NON-COMMUTATIVE GREEN FUNCTION FOR TWO COMPONENTS RELATIVISTIC EQUATION

L. Chetouani[1,a], M. Merad[b], T. Boudjedaa[c], A. Lecheheb[a]
[a] *Département de Physique,Faculté des Sciences, Université Mentouri 25000 Constantine, Algéria*
[b] *Département de Physique, Centre Universitaire de Oum-El-Bouaghi 04000 Oum-El-Bouaghi, Algéria*
[c] *Laboratoire de Physique Théorique, Faculté des Sciences Université de Jijel BP 98, Ouled Aissa, 18000 Jijel Algeria*

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A calculation of non-commutative Green function for the two components relativistic equation is presented using the analogous Klein-Gordon one. The result is the sum of two contributions: one is regular and having semi relativistic expression and the second is irregular playing the role of source. An illustration of calculation is given in the case of linear potential.

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1 Introduction

Since Heisenberg has suggested the use of a non-commutative algebra in quantum mechanics, the non-commutativity does not stop invading the various domains of the physics. It was Snyder who the first developed systematically this idea [1]. Now, non-commutative geometry plays an important role in modern physics and has sustained great interest, especially after the reproof of this phenomenon within the string theory with D-branes [2], where all the uncertainties about space-time has found its origin. In this aim, non-commutative field theory has been constructed by introducing the Moyal product in space of ordinary functions. This allows a manifestation of non local phenomena which has a considerable effects in high energy physics, but we could in principle observe these effects at low energy level. Accordingly, some experimental tests of this non-commutative effects has been proposed. It turns out that a sufficient test would be in the quantum mechanic framework. From theoretical point of view, many works have been attempted in the hope to give a concrete aspect to this space non-commutativity. Their application domain extends from the quantum fields theory to quantum mechanics. Among the works achieved in this domain, we cite the non-commutative Φ^4 theory [3] in which the effective action and the two points Green functions have been recalculated. Similarly, in the non-commutative QED theory, it was question to calculate the magnetic moments anomaly and the pair production [4].

[1] E-mail address: chetoua1@caramail.com

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In addition, in quantum mechanics, the spectrum and the wave functions of the particles have been corrected [5,6].

It is obvious that the principles of the relativity require charge symmetry. For the case of the relativistic spinless particle, this is shown using the two components theory described by Feshbach- Villars (FV) equation [7] where the both signs of the charges are involved and the negative energy solutions are interpreted as antiparticles. In this paper, we want to determine the Green function of the FV equation (spin 0) in interaction with a scalar potential in a non-commutative space using the Klein-Gordon (KG) equation. This work represents a non-commutative generalization, and a simplification of the resulting calculation proposed by the perturbation method in the case of a commutative space which used a heavy matrix algebra [8,9].

Section II contains a review of the FV equation and some relations of the non-commutative space necessary to calculation. Section III is devoted to the calculation of the non-commutative Green function relative to FV equation. It is demonstrated that the non-commutative Green function consists of two parts: one is regular and semi relativistic containing the corrections made by non-commutativity, the other is irregular and identical to that of the commutative case. The technique of calculation is purely algebraic and thus avoids the heaviness of calculations which are already made via the perturbation method. In section IV, an illustrative example is provided.

2 Review and notations

On commutative space, quantum mechanics is usually formulated by imposing the commutation relations generating the following Heisenberg algebra ($\hbar = 1$)

$$[X^\mu, P^\nu] = ig^{\mu\nu}, \qquad [X^\mu, X^\nu] = 0, \qquad \text{and} \qquad [P^\mu, P^\nu] = 0. \tag{1}$$

According to these relations, the relativistic equation describing the dynamics of the particles of spin zero in the presence of a scalar potential is set up via the correspondence rule and is known as KG equation

$$\left[\left(\frac{\partial}{\partial t} + ieV(x)\right)^2 - \nabla^2 + m^2\right]\Phi(x) = 0. \tag{2}$$

The main difficulty of this equation is the presence of the second-order time derivatives which generates the non positive density of probability. The FV procedure is an attempt to avoid this difficulty by considering a two components wave equation. The reduction of the order of the time derivative is allowed according to the following linearization

$$\Psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \Phi + \frac{1}{m}\left(i\frac{\partial}{\partial t} - eV\right)\Phi \\ \Phi - \frac{1}{m}\left(i\frac{\partial}{\partial t} - eV\right)\Phi \end{pmatrix}. \tag{3}$$

where Φ is the KG wave function and Ψ is that of FV.

