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DUAL QUATERNIONIC REFORMULATION OF ELECTROMAGNETISM

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It can be said that Maxwell's equations are one of the well-known equations of Physics. Despite the fact that this equations are more than hundred years old (1865), they still are the cornerstone of electrodynamics. Although they have several notations in the literature, this paper gives new representational method that based on dual quaternions. We have aimed the reformulation of classical electrodynamics. Therefore new dual quaternionic equations related with electromagnetism are derived.

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1 introduction

For electromagnetism several representational methods have been found in the literature. One of them, except vectors, is using complex numbers. Shen and Kong [1] explained in detail how to represent a time harmonic real physical quatity by a complex one, especially when the physical quantity is a vector and gave Maxwell's equations in phasor form. The second method is using four vectors. Kyrala [2] formulated relativistic electromagnetism by four vectors and investigated how the descriptions of the electromagnetic fields change under the Lorentz transformations. The third method is based on complex quaternions (also named biquaternions). Imaeda [3] investigated the classical electrodynamics and the theory of quaternions while Negi et al. [4] studied classical electrodynamics by complex quaternions and derived Maxwell's equations in compact and expanded forms, respectively, for quaternions and matrix representations. Lambek [5] employed biquaternions in special relativity and Maxwell's equations. Gsponer and Hurni [6] noted that the use of Clifford algebras particularly of biquaternions, can lead to a satisfactory formulation of elementary particle physics and and Maxwell's equations.

In general dual quatenions have dynamic and kinematical applications [7-16]. In this paper we have tried the reformulation of classical electrodynamics and Maxwell's equations by using dual quaternions. Starting with preliminaries of real and dual quaternion algebra, Maxwell's equations have been derived in terms of dual quaternions. Then dual quaternionic equations related with electromagnetism have been developed.

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2 Real Quaternions

Similar to a complex number $z = q_0 + iq_1$ where $i^2 = -1$, a real quaternion $\mathbf{q}$ which is a four component number can be defined as

$$\mathbf{q} = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3. \tag{1}$$

Here q_0, q_1, q_2 and q_3 are scalars and $e_0 = 1$. Imaginar basis elements e_1, e_2, e_3 satisfy the following conditions

$$e_0^2 = 1 \quad e_j e_k = -\delta_{jk} e_0 + \epsilon_{jkl} e_l \quad (j, k, l = 1, 2, 3) \tag{2}$$

where δ_{jk} and ϵ_{jkl} are the Kronecker delta and three-index Levi-Civita symbol, respectively. e_1, e_2, e_3 can be seen as orthogonal unit spatial vectors. With ideas from the both real and vector algebra, the quaternion $\mathbf{q}$ may be viewed as a linear combination of a scalar q_0 and a spatial vector $\vec{q}$:

$$\mathbf{q} = q_0 + \vec{q} = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3. \tag{3}$$

If $q_0 = 0$, the quaternion $\mathbf{q}$ becomes

$$\mathbf{q} = \vec{q} = q_1 e_1 + q_2 e_2 + q_3 e_3 \tag{4}$$

which is called vector quaternion. When $\vec{q} = \vec{0}$, $\mathbf{q}$ is a real number and is called a scalar quaternion. As seen, scalars and spatial vectors are quaternions and they are a subspace of quaternions.

Addition and subtraction of two quaternions $\mathbf{p}$ and $\mathbf{q}$ are defined as

$$\mathbf{p} \pm \mathbf{q} = (p_0 \pm q_0) + (\vec{p} \pm \vec{q}) = (p_0 \pm q_0) + (p_1 \pm q_1)e_1 + (p_2 \pm q_2)e_2 + (p_3 \pm q_3)e_3. \tag{5}$$

The quaternion addition and subtraction obey associative and commutative laws.

For a quaternion multiplication, a prodecure called quaternion product, can be defined. Because a vector is special case of a quaternion, care should be taken to distinguish the quaternion product from either the dot product or the cross product. Product of two quaternions $\mathbf{p}$ and $\mathbf{q}$ is

$$\mathbf{pq} = [p_0 + \vec{p}][q_0 + \vec{q}] = p_0 q_0 + p_0 \vec{q} + q_0 \vec{p} - \vec{p} \cdot \vec{q} + \vec{p} \times \vec{q} \tag{6}$$

where the dot and cross indicate, respectively, the usual three-dimensional scalar and vector products. The quaternion product is associative and distributive but not commutative.

