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QUATERNION FORMULATION OF THE GALILEAN SPACE-TIME TRANSFORMATION[1]

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After a brief review of the different types of quaternions, we express the special Galilean space-time transformation in an algebraic ring of the exotic four-component numbers forming the system of dual quaternions.

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1 Introduction

Mathematically, quaternions represent the natural extension of complex numbers, forming an associative algebra under addition and multiplication [1]. Quaternions were invented by Sir William Rowan Hamilton [2] who immediately tried to use them in physics [2]. The attempts at formulation of physical laws, by means of quaternions and octonions, also have a deep mathematical meaning in the generalized Frobenius theorem [3]. This theorem asserts that the four number systems called real numbers, complex numbers, quaternions and octonions, have an *exceptional position* within the algebras, because every real alternative algebra with division is isomorphic to one of these number systems. Therefore, use of the original, real (Hamiltonian) quaternions in physics has a long history dating back to Hamilton (1843) and Maxwell. As is well-known already, Maxwell used real quaternions (1867) for the formulation of his equations in his celebrated book " Treatise on Electricity and Magnetism". However, he has used quaternions in such a way that they only substituted for the common vector calculus. This made his field equations difficult for his contemporaries (see, e.g. [4]) because the quaternionic formulation in 3-space brings several complications. The particular field of applicability of *real* quaternions is Euclidean 4-space. Therefore, the turning point, in using quaternions in theoretical physics, was the creation of special relativity which units space and time, forming a 4-dimensional space-time. Since a quaternion has four components, all of the components of a 4-vector can be included in it. The best formalism should include 4-space and 5-space. For 4-space, one component must be a purely imaginary number, so it is *no longer* a quaternion. Such pseudo-quaternions are

[1] *Dedicated to Prof. D. Ilkovič, founder of Institute of Physics of Slovak Academy of Sciences, on the occasion of the 25th anniversary of his death*

[2] Gauss invented them in the early 1800's but did not publish

called Minkowski quaternions [5]. The difficulty with Minkowski quaternions is that they do not form an algebraic ring, this means that the product of two Minkowski quaternions is not always a Minkowski quaternion. The formulation of physical laws, using real quaternions, has been replaced by complex ones, and it has been recognized that the ring complex quaternions represents a powerful instrument in formulating classical physical laws (see, e.g. [6]). Complex quaternions are rings having a number of desirable properties which allow the powerful theorems of modern algebra to be applied, and this leads to a certain elegance of formulation of the physical laws.

An important extension of Hamiltonian quaternions represents the so-called binary (hyperbolic) quaternions and the dual quaternions. The different types of quaternions are suitable algebraic instruments for expressing important space-time transformations as well as description of the classical and quantum fields [6]. As has been shown in [7] the general Lorenz space-time transformation can be expressed in terms of binary quaternions. In our further consideration we introduce the dual quaternions which form an algebraic ring possible suitable for expressing the Galilean transformation. In terms of *dual quaternions* this transformation gets an elegant, economical and compact form explicitly showing its underlying algebraic properties.

The organization of the article is as following. In Part 1, we briefly review the algebraic properties of the Hamiltonian, binary and dual quaternions. In Part 2, we express the special Galilean transformation in terms of dual quaternions.

2 Quaternionic algebra

A real (or Hamiltonian) quaternion can be written in the form [1]

$$Q = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3$$

with the following multiplication schema for the quaternion units

$$e_i e_j = -\delta_{ij} + \epsilon_{ijk} e_k \qquad i, j, k = 1, 2, 3,$$

where ϵ_{ijk} are components of the totally antisymetric tensor.

We note that the quaternion units e_1, e_2, e_3 and the Pauli matrices $(\sigma_1, \sigma_2, \sigma_3)$ are almost algebraically isomorphic, i.e., $i \equiv -i\sigma_1 \quad j \equiv -i\sigma_2 \quad k \equiv -i\sigma_3$ [8]. With these rules the full product of two quaternions $Q = a + be_1 + ce_2 + de_3$ and $Q' = a^{'} + b^{'} e_1 + c^{'} e_2 + d^{'} e_3$ has the form

$$QQ' = (aa^{'} - bb^{'} - cc^{'} - dd^{'}) + (ab^{'} + ba^{'} + cd^{'} - dc^{'})i$$
$$+(ac^{'} + ca^{'} + db^{'} - bd^{'})j + (ad^{'} + da^{'} + bc^{'} - cb^{'})k.$$

An Hermitian conjugate quaternion assigned to Q is defined as

$$Q^* = a - be_1 - ce_2 - de_3$$

and the norm of a quaternion is

$$N(Q) = QQ^* = Q^*Q = a^2 + b^2 + c^2 + d^2.$$

For two quaternions, the law of moduli holds: The norm of products is product of the norm.