Consequently, Ψ satisfies the following Schrödinger equation

$$i\frac{\partial\Psi}{\partial t} = H(e)\Psi, \tag{4}$$

with

$$H(e) = -\frac{1}{2m}\nabla^2\left(\tau_3 + i\tau_2\right) + m\tau_3 + eV, \tag{5}$$

$V\left(x\right)$ represents the time independent scalar potential and $\left(\tau_1, \tau_2, \tau_3\right)$ are the Pauli matrices describing the charge symmetry which are usually defined by

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6}$$

Therefore, the equation exhibits the charge symmetry required by the relativistic principles and the negative energy is interpreted as the antiparticle.

In order to describe a non-commutative space, the above commutations relations would be changed as

$$[X^\mu, P^\nu] = ig^{\mu\nu}, \qquad [X^\mu, X^\nu] = i\theta^{\mu\nu}, \qquad \text{and} \qquad [P^\mu, P^\nu] = 0. \tag{7}$$

where $\theta^{\mu\nu}$ is a constant antisymmetric tensor parameter representing the non-commutation of space-time. To preserve the unitarity of the theory, we choose $\theta^{0\nu} = 0$, which implies that the time remains as a parameter and the non-commutativity affects only the physical space. In the context of this deformation the product of any two functions is equivalent to the star Moyal product defined by

$$\left(f * g\right)(x) = \exp\left(\frac{i}{2}\theta_{\mu\nu}\partial x_\mu \partial y_\nu\right) f(x)g(y) \mid_{x=y} . \tag{8}$$

Consequently, the non-commutative KG equation will be

$$\left[\left(\frac{\partial}{\partial t} + ieV(x)\right)^2 - \nabla^2 + m^2\right] * \Phi(x) = 0. \tag{9}$$

Using now the same technique of linearization given by

$$\Psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \Phi + \frac{1}{m}\left(i\frac{\partial}{\partial t} - eV\right) * \Phi \\ \Phi - \frac{1}{m}\left(i\frac{\partial}{\partial t} - eV\right) * \Phi \end{pmatrix},$$

the non-commutative FV equation becomes

$$i\frac{\partial\Psi(x)}{\partial t} = \left[-\frac{1}{2m}\nabla^2\left(\tau_3 + i\tau_2\right) + m\tau_3 + eV\right] * \Psi(x). \tag{10}$$

In what follows, our interest is to calculate the energy dependent Green function relative to the equation (10). The method of calculation is algebraic and will be based on the nilpotent property of the matrix $\left(\tau_3 + i\tau_2\right)$.

3 Non-commutative Feshbach- Villars Green Function

Before beginning the calculation, let us show that the star product inducing the non-commutativity is replaced by the usual product plus a non local correction in the scalar potential. In effect, it is easy to show that

$$\nabla^2 * \Psi(x) = \nabla^2 \Psi(x) \,, \tag{11}$$

and

$$V(x) * \Psi(x) = \exp\left(\frac{i}{2}\theta_{\mu\nu}\partial x_\mu \partial y_\nu\right) V(x)\Psi(y) \mid_{x=y} = V^*(x,\nabla)\Psi(x), \tag{12}$$

with

$$V^*(x,\nabla) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\prod_{k=1}^{n} \frac{i}{2}\theta^{i_k j_k}\nabla_{i_k}\right) V(x) \prod_{k=1}^{n} \nabla_{j_k}, \tag{13}$$

where i_k and j_k are the indices of the space components.

