MULTI-SPECIES ANYONS SUPERSYMMETRY ON TWO-DIMENSIONAL LATTICE

J. Douari
Université Mohamed V, Faculté des Sciences, Département de Physique, LPT, B.P. 1014, Av. Ibn Battota, Agdal, Rabat, Morocco

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The algebra of multi-species anyons characterized by different statistical parameters $\nu_{ij} = e_i e_j \Phi_i \Phi_j/(2\pi)$, $i, j = 1, ..., n$ is redefined by basing on fermions and k_i-fermions ($k_i \in \mathbf{N}/\{0, 1\}$ with $i \in \mathbf{N}$) and its superalgebra is constructed. The so-called fractional supersymmetry of multi-species anyons is realized on 2d lattice.

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1 Introduction

The idea of supersymmetry (SUSY) has stimulated new approaches in many branches of physics. An evidence example has been found for a dynamical SUSY related even-even and even-odd nuclei [1,2]. SUSY is a theoretically attractive possibility for several reasons. It is the unique possibility for non-trivial extension of the known symmetries of space and time [3]. Many Physicists have developed theories of SUSY, particularly in the context of Grand Unified Theories, which successfully attempt to combine the strong and electroweak interactions [4].

In SUSY quantum mechanics one is considering a simple realization of SUSY algebra, involving the fermionic and bosonic operators [5], which had to move beyond Lie algebras to "graded" Lie algebras. Graded Lie algebras are just like Lie algebras except they use anti-commutation relations and commutations relations.

In view of the fact that the SUSY provided us with an elegant symmetry between fermions and bosons, it was natural to enquire if there exists a generalization which includes the exotic statistics. Various kinds of such extensions were realized: paraSUSY (with parafermions), fractional SUSY (using the q-bosons with q a root of unity) and nonlinear SUSY (bosonization of SUSY quantum mechanics) [6–14]. Another generalization can be treated concerning the case of anyons. An attempt in this sense is considered, in this work, combining two different kinds of anyons.

On other hand, during roughly 20 years, anyons have attracted a great attention to understand the physics of lower dimensions. Wilczek is generally credited to others earlier [15, 16]. He defined these new particles as being a vortex of the gauge-field which are the intersection of a plane and tubes of magnetic flux electrically charged [17]. Owing the work [18] of Leinaas and Merhyeim on identical particles, we all know that anyons have a lot to do with braids. Thus, the quantum algebras seems to be a good candidates describing their symmetries [19–21]. The realization of these algebras had been started by Lerda and Sciuto [20] and others.

Now, let us pay close attention to extend the notion of symmetry for these new systems. So, what about "super"-symmetry (SUSY) of anyons? and what kind of SUSY could exist in 2d space? In our present work, we will consider a system of n types of anyons (those we can call multi-species anyons [21]) characterized by different statistical parameters denoted

$$\nu_{ij} = e_i e_j \Phi_i \Phi_j/(2\pi), \qquad i,j = 1,...,n$$

e_i is charge of the i^{th} particle and Φ_i is its flux. Each statistical parameter ν_{ij} describes the interaction of the i^{th} anyon and the j^{th} one and defines the existence of different fractional statistics.

In this context, this work discuss the construction of the SUSY describing a system of multi-species anyons. First, we redefine the algebra of anyons by basing on fermions and k_i-fermions [23, 24] ($k_i \in \mathbf{N}/\{0,1\}$ with $i \in \mathbf{N}$) in 2d square lattice. Let us denote here that the spatial coordinates x are restricted to a multiple of a lattice spacing a, i.e. $x = na$ with n an integer. By removing the lattice structure the algebraic results don't change. Second, we generalize the definition of anyonic algebra by taking into account all kind of anyons. So we construct the "super"-algebra associated to our system. By introducing supercharges in terms of two different anyonic operators we realize the SUSY of multi-species anyons system which can be called as fractional one since the construction is based on the nature of anyons.

This paper is organized as follows: In section 2, we introduce the definition of multi-species anyonic oscillators and their algebras on 2d square lattice based on fermionic ones. In section 3, we define the k_i-fermions and we extend the Lerda-Sciuto definition to construct the multi-species k_i-fermionic anyons and their algebras. in section 4, we construct the anyonic superalgebra by considering a system of different species of anyons. In section 5, we use the generators of anyonic superalgebra to construct the supercharges of multi-species anyons SUSY. In section 6, we discuss the irreducible representations of anyonic algebras and superalgebra. In the last section, we summarize the main result of the paper.

