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BIQUATERNIONIC REPRESENTATIONS OF ANGULAR MOMENTUM AND DIRAC EQUATION

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In the present article, after defining biquaternions, the general properties of biquaternion's algebra are introduced. The matrix representations of biquaternions are presented, as well. Then, the biquaternionic angular momentum is reformulated in terms of biquaternionic product. A new biquaternionic definition of the Dirac equation and its solution are given by the use of biquaternion's basis..

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1 Introduction

The use of algebraical structures in the explaining of physical events is needed. Biquaternions, which belong to Clifford algebra, have such an algebraical structure. This algebraical structure of biquaternions can be used for the generalization of quantum mechanics. Biquaternions, which are also named the complex quaternions, are useful numbers in the justification of the postulates in the special and general relativity, and quantum mechanics. There are a lot of studies done with quaternions and biquaternions in physics. Some of them, for example, are Revisiting Quaternion Formulation and Electromagnetism [1], If Hamilton Had Prevailed: Quaternions in Physics [2], Quaternions and Simple D = 4 Supergravity [3], Quaternionic Formulation of the Classical Fields [4], Quaternions and the Heuristic Role of Mathematical Structures in Physics [5], Quaternionic Electron Theory: Dirac's Equation [6], A One-Component Dirac Equation [7] and On a Generalization of Quantum Mechanics by Biquaternions [8] is another example.

The fundamental equation of relativistic quantum mechanics is the Dirac equation. Firstly, A. W. Conway investigated the Dirac's relativistic equation and used the complex quaternions [9]. The Dirac equation accomplishes the beginning of a theory, which is known to be the quantum field theory, as well.

The organization of the paper is as follows: Section 2 reveals biquaternion algebra with notations and preliminaries. The matrix representations of biquaternions are showed in section 3. Section 4 implies the biquaternionic angular momentum and section 5 derives the biquaternionic Dirac equation and its solution. A summary and perspective of our work are given in the final section.

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2 Biquaternion Algebra

The algebra of biquaternions, which are hypercomplex numbers, is not commutative [10, 11]. A biquaternion is a quantity represented symbolically by $\mathbf{Q}$ and it is defined by the following equation

$$\mathbf{Q} = \mathbb{Q}_0 \mathbf{e}_0 + \mathbb{Q}_1 \mathbf{e}_1 + \mathbb{Q}_2 \mathbf{e}_2 + \mathbb{Q}_3 \mathbf{e}_3, \tag{1}$$

where $\mathbb{Q}_m$'s $(m = 0, 1, 2, 3)$ denote complex numbers and $\mathbf{e}_i^{'}$s($i = 1, 2, 3$) are non-commutative triples with $\mathbf{e}_i^2 = -1$. The multiplication of $\mathbf{e}_i\mathbf{e}_j$ is defined as follows:

$$\mathbf{e}_i\mathbf{e}_j = -\delta_{ij}\mathbf{e}_0 + \varepsilon_{ijk}\mathbf{e}_k, (\mathbf{e}_0 = 1;\ i,j,k = 1,2,3) \tag{2}$$

where δ_{ij} and ε_{ijk} are the Kronecker delta and the three-index Levi-Civita symbol, respectively. A biquaternion is expressed as a tetrad of complex numbers, i.e.

$$\mathbf{Q} = (\mathrm{a}_0+\mathrm{ib}_0)\mathbf{e}_0 + (\mathrm{a}_1+\mathrm{ib}_1)\mathbf{e}_1 + (\mathrm{a}_2+\mathrm{ib}_2)\mathbf{e}_2 + (\mathrm{a}_3+\mathrm{ib}_3)\mathbf{e}_3 = (\mathbb{Q}_0, \mathbb{Q}), \tag{3}$$

where $\mathrm{a}_0,\mathrm{a}_1,\mathrm{a}_2,\mathrm{a}_3,\mathrm{b}_0,\mathrm{b}_1,\mathrm{b}_2,\mathrm{b}_3$ are all real numbers and $\mathrm{i}^2 = -1$. The scalar and vectorial parts of $\mathbf{Q}$ are defined, respectively, by

$$\mathbb{Q}_0 = (\mathrm{a}_0 + \mathrm{i}\ \mathrm{b}_0), \quad \mathbb{Q} = (\mathrm{a} + \mathrm{i}\ \mathrm{b}). \tag{4}$$

