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ON THE PROBLEM OF EVALUATION OF SUM AND DIFFERENCE OF π^0 POLARIZABILITIES

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The problem of electric and magnetic polarizabilities of pions is illustrated and their extraction from dispersion relations with one subtraction is given. Unambiguous determination of σ meson parameters and the decay of σ meson into two photons are carried out in order to determine the most realistic values of sum and difference of pion polarizabilities.

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1 Introduction

Some years ago, Fil'kov and Kashevarov [1] investigated $\gamma\pi^\pm \to \gamma\pi^\pm$ process by using the available experimental information on cross-section of the process $\gamma\gamma \to \pi^0\pi^0$ by means of the dispersion relations (DR's) with the subtraction to be expressed through difference of neutral pion polarizabilities $\alpha_{\pi^0}, \beta_{\pi^0}$.

The process $\gamma\pi^\pm \to \gamma\pi^\pm$ is described by the helicity amplitude $M_{++}(s,t)$ and $M_{+-}(s,t)$ and the corresponding cross-section takes the form

$$\frac{d\sigma(\gamma\pi^\pm \to \gamma\pi^\pm)}{d\Omega} = \frac{1}{256\pi^2}\frac{(s-m_\pi^2)^4}{s^3}\{(1-z)^2|M_{++}|^2 + s^2(1+z)^2|M_{+-}|^2\}. \tag{1}$$

For M_{++} one can write down DR at fixed t with one subtraction

$$ReM_{++}(s,t) = Re\bar{M}_{++}(s=m_\pi^2,t) + B_{++} + \frac{(s-m_\pi^2)}{\pi}$$
$$\times P\int_{4m_\pi^2}^{\infty} ds' ImM_{++}(s',t)\Big[\frac{1}{(s'-s)(s'-m_\pi^2)} - \frac{1}{(s'-u)(s'-m_\pi^2+t)}\Big], \tag{2}$$

where the Born term takes the form

$$B_{++} = \frac{2e^2m_\pi^2}{(s-m_\pi^2)(u-m_\pi^2)}, \tag{3}$$

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and the subtraction is at $s = m_\pi^2 : Re\bar{M}_{++}(s = m_\pi^2, t = 0) = 2\pi m_\pi(\alpha - \beta)_{\pi^\pm}$.

The DR for $M_{+-}(s,t)$ has the same expression with substitutions:

$$ImM_{++}(s,t) \rightarrow ImM_{+-}(s,t), \tag{4}$$

$$B_{++} \rightarrow B_{+-} = B_{++}/m_\pi^2, \tag{5}$$

$$2\pi m_\pi(\alpha-\beta)_{\pi^\pm} \rightarrow 2\pi/m_\pi(\alpha+\beta)_{\pi^\pm}. \tag{6}$$

So, knowing Im parts of M_{++} and M_{+-}, the Re parts of M_{++} and M_{+-} can be calculated by means of DR's and from there a behaviour of cross-section can be predicted. The Im parts are found with the help of the expression

$$ImM_{++}^{(V)}(s,t) = \mp sImM_{+-}^{(V)}(s,t) = \sum_V \mp 4g_V^2 \frac{s.\Gamma_0}{(m_V^2 - s)^2 + \Gamma_0^2}, \tag{7}$$

where $m_V = m_\rho, m_{b_1}, m_{a_1}, m_{a_2}$ are saturating resonances in s and u-channels and m_σ, m_{f_0}, m_{f_2} in t-channel. For g_V and Γ_0 holds

$$g_V^2 = 6\pi\sqrt{\frac{m_V^2}{s}}\left(\frac{m_V^2}{m_V^2 - m_\pi^2}\right)^3 \Gamma(V \rightarrow \gamma\pi), \tag{8}$$

$$\Gamma_0 = \left(\frac{s - m_\pi^2}{m_V^2 - m_\pi^2}\right)^{3/2} m_V\Gamma_V. \tag{9}$$

Note: Since the Compton process $\gamma\pi^\pm \rightarrow \gamma\pi^\pm$ and $\gamma\gamma \rightarrow \pi^0\pi^0$ should be described by a common analytic function, the same DR's can be used for a description of $\gamma\gamma \rightarrow \pi^0\pi^0$. However, in this case they are saturated (through Im parts of the amplitudes) by contributions of $\rho(770)$, $\omega(782)$ and $\phi(1020)$ in s and u channels and σ, $f_0(980)$, $f_2(1270)$ mesons in t channel.