Quaternions not only have properties of scalars and vectors, but quaternion algebra also has similarities to complex algebra. For any quaternion $\mathbf{q}$ there exists a quaternion conjugate that is denoted by $\mathbf{q}^*$ and is defined by negating its vector part (or imaginary part)

$$\mathbf{q}^* = q_0 - \vec{q} = q_0 - q_1 e_1 - q_2 e_2 - q_3 e_3. \tag{7}$$

The norm of a quaternion $\mathbf{q}$, denoted by N_{q}, is a scalar quaternion and it is given as

$$N_{\mathrm{q}} = \mathbf{qq}^* = \mathbf{q}^*\mathbf{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2. \tag{8}$$

If norm of a quaternion $N_q = 1$ is called unit quaternion.

Inverse of a quaternion (non-zero norm), denoted by $\mathbf{q}^{-1}$, is defined as

$$\mathbf{q}^{-1} = \frac{\mathbf{q}^*}{N_{\mathrm{q}}}. \tag{9}$$

If $N_{\mathrm{q}} = 1$, $\mathbf{q}$ is unit quaternion, inverse of quaternion becomes

$$\mathbf{q}^{-1} = \mathbf{q}^*. \tag{10}$$

3 Dual Quaternions

A brief summary of dual numbers and dual quaternions is presented in this section to provide the necessary background for the mathematical formulation developed in this paper.

A dual number , as invented by Clifford [17] in 1873, is

$$Q = q + \epsilon q'. \tag{11}$$

Here ϵ is known as the dual unit having the property $\epsilon^2 = 0$. The real numbers q and q' are called the real and dual parts of Q, respectively. It should be emphasized that dual numbers are extension of real numbers.

A dual quaternion $\mathbf{Q}$ is defined in a similar way to dual numbers

$$\mathbf{Q} = \mathbf{q} + \epsilon \mathbf{q}' \tag{12}$$

where $\mathbf{q}$ and $\mathbf{q}'$ are real quaternions

$$\mathbf{q} = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 \tag{13}$$

and

$$\mathbf{q}' = q'_0 e_0 + q'_1 e_1 + q'_2 e_2 + q'_3 e_3. \tag{14}$$

Dual quaternion $\mathbf{Q}$ can be written in the more clear form as

$$\begin{aligned}\mathbf{Q} &= (q_0 + \epsilon q'_0)e_0 + (q_1 + \epsilon q'_1)e_1 + (q_2 + \epsilon q'_2)e_2 + (q_3 + \epsilon q'_3)e_3 \\ &= Q_0 e_0 + Q_1 e_1 + Q_2 e_2 + Q_3 e_3.\end{aligned} \tag{15}$$

Here Q_0, Q_1, Q_2, Q_3 are dual numbers.

A dual quaternion $\mathbf{Q}$ consists of the scalar part S_{Q} and vector part V_{Q} where

$$S_{\mathrm{Q}} = Q_0 \tag{16}$$

and

$$V_{\mathrm{Q}} = \vec{Q} = Q_1 e_1 + Q_2 e_2 + Q_3 e_3. \tag{17}$$

Addition and subtraction of two dual quaternions $\mathbf{P}$ and $\mathbf{Q}$ are defined as

$$\mathbf{P} + \mathbf{Q} = (P_0 \pm Q_0)e_0 + (P_1 \pm Q_1)e_1 + (P_2 \pm Q_2)e_2 + (P_3 \pm Q_3)e_3. \tag{18}$$

Dual quaternion addition and subtraction obey associative and commutative laws.

The product of two dual quaternions $\mathbf{P}$ and $\mathbf{Q}$ is

$$\mathbf{PQ} = (P_0 + \vec{P})(Q_0 + \vec{Q}) = P_0 Q_0 + P_0 \vec{Q} + Q_0 \vec{P} - \vec{P} \cdot \vec{Q} + \vec{P} \times \vec{Q} \tag{19}$$

where the dot and cross product indicate, respectively, the usual three-dimensional scalar and vector products.