The product of two biquaternions $\mathbf{Q}$ and $\mathbf{P}$ is given by [1]

$$\mathbf{PQ} = \mathrm{P}_0\mathrm{Q}_0 - \mathbf{P}\cdot\mathbf{Q} + \mathrm{P}_0\mathbf{Q} + \mathbf{P}\mathrm{Q}_0 + \mathrm{i}\,\mathbf{P}\times\mathbf{Q}, \tag{5}$$

where $'\cdot'$and $'\times'$ indicate the usual three-dimensional scalar and vector products, respectively. The result of the product $\mathbf{PQ}$ is also biquaternion. This product is not commutative but rather associative.

For any biquaternion there exists a quaternion conjugate

$$\overline{\mathbf{Q}} = \mathbb{Q}_0 \mathbf{e}_0 - \mathbb{Q}_1 \mathbf{e}_1 - \mathbb{Q}_2 \mathbf{e}_2 - \mathbb{Q}_3 \mathbf{e}_3 = (\mathbb{Q}_0, -\mathbb{Q}), \tag{6}$$

while the $\mathbf{Q}^*$ is defined as

$$\mathbf{Q}^* = \mathbb{Q}_0^* \mathbf{e}_0 + \mathbb{Q}_1^* \mathbf{e}_1 + \mathbb{Q}_2^* \mathbf{e}_2 + \mathbb{Q}_3^* \mathbf{e}_3, \tag{7}$$

in which $'*'$ denotes the complex conjugate. Quaternion conjugation is an automorphism of ring of biquaternions, i.e.

$$(\overline{\mathbf{PQ}}) = (\overline{\mathbf{Q}})(\overline{\mathbf{P}}). \tag{8}$$

The complex conjugation is described as an anti-automorphism

$$(\mathbf{PQ})^* = \mathbf{Q}^*\mathbf{P}^*. \tag{9}$$

In eq.(5), if P_0 and Q_0 are zero, the products of $\mathbf{P}\cdot\mathbf{Q}$ and $\mathbf{P}\times\mathbf{Q}$ can be written as follows:

$$\mathbf{P}\cdot\mathbf{Q} = -\frac{\mathbf{PQ} + \overline{\mathbf{PQ}}}{2}, \tag{10}$$

and

$$\mathbf{P} \times \mathbf{Q} = \frac{\mathbf{PQ} - \overline{\mathbf{PQ}}}{2\,\mathrm{i}}. \tag{11}$$

In general, the norm $N(\mathbf{Q})$ of a biquaternion is a complex scalar

$$N(\mathbf{Q}) = \mathbf{Q}\overline{\mathbf{Q}} = \overline{\mathbf{Q}}\mathbf{Q} = \mathbb{Q}_0^2 + \mathbb{Q}_1^2 + \mathbb{Q}_2^2 + \mathbb{Q}_3^2, \tag{12}$$

which obeys the composition algebra for the norms of two biquaternions $\mathbf{P}$ and $\mathbf{Q}$, i.e.

$$N(\mathbf{PQ}) = N(\mathbf{P})N(\mathbf{Q}). \tag{13}$$

But the norm of a given $\mathbf{Q}$, $N(\mathbf{Q})$, may be zero. So, biquaternions do not form division algebra.

The inverse of a biquaternion $\mathbf{Q}(non - zero\, norm)$ is defined as

$$\mathbf{Q}^{-1} = \frac{\overline{\mathbf{Q}}}{N(\mathbf{Q})}. \tag{14}$$

The inverse $\mathbf{Q}^{-1}$ is a biquaternion. The product of $\mathbf{QQ}^{-1}$ is equal 1, i.e.

$$\mathbf{QQ}^{-1} = \mathbf{Q}^{-1}\mathbf{Q} = \mathbf{1}. \tag{15}$$

Biquaternions of norm unity can also be written as the product of two biquaternions of norm unity.