This allows to write the FV equation in the non-commutative case as FV equation of the commutative type with a non local potential. In other words, we have to solve the following equation of the FV type

$$i\frac{\partial \Psi(x)}{\partial t} = H^*_{FV}(e)\Psi(x), \tag{14}$$

where H^*_{FV} denotes the Hamiltonian containing the non local potential. It is given by

$$H^*_{FV}(e) = \left[-\frac{1}{2m}\nabla^2\left(\tau_3 + i\tau_2\right) + m\tau_3 + eV^*(x,\nabla)\right]. \tag{15}$$

After that, let us define the energy dependent Green function corresponding to the non-commutative FV equation by

$$\left[E - H^*_{FV}(e)\right] G^*_{FV}(E) = i, \tag{16}$$

whose the symbolic solution is written as

$$G^*_{FV}(E) = \frac{i}{E - H^*_{FV}(e)}. \tag{17}$$

In order to pass by the corresponding energy dependent KG Green function, let us multiply by the conjugate operator of $H^*_{FV}(e)$ defined by $H^*_{FV}(-e)$. Then, we write the energy dependent Green function as

$$G^*_{FV}(E) = \frac{i\left(E + H^*_{FV}(-e)\right)}{\left(E - H^*_{FV}(e)\right)\left(E + H^*_{FV}(-e)\right)}. \tag{18}$$

A direct and simple calculation shows that

$$G_{FV}^{*}(E)=\frac{i\left(E+H_{FV}^{*}(-e)\right)}{O_{KG}^{*}-\left[eV^{*},-\frac{1}{2m}\nabla^{2}\right]\left(\tau_{3}+i\tau_{2}\right)}, \tag{19}$$

where O_{KG}^{*} is the corresponding non-commutative KG operator, defined as

$$O_{KG}^{*}=\left(E-eV^{*}\right)^{2}-\left(-\nabla^{2}+m^{2}\right). \tag{20}$$

Due to the fact that $(\tau_3+i\tau_2)^2=0$, the expression of $G_{FV}^{*}(E)$ given by (19) decomposes as follows

$$G_{FV}^{*}(E)=i\left(E+H_{FV}^{*}\right)\left\{O_{KG}^{*^{-1}}+O_{KG}^{*^{-1}}\left[eV^{*},-\frac{1}{2m}\nabla^{2}\right]O_{KG}^{*^{-1}}\left(\tau_{3}+i\tau_{2}\right)\right\}. \tag{21}$$

The use of

$$O_{KG}^{*}O_{KG}^{*^{-1}}=1, \tag{22}$$

and the replacement of

$$-\frac{1}{2m}\nabla^{2}=\frac{\left(E-eV^{*}\right)^{2}-O_{KG}^{*}-m^{2}}{2m}, \tag{23}$$

permits us to evaluate

$$O_{KG}^{*^{-1}}\left[eV^{*},-\frac{1}{2m}\nabla^{2}\right]O_{KG}^{*^{-1}}=\frac{1}{2m}\left[eV^{*},O_{KG}^{*^{-1}}\right] \tag{24}$$

which simplifies the expression of $G_{FV}^{*}(E)$ to

$$G_{FV}^{*}(E)=i\left(E+H_{FV}^{*}(-e)\right)\left\{O_{KG}^{*^{-1}}+\frac{1}{2m}\left[eV^{*},O_{KG}^{*^{-1}}\right]\left(\tau_{3}+i\tau_{2}\right)\right\}. \tag{25}$$

It is easy to show that the first term of $G_{FV}^{*}(E)$ becomes

$$\left(E+H_{FV}^{*}(-e)\right)O_{KG}^{*^{-1}}=-\frac{1}{2m}\left(\tau_{3}+i\tau_{2}\right)+$$

$$+\left[E-eV^{*}+m\tau_{3}+\frac{\left(E-eV^{*}\right)^{2}-m^{2}}{2m}\left(\tau_{3}+i\tau_{2}\right)\right]O_{KG}^{*^{-1}}, \tag{26}$$

