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CIRCULAR ORBITS OF A PARTICLE IN KERR FIELD: A STUDY USING EFFECTIVE ONE-BODY APPROACH

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Constraint Hamiltonian dynamics is used in an effective one-body approach to study relativistic gravitational two-body system with spin-orbit interaction. Radii of circular orbits and approximate location of innermost stable circular orbit (ISCO) of a non-spinning body around a slowly spinning body whose gravitational field is described by the Kerr metric are derived.

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1 Introduction

Astrophysical binary systems consisting of compact objects e.g. neutron star, pulsar, black hole, are very important sources of experimental data that will provide precision tests in favor of one or other theory of gravitation. They are also of much popular interest because we seek a complete understanding of nature. We know that the dynamics of binary systems are governed by the relativistic two-body problem. The relativistic two-body problem is not yet fully solved in the framework of general relativity. Various approximation methods are now in use in the study of relativistic two-body problem. Among these methods are the Post-Newtonian (PN) and Parametrized Post-Newtonian (PPN) approximations. Recently, a novel approach to the relativistic two-body problem is shown by Fiziev and Todorov [1] where the notion of an effective particle moving in an external field has been developed in much like the way of non-relativistic two-body problem. In this formalism, the well known reduced mass attributable to the effective particle depends on the total centre-of-mass (c.m.) energy of the two-body system. Another effective one-body approach to the general relativistic two-body problem is shown by Buonanno and Damour [2] where the relativistic two-body dynamics is mapped onto that of a test particle moving in an effective external metric. For the case of a non-spinning body moving in the field of another non-spinning central body (Schwarzchild metric case), both the methods [1,2] lead to identical results for, say, radius and angular frequency of circular orbits, location of ISCO etc.

The purpose of this article is to carry on the necessary calculations, following the way shown by Fiziev and Todorov [1], to find the locations of circular orbits and of ISCO for the case of a non-spinning body moving round a spinning central body (Kerr metric case). As far as we know, this is the first such calculation and as such, this is very important. We first write the Hamiltonian necessary for the study and then derive the equations of motion. Finally, we find the radii of circular orbits and location of ISCO.

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2 Hamiltonian Constraint and the Equations of Motion

We undertake the task of writing the Hamiltonian constraint suitable for description of motion of an effective particle in the field of a spinning Kerr object. Before we enter into this, we need to clarify some necessary mathematical terms. First, we need to know the effective metric in which the particle moves. Following Fiziev and Todorov [1] and Buonanno and Damour [2] we assume that the field is a deformed Kerr metric and following Fiziev and Todorov [1] we further assume that the metric is determined by a coupling parameter that depends on the two masses; the coupling parameter is given by $\alpha_G = m_1 m_2$, which replaces the mass of the source of the Kerr metric[2].

The radial coordinate has dimension of action, $r = Rm_w$, R being the invariant distance between the bodies in c.m. frame and m_w being the energy dependent reduced mass, $m_w = m_1 m_2/w$, w being the total c.m. energy of the two-body system. The effective particle c.m. energy is given by

$$E_w = \frac{w^2 - m_1^2 - m_2^2}{2w}\,. \tag{1}$$

In what follows, the four momentum of the effective particle are dimensionless. As such, we use $\in = E_w/m_w$ as the energy and $u_r = p_R/m_w$ as the radial momentum with p_R as the actual radial momentum. For a detailed knowledge of these terms and their necessity we refer to [1] (Note that we use geometrical units with $G = c = 1$, while the authors of [1] do not). To obtain the equation of motion of the effective particle, we first construct the constraint Hamiltonian, following Fiziev and Todorov [1], as

$$H = \frac{1}{2\lambda}\left[1 + g^{\mu\nu} u_\mu u_\nu\right] \approx 0\,, \tag{2}$$

where λ is a Lagrange multiplier. For the theory of constraint Hamiltonian dynamics we refer to [3]. For the Kerr field, the metric in Boyer-Lindquist form is