3 The Matrix Representation of Biquaternions

3.1 The 2×2 matrix representation of biquaternions

The orthogonal unit basis of a biquaternion are represented by Pauli matrices σ_1, σ_2, σ_3 divided by $\mathrm{i} = \sqrt{-1}$

$$e_1 \simeq \begin{bmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{bmatrix}, \tag{16}$$

$$e_2 \simeq \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{17}$$

$$e_3 \simeq \begin{bmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{bmatrix}, \tag{18}$$

$$I = e_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{19}$$

where e_0 is the real 2×2 unit matrix, I.

A biquaternion can be represented by the complex 2×2 matrices

$$Q = \begin{bmatrix} \mathbb{Q}_0 - \mathrm{i}\,\mathbb{Q}_3 & -\mathbb{Q}_2 - \mathrm{i}\,\mathbb{Q}_1 \\ \mathbb{Q}_2 - \mathrm{i}\,\mathbb{Q}_1 & \mathbb{Q}_0 + \mathrm{i}\,\mathbb{Q}_3 \end{bmatrix}. \tag{20}$$

The determinant of eq.(20) is equal to the norm of a given **Q**

$$\det Q = N(\mathbf{Q}). \tag{21}$$

3.2 The 4×4 matrix representation of biquaternions

Biquaternions can be represented by the 4×4 matrices. The basis of a biquaternion and $'\mathrm{i}'$ complex number are written in the matrix form as

$$e_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{22}$$

$$e_1 = \begin{bmatrix} 0 & \mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{bmatrix}, \tag{23}$$

$$e_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \tag{24}$$

$$e_3 = \begin{bmatrix} -\mathrm{i} & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & \mathrm{i} \end{bmatrix}, \tag{25}$$

$$i = \begin{bmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \end{bmatrix}. \tag{26}$$

Using by these matrices, a given biquaternion (eq.3) can be represented by the following 4×4 matrix

$$Q = \begin{bmatrix} \mathrm{a}_0\text{-}\mathrm{ia}_3 & \mathrm{a}_2\text{+}\mathrm{ia}_1 & \mathrm{b}_3\text{+}\mathrm{ib}_0 & \text{-}\mathrm{b}_1\text{+}\mathrm{ib}_2 \\ \text{-}\mathrm{a}_2\text{+}\mathrm{ia}_1 & \mathrm{a}_0\text{+}\mathrm{ia}_3 & \text{-}\mathrm{b}_1\text{-}\mathrm{ib}_2 & \text{-}\mathrm{b}_3\text{+}\mathrm{ib}_0 \\ \mathrm{b}_3\text{+}\mathrm{ib}_0 & \text{-}\mathrm{b}_1\text{+}\mathrm{ib}_2 & \mathrm{a}_0\text{-}\mathrm{ia}_3 & \mathrm{a}_2\text{+}\mathrm{ia}_1 \\ \text{-}\mathrm{b}_1\text{-}\mathrm{ib}_2 & \text{-}\mathrm{b}_3\text{+}\mathrm{ib}_0 & \text{-}\mathrm{a}_2\text{+}\mathrm{ia}_1 & \mathrm{a}_0\text{+}\mathrm{ia}_3 \end{bmatrix}. \tag{27}$$

In eq.(27), if c_0, c_1, c_2 and c_3 are taken as follows:

$$c_0 = a_0 - i\, a_3,\ \ c_1 = -a_2 + i\, a_1,\ \ c_2 = b_3 + i\, b_0,\ \ c_3 = -b_1 - i\, b_2. \tag{28}$$

Now, the Q matrix can be rewritten

$$Q = \begin{bmatrix} c_0 & -c_1^* & c_2 & c_3^* \\ c_1 & c_0^* & c_3 & -c_2^* \\ c_2 & c_3^* & c_0 & -c_1^* \\ c_3 & -c_2^* & c_1 & c_0^* \end{bmatrix}, \tag{29}$$

where c_0, c_1, c_2 and c_3 are complex numbers and $'*'$ denotes the complex conjugate of complex numbers [12, 13].