and the second term will be written as

$$\frac{1}{2m}\left(E+H_{FV}^{*}(-e)\right)\left[eV^{*},O_{KG}^{*^{-1}}\right]\left(\tau_{3}+i\tau_{2}\right)=\frac{1}{2m}\left(E-eV^{*}+m\tau_{3}\right)\times$$

$$\times \left[eV^{*}, O_{KG}^{*^{-1}} \right] (\tau_3 + i\tau_2) \,. \tag{27}$$

Finally, the Green function $G_{FV}^{*}(E)$ takes the following simple form

$$G_{FV}^{*}(E) = -\frac{i}{2m}(\tau_3 + i\tau_2) + \frac{i}{2m}(E - eV^{*} + m\tau_3)\left[eV^{*}, O_{KG}^{*^{-1}} \right](\tau_3 + i\tau_2) + \\ + i\left[E - eV^{*} + m\tau_3 + \frac{(E - eV^{*})^2 - m^2}{2m}(\tau_3 + i\tau_2) \right] O_{KG}^{*^{-1}}. \tag{28}$$

In fact, the first term represents the irregular part which is purely relativistic and is responsible for the appearance of the square of the potential in the Klein-Gordon theory [9]. The second term is semi relativistic and is calculated by means of the corresponding non-commutative energy dependent KG Green function. Regarding this, our main result will be expressed in the configuration base as

$$\begin{aligned} \langle x_b \mid G_{FV}^{*}(E) \mid x_a \rangle &= i\tau^{*}(E, x_b, x_a, \nabla_b, \nabla_a)\langle x_b \mid O_{KG}^{*^{-1}} \mid x_a \rangle - \\ &- \frac{i}{2m}(\tau_3 + i\tau_2)\, \delta(x_b - x_a) \end{aligned} \tag{29}$$

with the operator

$$\begin{aligned} &\tau^{*}(E, x_b, x_a, \nabla_b, \nabla_a) = \Bigg[E - eV^{*}(x_b, \nabla_b) + m\tau_3 + \\ &+ \frac{(E - eV^{*}(x_b, \nabla_b))^2 - m^2}{2m}(\tau_3 + i\tau_2) \Bigg] + \\ &+ \frac{e}{2m}(E - eV^{*}(x_b, \nabla_b) + m\tau_3)\left[V^{*}(x_b, \nabla_b) - V^{*}(x_a, \nabla_a) \right](\tau_3 + i\tau_2) \,, \end{aligned} \tag{30}$$

and $\langle x_b \mid O_{KG}^{*^{-1}} \mid x_a \rangle$ is the non-commutative energy dependent Green function of the KG type.

In the case of the commutative limit ($\theta \to 0$), we get the result of the commutative Green function calculated by means of the perturbation method [9]

$$\begin{aligned} \langle x_b \mid G_{FV}(E) \mid x_a \rangle &= i\tau(E, x_b, x_a)\langle x_b \mid O_{KG}^{-1}(E) \mid x_a \rangle - \\ &- \frac{i}{2m}\delta(x_b - x_a)(\tau_3 + i\tau_2), \end{aligned} \tag{31}$$

with $\langle x_b \mid O_{KG}^{-1}(E) \mid x_a \rangle$ is the commutative Green function of KG particle submitted to the scalar potential and $\tau(E, x_b, x_a)$ is the energy dependent matrix defined by

$$\tau(E, x_b, x_a) = \lim_{\theta \to 0} \tau^{*}(E, x_b, x_a, \nabla_b, \nabla_a), \tag{32}$$

which can also be written in symmetric form as

$$\tau(E, x_b, x_a) = \mathcal{U}(E, x_b)\mathcal{U}^{+}(E, x_a)\tau_3, \tag{33}$$

with

$$\mathcal{U}(E, x) = \tfrac{1}{\sqrt{2m}} \begin{pmatrix} m + E - eV(x) \\ m - E + eV(x) \end{pmatrix}. \tag{34}$$