$$N(QQ') = N(Q)N(Q').$$

We note that a quaternion can also be written in the form

$$Q = (\vec{a}, \alpha),$$

where $\vec{a} = a_1 e_1 + a_2 e_2 + a_3 e_3$ is the vector part of this quaternion and α is the scalar part of it. The sum and product of two quaternions $A = (\vec{a}, \alpha)$ and $B = (\vec{b}, \beta)$, in this notation are

$$(\vec{a}, \alpha) + (\vec{b}, \beta) = (\vec{a} + \vec{b}, \alpha + \beta)$$

and

$$(\vec{a}, \alpha)(\vec{b}, \beta) = (\vec{a} \times \vec{b} + \alpha \vec{b} + \beta \vec{a}, \alpha\beta - \vec{a}.\vec{b}),$$

respectively. The multiplication and ratio of two quaternions Q and Q' is again a quaternion Q'' so the set of real (Hamiltonian) quaternions form a division algebra under addition and multiplication.

The corresponding four-component numbers, which represent the four-dimensional extension of two-component binary numbers, are *binary (hyperbolic) quaternions*. A binary quaternion is mathematical quantity which can be written in the form

$$Q_b = q_0 + q_1 e_1^{'} + q_2 e_2^{'} + q_3 e_3^{'}.$$

The quaternion units of binary quaternions $e_1^{'}, e_2^{'}, e_3^{'}$ obey the scheme

$$e_i^{'} e_j^{'} = \delta_{ij} + \epsilon_{ijk} e_k^{'} \qquad i, j, k = 1, 2, 3,$$

where ϵ_{ijk} are components of the totally antisymmetrical tensor. By means of binary quaternions, the Lorenz transformation can be formulated in a compact form exhibiting, explicitly, its algebraic properties.

In what follows, we introduce new type of quaternions, the so-called *dual quaternions*. A dual quaternion can be written in the form [1]

$$Q_d = a + bi + cj + dk$$

with the following multiplication schema for the quaternion units

$$ii = jj = kk = 0, \qquad ij = ji = ki = ik = jk = kj = 0.$$

The interesting property of dual quaternions is that by means of them one can express the Galilean transformation in *one* quaternionic equation. The norm N of a dual quaternion

$$Q_d = a + bi + cj + dk$$

is

$$N(Q_d) = Q_d Q_d^* = a^2,$$

where $Q_d^* = q_0 - q_1 i - q_2 j - q_3 k$ is an Hermitian conjugate dual quaternion assigned to Q_d. Any dual quaternion, $Q = a + bi + cj + dk$, can be expressed in the form (see Appendix)

$$Q = a + bi + cj + dj = a \exp\left(b^* i + c^* j + d^* k\right).$$

More about the algebraic properties of dual quaternions we present in Appendix. The real, binary and dual quaternions, introduced in this section, represent simple systems of four-dimensional hypercomplex numbers. By means of them, many important equations of mathematical physics can be expressed in a compact and explicitly covariant form, e.g., the classical Maxwell-like field equations, the equation of quantum physics and the general space-time transformations [9].

3 The Galilean space-time transformation and dual quaternions

Dual quaternions represent the natural number system for the compact expression of the special Galilean transformation. Let $\vec{r} = (x, y, z)$, $\vec{r}' = (x', y', z')$, $\vec{v} = (v_1, v_2, v_3)$, where $\vec{r}$ and $\vec{r}'$ are the position vectors in the inertial systems moving with the velocity $\vec{v}$ relative to each other. To express the Galilean space-time transformation in the form of dual quaternions, we define the coordinate quaternions

$$\mathbf{X} = ct + xi + yj + zk \qquad \text{and} \qquad \mathbf{X}' = \mathrm{ct}' + \mathrm{x}'\mathrm{i} + \mathrm{y}'\mathrm{j} + \mathrm{z}'\mathrm{k},$$

where i, j, k obey the exotic multiplication rules of dual quaternion units. The special Galilean space-time transformation can be expressed in the form

$$\mathbf{X}' = \exp(-\Phi_1 i - \Phi_2 j - \Phi_3 k)\mathbf{X}$$

or, explicitly,

$$ct' + x'j + y'j + z'k = (1 - \Phi_1 i - \Phi_2 i - \Phi_3 k)(ct + xi + yj + zk) =$$
$$ct + (x - \Phi_1 ct)i + (y - \Phi_2 ct)j + (z - \Phi_3 ct)k. \qquad (1)$$

Separating the coefficients assigned to individual quaternion units in Eq. (1), we get

$$ct' = ct, \qquad x' = x - \Phi_1 ct, \qquad y' = y - \Phi_2 ct, \qquad z' = z - \Phi_3 ct. \qquad (2)$$

