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ON GRAVITATIONAL LENSING BY QUADRUPOLE POTENTIALS

D. Palle [1]
Zavod za teorijsku fiziku, Institut Rugjer Bošković
Pošt. Pret. 180, HR-10002 Zagreb, Croatia

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We study gravitational lensing by quadrupole potentials within the linearized gravity approximation and the integration over the unperturbed photon trajectory. It is well known that the quadrupole potential contribution to the deviation angle is much smaller than that of the monopole one. We show that quadrupole potentials can change the photon polarization vector, but there is no contribution from the monopole term to the first order. The effect is maximal when the axis of the quadrupole is tangential to the photon trajectory and it is proportional to the frequency (rate) of the quadrupole moment of the deflector. The second order correction of the monopole potential to the polarization is canceled away by the renormalization of the polarization vector.

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1 Introduction and motivation

The common belief in physics is that Einstein's general theory of relativity inevitably implies the appearance of gravitational waves. The mathematical structure of these waves is defined as quadrupole radiative fields.

However, one can question about two issues relevant to gravity and gravitational waves: (A) The existence problem: to prove the existence of gravitational waves, it is necessary to derive the exact wave equation from general relativity, but this is impossible because of the presence of at least Newtonian "monopole" terms that spoil the structure of the "wave" equation [1–4]; any approximate or even exact solution of Einstein field equations cannot isolate or neglect Newtonian from quadrupole terms. (B) The locality problem: assuming the presence of gravitational waves infers that a decomposition of the total tensor field contains terms affecting and implying the curvature of spacetime, essentially the nonlocal physical process, and at the same time the local radiative quadrupole field as gravitational waves; such a decomposition is difficult to comprehend both physically and mathematically.

The classical electrodynamics was the inspiration to introduce gravitational waves, but (a) the dipole radiative field is a solution of the exact Maxwell wave equations and (b) the electrodynamics is completely defined only by the local gauge field. As a consequence, the predicted dipole radiation has been verified experimentally.

[1] E-mail address: dpalle@comet.nmsu.edu

It is possible to understand the energy loss of binary systems as the effect of quadrupole potentials and not as a result of radiated gravitational waves [5].

In this paper we inspect gravitational lensing by quadrupole potentials which are parts of the metric tensor, such as monopole terms, affecting spacetime curvature. It should not be confused with the lensing of the assumed gravitational radiation by cosmological environment or some astrophysical sources that one can find in the literature.

2 Lensing equations

Gravitational lensing by quadrupoles is an old subject of investigation [6–9], but the first reliable calculation is due to Damour and Esposito-Farèse [10]. They have shown that the incorrect results of Ref. [11] arise from a naive "plane wave" approximation. Unfortunately, the same unreliable approximation is used in Ref. [12].

Let us briefly summarize the short-wave approximation of Maxwell equations in curved spacetime and the transport along the rays of the wave vector (ℓ^μ), the scalar amplitude (a) and the complex polarization vector (V^ν) [13]:

$$\ell^\mu \equiv \frac{dx^\mu}{d\xi},\ \ell^\beta \ell^\alpha_{;\beta} = 0, \qquad \dot{V_\alpha} \equiv \ell^\beta V_{\alpha;\beta} = 0,\ \dot{a} + \frac{1}{2} a \ell^\alpha_{;\alpha} = 0, \tag{1}$$

$$\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1),\ V^*_\nu V^\nu = 1,\ \ell^\alpha \ell_\alpha = 0,$$
$$\dot{a} \equiv \ell^\beta a_{,\beta},\ V_\alpha \ell^\alpha = 0.$$