2 Multi-species anyonic oscillators

Let Ω be a 2d square lattice with spacing $a = 1$. We give a two-component fermionic spinor field by

$$S^- = \begin{pmatrix} s_1^-(x) \\ s_2^-(x) \end{pmatrix}, \tag{1}$$

and its conjugate hermitian by

$$S^+ = (s_1^+(x), s_2^+(x)), \tag{2}$$

such that the components of these fields satisfy the following standard anti-commutation relations

$$\begin{array}{rcl} \{s_i^-(x), s_j^-(y)\} &=& 0 \\ \{s_i^+(x), s_j^+(y)\} &=& 0 \\ \{s_i^-(x), s_j^+(y)\} &=& \delta_{ij}\delta(x,y), \end{array} \tag{3}$$

$\forall i,j \in \{1,2\}$ and $\forall x,y \in \Omega$. Here, $\delta(x,y)$ is the conventional lattice δ-function: $\delta(x,y) = 1$ if $x = y$ and vanishes if $x \neq y$.

The expression of anyonic oscillators are given in terms of fermionic spinors as follows

$$\begin{array}{rcl} b^{-}_{ij}(x_{\pm}) &=& e^{i\nu_{ij}\Delta_i(x_{\pm})}s^{-}_{i}(x) \\ b^{+}_{ij}(x_{\pm}) &=& s^{+}_{i}(x)e^{-i\nu_{ij}\Delta_i(x_{\pm})}, \end{array} \tag{4}$$

where ν_{ij} are called statistical parameters and the elements $\Delta_i(x_{\pm})$ are given by

$$\Delta_i(x_{\pm}) = \sum_{y\in\Omega} s^{+}_{i}(y)\Theta_{\pm\Gamma_x}(x,y)s^{-}_{i}(y), \tag{5}$$

with $\Theta_{\pm\Gamma_x}(x,y)$ are the so-called angle functions and its definition on 2d square lattice was recited in the references [12] and [13], where γ_x is the curve associated to each site $x \in \Omega$ and the signs $+$ and $-$ indicate the two kinds of rotation direction on Ω.

The elements $\Delta_i(x_{\pm})$ satisfy the following commutation relations

$$\begin{aligned} &[\Delta_i(x_{\pm}),\ s^{-}_{j}(y)] = -\delta_{ij}\Theta_{\pm\Gamma_x}(x,y)s^{-}_{i}(y) \\ &[\Delta_i(x_{\pm}),\ s^{+}_{j}(y)] = \delta_{ij}\Theta_{\pm\Gamma_x}(x,y)s^{+}_{-}(y) \\ &[\Delta_i(x_{\pm}),\ \Delta_j(y_{\pm})] = 0\,. \end{aligned}$$

Now, we can show that the anyonic oscillators satisfy the following algebraic relations

$$\begin{array}{ll} [b^{-}_{ij}(x_{\pm}), b^{-}_{ik}(y_{\pm})]_{\Lambda^{\mp}_{ijk}} = 0, & x > y \\ [b^{-}_{ij}(x_{\pm}), b^{+}_{ik}(y_{\pm})]_{\Lambda^{\pm}_{ijk}} = 0, & x > y \\ [b^{+}_{ij}(x_{\pm}), b^{-}_{ik}(y_{\pm})]_{\Lambda^{\pm}_{ijk}} = 0, & x > y \\ [b^{+}_{ij}(x_{\pm}), b^{+}_{ik}(y_{\pm})]_{\Lambda^{\mp}_{ijk}} = 0, & x > y \\ [b^{-}_{ij}(x_{\pm}), b^{+}_{ik}(x_{\pm})] = 1, & \\ [b^{-}_{ij}(x_{\pm}), b^{+}_{kl}(y_{\pm})] = 0, & i \neq j \\ [b^{+}_{ij}(x_{\pm}), b^{-}_{kl}(y_{\pm})] = 0, & i \neq j \\ [b^{+}_{ij}(x_{\pm}), b^{+}_{kl}(y_{\pm})] = 0, & i \neq j \\ [b^{\pm}_{ij}(x_{-}), b^{\pm}_{kl}(y_{+})] = 0, & \forall i,j,k,l \\ [b^{-}_{ij}(x_{-}), b^{+}_{kl}(y_{+})] = \delta_{ik}\delta(x,y)\Gamma^{s^{+}_{i}(z)s^{-}_{i}(z)}_{ijl} & \\ [b^{-}_{ij}(x_{+}), b^{+}_{kl}(y_{-})] = \delta_{ik}\delta(x,y)\Gamma^{-s^{+}_{i}(z)s^{-}_{i}(z)}_{ijl}, & \end{array} \tag{6}$$

where

$$\begin{aligned} &\Lambda^{\pm}_{ijk} = \exp\left(\pm i(\nu_{ij}\Theta_{-\Gamma_x}(x,y) - \nu_{ik}\Theta_{+\Gamma_y}(y,x))\right) \\ &\Gamma_{ijl} = \exp\left(i\sum_{z\neq x}(\nu_{ij}\Theta_{-\Gamma_x}(x,z) - \nu_{il}\Theta_{+\Gamma_z}(z,x))\right) \\ &[X,Y]_{\Lambda} = XY + \Lambda YX \end{aligned}$$

and

$$x > y \Leftrightarrow \begin{cases} x_+ > y_+ \Leftrightarrow \begin{cases} x_2 > y_2 \\ x_1 > y_1, x_2 = x_1 \end{cases} \\ x_- < y_- \Leftrightarrow \begin{cases} x_2 < y_2 \\ x_1 < y_1, x_1 = x_2 \end{cases} \end{cases}$$

One obtains also

$$(b_{ij}^{\pm}(x_{\pm}))^2 = 0, \tag{7}$$

which is known as the hard core condition.