The parameters of $\rho, \omega, \phi, f_0, f_2$ are taken from Review of Particle Properties [9].

As a result in this approach $m_\sigma, \Gamma_\sigma, \Gamma(\sigma \rightarrow \gamma\gamma)$ and $(\alpha+\beta)_{\pi^0}$, $(\alpha-\beta)_{\pi^0}$ are left as **free parameters**.

They are determined in comparison of the corresponding cross-section with existing data, considering the following three cases:

a) all parameters are free
b) $(\alpha+\beta)_{\pi^0}$ and $(\alpha-\beta)_{\pi^0}$ are fixed at the values given by χPT [2]
c) $(\alpha+\beta)_{\pi^0}$ and $(\alpha-\beta)_{\pi^0}$ are taken from the papers [3, 4]

According to the values of χ^2 all three sets of parameters in Table 1 give perfect description of data. Question: which of the values of $(\alpha+\beta)_{\pi^0}$ and $(\alpha-\beta)_{\pi^0}$ are the most realistic?

Tab. 1. Values of determined parameters.

	$m_\sigma(MeV)$	$\Gamma_\sigma(MeV)$	$\Gamma_{\sigma\rightarrow\gamma\gamma}(keV)$	$(\alpha+\beta)_{\pi^0}$	$(\alpha-\beta)_{\pi^0}$	χ^2
a)	547 ± 45	1204 ± 362	0.62± 0.19	0.98 ± 0.03	-1.6 ± 2.2	0.30
b)	471 ± 23	706 ± 164	0.33 ± 0.07	1.15 ± 0.30	-1.9 ± 0.2	0.42
c)	584 ± 32	1378 ± 277	0.83 ± 0.16	1.00 ± 0.05	-0.6 ± 1.8	0.31

2 Determination of m_σ, Γ_σ

We try to specify one of them to be correct first by a determination of m_σ, Γ_σ in the framework of the scalar pion form factor (FF) by using the experimental data on the isoscalar S-wave $\pi\pi$ phase shift.

Pion scalar FF is defined

$$\langle \pi^i(p_2)|\hat{m}(\bar{u}u+\bar{d}d)|\pi^j(p_1)\rangle = \delta^{ij}\Gamma_\pi(t). \tag{10}$$

We require the pion scalar FF to be in the form of the Padè type approximation in $q=\sqrt{(t-4)/4}$ variable

$$\Gamma_\pi(t) = \frac{\sum_{n=0}^{M} a_n q^n}{\prod_{i=1}^{N}(q-q_i)}, \tag{11}$$

from where one can express

$$\tan\delta_\pi(t) = \frac{Im[\prod_{i=1}^{N}(q-q_i)^* \sum_{n=0}^{M} a_n q^n]}{Re[\prod_{i=1}^{N}(q-q_i)^* \sum_{n=0}^{M} a_n q^n]}. \tag{12}$$

It follows from the unitarity condition that $\delta_\pi \equiv \delta_0^0$ in elastic region. Then one finds

$$\tan\delta_0^0(t) = \frac{A_1 q + A_3 q^3 + A_5 q^5 + A_7 q^7 + \dots}{1 + A_2 q^2 + A_4 q^4 + A_6 q^6 + \dots}. \tag{13}$$