3.3 The 8×8 matrix representation of biquaternions

A biquaternion, which has eight-components, can be represented by the 8×8 matrix, as well. The Q matrix of a given **Q** biquaternion can be defined as follows:

$$Q = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ -\mathcal{B} & \mathcal{A} \end{bmatrix}, \tag{30}$$

where $\mathcal{A}$ and $\mathcal{B}$ are real quaternions. A and B are skew-symmetric matrices and are expressed as

$$A = a_0\Gamma_0 + a_1\Gamma_1 + a_2\Gamma_2 + a_3\Gamma_3, \tag{31}$$

$$B = b_0\Gamma_0 + b_1\Gamma_1 + b_2\Gamma_2 + b_3\Gamma_3. \tag{32}$$

According to eqs.(30, 31 and 32), the biquaternionic Q matrix is written as

$$Q = (a_0 + Jb_0)\alpha_0 + (a_1 + Jb_1)\alpha_1 + (a_2 + Jb_2)\alpha_2 + (a_3 + Jb_3)\alpha_3, \tag{33}$$

where

$$J = \varepsilon \otimes I_4 = \begin{bmatrix} 0 & I_4 \\ -I_4 & 0 \end{bmatrix}, \qquad \varepsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{34}$$

$$\alpha_0 = I_2 \otimes I_4 = \begin{bmatrix} I_4 & 0 \\ 0 & I_4 \end{bmatrix}, \qquad \alpha_j = I_2 \otimes \Gamma_j = \begin{bmatrix} \Gamma_j & 0 \\ 0 & \Gamma_j \end{bmatrix}. \tag{35}$$

I_2 and I_4 denote the unit matrices of order 2 and 4. The Γ_j matrices are

$$\Gamma_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{36}$$

$$\Gamma_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \tag{37}$$

$$\Gamma_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}. \tag{38}$$

In eq.(33), the 8×8 matrix form of a given $\mathbf{Q}$ biquaternion is given by α_0 and α_j matrices, in which the α_0 and α_j definitions give the rule of product as follows:

$$\alpha_0\alpha_j = \alpha_j\alpha_0 = \alpha_j, \tag{39}$$

$$\alpha_0^2 = -\alpha_j^2 = I_8 = a_0, \tag{40}$$

$$\alpha_j\alpha_k = -\delta_{jk}\alpha_0 + \varepsilon_{jkl}\alpha_l. \tag{41}$$

The matrix J of eq.(33) corresponds to the imaginary quantity. The basis elements α_0, α_1, α_2, α_3, $J\alpha_0$, $J\alpha_1$, $J\alpha_2$ and $J\alpha_3$ form a group as well as a division ring. Then, the 8×8 matrix representation of $\mathbf{Q}$ [1] can be expressed by eq.(42)

$$Q_{ij} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & b_0 & b_1 & b_2 & b_3 \\ -a_1 & a_0 & -a_3 & a_2 & -b_1 & b_0 & -b_3 & b_2 \\ -a_2 & a_3 & a_0 & -a_1 & -b_2 & b_3 & b_0 & -b_1 \\ -a_3 & -a_2 & a_1 & a_0 & -b_3 & -b_2 & b_1 & b_0 \\ -b_0 & -b_1 & -b_2 & -b_3 & a_0 & a_1 & a_2 & a_3 \\ b_1 & -b_0 & b_3 & -b_2 & -a_1 & a_0 & -a_3 & a_2 \\ b_2 & -b_3 & -b_0 & b_1 & -a_2 & a_3 & a_0 & -a_1 \\ b_3 & b_2 & -b_1 & -b_0 & -a_3 & -a_2 & a_1 & a_0 \end{bmatrix}_{(i,j=1,2,\dots,8)} . \tag{42}$$

The Q_{ij} matrix is skew-symmetric.