Let us remark that if we suppose $i = j$ in the expression of ν_{ij} given in section 1, we refind the algebraic relations of anyons constructed by Lerda and Sciuto in [12]. Also, we would like to stress that despite the deformation of our above algebraic relations, the anyonic oscillators don't have anything to do with the k_i-fermions which have deformed algebraic relations (will be discussed in the next section) for several reasons: (i) the k_i-fermions can be defined in any dimensions whereas the anyons are strictly two-dimensional objects, (ii) the anyons are non-local contrary to the k_i-fermions constitute a mathematical tool, introduced in the context of quantum algebras, which is used to go beyond the conventional statistics in any dimension and can take into account some perturbation (deformation) responsible of small deviations from the Fermi-Dirac and Bose-Einstein usual statistics.

3 Multi-species k_i-fermionic anyons on 2d square lattice

In this part of our work, we will construct the k_i-fermionic anyons on 2d square lattice Ω from the k_i-fermions.

To define the k_i-fermionic anyons, we extend the Lerda-Sciuto definition as

$$\begin{aligned} a_{ij}^{-}(x_{\pm}) &= e^{i\nu_{ij} D_i(x_{\pm})} f_i^{-}(x) \\ a_{ij}^{+}(x_{\pm}) &= f_i^{+}(x) e^{-i\nu_{ij} D_i(x_{\pm})}, \end{aligned} \tag{8}$$

where $D_i(x_{\pm})$ is give by

$$D_i(x_{\pm}) = \sum_{y\in\Omega} \Theta_{\pm\gamma_x}(x, y) N_i(y). \tag{9}$$

$N_i(y)$ is the number operator of k_i-fermions on 2d square lattice defined by $f_i^{-}(x)$ and $f_i^{+}(x)$ the k_i-fermionic annihilation and creation operators respectively as follows

$$\begin{aligned} f_i^{+}(x) f_i^{-}(x) &= [N_i(x)]_{q_i} \\ f_i^{-}(x) f_i^{+}(x) &= \mathbf{1} + [\mathrm{N_i(x)}]_{\mathrm{q_i}}, \end{aligned}$$

with $\mathbf{1}$ the identity, $q_i = e^{i\frac{2\pi}{k_i}}$ $(k_i \in \mathbf{N}/\{0,1\}, i \in \mathbf{N})$ and the notion $[x]_q = (q^x - 1)/(q - 1)$.

The k_i-fermionic operators satisfy the algebraic relations

$$\begin{array}{ll}
[f_i^-(x), f_j^+(y)]_{q_i^{\delta_{ij}}} = \delta_{ij}\delta(x,y) & \\
[f_i^-(x), f_j^-(y)]_{q_i^{\delta_{ij}}} = 0, & \forall x, y, \forall i, j \\
[f_i^+(x), f_j^+(y)]_{q_i^{\delta_{ij}}} = 0, & \forall x, y, \forall i, j \\
[N_i(x), f_j^-(y)] = -\delta_{ij}\delta(x,y) f_i^-(x) & \\
[N_i(x), f_j^+(y)] = \delta_{ij}\delta(x,y) f_i^+(x) & \\
(f_i^-(x))^{k_i} = (f_i^+(x))^{k_i} = 0. &
\end{array} \tag{10}$$

By using the previous tools, the operators $a_{ij}^{\pm}(x_\pm)$ constructed from k_i-fermionic oscillators satisfy the following algebraic relations

$$\begin{array}{ll}
[a_{ij}^-(x_\pm), a_{ik}^-(y_\pm)]_{p_{ijk}^{\mp}} = 0, & x > y \\
[a_{ij}^-(x_\pm), a_{ik}^+(y_\pm)]_{p_{ijk}^{\pm}} = 0, & x > y \\
[a_{ij}^+(x_\pm), a_{ik}^-(y_\pm)]_{p_{ijk}^{\pm}} = 0, & x > y \\
[a_{ij}^+(x_\pm), a_{ik}^+(y_\pm)]_{p_{ijk}^{\mp}} = 0, & x > y \\
[a_{ij}^-(x_\pm), a_{kl}^+(y_\pm)] = 0, & i \neq k \\
[a_{ij}^+(x_\pm), a_{kl}^-(y_\pm)] = 0, & i \neq k \\
[a_{ij}^+(x_\pm), a_{kl}^+(y_\pm)] = 0, & i \neq k \\
[a_{ij}^\pm(x_-), a_{kl}^\pm(y_+)] = 0, & \forall i, j \\
[a_{ij}^-(x_\pm), a_{ik}^+(x_\pm)]_{q_i} = 1, & \\
[a_{ij}^\pm(x_-), a_{kl}^\pm(y_+)]_{q_i} = 0, & x, y \in \Omega, \\
[a_{ij}^-(x_-), a_{kl}^+(y_+)]_{q_i^{\delta_{ik}}} = \delta_{ik}\delta(x,y)\Gamma_{ijl}^{N_i(z)}, & \forall i, j, k = 1, ..., n \\
[a_{ij}^-(x_+), a_{kl}^+(y_-)]_{q_i^{\delta_{ik}}} = \delta_{ik}\delta(x,y)\Gamma_{ijl}^{-N_i(z)}, & \forall i, j, k = 1, ..., n
\end{array} \tag{11}$$