If the degree of the numerator is higher than the degree of its denominator, then

$$\lim_{q\to\infty} \delta_0^0(t) = \frac{\pi}{2}. \tag{14}$$

If the degree of the numerator is lower than the degree of its denominator, then

$$\lim_{q\to\infty} \delta_0^0(t) = 0. \tag{15}$$

Applying the Cauchy formula together with the asymptotic condition the following dispersion relation without subtractions can be derived for the pion scalar FF

$$\Gamma_\pi(t) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \frac{Im\Gamma_\pi(t')}{t'-t}dt', \tag{16}$$

which together with the elastic unitarity condition gives the Omnes-Muskhelishvili integral equation. Its solution is just the pion scalar FF phase representation

$$\Gamma_\pi(t) = P_n(t) exp\left[\frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \frac{\delta_0^0(t')}{t'-t}dt'\right], \tag{17}$$

where $P_n(t)$ is arbitrary but normalized polynomial of the $n^{\rm th}$ degree. However, if $\delta_0^0(t)$ has the property given by the condition (14), then one has to start with the dispersion relation with one subtraction at the origin

$$\Gamma_\pi(t) = 1 + \frac{t}{\pi}\int_{4m_\pi^2}^{\infty} \frac{Im\Gamma_\pi(t')}{t'(t'-t)}dt', \tag{18}$$

Tab. 2. Coefficients of polynomial.

A_1	A_2	A_3	A_4	A_5	A_6	χ^2/ndf
0.31737	-0.11279	0.15783	—	—	—	1.6
0.31737	0.38323	0.21093	-0.056202	—	—	1.56
0.24875	0.12608	0.20096	-0.027166	-0.015037	—	1.409
0.26218	0.058832	0.14748	-0.030127	-0.014064	0.001217	1.426

which leads to the pion scalar FF phase representation as follows

$$\Gamma_\pi(t) = P_n(t) exp \left[\frac{t}{\pi} \int_{4m_\pi^2}^{\infty} \frac{\delta_0^0(t')}{t'(t'-t)} dt' \right]. \tag{19}$$

In order to find the degrees of the numerator and denominator in Eq. (13) and to determine the numerical values of the corresponding parameters $A_1, A_2, A_3, \ldots$ we have collected 66 experimental points on the S-wave isoscalar $\pi\pi$ phase shift $\delta_0^0(t)$ at the elastic region $4m_\pi^2 \le t \le 1\mathrm{GeV}^2$ from papers [5–8] and carried out the fit of all existing data. We found the best χ^2/NDF for five coefficients of the numerator. Results are presented in Table 2. Thus we considered the pion scalar FF phase representation (19) with one subtraction. Transforming (13) with five nonzero coefficients into the form

$$\delta_0^0(t) = \frac{1}{2i} \ln \frac{(1 + A_2 q^2 + A_4 q^4) + i(A_1 q + A_3 q^3 + A_5 q^5)}{(1 + A_2 q^2 + A_4 q^4) - i(A_1 q + A_3 q^3 + A_5 q^5)}, \tag{20}$$

and inserting into (19), the calculation was carried out by theory of residues. So, finally the scalar pion FF has the form

$$\Gamma_\pi(t) = P_n(t) \frac{(q-q_1)}{(q-q_2^*)(q-q_3^*)(q-q_4^*)(q-q_5^*)} \times \frac{(i-q_2^*)(i-q_3^*)(i-q_4^*)(i-q_5^*)}{(i-q_1)}, \tag{21}$$

with q_i and q_i^* parameters given in Table 3, from where one can determine the σ-meson parameters (q_2^* and q_4^* correspond to σ-meson) by using the formula

$$t_\sigma = (m_\sigma - i\frac{\Gamma_\sigma}{2})^2, \tag{22}$$

Tab. 3. Roots of numerator and denominator.

q_1	-3.64061 + 0.998237i
q_2	-1.36100 + 0.825791i
q_3	-1.84145i
q_4	1.36100 + 0.825791i
q_5	3.64061 + 0.998237i

q_1^*	-3.64061 - 0.998237i
q_2^*	-1.36100 - 0.825791i
q_3^*	1.84145i
q_4^*	1.36100 - 0.825791i
q_5^*	3.64061 - 0.998237i

