+V\left(X\right)+mX\ddot{f}\right)\right]\\
\text{with }B &= b-f(T);\quad A=a-f(0);\quad X(t)=x(t)-f(t) \qquad (43)
\end{aligned}$$

is the restricted propagator.

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