

FISHER INFORMATION: QUANTUM UNCERTAINTY RELATION**I. Chakrabarty¹***Department of Mathematics, Heritage Institute of Technology
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The paper deals with the reformulation of quantum uncertainty relation involving position and momentum of a particle on the basis of the Kerridge measure of inaccuracy and the Fisher information.

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1 Introduction

A basic problem in quantum physics is to find the limits placed on the joint measurability of non-commuting variables which may be, e. g., the optical phase and the photon numbers in optics, the position and momentum of a free and bound particle in atomic physics. The fact that two non-commuting observables A and B cannot simultaneously obtain sharp eigenvalues represents the cornerstone of the principle of uncertainty in quantum mechanics and this can be expressed in different forms, commonly called uncertainty relation [1]. An uncertainty relation provides an estimate of the minimum uncertainty expected in the outcome of a measurement of an observable, given the uncertainty in outcome of a measurement of another observable. The limits of the measurability placed on the concrete quantum system are commonly given by standard and entropic uncertainty relations [2–5]. The standard uncertainty relation of two non-commuting observables represents the product of their standard deviations whereas the entropic uncertainty relation is given by the sum of their Shannon entropies.

In the present paper, we shall use the concept of statistical inference and the Fisher information to reformulate the uncertainty relation of non-commuting observables. The problem is to discuss the quantum uncertainty relation involving position and momentum of a particle on the basis of the Kerridge measure of inaccuracy (or the Kullback relation information) and the Fisher information [6–8].

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2 Quantum mechanics and the Fisher information

A fundamental problem of statistical inference is the problem of deciding how well a set of outcomes, obtained in independent measurements, fits to a proposed probability distribution [9] if the probability is characterized by one or more parameters. This problem is equivalent to inferring the value of its parameter(s) from the observed measurement outcomes x . Perhaps the simplest and the most well-known approach to the studied problem is the theory of estimation developed by R.A. Fisher [10]. In this approach, it is assumed that one out of a family of distribution functions $\{P_\theta(x), \theta \in R\}$ is the true one, the parameter θ being unknown. To make inferences about the parameter θ , one constructs estimators, i.e. the functions $T(x_1, x_2, \dots, x_n)$ of the outcomes of n independent repeated measurements. The value of the function is intended to represent the best guess for θ . Several criteria are imposed on these estimators in order to ensure that the values do in fact constitute 'good' estimates of the parameter θ . One criterion is unbiasedness. For example, if

$$\langle T \rangle = \int_R T(x_1, x_2, \dots, x_n) \prod_{i=1}^n p_\theta(x_i) dx_i = \theta \quad (1)$$

for all θ , that is, if the expectation value of T represents that value, we call the estimator $T(x_1, x_2, \dots, x_n)$ to be an unbiased estimator of θ . Again, if the standard deviation $\sigma(T)$ is as small as possible, the estimator is called efficient. The famous Cramer-Rao inequality puts to a bound to the efficiency of an arbitrary estimator [6]:

$$\text{var}(T) = \sigma^2(T) \geq \frac{(\frac{d\langle T \rangle}{d\theta})^2}{nI(\theta)}, \quad (2)$$

where

$$I(\theta) = \int_R \left(\frac{\partial \ln p_\theta(x)}{\partial \theta} \right)^2 p_\theta(x) dx \quad (3)$$

is a quantity depending only on the family $\{p_\theta(x), \theta \in R\}$, known as the Fisher information. According to Fisher [10], $I(\theta)$ is the amount of information about the parameter θ contained in the random variable \tilde{x} [10]. in the case of a single observation (2) reduces to

$$\text{var}(T) = \sigma^2(T) \geq \frac{(\frac{d\langle T \rangle}{d\theta})^2}{I(\theta)} \quad (4)$$

and, finally, if the estimator T is unbiased, the inequality (4) becomes

$$\text{var}(T) = \sigma^2(T) \geq \frac{1}{I(\theta)}. \quad (5)$$

In quantum mechanics, the probability amplitudes, and not the probability densities, are the fundamental quantities. Accordingly, we define the Fisher information in quantum mechanics as follow :