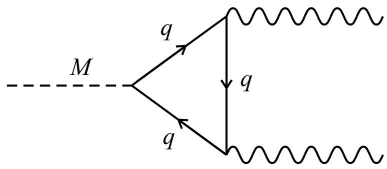

Fig. 1. The M decay into $\gamma\gamma$; $M = \pi^0, \sigma$

which are

$$m_\sigma = 456\,\mathrm{MeV}, \qquad \Gamma_\sigma = 387\,\mathrm{MeV}. \tag{23}$$

3 Prediction of $\Gamma_{\sigma\to\gamma\gamma}$

Another way to specify electric and magnetic polarizabilities is via calculation of $\Gamma_{\sigma\to\gamma\gamma}$, corresponding to the σ-meson masses determined by Fil'kov and Kashevarov, in the framework of the linearized Nambu-Jona-Lasinio model type Lagrangian which is of the form

$$\mathcal{L}_{q\bar{q}M} = g_M \bar{q}(x)[\sigma(x) + i\pi(x)\gamma_5]q(x). \tag{24}$$

The two-photon decay width of the σ-meson is calculated by means of the constituent quark triangle loop (Fig. 1.) with colourless and flavourless quarks with charge equal to the electron one. The mass of the quark in the triangle loop is taken to be $m_u = m_d = m_q = (280 \pm 20)$ MeV. The standard procedure for calculating of the processes gives us the amplitude

$$iM(\sigma \to \gamma\gamma) = ig_\sigma e^2 Q^2 \varepsilon^*_\mu(k_1)\varepsilon^*_\nu(k_2)$$
$$\times \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p+k_1)^2 - m_q^2} \frac{1}{p^2 - m_q^2} \frac{1}{(p-k_2)^2 - m_q^2} L^{\mu\nu}_\sigma(p, k_1, k_2). \tag{25}$$

with

$$L^{\mu\nu}_\sigma(p, k_1, k_2) = Tr[((\not{p} + \not{k_1}) + m_q)\gamma^\mu(\not{p} + m_q)\gamma^\nu((\not{p} - \not{k_2}) + m_q)]. \tag{26}$$

Just from the expression of the Lagrangian it follows that the σ-meson coupling constant with quarks is equal π^0 coupling constant. g_{π^0} was determined from the world average value of $\Gamma(\pi^0 \to \gamma\gamma) = (0.008 \pm 0)$ keV [9]. The amplitude (Eq. (25)) leads to

$$\Gamma(\sigma \to \gamma\gamma) = \frac{\alpha^2 m_\sigma^3 g_\sigma^2}{16\pi^3 m_q^3} \left[\int_0^1 dx \int_0^{1-x} dy \frac{1 - 4xy}{1 - (m_\sigma/m_q)^2 xy} \right]^2. \tag{27}$$

The corresponding results for $\Gamma(\sigma \to 2\gamma)$ are presented in the Table 4.

Tab. 4. Theoretical results for the decay width of σ into $\gamma\gamma$.

m_σ (MeV)	$\Gamma_{\sigma\to\gamma\gamma}$ (keV)
547	0.46675
471	0.2247
584	0.7376

4 Conclusion

We have carried out the unambiguous determination of the σ-meson parameters m_σ and Γ_σ in the framework of the isoscalar pion FF by using the data on the isoscalar S-wave $\pi\pi$ phase shift and also $\Gamma_{\sigma\to\gamma\gamma}$ in the framework of the linearized Nambu-Jona-Lasinio model in order to resolve the problem of evaluation of neutral pion polarizabilities, which take the most probable values

$$\alpha_{\pi^0} = -0.38 \times 10^{-4} fm^3; \qquad \beta_{\pi^0} = 1.53 \times 10^{-4} fm^3, \tag{28}$$

to be consistent with χPT predictions.

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