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THE QUATERNIONIC ENERGY CONSERVATION EQUATION FOR ACOUSTIC

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After introducing the quaternions and its algebra, the equations of the linear acoustics are defined. The local conservation equation for energy of the linear acoustics using quaternions and a quaternionic first-order Lagrangian description is then formulated. Using the variational principle, the local conservation equation for the energy is derived from the quaternionic gauge transformation. The purpose is to provide an alternative for the usual derivations.

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1 Introduction

Quaternions as generalization of complex numbers were first invented by Sir W. R. Hamilton [1] after a lengthy struggle to extend the theory of complex numbers to three dimensions. Clifford extended Hamilton's notation of a quaternion as the ratio of two vectors. Quaternions are divisible algebraically. This property is advantageous for physicists. Quaternions play an important role for the justification of the postulates in the special relativity, quantum and classical mechanics as well as solving high energy physics' problems. They can also be used to represent physical quantities. There are a lot of studies with quaternions in physics. Some examples include Revisiting Quaternion Formulation and Electromagnetism [2], If Hamilton Had Prevailed: Quaternions in Physics [3], Quaternionic Electron Theory: Dirac's Equation [4], Quaternionic Formulation of the Classical Fields [5], Quaternions and Simple D=4 Supergravity [6], Dimensional-Directional Analysis by a Quaternionic Representation of Physical Quantities [7], Quaternions and the Heuristic Role of Mathematical Structures in Physics [8], and Molecular Symmetry with Quaternions [9].

In the analysis of linear acoustics, the two basic variables are pressure and particle velocity. In general, both p and $\overrightarrow{u}$ are functions of position and time. The usual derivations of the local conservation equations for energy, linear momentum and angular momentum of acoustics are based on time translation, space translation and space rotation invariance.

The rest of the paper is organized within 3 sections. Section 2 reveals quaternion algebra with notations and preliminaries. In section 3, the gauge transformation, Lagrange description, linear acoustics and their definitions are given, in which the quaternionic energy conservation equation for the acoustics is also defined. Conclusion is drawn in the last section.

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2 Quaternion Algebra

A quaternion is a quantity represented symbolically by Q and it is defined through the following equation

$$Q = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 = [q_0, q_1, q_2, q_3] = \sum q_k e_k, \tag{1}$$

where the real numbers q_k denote the components of Q relative to the unitary quaternion $e_k (k = 0, 1, 2, 3)$. The scalar and vectorial parts of Q are designated, respectively, by $(Q)_s$ and $(Q)_v$, and defined as

$$(Q)_s = q_0 e_0, \quad (Q)_v = q_1 e_1 + q_2 e_2 + q_3 e_3. \tag{2}$$

A quaternion is a scalar(or vector) quaternion if its vectorial(or scalar) part is equal to zero. The unitary quaternions satisfy the Hamilton and Taif multiplication table as follows:

	e_0	e_1	e_2	e_3
e_0	1	e_1	e_2	e_3
e_1	e_1	-1	e_3	-e_2
e_2	e_2	-e_3	-1	e_1
e_3	e_3	e_2	-e_1	-1

The quaternion conjugate $Q^*$of a given quaternion Q is defined as

$$Q^* = q_0 e_0 - q_1 e_1 - q_2 e_2 - q_3 e_3 = [q_0, -q_1, -q_2, -q_3]. \tag{3}$$

The product of two quaternions, namely Q and P, with components q_k and p_k is given by [10]

$$QP = [q_0 p_0 - (q_1 p_1 + q_2 p_2 + q_3 p_3)]\, e_0 + [q_0 p_1 + q_1 p_0 + (q_2 p_3 - q_3 p_2)]\, e_1$$

$$+ [q_0 p_2 + q_2 p_0 + (q_3 p_1 - q_1 p_3)]\, e_2 + [q_0 p_3 + q_3 p_0 + (q_1 p_2 - q_2 p_1)]\, e_3. \tag{4}$$

It must be noted that the product of quaternions is not commutative $(QP \neq PQ)$, but associative

$$P(QR) = (PQ)R. \tag{5}$$

The inverse Q^{-1} of a quaternion Q, the norm N_Q of which is different from zero, is given by

$$Q^{-1} = \frac{Q^*}{N_Q}, \tag{6}$$

where the norm of a given Q is defined as $N_Q = \sqrt{QQ^*}$. The quotient between quaternions P and Q with $N_Q \neq 0$ is defined as

$$\frac{P}{Q} = PQ^{-1} = \frac{PQ^*}{N_Q}. \tag{7}$$

The vector quaternion P with components $[0, p_1, p_2, p_3]$ and a vector P of the Euclidean tridimensional space with components (p_1, p_2, p_3) are reciprocally associated.