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{A}{\Sigma}\sin^2\theta d\varphi^2 - \frac{4Ma}{\Sigma} r \sin^2\theta \; dt d\varphi\,, \tag{3}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2Mr + a^2, \quad A = \left(r^2 + a^2\right)^2 - a^2\Delta\sin^2\theta \tag{4}$$

and M is the mass of the source of the Kerr metric, a is the spin angular momentum per unit mass of the source. It is known that motion in a plane around a Kerr object is possible only on the equatorial plane, $\theta = \pi/2, d\theta = 0$. The square of the four-gradient operator corresponding to the metric (3) applicable for motion on the equatorial plane can be written as [4]

$$\begin{aligned} g^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} = &-\frac{1}{r^2 - 2Mr + a^2}\left(r^2 + a^2 + \frac{2Mra^2}{r^2}\right)\left(\frac{\partial}{\partial t}\right)^2 + \frac{r^2 - 2Mr + a^2}{r^2}\left(\frac{\partial}{\partial r}\right)^2 + \\ &+ \frac{1}{r^2 - 2Mr + a^2}\left(1 - \frac{2M}{r}\right)\left(\frac{\partial}{\partial\varphi}\right)^2 - \frac{4Mra}{r^2\left(r^2 - 2Mr + a^2\right)}\left(\frac{\partial}{\partial\varphi}\right)\left(\frac{\partial}{\partial t}\right)\,. \end{aligned} \tag{5}$$

[2]In conventional units $\alpha_G = Gm_1m_2/c$, having the dimension of action. This, when divided by $m_w c$, gives the true radius in dimension of a length , which replaces the term GM/c^2 that usually appears in the metric. See the following.

We now write the Hamiltonian constraint (2) explicitly from (5) replacing M by α_G and using the transformations $\partial/\partial t \to -\in$, $\partial/\partial r \to u_r$, $\partial/\partial\varphi \to J$:

$$H = \frac{1}{2\lambda}\left[1 + \left(1 - \frac{2\alpha_G}{r} + \frac{a^2}{r^2}\right)u_r^2 + \frac{1 - \frac{2\alpha_G}{r}}{1 - \frac{2\alpha_G}{r} + \frac{a^2}{r^2}}\frac{J^2}{r^2} - \left\{1 + \frac{\frac{2\alpha_G}{r^3}\left(r^2 + a^2\right)}{1 - \frac{2\alpha_G}{r} + \frac{a^2}{r^2}}\right\}\in^2 + \frac{4\alpha_G a}{r^3\left(1 - \frac{2\alpha_G}{r} + \frac{a^2}{r^2}\right)}J\in\right] \approx 0\,. \quad (6)$$

That this is the correct Hamiltonian describing motion of a nonspinning effective particle around a spinning particle can be verified by the correspondence set by the requirement that when the spin is made off, the Kerr metric reduces to the Schwarzchild metric which is here translated as that the Hamiltonian should reduce to the one for Schwarzchild case. One sees that this is indeed so: $a \to 0$ reduces (6) to

$$H = \frac{1}{2\lambda}\left[1 + \left(1 - \frac{2\alpha_G}{r}\right)u_r^2 + \frac{J^2}{r^2} - \frac{\in^2}{1 - \frac{2\alpha_G}{r}}\right] \approx 0\,, \quad (7)$$

which is the Hamiltonian found by Fiziev and Todorov [1] for the Schwarzchild case. Now, we write the following equations of motion:

$$\dot{r} = \frac{\partial H}{\partial u_r} = \frac{f u_r}{\lambda} \quad (8)$$

$$\dot{\varphi} = \frac{\partial H}{\partial J} = g f^{-1}\frac{J}{\lambda r^2} + \frac{2\alpha_G a}{\lambda r^3}f^{-1}\in \quad (9)$$

$$\dot{t} = -\frac{\partial H}{\partial \in} = \frac{\in}{\lambda}\left\{1 + \frac{2\alpha_G}{r^3}\left(r^2 + a^2\right)f^{-1}\right\} - \frac{2\alpha_G a}{\lambda r^3}f^{-1}J\,, \quad (10)$$

where

$$f = \left(1 - \frac{2\alpha_G}{r} + \frac{a^2}{r^2}\right), \qquad g = \left(1 - \frac{2\alpha_G}{r}\right). \quad (11)$$