4 Illustrative example

To show how to deal with the formula (29), let us illustrate the case of linear potential

$$V(x) = \sum_k \lambda_k x_k, \tag{35}$$

which gives for the non local potential the following expression

$$V^*(x, \nabla) = \sum_k (\lambda_k x_k + i\mu_k \nabla_k), \tag{36}$$

with

$$\mu_k = \frac{1}{2} \sum_l \theta_{lk} \lambda_l, \tag{37}$$

To calculate the Green function $\langle x_b \mid G^*_{FV}(E) \mid x_a \rangle$ we need to evaluate $\langle x_b \mid O^{*^{-1}}_{KG} \mid x_a \rangle$. To this purpose let us write

$$\langle x_b \mid O^{*^{-1}}_{KG} \mid x_a \rangle = \frac{1}{i} \int_0^\infty dT \exp\left[i\left(E^2 - m^2\right)\right] \langle x_b \mid \exp\left(-i\widehat{H}T\right) \mid x_a \rangle, \tag{38}$$

where

$$\widehat{H} = \sum_k \left[-\left(\nabla_k\right)^2 + 2eEV^*(x, \nabla) - e^2 V^{*2}(x, \nabla)\right]. \tag{39}$$

It is easy to show that the propagator $\langle x_b \mid \exp\left(-i\widehat{H}T\right) \mid x_a \rangle$ writes in the following path integral form

$$\langle x_b \mid \exp\left(-i\widehat{H}T\right) \mid x_a \rangle = \int \mathcal{D}x\mathcal{D}p \exp\left[i \int_0^T \left(p\frac{dx}{ds} - H(p, x)\right) ds\right], \tag{40}$$

with

$$\begin{aligned} H(p, x) \ = \ & \sum_{k,l} \left[\left(\delta_{kl} - e^2 \mu_k \mu_l\right) p_k p_l + 2e\lambda_k \mu_l x_k p_l - e^2 \lambda_k \lambda_l x_k x_l\right] + \\ & + \sum_k 2eE(\lambda_k x_k - \mu_k p_k), \end{aligned} \tag{41}$$

where we have used the antisymmetric property of θ_{kl} .

It is remarkable that this path integral can be done exactly because the Lagrangian is quadratic in x and p. There are many methods for doing this. In our case we take into account the semi classical one. The result of the propagator via this method is given as

$$\langle x_b \mid \exp\left(-i\widehat{H}T\right) \mid x_a \rangle = \left(\frac{1}{2\pi i}\right)^{-\frac{3}{2}} \left[-\frac{\partial S(x_b, x_a)}{\partial x_b \partial x_a}\right]^{1/2} \exp\left[iS(x_b, x_a)\right], \tag{42}$$

where $S(x_b, x_a)$ is the classical action

$$S(x_b, x_a) = \int_0^T \left(p_{cl} \frac{dx_{cl}}{ds} - H(p_{cl}, x_{cl}) \right), \tag{43}$$

with p_{cl} and x_{cl} checking the following system of differential equations

$$\begin{aligned} \frac{1}{2}\frac{dx_k}{ds} &= p_k - e^2\mu_k \sum_l \mu_l p_l + e\mu_k \sum_l \lambda_l x_l - eE\mu_k, \\ \frac{1}{2}\frac{dp_k}{ds} &= -e\lambda_k \sum_l \mu_l p_l + e^2\lambda_k \sum_l \lambda_l x_l - eE\lambda_k, \end{aligned} \tag{44}$$

and $x(0) = x_a$ and $x(T) = x_b$.

The exact expression of this propagator is easily deduced from this classical system (44) and the Feshbach- Villars spectrum of this problem can be determined from $(31, 38)$.

5 Conclusion

In this paper, we have evaluated the Green function of spin zero in non-commutative FV formalism by means of the non-commutative KG type. Therefore, we have used an algebraic technique providing the necessary tools to treat explicitly the problem. The corrections brought by the non-commutativity of space were gathered in a non local potential. The non-commutative FV Green function is equal to that of non-commutative KG times a matrix containing the non local potential plus a singular part. This latter remains unchanged. The regular part is affected by the non-commutativity. To illustrate the calculations we have treated explicitly the case of the linear potential.

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