We can write the transport equations more explicitly using Christoffel symbols:

$$\frac{d\ell_\mu}{d\xi} = \frac{1}{2} \ell^\alpha \ell^\beta \partial_\mu g_{\alpha\beta}, \tag{2}$$

$$\frac{1}{a}\frac{da}{d\xi} = -\frac{1}{2}(\partial_\beta \ell^\beta + \Gamma^\alpha_{\alpha\lambda} \ell^\lambda), \tag{3}$$

$$\frac{dV_\mu}{d\xi} = \ell^\alpha V_\beta \Gamma^\beta_{\mu\alpha}. \tag{4}$$

The linearized gravity approximation and the integration over the unperturbed photon trajectory [13] of the transport equations, lead to the following lensing equations:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},\ b^\alpha \equiv \text{impact vector},\ b^\mu \ell_\mu = 0,$$

$$\Delta \ell_\mu \equiv \ell_\mu(out) - \ell_\mu(in) = \frac{1}{2} \ell^\alpha \ell^\beta \int_{-\infty}^{+\infty} d\xi \partial_\mu h_{\alpha\beta}(b^\lambda + \xi \ell^\lambda)\ , \tag{5}$$

$$\Delta \ln\ a \equiv \ln \frac{a(out)}{a(in)} = -\frac{1}{4} \ell^\lambda \int_{-\infty}^{+\infty} d\xi \partial_\lambda h(b^\mu + \xi \ell^\mu),\ h \equiv \eta^{\alpha\beta} h_{\alpha\beta}, \tag{6}$$

$$\Delta V_\alpha \equiv V_\alpha(out) - V_\alpha(in) = \frac{1}{2} \ell^\beta V^\gamma \int_{-\infty}^{+\infty} d\xi (\partial_\alpha h_{\beta\gamma} + \partial_\beta h_{\alpha\gamma} - \partial_\gamma h_{\alpha\beta})(b^\mu + \xi \ell^\mu). \tag{7}$$

A direct evaluation of the lensing equations with the monopole potential straightforwardly gives

$$h^{(1)}_{\alpha\beta}(x^\mu) = \frac{2G_N M}{\mid \vec{x} \mid}\delta_{\alpha\beta} \Longrightarrow \Delta\ell_\mu = -\frac{4G_N M}{b^2}b_\mu,\ \Delta \ln\ a = 0,\ t^\mu \Delta V_\mu = 0, \qquad (8)$$

$$t^\mu = \text{any spacelike vector orthogonal to } \ell^\mu.$$

Thus, besides the resulting nonvanishing Einstein deflection angle [13], the monopole potential does not influence the transport of the scalar amplitude or of the polarization vector to the first order.

For the calculus with quadrupole potentials, we adopt the formalism of Damour and Esposito-Farèse [10] performing the integration in the Fourier space. They evaluated the lensing of the wave vector, which we therefore omit and concentrate on the lensing of the scalar amplitude and the polarization vector.

One can read the Fourier transform of the potentials where the energy-momentum conservation is also included [10]:

$$h_{\mu\nu}(x^\lambda) = \int \frac{d^4k}{(2\pi)^4}\widehat{h}_{\mu\nu}(k^\lambda)e^{ik\cdot x}, k\cdot x \equiv k^\nu x_\nu, \widehat{T} \equiv \eta^{\mu\nu}\widehat{T}_{\mu\nu},$$

$$\widehat{h}_{\mu\nu}(k^\lambda) = 16\pi G_N \frac{\widehat{T}_{\mu\nu}(k^\lambda) - \frac{1}{2}\eta_{\mu\nu}\widehat{T}(k^\lambda)}{k\cdot k - i\epsilon k^0}, \qquad \widehat{T}_{0i} = -\frac{k^j}{k^0}\widehat{T}_{ij},\ \widehat{T}_{00} = \frac{k^i k^j}{(k^0)^2}\widehat{T}_{ij}. \qquad (9)$$

We can fix the constant wave (ℓ^μ), impact (b^ν) and polarization (V^α) vectors:

$$\ell^\mu = (1,0,0,1),\ b^\mu = (0,b,0,0),\ V^\mu = (0,V_1,V_2,0). \qquad (10)$$

There is no influence of the quadrupole as a deflector to the scalar amplitude:

$$\Delta \ln\ a(quadrupole) = 0. \qquad (11)$$

Changing the integration variable $k \equiv\mid \vec{k} \mid$ to $u = \sqrt{k^2 - \omega^2}$ [10], one obtains the expression for the lensing of polarization:

$$k^\mu \equiv (\omega, k\sin\vartheta\cos\phi, k\sin\vartheta\sin\phi, k\cos\vartheta),\ k\cdot V \equiv k^\nu V_\nu,$$