$\forall i, j, k, l = 1, ..., n$. In this equation

$$p_{ijk}^- = q_i \exp\left[-i(\nu_{ij}\Theta_{-\Gamma_x}(x,y) - \nu_{ik}\Theta_{+\Gamma_y}(y,x))\right]$$

and

$$p_{ijk}^+ = q_i \exp\left[i(\nu_{ij}\Theta_{-\Gamma_x}(x,y) - \nu_{ik}\Theta_{+\Gamma_y}(y,x))\right].$$

We also have the following nilpotency condition

$$(a_{ij}^\pm(x_\pm))^{k_i} = 0\,, \tag{12}$$

which can be interpreted as a hard core condition generalizing the Pauli exclusion principle. In the particular case $k_i = 2$ (undeformed fermions), we recover the multi-species anyonic algebra of section 2.

4 Anyonic Superalgebra

In this section, we will consider n species of k_i-fermionic anyons ($i = 1, 2, ..n-1$) having different fractional spin and characterized by different fractional statistical parameters ν_{ij}. To construct the associate algebra the new generators will be defined as direct sum of k_i-fermionic anyons oscillators given by the equations (8). This definition will be in a cyclic order taking into account all kind of anyons can exist in the combined system. So, the constructed algebra will be in a "graded" form, and we will call it anyonic superalgebra. We define its generators as follows

$$\begin{aligned} A^-_{ij}(x_\pm) &= a^-_{ij}(x_\pm) \oplus a^-_{i+1,j}(x_\pm) \oplus ... \oplus a^-_{i+(n-1),j}(x_\pm) \\ A^+_{ij}(x_\pm) &= a^+_{ij}(x_\pm) \oplus a^+_{i+1,j}(x_\pm) \oplus ... \oplus a^+_{i+(n-1),j}(x_\pm). \end{aligned} \tag{13}$$

In a straightforward calculation, we prove that these operators obey to the following commutation relations

$$\begin{aligned} &[A^-_{ij}(x_\pm), A^-_{ir}(y_\pm)]_{P^\mp_{ijr}} = 0, && x > y \\ &[A^+_{ij}(x_\pm), A^+_{ir}(y_\pm)]_{P^\mp_{ijr}} = 0, && x > y \\ &[A^-_{ij}(x_\pm), A^+_{ir}(y_\pm)]_{P^\pm_{ijr}} = 0, && x > y \\ &[A^+_{ij}(x_\pm), A^-_{ir}(y_\pm)]_{P^\pm_{ijr}} = 0, && x > y \\ &[A^-_{ij}(x_\pm), A^+_{ir}(x_\pm)]_{Q_i} = 1\!\!1\,, && \\ &[A^\pm_{ij}(x_-), A^\pm_{rl}(y_+)]_{Q_i} = 0, && \forall x, y \in \Omega \\ &[A^-_{ij}(x_-), A^+_{rl}(y_+)]_{Q_i} = \delta_{ir}\delta(x,y)\Gamma_{[ijl]}, && \forall i,j,r,l = 1, ..., n \\ &[A^-_{ij}(x_+), A^+_{rl}(y_-)]_{Q_i} = \delta_{ir}\delta(x,y)\Gamma^{-1}_{[ijl]}, && \forall i,j,r,l = 1, ..., n. \end{aligned} \tag{14}$$

Let us denote here that $Q_i^k = 1\!\!1$, $k = k_0 k_1 ... k_{n-1}$. The new operators $A^-_{ij}(x_\pm)$ and $A^+_{ij}(x_\pm)$ satisfy the following nilpotency condition

$$(A^-_{ij}(x_\pm))^k = (A^+_{ij}(x_\pm))^k = 0, \quad k \in \mathbf{N}^* \tag{15}$$

with $k = k_0....k_{n-1}$, and $\forall i, j = 1, ..., n$, with

$$\Gamma_{[ijl]} = \begin{pmatrix} \exp\,[i \sum\limits_{z\neq x} (\nu_{ij}\Theta_{-\Gamma_x(x,z)} \\ \quad -\nu_{il}\Theta_{+\Gamma_x(x,z)})N_i(z)] & & \\ & \ddots & \\ & & \exp\,[i \sum\limits_{z\neq x} (\nu_{i+(n-1),j}\Theta_{-\Gamma_x(x,z)} \\ & & \quad -\nu_{i+(n-1),l}\Theta_{+\Gamma_x(x,z)})N_i(z)] \end{pmatrix}$$