If we set in Eq. 2 $\Phi = v_1/c$, $\Phi_2 = v_2/c$ and $\Phi_3 = v_3/c$ we obtain the familiar form of the Galilean transformation

$$x' = x - v_1 t, \qquad y' = y - v_2 t, \qquad z' = z - v_3 t, \qquad t' = t.$$

The norms of the coordinate quaternions X and X' remains under a Galilean transformation unchanged, i.e. $\mathbf{X}\mathbf{X}^* = \mathbf{X}'\mathbf{X}'^* = c^2t^2$. This is in agreement with the assumption made in Newtonian physics, requiring the existence of absolute time. Eq. (1) can be rewritten into the form

$$\mathbf{X}' = G(\Phi)\mathbf{X},$$

where $\Phi = -\Phi_1 i - \Phi_2 j - \Phi_3 k$. $G(\Phi)$ stands for the operator of Galilean transformation. The successive application of two Galilean transformations, described by operators $G(\Phi^{(1)})$ and $G(\Phi^{(2)})$,

$$\mathbf{X}' = G(\Phi^{(1)})\mathbf{X}, \qquad \mathbf{X}'' = G(\Phi^{(2)})\mathbf{X}'$$

is equivalent to the application of a single operator

$$\mathbf{X}'' = G(\Phi^{(3)})\mathbf{X},$$

where $G(\Phi^{(3)}) = G(\Phi^{(1)})G(\Phi^{(2)}) = G(\Phi^{(2)})G(\Phi^{(1)}) = G(\Phi_1 + \Phi_2)$. This implies the addition theorem for the velocity in a Galilean transformation.

Eq. (1) resembles the formula for the rotation of a complex vector, $Z = x + iy$, $Z' = R(\phi)Z$, where $R(\phi) = \exp i\phi$ represents the operator of rotation and ϕ is the angle of rotation. As is well-known the set of operators $R(\phi)$ forms a group. Similarly, the set of Galilean operators $G(\Phi)$ forms a group with the continuous parameter v.

The formulation of Galilean transformation by means of *dual quaternions* has not only a certain elegance and aesthetic appeal, but it also shows that the linkage between space and time exists also in the Newtonian physics. Moreover, it may have a considerable heuristic value for the study of the underlying mathematical formalism of the general space-time transformations [10] [11].

4 Appendix

Some important algebraic properties of the dual quaternions.
(i) Any dual quaternion, $Q = a + bi + cj + dk$, can be expressed in the form

$$Q = a \exp{(b^* i + c^* j + d^* k)},$$

where $b^* = b/a, c^* = c/a$ and $d^* = d/a$. This can be shown taking into account the multiplication scheme for the dual quaternion units according to which it holds $i^n = j^n = k^n = 0$ for $n > 1$, therefore, the expressions of the form $(ai + bj + ck)^n$ for $n > 1$ also become equal to zero. So, we can expand $\exp{(ai + bj + ck)}$ into a series and get

$$\exp{(a^* i + b^* j + c^* k)} = 1 + (a^* i + b * j + c^* k) + \frac{(a^* i + b^* j + c^* k)^2}{2!} + ...$$

$$= 1 + (a^* i + b^* j + c^* k) + 0.$$

Due to this series expansion of $exp(a^* i + b^* j + c^* k)$, we have immediately

$$a \exp{(a^* i + b^* j + v^* k)} = a + bi + cj + dk.$$

(ii) Using the exponential form of dual quaternions one can easily prove that they obey the commutative and associative laws.
The associative law can also be verified by direct calculation. Take

$$Q = a + bi + cj + dk \qquad Q' = a' + b'i + c'j + d'k$$

$$Q'' = a'' + b''i + c''j + d''k$$

then it holds

$$(QQ')Q'' = aa'a'' + [a''(a' + ab') + aa'b'']i$$
$$+[a''(a'c + ac') + aa'c'']j + [a''(a'd + ad') + aa'd'']k$$

and

$$Q(Q'Q'') = aa'a'' + [a(b'a'' + a'b'') + ba'a'']i$$
$$+[a(c'a'' + a'c'') + ca'a'']j + [a(d'a'' + a'd'') + da'a'']k.$$

We see that in both cases the coefficients assigned to the individual quaternionic units are equal.

(iii) The ratio of two dual quaternions $Q = a + bi + cj + dk$ and $Q' = a' + b'i + c'j + d'k$ is again a dual quaternion Q''

$$Q'' = \frac{Q}{Q'} = \frac{a}{a'} \exp\left((b^* - b'^*)i + (c^* - c'^*)j + (d^* - d'^*)k\right) \qquad a' \neq 0.$$

(iv) Since the multiplication and ratio of two dual quaternions Q and Q' is again a dual quaternion Q'' the set of dual quaternions forms a division algebra under addition and multiplication.
(v) The multiplication scheme for the corresponding quaternion units of all three types of quaternions can be written by *one* formula

$$e_i e_j = a\delta_{ij} + b\epsilon_{ijk} e_k,$$

where ϵ_{ijk} represent components of the totally antisymmetric tensor. If $a = -1, b = 1$, $a = 1, b = 1$ and $a = 0, b = 0$ we get the multiplication rules for the quaternion units of the Hamiltonian, binary and dual quaternions, respectively.

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