$$\begin{aligned}\Delta V_\alpha = \frac{iG_N}{\pi^2}\int_{-\infty}^{+\infty} d\omega \int_0^{+\infty} du \int_0^{2\pi} d\phi u e^{ibu\cos\phi}(u^2 - i\varepsilon\omega)^{-1}\\ \times\{k_\alpha \widehat{T}_{ij}(-k^j V_i \omega^{-1} + \delta_{i3}V_j) - (k\cdot V)\widehat{T}_{ij}[\delta_{\alpha 0}(k^i k^j \omega^{-2} - \delta_{i3}k^j\omega^{-1})\\ +\delta_{\alpha i}(-k^j\omega^{-1} + \delta_{j3})] + \frac{1}{2}(k\cdot V)\ell_\alpha(-k^i k^j \omega^{-2}\widehat{T}_{ij} + \widehat{T}_{ii})\}.\end{aligned} \qquad (12)$$

Neglecting the $\vec{k}$ dependence (quadrupole approximation) of the deflector field [10]

$$\widehat{T}_{ij}(\omega,\vec{k}) \simeq \widehat{T}_{ij}(\omega,\vec{0}) = -\frac{\omega^2}{2}D_{ij}(\omega), \qquad (13)$$

$$D_{ij}(\omega) \equiv \int d^3x x^i x^j T^{00}(\omega,\vec{x}),\ D_{ij}(\omega) \equiv \int dt e^{i\omega t}D_{ij}(t),$$

the rest of integrations could be performed with elementary integrals but with a careful regularization procedure [10].

The result for the lensing of polarization is of the form

$$\Delta V_1(Q) = \frac{4G_N V_2}{b^2}\frac{\partial D_{12}(t)}{\partial t}\mid_{t=0},\ \Delta V_2(Q) = -\frac{4G_N V_1}{b^2}\frac{\partial D_{12}(t)}{\partial t}\mid_{t=0}, \tag{14}$$

$$\begin{aligned}\Delta V_0(Q) &= -\frac{2G_N}{b^3}[V_1(D_{22}(0) - D_{11}(0) + \frac{b^2}{2}\frac{\partial^2 D_{11}(t)}{\partial t^2}\mid_{t=0} - \frac{b^2}{2}\frac{\partial^2 D_{22}(t)}{\partial t^2}\mid_{t=0})\\ &+ V_2(2D_{12}(0) + b^2\frac{\partial^2 D_{12}(t)}{\partial t^2}\mid_{t=0})],\end{aligned}$$

$$\begin{aligned}\Delta V_3(Q) &= -\frac{2G_N}{b^3}[V_1(D_{22}(0) - D_{11}(0) - \frac{b^2}{2}\frac{\partial^2 D_{11}(t)}{\partial t^2}\mid_{t=0} + \frac{b^2}{2}\frac{\partial^2 D_{22}(t)}{\partial t^2}\mid_{t=0})\\ &+ V_2(2D_{12}(0) - b^2\frac{\partial^2 D_{12}(t)}{\partial t^2}\mid_{t=0})].\end{aligned}$$

One can easily verify the following relation

$$V^*_\mu \Delta V^\mu(Q) + V^\mu \Delta V^*_\mu(Q) = 0.$$

From the exact integral for the evolution along the geodesic (C) of the polarization vector

$$\Delta V_\mu = \int_C d\xi \ell^\alpha V_\beta \Gamma^\beta_{\mu\alpha}, \tag{15}$$

one can easily deduce the second order corrections perturbing Christoffel symbols [2], wave and polarization vectors, as well as the photon trajectory with regard to the monopole potential:

$$\Delta V_\alpha(M) = \mathcal{I}_\alpha + \mathcal{J}_\alpha + \mathcal{K}_\alpha + \mathcal{L}_\alpha + \mathcal{M}_\alpha, \tag{16}$$

$$\begin{aligned}\mathcal{I}_\alpha &= \frac{1}{2}\ell^\beta V_\lambda \int_{C_0} d\xi h^{(1)\lambda\kappa}(\partial_\alpha h^{(1)}_{\beta\kappa} + \partial_\beta h^{(1)}_{\alpha\kappa} - \partial_\kappa h^{(1)}_{\alpha\beta}), \qquad (17)\\ h^{(1)}_{\mu\nu} &= \frac{2G_N M}{\mid \vec{x} \mid}\delta_{\mu\nu},\ h^{(1)\kappa\gamma} = -\eta^{\beta\gamma}\eta^{\alpha\kappa}h^{(1)}_{\alpha\beta},\ C_0 = \text{unperturbed trajectory},\end{aligned}$$