4 The Biquaternionic Angular Momentum

It is well known that the angular momentum is the vector quantity represented in the following manner

$$\overrightarrow{L} = \overrightarrow{r} \times \overrightarrow{p}, \tag{43}$$

where $\overrightarrow{r}$ and $\overrightarrow{p}$ are the position and momentum vectors, respectively. If $\overrightarrow{r}$ and $\overrightarrow{p}$ vectors are written in the biquaternionic notation, the following equations are get

$$\mathbf{r} = \mathrm{ix}\,\mathbf{e}_1 + \mathrm{iy}\,\mathbf{e}_2 + \mathrm{iz}\,\mathbf{e}_3, \tag{44}$$

and

$$\mathbf{p} = \mathrm{ip}_x\mathbf{e}_1 + \mathrm{ip}_y\mathbf{e}_2 + \mathrm{ip}_z\mathbf{e}_3. \tag{45}$$

Using by eq.(11), the biquaternionic angular momentum can be written as

$$\mathbf{L} = \frac{\mathbf{rp}-\overline{\mathbf{rp}}}{2\,\mathrm{i}}, \tag{46}$$

and the biquaternionic angular momentum is defined

$$\mathbf{L} = \mathrm{i}\,(\mathrm{yp}_z\text{-}\mathrm{zp}_y)\,\mathbf{e}_1 + \mathrm{i}\,(\mathrm{zp}_x\text{-}\mathrm{xp}_z)\,\mathbf{e}_2 + \mathrm{i}\,(\mathrm{xp}_y\text{-}\mathrm{yp}_x)\,\mathbf{e}_3. \tag{47}$$

L_x, L_y and L_z called the components of the angular momentum can be written as

$$\mathrm{L}_x\text{=}\mathrm{yp}_z\text{-}\mathrm{zp}_y,\quad \mathrm{L}_y\text{=}\mathrm{zp}_x\text{-}\mathrm{xp}_z,\quad \mathrm{L}_z\text{=}\mathrm{xp}_y\text{-}\mathrm{yp}_x. \tag{48}$$

The biquaternionic angular momentum with its components is

$$\mathbf{L} = \mathrm{iL}_x\mathbf{e}_1 + \mathrm{iL}_y\mathbf{e}_2 + \mathrm{iL}_z\mathbf{e}_3. \tag{49}$$

5 The Biquaternionic Dirac Equation

Some observed events in the atomic spectrums can be explained when electrons are supposed to rotate around an axis which passes through the center of electron's mass. However, the Schrödinger equation, which is the basic equation of quantum mechanics, gives the spin to be a result. Moreover, the Schrödinger equation can be applied to the low energy systems. So, we have to write a wave equation, which can also be applied to the high energy systems. This wave equation has to include the spin of electron and to concur the special relativity. Such an equation was discovered by Dirac. Now, the new biquaternionic definition of the Dirac equation for a free-electron will be investigated.

It is well known that the classical Dirac equation for free-particle is [14]

$$(\mathrm{E} - \mathrm{c}\overrightarrow{p} \cdot \overrightarrow{\alpha} - \mathrm{mc}^2\beta)\Psi = 0, \tag{50}$$

where E, $\overrightarrow{p}$ and m are the energy, momentum and mass of particle, respectively. Also, Ψ is the spin wave function of particle. α and β are useful matrices (the usual Dirac matrices) to investigate non-relativistic limit of the Dirac equation and

$$\alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix}, (i = 1, 2, 3) \qquad \beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \tag{51}$$

where I is the 2×2 unit matrix and $\sigma_i' s$ are the 2×2 Pauli-spin matrices as follows:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{52}$$

$\alpha_1, \alpha_2, \alpha_3$ and β matrices are written as

$$\alpha_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \tag{53}$$

$$\alpha_2 = \begin{bmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & -\mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{bmatrix}, \tag{54}$$

$$\alpha_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \tag{55}$$

$$\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{56}$$

In the matrix notation, α_i can be defined with the basis of biquaternion

$$\alpha_1 = -i\, e_1, \tag{57}$$

$$\alpha_2 = -i\, e_2, \tag{58}$$

$$\alpha_3 = i\, e_3. \tag{59}$$

(with $\alpha_i^2 = 1$)