$$P^-_{ijr} = \begin{pmatrix} p^-_{ijr} & & & & \\ & p^-_{i+1,jr} & & & \\ & & p^-_{i+2,jr} & & \\ & & & \ddots & \\ & & & & p^-_{i+(n-1),jr} \end{pmatrix} \tag{16}$$

$$P^+_{ijr} = \begin{pmatrix} p^+_{ijr} & & & & \\ & p^+_{i+1,jr} & & & \\ & & p^+_{i+2,jr} & & \\ & & & \ddots & \\ & & & & p^+_{i+(n-1),jr} \end{pmatrix}$$

$$Q_i = \begin{pmatrix} q_i & & & & \\ & q_{i+1} & & & \\ & & q_{i+2} & & \\ & & & \ddots & \\ & & & & q_{i+n-1} \end{pmatrix}.$$

The equation (15) generalizes the hard core condition for combined anyonic system. This means that no more that $(k_i - 1)$ particles can live in the same state of anyons constructed from k_i-fermions $(i = 0, 1, ..., n - 1)$.

5 Multi-species k_i-fermionic anyons supersymmetry

In this section, we still consider a combined system of multi-species k_i-fermionic anyons. We define the supercharges of the SUSY associated to the present system in terms of the generators of anyonic superalgebra given in section 4 as follows

$$\begin{array}{ll} C^{--}_{ijrs}(x) & = A^-_{ij}(x_-)A^+_{rs}(x_+) \\ C^{+-}_{ijrs}(x) & = A^+_{ij}(x_+)A^-_{rs}(x_-) \end{array} \tag{17}$$

with $i, j, r, s = 1, ..., n$ and $i \neq r$.

Using the above tools, these new supercharges obey to the following commutation relation

$$Q_i C^{+-}_{ijrs}(x) C^{--}_{ijrs}(y) - Q_r C^{--}_{ijrs}(y) C^{+-}_{ijrs}(x) = \delta(x, y)[B_r Q_i [\aleph_i(x)]_{Q_i} - B_i Q_r [\aleph_r(x)]_{Q_r}] \tag{18}$$

with

$$B_i = \begin{pmatrix} d_{ij}^{(\sum\limits_{z>x} - \sum\limits_{z<x}) N_i(z)} & & & \\ & d_{i+1,j}^{(\sum\limits_{z>x} - \sum\limits_{z<x}) N_{i+1}(z)} & & \\ & & \ddots & \\ & & & d_{i+n-1,j}^{(\sum\limits_{z>x} - \sum\limits_{z<x}) N_{i+n-1}(z)} \end{pmatrix}$$

$$[\aleph_i(x)]_{Q_i} = \begin{pmatrix} [N_i(x)]_{q_i} & & & \\ & [N_{i+1}(x)]_{q_{i+1}} & & \\ & & \ddots & \\ & & & [N_{i+n-1}(x)]_{q_{i+n-1}} \end{pmatrix} \quad (19)$$

$$d_{ij} = e^{i\nu_{ij}\pi},$$

here $N_i(x) = a_{ij}^{+}(x_{\pm})a_{ij}^{-}(x_{\pm})$.

To have an invariant expression under the hermitian conjugate we are doing the following step by computing the hermitian conjugate of Eq. (18)

$$Q_i^{-1}C_{ijrs}^{-+}(y)C_{ijrs}^{++}(x) - Q_r^{-1}C_{ijrs}^{++}(x)C_{ijrs}^{-+}(y) = \delta(x,y)[B_r^{-1}Q_i^{-1}[\aleph_i(x)]_{Q_i^{-1}} - [B_i^{-1}Q_r^{-1}[\aleph_r(x)]_{Q_r^{-1}}], \quad (20)$$

where $C_{ijrs}^{\pm+}(x)$ are the hermitian conjugates of the generalized supercharges $C_{ijrs}^{\pm-}(x)$. We write

$$C_{ijrs}^{-+}(x) = A_{rs}^{-}(x_+)A_{ij}^{+}(x_-)$$
$$C_{ijrs}^{++}(x) = A_{rs}^{+}(x_-)A_{ij}^{-}(x_+). \quad (21)$$

Let us remark here that these generators satisfy

$$(C_{ijrs}^{\pm+}(x))^k = (C_{ijrs}^{\pm-}(x))^k = 0, \quad k = k_0k_1...k_{n-1}. \quad (22)$$

Now we introduce a hermitian operator denoted $H(x)$ as the sum of the equalities (18) and (20), then we get

$$H(x) = B_rQ_i[\aleph_i(x)]_{Q_i} - B_iQ_r[\aleph_r(x)]_{Q_r} + B_r^{-1}Q_i^{-1}[\aleph_i(x)]_{Q_i^{-1}} - B_i^{-1}Q_r^{-1}[\aleph_r(x)]_{Q_r^{-1}}. \quad (23)$$