If P and Q are the vectors associated with the quaternion vectors P and Q, the scalar and vectorial products of these vectors can be expressed as

$$P \cdot Q = (PQ)_s,$$
$$P \times Q = (PQ)_v.$$

It must be noticed that

$$PQ = -P \cdot Q + P \times Q. \tag{8}$$

The quaternion notation of the ∇ operator in the Hamilton's notation can be written as

$$\nabla = e_i \nabla_i. \tag{9}$$

The divergence and curl operators are expressed as [11]

$$\nabla F(x) = -\nabla \cdot F(x) + \nabla \times F(x) = -divF(x) + curlF(x), \tag{10}$$

where '$\cdot$' and '$\times$' are the dot and cross product of two vector quaternions, respectively. The Laplace operator can also be defined as follows:

$$N(\nabla) = \nabla_i \nabla_i. \tag{11}$$

The inner (dot) product of two vector quaternions is simply written

$$P \cdot Q = -\frac{1}{2}[PQ + (PQ)^*], \tag{12}$$

and the cross (vector) product of two vector quaternions is

$$P \times Q = \frac{1}{2}[PQ - (PQ)^*]. \tag{13}$$

3 Linear Acoustics

The equations of linear acoustics are expressed in the following form

$$p = c^2 \rho, c^2 = (\frac{\partial p}{\partial \rho})_0, \tag{14}$$

$$\frac{\partial \rho}{\partial t} = -\rho_0 \nabla \cdot \overrightarrow{u}, \tag{15}$$

$$\rho_0 \frac{\partial \overrightarrow{u}}{\partial t} = -\overrightarrow{\nabla}(\rho), \tag{16}$$

where ρ_0 is the ambient fluid density, $\rho(\overrightarrow{x}, t)$ is the small deviation from ambient density, $p(\overrightarrow{x}, t)$ is the small deviation from ambient pressure, and $\overrightarrow{u}(\overrightarrow{x}, t) = (u, v, w)$ is the fluid velocity vector [12]. Eq.14 is the adiabatic equation of state relating density to pressure, Eq.15 is the conservation of mass equation, and Eq.16 is the fluid linear momentum equation. Using Eq.14,Eq.15 and Eq.16, the equations of linear acoustics can be written as follows:

$$\frac{\partial(\rho/\rho_0)}{\partial(ct)} = -\overrightarrow{\nabla} \cdot (\overrightarrow{u}/c), \tag{17}$$

$$\frac{\partial(\overrightarrow{u}/c)}{\partial(ct)} = -\overrightarrow{\nabla}(\rho/\rho_0). \tag{18}$$

By changing variables to

$$\rho' = \rho/\rho_0, \tag{19}$$

$$t' = ct, \tag{20}$$

$$\overrightarrow{u'} = \overrightarrow{u}/c, \tag{21}$$

and dropping primes, Eq.17 and Eq.18 then become

$$\frac{\partial\rho}{\partial t} = -\overrightarrow{\nabla} \cdot \overrightarrow{u}, \tag{22}$$

$$\frac{\partial\overrightarrow{u}}{\partial t} = -\overrightarrow{\nabla}\rho, \tag{23}$$

which are the form of the acoustic equations [13]. By taking the curl of Eq.23 and assuming that $\overrightarrow{u}$ and ρ are twice differentiable in space and time, it can be written as

$$\frac{\partial}{\partial t}(\overrightarrow{\nabla} \times \overrightarrow{u}) = 0. \tag{24}$$

