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NEW CONSTRUCTION OF EXOTIC PARTICLES ALGEBRA AND NON-COMMUTATIVE GEOMETRY

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This letter establishes a procedure which determines an algebra of exotic particles by using a non-commuting coordinates. These particles obey fractional statistics and live in two-dimensional space.

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1 Introduction

The goal of this work is mainly to obtain an algebra describing anyons. These particles are also known as excitations, quasi-particles or exotic particles; i.e. fermions (bosons) carrying odd (even) number of elementary magnetic flux quanta [1–12]. They are living in two-dimensional space as composite particles having arbitrary spin, and they are characterized by fractional statistics which is interpolating between bosonic statistics and fermionic one. Experimentally, it was proved in fractional quantum Hall effect that electrons don't act as a gas like they do in normal metals. They have condensed to form a new type of quantum fluid; each electron combines with a quantum unit of magnetic flux. This was the proposition of Robert Laughlin [13–15]. The unique property of his quantum fluid is that if one electron is added the fluid is excited creating quasi-particles that have a fraction of the charge of an electron. These are not normal particles, but just the coordinated motion of several electrons in the fluid called anyons. Several works were done to find out the quantum theory of this kind of particles and its right model is not yet reached.

The construction of the exotic particles algebra we give in this letter is different from the one introduced by Lerda and Sciuto in the ref. [16]. As result, the obtained symmetry is interpolating between bosonic and deformed fermionic algebras. The procedure is based on considering a non-commutative geometry depending on the statistical parameter characterizing anyonic system. First we review the non-commutative geometry as known in the literature which is one of the approaches to describe the non-commutative gauge theories which are the topic of the most recent interest [17–25]. They are described through two ways: by considering the standard commutative gauge action and re-interpreting any product of arbitrary fields in terms of the Moyal

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product, and by re-interpreting the fields as operators in the Hilbert space which provides for a representation of the fundamental algebra that defines the non-commutative geometry [26–31]. Then we suggest that the commutation relations of coordinates should be depending on statistical parameters ν characterizing exotic particles. The obtained non-commutative geometry is used to construct an annihilation and creation operators generating the exotic particles algebra. An interesting remark is that the latter symmetry has two extremes the bosonic algebra for $\nu = 0$ and deformed fermionic algebra for $\nu = 1$ which means that the discussed planar system is gotten from bosons as origin and does not have any thing to do with fermions.

The letter is organized as follows: In section 2, we briefly review the origin of anyons and its statistics. The section 3 will be devoted to construct an exotic particles algebra basing on the non-commuting coordinates which generate the algebra underlying the non-commutative geometry depending on the statistical parameter. In section 4, we conclude.

2 Exotic Particles

Let us start by recalling the origin of anyons [32–39]. Firstly, to see how come anyons as a theory, the configuration space M_N^d of N identical particles in d-dimensional space $(\Re^d)^N$ is given as follows

$$M_N^d = \frac{(\Re^d)^N - \Delta}{S_N}$$

by removing the diagonal Δ defined by the set

$$\Delta = \{(x_1, ..., x_N) \in (\Re^d)^N / x_i = x_j\}$$

such that $x_i = x_j$ for at least one pair. Here we can imagine that there is a hard core interaction between particles keeping them apart. Then we identify the elements of the configuration space $(x_1, ..., x_N)$ and $(x_{\pi(1)}, ..., x_{\pi(N)})$, for any element π of the symmetry group S_N since particles are identical.

In the special case two-dimensional space with two identical particles, the configuration space M_2^2

$$M_2^2 = \Re^2 \times \{\text{cone without the tip}\}$$

is infinitely connected. It is constructed by replacing the coordinates x_1 and x_2 by the center of mass coordinate $X = \frac{x_1+x_2}{2}$ and the relative coordinate $x = x_1 - x_2$ and by removing the diagonal $x_1 = x_2$ means leaving out the origin of the x-plane and "modding" by S_2 means identifying x and $-x$. Then, the resulting construction is the surface of a cone with the tip $x = 0$ excluded. Consequently, any closed loop on the mantle of the cone encircling the tip can not be shrunk to a point. Thus M_2^2 is multiply connected.