In a straightforward computation we get the following relation

$$[N_i(x)]_{q_i^{-1}} = q_i^{1-N_i(x)}[N_i(x)]_{q_i}, \quad (24)$$

then the operator $[\aleph_i(x)]_{Q_i^{-1}}$ (Eq. (19)) can be written as

$$\begin{aligned}[\aleph_i(x)]_{Q_i^{-1}} &= \begin{pmatrix} q_i^{1-N_i(x)} & & & \\ & q_{i+1}^{1-N_{i+1}(x)} & & \\ & & \ddots & \\ & & & q_{i+n-1}^{1-N_{i+n-1}(x)} \end{pmatrix} [\aleph_i(x)]_{Q_i} \\ &= A_i[\aleph_i(x)]_{Q_i}\,. \end{aligned} \tag{25}$$

Thus the equality (23) will be rewritten as

$$\begin{aligned} H(x) &= (B_r Q_i + B_r^{-1} J_i)[\aleph_i(x)]_{Q_i} \\ &\quad -(B_i Q_r + B_i^{-1} J_r)[\aleph_{i+1}(x)]_{Q_{i+1}}, \end{aligned} \tag{26}$$

where

$$J_i = \begin{pmatrix} q_i^{-N_i(x)} & & & \\ & q_{i+1}^{-N_{i+1}(x)} & & \\ & & \ddots & \\ & & & q_{i+n-1}^{-N_{i+n-1}(x)} \end{pmatrix}. \tag{27}$$

To extend these results to 2d continuum space, it is sufficient to summon on all the sites of 2d lattice

$$\begin{aligned} C_{ijrs}^{\pm\pm} &= \sum_{x\in\Omega} C_{ijrs}^{\pm\pm}(x) \\ H &= \sum_{x\in\Omega} H(x). \end{aligned} \tag{28}$$

In the result (26), we remark that the hermitian operator $H(x)$ looks like a deformed supersymmetry Hamiltonian operator describing a peculiar particles constructed from k_i-fermions and defined on 2d space. The deformation, in this case, looks be normal since the basis of our construction is deformed system (k_i-fermionic one) which generalizes the bosonic and fermionic ones and also the presence of special topological effects of 2d space in which anyons live.

6 Irreducible Representations of anyonic algebras and superalgebra

To construct the representations of anyonic algebras treated above, we will consider a Fock space. In first of all, let us give a local irreducible representation on a Fock space of k_i-fermionic algebra. We introduce this space of this algebra by the set

$$F_{i_x} = \{|n_{i_x}\rangle, n_{i_x} = 0, 1, ..., k_i - 1\}\,, \tag{29}$$

where the notation i_x means that this Fock space is introduced in each site x of the lattice Ω.

The action of k_i-fermionic operators $f_i^-(x)$ and $f_i^+(x)$ on F_{i_x} is expressed by the following equalities

$$\begin{aligned} f_i^+(x)|n_{i_x}\rangle &= |n_{i_x}+1\rangle, \qquad & f_i^+(x)|k_i-1\rangle = 0, \\ f_i^-(x)|n_{i_x}\rangle &= [n_{i_x}]_{q_i}|n_{i_x}-1\rangle, \qquad & f_i^-(x)|0\rangle = 0. \end{aligned} \tag{30}$$

Then, the operator $f_i^+(x)$ is called a creation k_i-fermionic operator on the site x and $f_i^-(x)$ annihilation one. These generators also satisfy the following nilpotency condition which is coherent with the above equalities. So, we have

$$(f_i^-(x))^{k_i} = (f_i^+(x))^{k_i} = 0\,, \tag{31}$$

which generalizes the Pauli exclusion principle; i.e. we can not find in one state more than $k_i - 1$ particles of the i^{th} kind.

Owing to the definition of anyonic operator given by Eq. (8) the irreducible representation space is the same one of k_i-fermionic system F_{i_x}. Thus, we can prove that the algebraic relations of Eq. (11) are coherent with the action of anyonic operators $a_{ij}^{\pm}(x_\pm)$ on the Fock space F_{i_x}. We get

$$\begin{aligned} a_{ij}^+(x_\pm)|n_{i_x}\rangle &= e^{i\frac{\nu_{ij}}{2}\sum\limits_{y\neq x}\Theta_{\pm\Gamma_x}(x,y)}|n_{i_x}+1\rangle \\ a_{ij}^-(x_\pm)|n_{i_x}\rangle &= [n_{i_x}]_{q_i}e^{-i\frac{\nu_{ij}}{2}\sum\limits_{y\neq x}\Theta_{\pm\Gamma_x}(x,y)}|n_{i_x}-1\rangle \\ a_{ij}^+(x_\pm)|k_i-1\rangle &= 0 \\ a_{ij}^-(x_\pm)|0\rangle &= 0. \end{aligned} \tag{32}$$

According to these relations we see the operators $a_{ij}^+(x_\pm)$ and $a_{ij}^-(x_\pm)$ as a creation and annihilation anyonic operators respectively.