Equations (8)–(10) should be compared with the equations of Carter [5] for geodetical motion in the Kerr field. We note that Eqs. (8)–(10) reduce to Carter equations when λ is set equal to unity and the transformation $\alpha_G \to M$ is applied [4,5]. Also, in our equations mass of the orbiting particle does not appear explicitly which is explicit in Carter equations (For clarification see Refs. [1,4,5]). The parallel between our equations of motion and Carter equations is a sign of correctness of the approach presented here. However, Carter's equations are applicable for geodetical motion, i.e., for motion of a particle whose gravitational field does not disturb the field of the central body. As such, the metric in geodetical motion is determined by the central body alone, whereas in the effective one-body approach we are following here, the metric is determined by the two bodies of the binary. Hence, Carter's results are applicable in the test mass limit whereas in the effective one-body approach mass of the orbiting body can be comparable to the mass of the central body.

Now, solving (6) for u_r, we get

$$u_r = \left[\left(f^{-1} + \frac{2\alpha_G}{r^3}\left(r^2+a^2\right) f^{-2}\right) \in^2 - \frac{4\alpha_G a}{r^3} f^{-2} J \in - g f^{-2} \frac{J^2}{r^2} - f^{-1}\right]^{1/2} . \quad (12)$$

Substituting (12) in (8) and dividing (8) by (9), we get the λ—independent equation of motion

$$\frac{J}{r^2}\frac{dr}{d\varphi} =$$
$$= \left[\begin{array}{l} \frac{2\alpha_G}{r^3}\left(r^2+a^2\right) \frac{\left(1-\frac{2\alpha_G}{r}+\frac{a^2}{r^2}\right)^2}{\left(1-\frac{2\alpha_G}{r}\right)^2} \in^2 - \frac{4\alpha_G a}{r^3} \frac{\left(1-\frac{2\alpha_G}{r}+\frac{a^2}{r^2}\right)^2}{\left(1-\frac{2\alpha_G}{r}\right)^2} J \in \\ -\frac{\left(1-\frac{2\alpha_G}{r}+\frac{a^2}{r^2}\right)^2}{1-\frac{2\alpha_G}{r}} \frac{J^2}{r^2} - \frac{\left(1-\frac{2\alpha_G}{r}+\frac{a^2}{r^2}\right)^3}{\left(1-\frac{2\alpha_G}{r}\right)^2} \left(1- \in^2\right) \end{array}\right]^{1/2} \left(1 - \frac{2\alpha_G a \in}{Jrg}\right) . \quad (13)$$

This equation governs motion of the effective particle on the equatorial plane. Solution of this equation seems to be extremely difficult and this forces us to take resort to some approximations. We, therefore, assume a slowly spinning central body with $a << 1$. In the subsequent equations, then, we neglect terms proportional to a^3 and higher, and terms proportional to $1/r^4$ and higher. Moreover, the last term in Eq. (13) is neglected since both $a/r, \in$ are much smaller than unity. The resulting equation can then be put in the form

$$-\frac{dy}{d\varphi} = \left[\alpha y^3 - \gamma y^2 + \rho y - \beta\right]^{1/2} , \quad (14)$$

where

$$y = \frac{J}{r}, \quad \rho = \frac{2\alpha_G}{J}, \quad \mu = \frac{a}{J}, \quad \beta = 1- \in^2,$$
$$\alpha = \rho\left(1 - 2\mu \in + 3\mu^2 \in^2\right), \quad \gamma = 1 + 3\beta\mu^2 . \quad (15)$$

The substitution $\mu = 0$ reduces $\alpha \to \rho, \gamma \to 1$ and we get

$$-\frac{dy}{d\varphi} = \left[\rho y^3 - y^2 + \rho y - \beta\right]^{1/2} , \quad (16)$$

which is exactly Eq. (62) of Ref. [1], the equation of motion for the Schwarzchild case. Now, we study the general features of motion defined by Eq. (14). We assume that all three roots y_0, y_1, y_2 of the polynomial under the square root of Eq. (14) are positive reals, since this is required for a bounded orbit:

$$P_3(y) = \alpha y^3 - \gamma y^2 + \rho y - \beta = \alpha (y - y_0)(y - y_1)(y - y_2) , \qquad 0 < y_2 \le y_1 < y_0 . \quad (17)$$