Secondly, the connection to statistics comes through recognizing that the class of closed loops σ_i corresponds to an interchange of particles i and $i + 1$ (Figure 1). In $d \geq 3$, these loops can be deformed into each other; e.g. by rotating the loop around a diameter of a sphere, then $\sigma_i = \sigma_i^{-1}$. In $d = 2$, this can be done in two homotopically inequivalent ways which can be represented by the loop C_iC_{i+1} where the two particles move either counter clockwise (corresponding to σ_i) or clockwise (corresponding to σ_i^{-1}) interchanging their places, so $\sigma_i \neq \sigma_i^{-1}$ and they are elements of the braid group B_N. The latter condition is the difference between B_N and S_N.

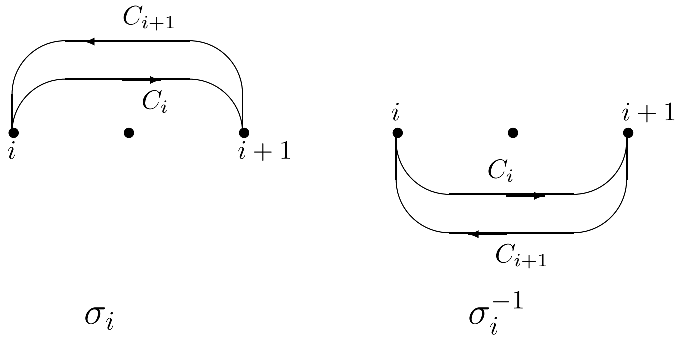

Figure 1: The interchange of two particles i and $i+1$ along the closed loop C_iC_{i+1}.

Now to look for a unitary one-dimensional representations of B_N, we pose $\chi(\sigma_i) = e^{i\phi_i}$ and we have the following constraint

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$$

from the Feynman propagator

$$K = \sum_{\alpha \in B_N} \chi(\alpha) K^\alpha$$

requiring $\chi(\sigma_i)\chi(\sigma_j) = \chi(\sigma_i\sigma_j)$. The representations $\chi(\alpha)$ are the weights of different classes α and the sum runs over all classes. K^α denotes the integral over all paths in the class α. The constraint $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ requires that $\phi_i = \phi_{i+1}$. Thus it is customary to write

$$\chi(\sigma_i) = e^{-i\nu\pi}, \quad \chi(\sigma_i^{-1}) = e^{i\nu\pi}, \qquad \nu \in [0,2) \tag{1}$$

with ν is called statistical parameter. If $\nu = 0$, the particles are bosons that obey Bose-Einstein statistics and if $\nu = 1$ the particles are fermions with Fermi-Dirac statistics.

After reviewing in brief the planar system and its statistics, we give in what follows its associated symmetry interpolating between bosonic and fermionic symmetries.

3 Exotic Particles Algebra

First, we briefly recall the non-commutative geometry. Its most simple example consists of the geometric space described by non-commutative hermitian operator coordinates x_i, and by considering the non-commutative momentum operators $p_i = i\partial_{x_i}$ (∂_{x_i} the corresponding derivative of x_i). These operators satisfy the following algebra

$$\begin{gathered}[x_i, x_j] = i\theta\epsilon_{ij}, \quad [p_i, p_j] = -i\theta^{-1}\epsilon_{ij}, \quad [p_i, x_j] = -i\delta_{ij} \\ [p_i, t] = 0 = [x_i, t], \quad [p_i, \partial_t] = 0 = [x_i, \partial_t],\end{gathered} \tag{2}$$

with t the physical time and ∂_t its corresponding derivative.