Let $\{\psi_\theta(x) : \theta \in R\}$ be the family of Schrödinger wave functions sufficiently well behaved with respect to the parameter θ . The parameter θ may be interpreted as temporal a spatial shift or

any other physical parameter. According to the statistical interpretation of wave function $p_\theta(x) = |\psi_\theta(x)|^2$ describes the probability density of the particle, if $\psi_\theta(x)$ is normalized. The wave function $\psi_\theta(x)$ is a probability amplitude corresponding to $p_\theta(x)$ (for any real ϕ , $\exp(i\phi)\psi_\theta(x)$ is also a probability amplitude corresponding to $p_\theta(x)$). The Fisher information of $\{\psi_\theta(x) : \theta \in R\}$ with respect to the parameter θ , is defined according to (3) as [11,12]

$$I(\psi_\theta) = \int_R \left(\frac{\partial \ln |\psi_\theta(x)|^2}{\partial \theta} \right)^2 |\psi_\theta(x)|^2 dx = 4 \int_R \left(\frac{\partial |\psi_\theta(x)|}{\partial \theta} \right)^2 dx \quad (6)$$

provided the integral is finite. This is essentially the Fisher information of the family of likelihood functions $p_\theta(x) = |\psi_\theta(x)|^2$. In particular, if we assume the invariance of the wave function under the shift of location parameter x that is, if $\psi_\theta(x) = \psi(x + \theta)$ then (6) becomes

$$I(\psi_\theta) = 4 \int_R \left(\frac{\partial |\psi(x + \theta)|}{\partial \theta} \right)^2 dx = 4 \int_R \left(\frac{d|\psi(x)|}{dx} \right)^2 dx \quad (7)$$

which is now independent of the parameter θ and henceforth we shall denote it by $I(\psi)$. In the next section we shall study the deep significance of the Fisher information $I(\psi)$ and the Cramer-Rao inequality in relation to uncertainty relation.

3 Fisher information: uncertainty relation

For the sake of simplicity, we consider a one-dimensional system — a particle whose quantum state is represented by Schrödinger wave function $\psi(x)$. Particle's co-ordinate x , in the statistical interpretation of the wave function $\psi(x)$, is a continuous variable with probability density

$$P(x) = \psi^*(x)\psi(x) = |\psi(x)|^2. \quad (8)$$

The co-ordinate x and momentum p of the particle, according to the Heisenberg uncertainty principle, are subject to the uncertainty relation

$$(\Delta x)(\Delta p) \geq \frac{\hbar}{2}, \quad (9)$$

where (Δx) and (Δp) are the standard deviations of the position (location) x and momentum p respectively. For simplicity we assume that the centre of the wave packet is at $x = 0$, that is, $\langle x \rangle = 0$ and let $\langle p \rangle = 0$. Then [13,14]

$$(\Delta x)^2 = \langle x^2 \rangle = \int_R x^2 |\psi(x)|^2 dx, \quad (10)$$

$$(\Delta p)^2 = \langle p^2 \rangle = \int_R \left| \frac{\hbar}{i} \frac{d\psi(x)}{dx} \right|^2 dx. \quad (11)$$

Let us now approach to the uncertainty relation (9) by a route based on the Fisher information and the Cramer-Rao inequality developed in the statistical theory of estimation [6,7,10]. Stam [15] was the first who pointed out the importance of the Fisher information and the Cramer-Rao

inequality in the study of quantum uncertainty relation. We are going to do this but with a difference. Our method is based on the Kullback relative information [7] and the Kerridge measure of inaccuracy in the choice of correct probability density [8].