Bounded motion belongs to the range $y_2 \le y \le y_1$ for which $P_3(y)$ is non-negative. The necessary and sufficient conditions for the constants appearing in (17) for which all zeros of $P_3(y)$ are positive are (i) positivity of β (in that case there is no change of sign of the terms in $P_3(-y)$):

$$0 < \beta \left(= 1- \in^2\right) < 1 \quad (18)$$

and the condition that the discriminant D satisfy

$$D = Q^3 + R^2 \leq 0\,, \tag{19}$$

where

$$Q = \frac{3\frac{\rho}{\alpha} - \left(\frac{\gamma}{\alpha}\right)^2}{9}\,,$$
$$R = \frac{9\left(-\frac{\gamma}{\alpha}\right)\left(\frac{\rho}{\alpha}\right) - 27\left(-\frac{\beta}{\alpha}\right) - 2\left(-\frac{\gamma}{\alpha}\right)^3}{54}\,, \tag{20}$$

which follows from the theory of polynomials [6]. After some algebra, Eq. (19) gives us

$$4\left(1 - \frac{3\rho\alpha}{\gamma^2}\right) \geq 729\left[\beta\frac{\alpha^2}{\gamma^3} + \frac{1}{3}\left(\frac{2}{9} - \frac{\alpha\rho}{\gamma^2}\right)\right]^2, \qquad \frac{3\rho\alpha}{\gamma^2} \leq 1\,. \tag{21}$$

Equation (21) can be compared with Eq. (65) of Ref. [1] to which Eq. (21) reduces in the limit $a \to 0$. Now, when the discriminant vanishes, the two roots y_1 and y_2 become equal and corresponds to circular orbit with

$$y_1 = y_2 = y_c = \frac{\gamma}{3\alpha} - \sqrt[3]{R} = \frac{\gamma\left(1 - \left(1 - \frac{3\rho\alpha}{\gamma^2}\right)^{1/2}\right)}{3\alpha}\,. \tag{22}$$

In the limit $a \to 0$, we get the circular orbit of a nonspinning effective particle around a nonspinning central body (Schwarzchild case):

$$y_c\,(S) = \frac{1 - \left(1 - 3\rho^2\right)^{1/2}}{3\rho}\,, \tag{23}$$

which coincides with Eq. (66) of Ref. [1]. We note that for circular orbit

$$\beta = \left(\alpha y_c^2 - \gamma y_c + \rho\right) y_c = \frac{2\gamma^3\left(1 - \frac{3\rho\alpha}{\gamma^2}\right)^{1/2} - 2\gamma^3 + 9\alpha\rho\gamma - 6\alpha\rho\gamma\left(1 - \frac{3\rho\alpha}{\gamma^2}\right)^{1/2}}{27\alpha^2}\,. \tag{24}$$

The frequency in a circular orbit is

$$\omega = \frac{d\varphi}{dt} = \frac{\dot{\varphi}}{\dot{t}} = \frac{y_c^2\,(1 - \rho y_c)}{J \in \left[1 + \mu^2 y_c^2 + \mu^2\rho y_c^3\right] - J\rho\mu y_c^3}\left(1 + \frac{\rho\mu y_c \in}{1 - \rho y_c}\right)\,, \tag{25}$$

which reduces to Eq. (68) of Ref. [1] in the limit $a \to 0$. To find the actual radius of a circular orbit given by the solution (22), we have to use appropriate transformation described in the beginning of this article.