Let us now define the Fock-like space of combined anyonic system as a direct sum of Fock spaces F_{i_x} defined in the equality (29). We write

$$F_x = \bigoplus_{i=0}^{n-1} F_{i_x}. \tag{33}$$

In this space, the vacuum state and n-particles state are described on each x on the lattice Ω, and denoted, respectively, by

$$(|0\rangle)_i \equiv \begin{pmatrix} |0\rangle_i \\ |0\rangle_{i+1} \\ \vdots \\ |0\rangle_{i+n-1} \end{pmatrix}, \quad (|n\rangle)_i \equiv \begin{pmatrix} |n\rangle_i \\ |n\rangle_{i+1} \\ \vdots \\ |n\rangle_{i+n-1} \end{pmatrix} \tag{34}$$

with index $i = 0, 1, ..., n-1$ of components in cyclic order. We remark, owing to equation (32), that the action of $A_{ij}(x_\alpha)$ and $A^\dagger_{ij}(x_\alpha)$ defined in equations (13) on F_x will be given by

$$\begin{aligned} &A^-_{ij}(x_\pm)(|0\rangle)_i = 0, \qquad && A^+_{ij}(x_\pm)(|k_i - 1\rangle)_i = 0 \\ &A^-_{ij}(x_\pm)(|n\rangle)_i = A(|n-1\rangle)_i, \quad && A^+_{ij}(x_\pm)(|n\rangle)_i = B(|n+1\rangle)_i \end{aligned} \tag{35}$$

with

$$A = \begin{pmatrix} \begin{matrix}[n]_{q_i} \exp[-i\frac{\nu_{ij}}{2} \times \\ \times \sum\limits_{y\neq x} \Theta_{\pm\Gamma_x}(x,y)]\end{matrix} & & & \\ & \begin{matrix}[n]_{q_{i+1}} \exp[-i\frac{\nu_{i+1,j}}{2} \times \\ \times \sum\limits_{y\neq x} \Theta_{\pm\Gamma_x}(x,y)]\end{matrix} & & \\ & & \ddots & \\ & & & \begin{matrix}[n]_{q_{i+n-1}} \exp[-i\frac{\nu_{i+n-1,j}}{2} \times \\ \times \sum\limits_{y\neq x} \Theta_{\pm\Gamma_x}(x,y)]\end{matrix} \end{pmatrix}$$

$$B = \begin{pmatrix} \begin{matrix}\exp[i\frac{\nu_{ij}}{2} \times \\ \times \sum\limits_{y\neq x} \Theta_{\pm\Gamma_x}(x,y)]\end{matrix} & & & \\ & \begin{matrix}\exp[i\frac{\nu_{i+1,j}}{2} \times \\ \times \sum\limits_{y\neq x} \Theta_{\pm\Gamma_x}(x,y)]\end{matrix} & & \\ & & \ddots & \\ & & & \begin{matrix}\exp[i\frac{\nu_{i+n-1,j}}{2} \times \\ \times \sum\limits_{y\neq x} \Theta_{\pm\Gamma_x}(x,y)]\end{matrix} \end{pmatrix}. \tag{36}$$

Then, the relations of Eq. (35) are compatible with the nilpotency condition Eq. (15), and we can call $A^-_{ij}(x_\pm)$ annihilation operator and $A^+_{ij}(x_\pm)$ creation one.

Now, let us give the representation of the supersymmetry constructed on 2d lattice. The action of the supercharges $C^{\pm-}_{ijrs}(x)$ on the associated Fock-like space that we define as

$$F = \bigoplus_{\substack{i,r=0 \\ i\neq r}}^{n-1} (F_{i_x} \otimes F_{r_x}) \tag{37}$$

is given by the equalities

$$\begin{aligned} C^{\pm-}_{ijrs}(x)(|0\rangle)_i \otimes (|0\rangle)_r &= 0 \\ C^{+-}_{ijrs}(x)(|n\rangle)_i \otimes (|n\rangle)_r &= C(|n+1\rangle)_i \otimes (|n-1\rangle)_r \\ C^{--}_{ijrs}(x)(|n\rangle)_i \otimes (|n\rangle)_r &= D(|n-1\rangle)_i \otimes (|n+1\rangle)_r \\ C^{\pm-}_{ijrs}(x)(|k_i-1\rangle)_i \otimes (|k_i-1\rangle)_r &= 0 \end{aligned} \tag{38}$$

and their hermitian conjugates $C^{\pm+}_{ijrs}(x)$ act on the Fock like-space

$$F' = \bigoplus_{\substack{r=0 \\ i\neq r}}^{n-1} (F_{r_x} \otimes F_{i_x}) \tag{39}$$