For any dual quaternion $\mathbf{Q}$ there exists a complex conjugate,

$$\mathbf{Q}^* = S_{\mathrm{Q}} - V_{\mathrm{Q}} = Q_0 - Q_1 e_1 - Q_2 e_2 - Q_3 e_3. \tag{20}$$

while the dual conjugate $\mathbf{Q}^{\mathrm{c}}$ is given by

$$\begin{aligned}\mathbf{Q}^{\mathrm{c}} = S_{\mathrm{Q}}^{\mathrm{c}} + V_{\mathrm{Q}}^{\mathrm{c}} = (q_0 - \epsilon q_0')e_0 + (q_1 - \epsilon q_1')e_1 + (q_2 - \epsilon q_2')e_2 + (q_3 - \epsilon q_3')e_3 \\ = Q_0^{\mathrm{c}} e_0 + Q_1^{\mathrm{c}} e_1 + Q_2^{\mathrm{c}} e_2 + Q_3^{\mathrm{c}} e_3\end{aligned} \tag{21}$$

in which c denotes dual conjugate. Complex conjugation and dual conjugation are also the dual quaternions.

The norm of a dual quaternion in general is a dual scalar,

$$N_{\mathrm{Q}} = \mathbf{QQ}^* = \mathbf{Q}^*\mathbf{Q} = Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2. \tag{22}$$

The inverse of a dual quaternion $\mathbf{Q}$ (non-zero norm) is also the dual quaternions and can be defined as

$$\mathbf{Q}^{-1} = \frac{\mathbf{Q}^*}{N_{\mathrm{Q}}}. \tag{23}$$

Dual quaternions of norm unity are called unit dual quaternions.

4 Maxwell's Equations in Dual Quaternionic Form

While studing of electromagnetics we are concerned with four vector quantities called electromagnetic fields. They are electric field strength $\vec{E}(V/m)$,

$$\vec{E} = E_1 e_1 + E_2 e_2 + E_3 e_3 \tag{24}$$

electric flux density $\vec{D}(C/m^2)$,

$$\vec{D} = D_1 e_1 + D_2 e_2 + D_3 e_3 \tag{25}$$

magnetic field strength $\vec{H}(A/m)$,

$$\vec{H} = H_1 e_1 + H_2 e_2 + H_3 e_3 \tag{26}$$

and magnetic flux density $\vec{B}(T)$,

$$\vec{B} = B_1 e_1 + B_2 e_2 + B_3 e_3 \tag{27}$$

and they are functions of space and time. Here e_1, e_2 and e_3 which satisfy (2), can be seen as three dimensional cartesian coordinate basis. Fundamental theory of the these electromagnetic fields based on Maxwell's equations which are given by

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{28}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{29}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{30}$$

$$\vec{\nabla} \cdot \vec{D} = \rho_v \tag{31}$$

where $\vec{J}$ is electric current density,

$$\vec{J} = J_1 e_1 + J_2 e_2 + J_3 e_3. \tag{32}$$

J_1, J_2 and J_3 are three components of the $\vec{J}$ along the x, y and z direction respectively. The del operator is

$$\vec{\nabla} = \frac{\partial}{\partial x} e_1 + \frac{\partial}{\partial y} e_2 + \frac{\partial}{\partial z} e_3 \tag{33}$$

and ρ_{v} is electric charge density. Equations (24) - (32) express the physical laws governing the $\vec{E}$, $\vec{D}$, $\vec{H}$ and $\vec{B}$ fields and sources $\vec{J}$ and ρ_{v} at every point in space and at all times. So far there has been no experimental evidence of an electromagnetic field that does not satisfy all four of Maxwell's equations.

Maxwell's first equation (28) is known as Faraday's law of induction. It is discovered by Michael Faraday (1791-1867). Maxwell's second equation (29) shows Ampére's law. Although Ampére (1775-1836) did not contain the displacement current $\frac{\partial \vec{D}}{\partial t}$ term James Clerk Maxwell (1831-1879) first proposed to the addition of this term to the conduction or convection current $\vec{J}$ in Ampére's law. The induction of this extra term was very significant because it made it possible to predict the existence of electromagnetic waves. Equations (30) and (31) are known respectively as magnetic Gauss' law and electric Gauss' law [1].