Any measurement of the position x is always a subject to an error. Let $\hat{x} = x + \delta x$ be the observed value of the position x , where δx is the inaccuracy in the location parameter x resulting from the measurement. Then according to Kerridge [8], the error occurred about the state of the system (particle) in accepting the probability density $P(x + \delta x)$ in place of $P(x)$ is given by [7]

$$K(x|x + \delta x) = \int_R P(x) \ln \frac{P(x)}{P(x + \delta x)} dx = \int_R |\psi(x)|^2 \ln \frac{|\psi(x)|^2}{|\psi(x + \delta x)|^2} dx. \quad (12)$$

The expression (12), known as the Kullback-Leibler discrimination information or simply the Kullback relative information, gives a measure of directed divergence between the probability densities $P(x)$ and $P(x + \delta x)$. For small δx , expanding $K(x|x + \delta x)$ in powers of δx , we have

$$K(x|x + \delta x) = \frac{1}{2} I(\psi) (\delta x)^2, \quad (13)$$

where

$$I(\psi) = \int_R \left[\frac{d}{dx} \ln |\psi(x)|^2 \right]^2 |\psi(x)|^2 dx = 4 \int_R \left[\frac{d}{dx} |\psi(x)| \right]^2 dx \quad (14)$$

is the Fisher information with respect to the position x . In general $\psi(x)$ is a complex function, but in the particular case when it may be a real function [12]

$$I(\psi) = 4 \int_R \left| \frac{d\psi(x)}{dx} \right|^2 dx. \quad (15)$$

Our basic problem is to reduce the error about the state of the system given by (13) resulting from the measurement. This can be achieved by the Cramer-Rao inequality [6]

$$I(\psi) (\delta x)^2 \geq \frac{(\delta x)^2}{(\Delta x)^2}, \quad (16)$$

where $(\Delta x)^2 = \langle (x - \langle x \rangle)^2 \rangle = \langle x^2 \rangle$ is the mean square deviation of the position of the particle. Since

$$\left(\frac{\hbar}{i} \right)^2 I(\psi) = 4 \int_R \left| \frac{\hbar}{i} \frac{d\psi(x)}{dx} \right|^2 dx = 4 \langle p^2 \rangle, \quad (17)$$

we can reduce the inequality (16) to the usual form of the Heisenberg uncertainty relation :

$$\langle x^2 \rangle \langle p^2 \rangle \geq \frac{\hbar^2}{4} \quad (18)$$

or

$$(\Delta x)(\Delta p) \geq \frac{\hbar}{2} \quad (19)$$

provided we take the inaccuracy δx in the location parameter x to be equal to the standard deviation (Δx) . The equality in (18) corresponds to the equality in (16). The later holds when

$$\frac{d}{dx} \ln |\psi(x)|^2 = \alpha(x - \langle x \rangle) = \alpha x \quad (20)$$

or

$$|\psi(x)|^2 = A \exp(-\lambda x^2). \quad (21)$$

Adjusting the constants A and λ by the normalization condition we have the Gaussian wave packet [13,14]

$$\psi(x) = \frac{1}{\sqrt{2\pi(\Delta x)^2}} \exp \left[-\frac{x^2}{4(\Delta x)^2} \right] \quad (22)$$

which corresponds to the wave packet having minimum uncertainty product

$$(\Delta x)(\Delta p) = \frac{\hbar}{2}. \quad (23)$$

The above approach is different from that of Stam [15] and others [11,12]. In the present case the Fisher information is not the starting concept, it results from the Kerridge measure of inaccuracy in terms of the Kullback relative information. The uncertainty relation (18) results from the requirement of reducing the inaccuracy in the measurement process.

4 Conclusion

There exists extensive literature on the different forms of the uncertainty relations in quantum mechanics [16]. The present paper is an attempt to reinterpret the traditional quantum uncertainty relation in terms of the statistical theory of information and inference. The basis of the present approach is the Kerridge's interpretation of the Kullback relative information (12) as a measure of inaccuracy. The Kullback relative information $K(x|x + \delta x)$ given by (13) introducing the Fisher information $I(\psi)$ defines a metric - a statistical distance on the parametric space. The importance of statistical distance in the study of uncertainty relations (both thermodynamical and quantum mechanical) was stressed by Uffink and Van Tith [17].

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