$$\begin{aligned}\mathcal{J}_\alpha &= \frac{1}{2}\ell^\beta V^\kappa \int_{C_0} d\xi(\partial_\alpha h^{(2)}_{\beta\kappa} + \partial_\beta h^{(2)}_{\alpha\kappa} - \partial_\kappa h^{(2)}_{\alpha\beta}), \qquad (18)\\ h^{(2)}_{00} &= -\frac{2G_N^2 M^2}{\mid \vec{x} \mid^2},\ h^{(2)}_{ij} = \frac{G_N^2 M^2}{\mid \vec{x} \mid^2}(\delta_{ij} + \frac{x_i x_j}{\mid \vec{x} \mid^2}),\ h^{(2)}_{i0} = 0,\end{aligned}$$

$$\mathcal{K}_\alpha = \frac{1}{2}V^\kappa \int_{C_0} d\xi \Delta\ell^\beta(\partial_\alpha h^{(1)}_{\beta\kappa} + \partial_\beta h^{(1)}_{\alpha\kappa} - \partial_\kappa h^{(1)}_{\alpha\beta}), \tag{19}$$

$$\Delta\ell_\alpha(\xi) = \frac{1}{2}\ell^\mu\ell^\nu \int_{-\infty}^{\xi} d\xi \partial_\alpha h^{(1)}_{\mu\nu}(b^\lambda + \xi\ell^\lambda),$$

$$\Longrightarrow \Delta\ell_\alpha(\xi) = G_N(0, \frac{-2M\xi}{b\sqrt{b^2+\xi^2}} - \frac{2M}{b}, 0, \frac{2M}{\sqrt{b^2+\xi^2}}),$$

$$\begin{aligned}
\mathcal{L}_\alpha &= \frac{1}{2}\ell^\beta \int_{C_0} d\xi \Delta V^\kappa (\partial_\alpha h^{(1)}_{\beta\kappa} + \partial_\beta h^{(1)}_{\alpha\kappa} - \partial_\kappa h^{(1)}_{\alpha\beta}), \qquad (20)\\
\Delta V_\lambda &= \frac{1}{2}\ell^\beta V^\kappa \int_{-\infty}^{\xi} d\xi (\partial_\lambda h^{(1)}_{\beta\kappa} + \partial_\beta h^{(1)}_{\lambda\kappa} - \partial_\kappa h^{(1)}_{\lambda\beta})(b^\nu + \xi\ell^\nu),
\end{aligned}$$

$$\Rightarrow \Delta V_\lambda = G_N \left[\tfrac{MV_1}{b}(1 + \xi/\sqrt{b^2+\xi^2}), \tfrac{MV_1}{\sqrt{b^2+\xi^2}}, \tfrac{MV_2}{\sqrt{b^2+\xi^2}}, \tfrac{MV_1}{b}(1 + \xi/\sqrt{b^2+\xi^2}) \right],$$

$$\begin{aligned}
\mathcal{M}_\alpha &= \frac{1}{2}\ell^\beta V^\kappa \int_{C_1} d\xi (\partial_\alpha h^{(1)}_{\beta\kappa} + \partial_\beta h^{(1)}_{\alpha\kappa} - \partial_\kappa h^{(1)}_{\alpha\beta}), \qquad (21)\\
C_1 &= \text{perturbed trajectory},\\
\Rightarrow \mathcal{M}_\alpha &= \frac{1}{2}\ell^\beta V^\kappa \int_{C_0} d\xi \Delta x^\rho \frac{\partial}{\partial x^\rho}[\partial_\alpha h^{(1)}_{\beta\kappa} + \partial_\beta h^{(1)}_{\alpha\kappa} - \partial_\kappa h^{(1)}_{\alpha\beta}],\\
\Delta x_\alpha &= \frac{1}{2}\ell^\beta \int_{-\infty}^{\xi} d\xi x^\kappa (\partial_\alpha h^{(1)}_{\beta\kappa} + \partial_\beta h^{(1)}_{\alpha\kappa} - \partial_\kappa h^{(1)}_{\alpha\beta}),
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \Delta x_\alpha = G_N(M(1 + \xi/\sqrt{b^2+\xi^2}), \frac{3bM}{\sqrt{b^2+\xi^2}}, 0, 3M(1 + \xi/\sqrt{b^2+\xi^2})\\
-2M \ln \frac{\xi + \sqrt{b^2+\xi^2}}{\epsilon}), \epsilon = \text{small positive real cut} - \text{off}.
\end{aligned}$$