In order to represent Maxwell's equations in dual quaternionic form, let's define two basic dual quaternion. One of them is dual quaternionic differential operator that is given by

$$\mathbf{D} = \vec{\nabla} + \epsilon \frac{\partial}{\partial t} = \left[\frac{\partial}{\partial x} e_1 + \frac{\partial}{\partial y} e_2 + \frac{\partial}{\partial z} e_3 \right] + \epsilon \frac{\partial}{\partial t}. \tag{34}$$

The other is dual quaternion $\mathbf{M}$ which include both electric field strength $\vec{E}$ and magnetic field strength $\vec{H}$,

$$\mathbf{M} = -\vec{E} + \epsilon \vec{H} = -[E_1 e_1 + E_2 e_2 + E_3 e_3] + \epsilon [H_1 e_1 + H_2 e_2 + H_3 e_3]. \tag{35}$$

Using Gaussian and natural units $\varepsilon = \mu = c = 1$, we can operate $\mathbf{D}$ on $\mathbf{M}$ as

$$\begin{aligned} \mathbf{DM} = \left[\vec{\nabla} + \epsilon \frac{\partial}{\partial t} \right] [-\vec{E} + \epsilon \vec{H}] &= \vec{\nabla} \cdot \vec{E} - \vec{\nabla} \times \vec{E} + \epsilon \left[-\frac{\partial \vec{E}}{\partial t} - \vec{\nabla} \cdot \vec{H} + \vec{\nabla} \times \vec{H} \right] \\ &= \frac{\partial \vec{H}}{\partial t} + \left[\rho_v + \epsilon \vec{J} \right]. \end{aligned} \tag{36}$$

4.1 Static Fields

The electromagnetic fields are generally functions of space and time. In the special case in which they are time-invariant, by setting the time derivative equal to zero Maxwell's equations become

$$\vec{\nabla} \times \vec{E} = 0 \tag{37}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} \tag{38}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{39}$$

$$\vec{\nabla} \cdot \vec{D} = \rho_{\mathrm{v}} \tag{40}$$

$\frac{\partial \vec{E}}{\partial t}$ and $\frac{\partial \vec{H}}{\partial t}$ are vanishes. Therefore dual quaternionic equation (36) gets new form as

$$\mathbf{DM} = \left[\vec{\nabla} + \epsilon \frac{\partial}{\partial t}\right] [-\vec{E} + \epsilon \vec{H}] = \rho_v + \epsilon \vec{J}. \tag{41}$$

4.2 Dual Quaternionic Current Density and Constituve Relations

The sources $\vec{J}$ and ρ_v have been defined. These sources can be combined in a dual quaternionic form that is

$$\mathbf{S} = \rho_v + \epsilon \vec{J} = \rho_v + \epsilon [J_1 e_1 + J_2 e_2 + J_3 e_3]. \tag{42}$$

Dual quaternion $\mathbf{S}$ contains two basic sources that are electric current density $\vec{J}$ and electric charge density ρ_v. It's called dual quaternionic current density. Thus equations (36) and (41) get more compact form;

$$\mathbf{DM} = \frac{\partial \vec{H}}{\partial t} + \mathbf{S} \tag{43}$$

and

$$\mathbf{DM} = \mathbf{S}. \tag{44}$$

These are the reformulation of Maxwell's equations of classical electrodynamics in dual quaternionic form.

On the other hand conjugate of dual quaternionic differential operator $\mathbf{D}$ is denoted by $\mathbf{D}^*$ and is given as

$$\mathbf{D}^* = \vec{\nabla}^* + \epsilon \frac{\partial}{\partial t} = \left[-\frac{\partial}{\partial x} e_1 - \frac{\partial}{\partial y} e_2 - \frac{\partial}{\partial z} e_3\right] + \epsilon \frac{\partial}{\partial t}. \tag{45}$$

Operating of $\mathbf{D}^*$ on $\mathbf{S}$ yields

$$\mathbf{D}^*\mathbf{S} = \left[\vec{\nabla}^* + \epsilon \frac{\partial}{\partial t}\right] [\rho_v + \epsilon \vec{J}] = \epsilon \left[\frac{\partial \rho_v}{\partial t} + \vec{\nabla} \cdot \vec{J} - \vec{\nabla} \times \vec{J}\right]. \tag{46}$$