We consider two-dimensional harmonic oscillator which can be decomposed into one-dimensional oscillators. It is known that the algebra (2) allows to define, for each dimension, the representation of annihilation and creation operators as follows

$$\begin{aligned} a_i &= \sqrt{\tfrac{\mu\omega}{2}}(x_i + \tfrac{i}{\mu\omega}p_i) \\ a_i^\dagger &= \sqrt{\tfrac{\mu\omega}{2}}(x_i - \tfrac{i}{\mu\omega}p_i). \end{aligned} \tag{3}$$

with μ is the mass and ω the frequency. These operators satisfy

$$[a_i, a_i^\dagger] = 1,$$

defining the Heisenberg algebra. In the simultaneously non-commutative space-space and non-commutative momentum-momentum, the bosonic statistics should be maintained; i.e, the operators $a_i^\dagger$ and $a_j^\dagger$ are commuting for $i \neq j$. Thus, the deformation parameter θ is required to satisfy the condition

$$\theta = -(\frac{1}{\mu\omega})^2\theta^{-1}.$$

To find out an algebra describing the planar system we start by introducing the non-commutative geometry depending on the statistical parameter $\nu \in \mathbf{R}$. The fundamental algebra is defined by the coordinates x_i and the momentum p_i satisfying

Proposition 1

$$\begin{gathered}[x_i, x_j]_\chi = i\theta\epsilon_{ij}, \quad [p_i, p_j]_\chi = -i\theta(\mu\omega)^2\epsilon_{ij}, \quad [p_i, x_j] = -i\delta_{ij} \\ [p_i, t] = 0 = [x_i, t], \quad [p_i, \partial_t] = 0 = [x_i, \partial_t]\end{gathered} \tag{4}$$

and by straightforward calculations we obtain

$$[x_i, p_j]_\chi = i\delta_{ij} + C_{ji}, \quad [p_i, x_j]_\chi = -i\delta_{ij} + D_{ji}, \tag{5}$$

where $C_{ji} = (1-\chi)p_j x_i$ and $D_{ji} = (1-\chi)x_j p_i$ and the second deformation parameter χ is given by

Definition 1

$$\chi = e^{\pm i\nu\pi}, \tag{6}$$

where ± sign indicates the two rotation directions on two-dimensional space. θ in (4) is a non-commutative parameter depending on statistical parameter ν as we will see later and the notation $[x, y]_q = xy - qyx$.

Then, we introduce an operator ξ_i acting on the momentum direction in the phase-space. We assume that ξ_i satisfies the following commutation relation

Proposition 2

$$[\xi_i, x_j] = 0 \quad \forall i, j. \tag{7}$$

In this case, we define the annihilation and the creation operators by

Definition 2

$$\begin{aligned} b_i^- &= \sqrt{\tfrac{\mu\omega}{2}}(x_i + \tfrac{i}{\mu\omega}\xi_i p_i) \\ b_i^+ &= \sqrt{\tfrac{\mu\omega}{2}}(x_i - \tfrac{i}{\mu\omega}\xi_i^{-1} p_i), \end{aligned} \tag{8}$$

with ξ_i is defined in terms of statistical parameter ν and an operator K_i which could be a function of the number operator N

Definition 3

$$\xi_i = e^{i\nu\pi K_i}, \tag{9}$$

with K_i an arbitrary operator.

The non-commutative geometry defined by (4-5) leads to a deformed Heisenberg algebra generated by the operators (8) and defined by the following commutation relations

$$\begin{aligned} [b_i^-, b_j^+]_\chi &= \tfrac{1}{2}(\xi_i + \xi_j^{-1})\delta_{ij} + i\tfrac{\mu\omega}{2}\theta(I + \xi_i\xi_j^{-1})\epsilon_{ij} - \tfrac{i}{2}(\xi_j^{-1}C_{ji} - \xi_i D_{ji}), \\ [b_i^+, b_j^+]_\chi &= \tfrac{1}{2}(\xi_j^{-1} - \xi_i^{-1})\delta_{ij} + i\tfrac{\mu\omega}{2}\theta(I - \xi_i^{-1}\xi_j^{-1})\epsilon_{ij} - \tfrac{i}{2}(\xi_j^{-1}C_{ji} + \xi_i^{-1}D_{ji}), \\ [b_i^-, b_j^-]_\chi &= \tfrac{1}{2}(\xi_i - \xi_j)\delta_{ij} + i\tfrac{\mu\omega}{2}\theta(I - \xi_i\xi_j)\epsilon_{ij} + \tfrac{i}{2}(\xi_j C_{ji} + \xi_i D_{ji}), \end{aligned} \tag{10}$$

with I is the identity.