This shows that the curl of $\overrightarrow{u}$ is constant in time . It is accepted as the usual acoustic assumption that $\overrightarrow{\nabla} \times \overrightarrow{u} = 0$. The acoustic state vector can be written by quaternions as

$$\Psi = [\rho, u, v, w] = \rho e_0 + u e_1 + v e_2 + w e_3. \tag{25}$$

The Eq.22 and Eq.23 can be expressed as

$$\boxdot\Psi = (\nabla\Psi)^*, \tag{26}$$

where

$$\nabla = [0, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}], \tag{27}$$

and

$$\boxdot = [\frac{\partial}{\partial t}, 0, 0, 0]. \tag{28}$$

The quaternionic Lagrangian density can be derived from Eq.26 as

$$\pounds = \Psi^* \cdot [\boxdot\Psi - (\nabla\Psi)^*], \tag{29}$$

and the variational principle is [14]

$$\delta A = \delta \int dt \int dx\ \pounds = 0. \tag{30}$$

Ψ and Ψ^* are considered as independent variables in the calculus of variations.

The conservation law associated with the acoustic field can be obtained via quaternionic transformation of the form

$$\Psi \longrightarrow \Psi' = e^{i\alpha}\Psi \approx [1+\alpha, 0, 0, 0]\Psi, \tag{31}$$

where α is an arbitrary infinitesimal function of x and t. The type of transformation in Eq.31 can be called a quaternionic gauge transformation of the first kind although the properties of the quantity α in this study are not considered. When the quaternionic transformation defined by Eq.31 is applied to Eq.29, the quaternionic Lagrangian density can be given the following form

$$\pounds'(\Psi') = \pounds(\Psi + [1+\alpha, 0, 0, 0]\Psi) = \pounds + \delta\pounds, \tag{32}$$

where

$$\delta\pounds = \Psi^* \cdot \{\boxdot([\alpha, 0, 0, 0]\Psi) - (\nabla([\alpha, 0, 0, 0]\Psi))^*\}. \tag{33}$$

Expanding the derivatives in Eq.33 gives

$$\delta\pounds = \Psi^* \cdot \{(\boxdot[\alpha, 0, 0, 0])\Psi - ((\nabla[\alpha, 0, 0, 0])\Psi)^* + [\alpha, 0, 0, 0](\boxdot\Psi - (\nabla\Psi)^*)\}. \tag{34}$$

According to Eq.26, the terms multiplying α in Eq.34 vanish, and the variation in the action A becomes

$$\delta A = -\int dt \int dx \{\boxdot[\alpha, 0, 0, 0](\Psi^* \cdot \Psi) - \Psi^* \cdot ((\nabla[\alpha, 0, 0, 0])\Psi)^*\}. \tag{35}$$

After an integration by parts in space and time, the requirement that the variation in the action must vanish for arbitrary α leads to the quaternionic equation

$$\boxdot(\Psi^* \cdot \Psi) - \Psi^* \cdot (\nabla\Psi)^* = 0. \tag{36}$$

It has been assumed here that the boundary terms in space and time are vanishing. Eq.36 is a quaternionic local conservation equation of energy for acoustics, and it can be written as

$$\frac{\partial}{\partial t}\left(\frac{\rho^2}{2} + \frac{u^2 + v^2 + w^2}{2}\right) + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0. \tag{37}$$

Eq.37 is the energy conservation equation for acoustics. The quantity W is the acoustic energy density and it has the following form

$$W = \frac{\rho^2}{2} + \frac{u^2 + v^2 + w^2}{2}, \tag{38}$$

and the vector I,

$$I = (\rho u, \rho u, \rho w), \tag{39}$$

is the acoustic energy flux.

4 Conclusions

The usual method for deriving the local energy conservation law is to apply the Noether's theorem by making use of the translational and rotational invariances of a second-order Lagrangian [13]. In the present study, the local energy conservation equation for the linear acoustics using quaternionic Lagrangian and the first kind of gauge transformation have been reformulated by eliminating the use of translational and rotational invariance. The quaternionic equations defined above for the acoustics have compact representation, and the results are the same as the results found by Nagem et al. These equations can be easily used for similar calculations. The procedure used here to derive the local energy conservation equation may be applied to derive the linear momentum equation for the acoustics and the local conservation equation for the acoustic angular momentum.

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