The coordinate singularity in Eq.(21), regulated by the cut-off ϵ, disappears after the integration over the unperturbed photon trajectory, as one should expect for an observable. After performing integrations, only one term contributes to the deviation of the polarization vector:

$$\begin{aligned}
\mathcal{I}_\alpha &= G_N^2(-\frac{2\pi M^2}{b^2}V_1, 0, 0, -\frac{2\pi M^2}{b^2}V_1),\\
\mathcal{J}_\alpha &= G_N^2(-\frac{\pi M^2}{2b^2}V_1, 0, 0, \frac{3\pi M^2}{8b^2}V_1),\\
\mathcal{K}_\alpha &= G_N^2(0, \frac{4M^2}{b^2}V_1, \frac{4M^2}{b^2}V_2, \frac{2\pi M^2}{b^2}V_1),\\
\mathcal{L}_\alpha &= G_N^2(\frac{3\pi M^2}{2b^2}V_1, 0, 0, \frac{\pi M^2}{2b^2}V_1),\\
\mathcal{M}_\alpha &= G_N^2(-\frac{2\pi M^2}{b^2}V_1, 0, 0, -\frac{2\pi M^2}{b^2}V_1),\\
&\Rightarrow \Delta V_1(M) = \frac{4G_N^2 M^2}{b^2}V_1,\ \Delta V_2(M) = \frac{4G_N^2 M^2}{b^2}V_2. \qquad (22)
\end{aligned}$$

However, this contribution is canceled away by the renormalization of the polarization vector:

$$\Delta V_\mu \equiv \Delta V_\mu(Q) + \Delta V_\mu(M),\ (V^\mu + \Delta V^\mu)^*(V_\mu + \Delta V_\mu)$$

$$
\begin{aligned}
&= 1+\frac{8G_N^2M^2}{b^2}+O((\Delta V)^2),\\
\tilde{V}^\mu &\equiv (1+\frac{8G_N^2M^2}{b^2})^{-\frac{1}{2}}(V^\mu+\Delta V^\mu),\ \tilde{V}^\mu\tilde{V}_\mu=1,\\
\tilde{V}^\mu &= V^\mu-\frac{4G_N^2M^2}{b^2}V^\mu+\Delta V^\mu+O((\Delta V)^2)\\
&= V^\mu+\Delta V^\mu(Q)+O((\Delta V)^2),\ \mu=1,2.
\end{aligned}
$$

3 Results and discussion

For example, take a binary system and its quadrupole moment when a surface of motion is in the xy plane and the z axis is defined by the direction of the incoming photon. In this case, the deflection of polarization is maximal. Generally, the Euler rotation matrix R projects the quadrupole moment of arbitrary orientation to the frame defined by the wave vector of the deflected photon

$$
\begin{gathered}
\tilde{D}_{ij}(t)=\frac{m_1m_2}{m_1+m_2}d^2\begin{bmatrix}\cos^2(\Omega t) & \frac{1}{2}\sin(2\Omega t) & 0\\ \frac{1}{2}\sin(2\Omega t) & \sin^2(\Omega t) & 0\\ 0 & 0 & 0\end{bmatrix},\\
\Omega\equiv\frac{2\pi}{P},\ P=\text{orbital period},\\
\text{circular orbit}: x=d\cos(\Omega t),\ y=d\sin(\Omega t),\\
D_{ij}(t)=R(\theta_E,\phi_E)_{3\times3}\tilde{D}_{ij}(t).
\end{gathered}
\tag{23}
$$