Now, the Dirac equation for free particle can be easily defined. The biquaternionic equations of momentum and α for free particle, which has x, y and z coordinates, are

$$\mathbf{P} = \mathrm{p}_x\mathbf{e}_1 + \mathrm{p}_y\mathbf{e}_2 + \mathrm{p}_z\mathbf{e}_3, \tag{60}$$

and

$$\alpha = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3. \tag{61}$$

Then, the product of $\mathbf{P}\cdot\alpha$ can be written by eq.(10) as

$$\mathbf{P}\cdot\alpha = -\frac{\mathbf{P}\alpha + \overline{\mathbf{P}\alpha}}{2} = -(\mathrm{p}_x\alpha_1 + \mathrm{p}_y\alpha_2 + \mathrm{p}_z\alpha_3). \tag{62}$$

The new biquaternionic formulation for the Dirac equation is

$$(\mathrm{E} - \mathrm{c}\mathbf{P}\cdot\alpha - \mathrm{mc}^2\mathrm{b})\mathbf{e}_0\boldsymbol{\Psi} = 0, \tag{63}$$

and

$$(\mathrm{E} + \mathrm{cp}_x\alpha_1 + \mathrm{cp}_y\alpha_2 + \mathrm{cp}_z\alpha_3 - \mathrm{mc}^2\mathrm{b})\mathbf{e}_0\boldsymbol{\Psi} = 0, \tag{64}$$

in which b defines the solution between electron and positron with b = ∓ 1. If $\alpha_1, \alpha_2, \alpha_3$ are defined by *the unit basis of biquaternion*, using by eqs.(57, 58 and 59), the Dirac equation will be written as follows:

$$((\mathrm{E} - \mathrm{m\,c}^2\ \mathrm{b})\ e_0 - i\mathrm{cp}_x e_1 - i\mathrm{cp}_y e_2 + i\mathrm{cp}_z e_3)\ \boldsymbol{\Psi} = 0. \tag{65}$$

Then, in the biquaternionic notation, we get the following equation

$$((\mathrm{E}\text{-}\mathrm{mc}^2\mathrm{b})\ e_0 + \mathrm{c\ p}_z\ e_2 + \mathrm{c}\ (\mathrm{p}_x - \mathrm{ip}_y)\ e_3)(\Psi_1\ e_0 + \Psi_2\ e_1 + \Psi_3\ e_2 + \Psi_4\ e_3) = 0. \tag{66}$$

The solutions of eq.(66) are

$$(\mathrm{E} - \mathrm{m\,c}^2\ \mathrm{b})\ \Psi_1 - \mathrm{c\ p}_z\Psi_3 - \mathrm{c}\ (\mathrm{p}_x - \mathrm{i\ p}_y)\ \Psi_4 = 0, \tag{67}$$

$$(\mathrm{E} - \mathrm{m\,c}^2\ \mathrm{b})\ \Psi_2 + \mathrm{c\ p}_z\Psi_4 - \mathrm{c}\ (\mathrm{p}_x - \mathrm{i\ p}_y)\ \Psi_3 = 0, \tag{68}$$

$$(\mathrm{E} - \mathrm{m\,c}^2\ \mathrm{b})\ \Psi_3 + \mathrm{c\ p}_z\Psi_1 + \mathrm{c}\ (\mathrm{p}_x - \mathrm{i\ p}_y)\ \Psi_2 = 0, \tag{69}$$

$$(\mathrm{E} - \mathrm{m\,c}^2\ \mathrm{b})\ \Psi_4 - \mathrm{c\ p}_z\Psi_2 + \mathrm{c}\ (\mathrm{p}_x - \mathrm{i\ p}_y)\ \Psi_1 = 0. \tag{70}$$

6 Conclusions

We have presented an alternative way to look at the biquaternionic world. A biquaternion includes both of the quaternion and quaternion with complex components. Also, this property enlarges the use of biquaternions. All quaternionic equations in physics can be defined with biquaternions, so biquaternions can be used to define the equations in the general and special relativity, and quantum mechanics.

The Dirac equation describes relativistic systems. Alternative formulations of the Dirac relativistic equation on the biquaternionic field have been written.

The angular momentum, Dirac equation and the solution of the Dirac equation by using biquaternions and their unit basis were written. These new formulations for the angular momentum and Dirac equation have useful forms for biquaternionic quantum mechanics, as well.

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