Scalar part of $\mathbf{D}^*\mathbf{S}$ asserts that

$$Sc[\mathbf{D}^*\mathbf{S}] = \frac{\partial \rho_v}{\partial t} + \vec{\nabla} \cdot \vec{J} \tag{47}$$

is zero and it is called equation of continuity. This is one of the most important consequences of Maxwell's equation. Therefore equation (47) becomes

$$Sc[\mathbf{D}^*\mathbf{S}] = \left[\vec{\nabla}^* + \epsilon \frac{\partial}{\partial t}\right][\rho_v + \epsilon \vec{J}] = 0. \tag{48}$$

4.3 Dual Quaternionic Electromagnetic Potential

Another consequence of Maxwell's equations can be obtained by the existence of a electromagnetic potential. Similar to equation (42), it is possible to define the electromagnetic potential of an electrically charged particle as

$$\mathbf{P} = \vec{A} - \epsilon\varphi = [A_1 e_1 + A_2 e_2 + A_3 e_3] - \epsilon\varphi. \tag{49}$$

It's called dual quaternionic electromagnetic potential. Here $\vec{A}$ and φ are magnetic vector potential and electric field potential, respectively. By applying the dual quaternion product on dual conjugate form of differential operator $\mathbf{D}$ which is denoted $\mathbf{D}^c$ and dual quaternionic electromagnetic potential $\mathbf{P}$, we can obtain

$$\mathbf{D}^c\mathbf{P} = \left[\vec{\nabla} - \epsilon \frac{\partial}{\partial t}\right][\vec{A} - \epsilon\varphi] = \vec{\nabla} \times \vec{A} + \epsilon \left[-\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t}\right]. \tag{50}$$

Since

$$\vec{H} = \vec{\nabla} \times \vec{A} \tag{51}$$

and

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t} \tag{52}$$

are defined, respectively, magnetic vector field $\vec{H}$ and electric vector field $\vec{E}$, equation (50) becomes

$$\mathbf{D}^c\mathbf{P} = \left[\vec{\nabla} - \epsilon \frac{\partial}{\partial t}\right][\vec{A} - \epsilon\varphi] = \vec{H} + \epsilon\vec{E}. \tag{53}$$

As it has been shown that operation of $\mathbf{D}^c$ on $\mathbf{P}$ allows determination of the fields $\vec{H}$ and $\vec{E}$.

5 Conclusion

There are many representational methods for electromagnetism in the literature, but this paper offers a new alternative one. Negi [4] et al. and Lambek [5] investigated Maxwell's equations by using quaternions with complex components, known as biquaternions or complex quaternions. In the papers [3-5] biquaterionic differential operator is defined as $\frac{\partial}{\partial t} + i\vec{\nabla}$, hermitian biquaternions as $\vec{H} + i\vec{E}$ and $\varphi + i\vec{J}$, biquaternionic potential as $\varphi + i\vec{A}$ where $i^2 = -1$. Generally biquaternionic differential operator satisfies following equations: First one is $\left[\frac{\partial}{\partial t} - i\vec{\nabla}\right][\vec{H} + i\vec{E}][\varphi + i\vec{J}] = 0$. According to Lambek [5] this is the compact notation of Maxwell's equations and it was first pointed out independently by Conway (1911) and Silberstein (1912), although it might have been realized by Clerk Maxwell himself (1869). Second, scalar part of $\left[\frac{\partial}{\partial t} + i\vec{\nabla}\right][\varphi + i\vec{J}]$ is zero and this is known the equation of continuity. Finally, vector part of $\left[\frac{\partial}{\partial t} + i\vec{\nabla}\right][\varphi + i\vec{A}]$ is $\vec{H} + i\vec{E}$.

Although in general dual quatenions have dynamic and kinematics applications [7-16], first time we have investigated classical electrodynamics with dual quaternions. In the same way which is mentioned above, we have defined some new dual quaternionic quantities as dual quaternionic differential operator $\vec{\nabla} + \epsilon\frac{\partial}{\partial t}$, dual quaternion $-\vec{E} + \epsilon\vec{H}$, current density $\varphi + \epsilon\vec{J}$ and electromagnetic potential $\vec{A} - \epsilon\varphi$ where $\epsilon^2 = 0$. Using quaternions with dual components we have combined the Maxwell's four equations in a single equation (36) or (41). Then we have managed to derive constitutive relations and related equations (47), (50) and (53). These equations satisfy all the equations which were given by Negi [4] et al. and Lambek [5] using biquaternions. These dual quaternionic representations can be succesfully adapted to fit physical situations.

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