The complex polarization vector and its deflections could be cast into a more conventional form to recognize linear, circular, or elliptical polarization states (here β is the phase-difference between two independent polarization states) [14]

$$
\begin{aligned}
\vec{V} &= V_1\vec{x}_0+V_2\vec{y}_0=V_-\vec{\epsilon}_++V_+\vec{\epsilon}_-,\\
V_\pm &= \frac{1}{\sqrt{2}}(V_1\pm iV_2),\ \mid V_-\mid^2+\mid V_+\mid^2=1,\ \vec{\epsilon}_\pm=\frac{1}{\sqrt{2}}(\vec{x}_0\pm i\vec{y}_0),\\
V_+ &= \rho_+,\ V_-=\rho_-e^{i\beta},\ \hat{\rho}=\frac{\rho_-}{\rho_+}=\frac{\sqrt{1-\rho_+^2}}{\rho_+},\\
&\Longrightarrow \Re V_1=\frac{1}{\sqrt{2}}(\rho_++\rho_-\cos\beta),\ \Im V_1=\frac{1}{\sqrt{2}}\rho_-\sin\beta,\\
\Re V_2 &= -\frac{1}{\sqrt{2}}\rho_-\sin\beta,\ \Im V_2=\frac{1}{\sqrt{2}}(\rho_-\cos\beta-\rho_+),\\
\Delta\rho_+ &= \frac{1}{\sqrt{2}}(\Re\Delta V_1-\Im\Delta V_2), && (24)\\
\Delta\beta &= \frac{1}{\sqrt{2}\rho_-\cos\beta}(\Im\Delta V_1-\Re\Delta V_2+\sqrt{2}\sin\beta\frac{\rho_+}{\rho_-}\Delta\rho_+), && (25)\\
&\text{quadrupole contribution (Eq. (14))}:\\
\Delta\beta &= \frac{4G_N}{b^2}\frac{\partial D_{12}(t)}{\partial t}\mid_{t=0},\ \Delta\rho_+=\Delta\hat{\rho}=0. && (26)
\end{aligned}
$$

Finally, let us present some numerical estimates of the effect assuming high-frequency (high-rate) deflectors:

$$\text{binary neutron star system} : m = m_1 m_2/(m_1+m_2), \dot{D}_{12}(0) = md^2\Omega,$$
$$\text{Kepler's third law} : d = \Omega^{-2/3}(G_N(m_1+m_2))^{1/3},$$
$$m \simeq M_{\odot},\ d \simeq 10^{-4}s,\ \Omega \simeq 10^3 s^{-1},\ b \simeq 10^{-3}s \Longrightarrow \Delta\beta = O(10^{-4}),$$

and similarly for the accretion by supermassive black holes or coalescing black hole binaries.

Although the effect is small, the deviation of polarization (change of the phase-difference between the two independent states) by the quadrupole is free of any "background" monopole contribution and it is worth measuring.

References

[1] J.L. Synge: *Relativity: The General Theory*, North-Holland, Amsterdam, 1960
[2] S. Weinberg: *Gravitation and cosmology*, John Wiley, Sons, New York, 1972
[3] L.D. Landau, E.M. Lifšic: *Teorija polja*, Nauka, Moskva, 1973
[4] C.W. Misner, K.S. Thorne, J.A. Wheeler: *Gravitation*, Freeman, San Francisco, 1973
[5] D. Palle: *astro-ph/0211481*
[6] D.M. Zipoy: *Phys. Rev.* **142** (1966) 825
[7] F. Winterberg: *Nuovo Cimento B* **53** (1967) 264
[8] D.M. Zipoy, B. Bertotti: *Nuovo Cimento B* **56** (1968) 195
[9] B. Bertotti, D. Trevese: *Nuovo Cimento B* **7** (1971) 240
[10] T. Damour and G. Esposito-Farèse: *Phys. Rev. D* **58** (1998) 044003
[11] R. Durrer: *Phys. Rev. Lett.* **72** (1994) 3301
[12] S. Kopeikin, B. Mashhoon: *Phys. Rev. D* **65** (2002) 064025
[13] P. Schneider, J. Ehlers, E.E. Falco: *Gravitational Lenses*, Springer, Berlin, 1999
[14] J.D. Jackson: *Classical Electrodynamics*, John Wiley and Sons, New York, 1962