To be consistent with the hermiticity of the operators x_i and p_i the non-commutative geometry (4) leads to the fact that θ is an operator satisfying

$$\theta^\dagger = \chi^{-1}\theta$$

and we suggest the following definition

Definition 4

$$\theta = \nu(1+\chi)I \tag{11}$$

We remark that the algebra (10) is a deformed version of Heisenberg algebra satisfied by the operators given in (3). This new algebra describes the anyonic system for arbitrary statistical parameter ν.

Another important point is that the obtained deformed Heisenberg algebra (10) is interpolating between two extremes depending on the statistical parameter ν. We know that, in three or more dimensions, ν takes the values 0 or 1 and in two dimensions ν is arbitrary real number. The latter case characterizing exotic particles has already discussed above. In the case of three or more dimensions if $\nu = 0$ we get $\chi = 1$, $\xi_i = \xi_i^{-1} = I$, $\theta = 0$ and $C_{ji} = 0 = D_{ji}$, so the commutation relations in the algebra (10) becomes

$$[b_i^-, b_j^+] = \delta_{ij}, \quad [b_i^+, b_j^+] = 0, \quad [b_i^-, b_j^-] = 0. \tag{12}$$

These relations define the bosonic algebra and this is one extreme. Then the next interesting remark is that the algebra (10) will not have fermionic algebra as extreme for $\nu = 1$, $\chi = -1$, $\theta = 0$ and $C_{ji} \neq 0 \neq D_{ji}$ but we get a deformed fermionic algebra as second extreme defined by

$$\begin{aligned}
\{b_i^-, b_j^+\} &= \tfrac{1}{2}(e^{i\pi K_i} + e^{-i\pi K_j})\delta_{ij} - \tfrac{i}{2}(e^{-i\pi K_j}C_{ji} - e^{i\pi K_i}D_{ji}),\\
\{b_i^+, b_j^+\} &= \tfrac{1}{2}(e^{-i\pi K_j} - e^{-i\pi K_i})\delta_{ij} - \tfrac{i}{2}(e^{-i\pi K_j}C_{ji} + e^{-i\pi K_i}D_{ji}),\\
\{b_i^-, b_j^-\} &= \tfrac{1}{2}(e^{i\pi K_i} - e^{i\pi K_j})\delta_{ij} + \tfrac{i}{2}(e^{i\pi K_j}C_{ji} + e^{i\pi K_i}D_{ji}).
\end{aligned} \tag{13}$$

The main result we get from this investigation is that from exotic particles algebra we get the bosonic algebra as extreme this means that our system is originally gotten by exciting a bosonic system in two-dimensional space. On other hand, we remark that for arbitrary operator K_i if $\nu = 1$ we get a deformed fermionic algebra as a second extreme. It is known in the literature that anyons are interpolating between bosons and fermions and the statistical parameter ν equals to 1 describes fermionic system, but in our case, it is different and our system doesn't have any thing to do with fermions originally but it could be related to something else as deformed fermions which are known in the literature as quionic particles or k_i-fermions, k_i integer number introduced as a deformation parameter, and these kinds of particles are not physical particles. Consequently, in physics wise, our system has just one extreme which is a bosonic system with the statistical parameter $\nu = 0$.

4 Conclusion

In this paper, we started by giving a short review on exotic particles and then we assumed that the fundamental algebra satisfied by the non-commuting coordinates is depending on the statistical parameter ν characterizing the quasi-particles. Basing on this proposed algebra we defined an annihilation and creation operators in two-dimensional space. Then, by a straightforward calculation we found that the new operators are generators of a deformed Heisenberg algebra describing exotic particles. For arbitrary ξ_i which is introduced to define the "exotic" annihilation and creation operators the two extremes of the exotic particles algebra (10) are the bosonic algebra and the deformed fermionic one. We also get the same result if ξ_i is unitary and K_i is hermitian as a special case in which the operator b_i^+ becomes a complex conjugate of b_i^-.

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