Next, we find parameters for approximate innermost stable circular orbit (ISCO), which occurs when all three roots of $P_3\,(y)$ coincide. This happens when $Q = R = 0$ in Eqs. (19) and

(20) and both sides of Eq. (21) vanish. Some algebra gives us, for this approximate ISCO, the following:

$$\frac{3\rho\alpha}{\gamma^2} = 1, \qquad \beta\alpha^2 = \left(\frac{\gamma}{3}\right)^3 \tag{26}$$

$$y_1 = y_2 = y_c = y_0^{ISCO} = \frac{\gamma}{3\alpha} = \frac{\rho}{\gamma}\,. \tag{27}$$

In the limit $a \to 0$, we get the Schwarzchild values

$$y_0^{ISCO} = \frac{1}{3\rho} = \rho \qquad \Rightarrow \rho^2 = \frac{1}{3}, \qquad \rho = \frac{1}{\sqrt{3}} = \rho^{ISCO}\,(S) \tag{28}$$

which are the values found by Fiziev and Todorov [1]. The β corresponding to the ISCO defined by Eqs. (26) and (27) can be found from (24) and (27) as

$$\beta^{ISCO} = \frac{\sqrt{3}}{9}\sqrt{\frac{\rho^3}{\alpha}}\,. \tag{29}$$

Equation (29) leads to a polynomial equation of 4^{th} order whose solution gives the energy of the ISCO. We remark that the expression (29) correctly gives in the limit $a \to 0$, the Schwarzchild value

$$\beta^{ISCO}\,(S) = \frac{1}{9}\,. \tag{30}$$

A unique value for $\in$ can be found from this study by imposing the further condition that $\mu << 1$. In that case, we get $\gamma \approx 1, \alpha = \rho\,(1 - 2\mu \in)$ and Eq. (29) leads to the value

$$\in^{ISCO} = \frac{-\frac{\sqrt{3}}{9}\rho\mu + \left(\frac{3}{81}\rho^2\mu^2 - \frac{4\sqrt{3}}{9}\rho + 4\right)^{1/2}}{2}\,. \tag{31}$$

This solution possesses the Schwarzchild value

$$\in^{ISCO}\,(S) = \frac{\sqrt{8}}{3}\,, \tag{32}$$

which is found by Fiziev and Todorov [1]. The radius of the ISCO found from (27) is, in the limit $\mu << 1$,

$$r^{ISCO} = 6\alpha_G\,(1 - 2\mu \in)\,. \tag{33}$$

The actual radius is found by dividing (33) by m_w, where

$$m_w = \frac{m_1 m_2}{(m_1 + m_2)\sqrt{1 - 2\nu\,(1 - \in)}}, \qquad \nu = \frac{m_1 m_2}{(m_1 + m_2)^2} \tag{34}$$

(see Ref. [1] for details). We note that the radius given by (33) is in parallel with the test particle limit, i.e. we get the result in the test particle limit from (33) without using the division mentioned above. In that case Eq. (33) gives us the Schwarzchild value for the radius of ISCO in the limit $\mu \to 0$, as

$$r^{ISCO}(S) = 6M \,, \tag{35}$$

where we have used the transformation $\alpha_G \to M$ to go back to the test particle limit. Equation (35) gives the well known value of ISCO radius in the Schwarzchild field [7].

We have, thus, found radii of circular orbits (Eq. (22)), location of ISCO (Eqs. (27) and (33)) and energy of the ISCO (Eq. (31)) of an effective particle moving round a slowly spinning central Kerr object. The corresponding parameters for a body in the Schwarzchild field are given by Eq. (23), Eqs. (28) and (35), and Eq. (32).

3 Conclusion

In this paper, we have presented a fresh study of orbits of a body in the field of a spinning central body. Masses of the bodies are assumed to be comparable so that the results computed are not limited to the test particle limit. Our study introduced a new Hamiltonian constraint suitable for the study of relativistic two-body problem with spin-orbit interaction. The subsequent equations of motion are elegant and they reduce to those of the Schwarzchild case in the appropriate limit. Moreover, the equations of motion reduce to Carter equations [4,5] for geodetical motion when appropriate transformations are applied. This is indicated in the text.We have derived expression for radii of circular orbits of an equivalent effective particle in the field of the central body whose gravitational field is described by the Kerr metric. The procedure followed in the derivation is due to Fiziev and Todorov [1] who computed orbital parameters for an effective particle in the field described by the Schwarzchild metric. As far as we know, our study is the first effective one-body calculation of orbits of a nonspinning body round a slowly spinning central body along the line shown by Fiziev and Todorov [1]. As such we are in a somewhat lone position regarding accuracy of the results. Nevertheless, we can justify our results by their simple correspondence with the Schwarzchild case which is shown in the text.

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