as follows

$$\begin{aligned}
C^{\pm+}_{ijrs}(x)(|0\rangle)_r \otimes (|0\rangle)_i &= 0 \\
C^{-+}_{ijrs}(x)(|n\rangle)_r \otimes (|n\rangle)_i &= E(|n-1\rangle)_r \otimes (|n+1\rangle)_i \\
C^{++}_{ijrs}(x)(|n\rangle)_r \otimes (|n\rangle)_i &= F(|n+1\rangle)_r \otimes (|n-1\rangle)_i \\
C^{\pm+}_{ijrs}(x)(|k_i-1\rangle)_r \otimes (|k_i-1\rangle)_i &= 0
\end{aligned} \tag{40}$$

where

$$C = \begin{pmatrix} [n]_{q_r} \exp[\frac{i}{2}\sum_{y\neq x}(\nu_{ij}\Theta_{+\Gamma_x}(x,y) \\ \qquad -\nu_{rs}\Theta_{-\Gamma_x}(x,y))] & & \\ & \ddots & \\ & & [n]_{q_{r+n-1}} \exp[\frac{i}{2}\sum_{y\neq x}(\nu_{i+n-1,j}\Theta_{+\Gamma_x}(x,y) \\ & & \qquad -\nu_{r+n-1,s}\Theta_{-\Gamma_x}(x,y))] \end{pmatrix}$$

$$D = \begin{pmatrix} [n]_{q_i} \exp[-\frac{i}{2}\sum_{y\neq x}(\nu_{ij}\Theta_{+\Gamma_x}(x,y) \\ \qquad -\nu_{rs}\Theta_{-\Gamma_x}(x,y))] & & \\ & \ddots & \\ & & [n]_{q_{i+n-1}} \exp[-\frac{i}{2}\sum_{y\neq x}(\nu_{i+n-1,j}\Theta_{+\Gamma_x}(x,y) \\ & & \qquad -\nu_{r+n-1,s}\Theta_{-\Gamma_x}(x,y))] \end{pmatrix}$$

$$E = \begin{pmatrix} [n]_{q_r} \exp[\frac{i}{2}\sum_{y\neq x}(\nu_{ij}\Theta_{-\Gamma_x}(x,y) \\ \qquad -\nu_{rs}\Theta_{+\Gamma_x}(x,y))] & & \\ & \ddots & \\ & & [n]_{q_{r+n-1}} \exp[\frac{i}{2}\sum_{y\neq x}(\nu_{i+n-1,j}\Theta_{-\Gamma_x}(x,y) \\ & & \qquad -\nu_{r+n-1,s}\Theta_{+\Gamma_x}(x,y))] \end{pmatrix}$$

$$F = \begin{pmatrix} [n]_{q_i} \exp[-\frac{i}{2}\sum_{y\neq x}(\nu_{ij}\Theta_{+\Gamma_x}(x,y) & & \\ \quad -\nu_{rs}\Theta_{-\Gamma_x}(x,y))] & & \\ & \ddots & \\ & & [n]_{q_{i+n-1}} \exp[-\frac{i}{2}\sum_{y\neq x}(\nu_{i+n-1,j}\Theta_{+\Gamma_x}(x,y) \\ & & \quad -\nu_{r+n-1,s}\Theta_{-\Gamma_x}(x,y))] \end{pmatrix}.$$

Owing to these results the irreducible representations of anyonic algebras were considered and it was shown that they are related to generalized Pauli exclusion principle. Furthermore, we could see the supersymmetry of our combined system has deformation properties plus "fractional" properties coming from the basis particles (k_i-fermions) and the nature of anyons respectively.

Concluding remarks

To summarize, exotic statistics were introduced in physics as an exotic extension of bosonic and fermionic statistics, and the both statistics could be unified by SUSY. Recently, the existence of intimate relation between exotic statistics and SUSY was established by observation of hidden SUSY structure in purely parabosonic and purely parafermionic systems. So, the SUSY and exotic statistics can be unified in the form of paraSUSY for parafermions, and also in the form of so-called fractional SUSY for q-bosons where q is a root of unity and nonlinear SUSY for bosonization of SUSY quantum mechanics. These studies were formulated on four dimensional space-time.

For lower dimensions, we have discussed in this work what we could call fractional SUSY in a general case for k_i-fermionic anyons. As generalization, our work was for unification of different exotic statistics on 2d space. These kinds of statistics describe quasi-particles those we defined as deformed particles constructed from bosons or fermions. To generalize this construction we have considered k_i-fermions as basis to built our generalized anyons, those we call k_i-fermionic anyons. From our present construction, it is easy to remark that in the limit cases $k_i = 2$ we refind the anyonic oscillators defined in the reference [20] by Lerda and Sciuto, and for $k_i \longrightarrow \infty$ our anyons go to bosonic anyons.

These results are compatible with the fact that anyonic particles having different fractional charge and spin can not live peacefully, such that the commutation relations for anyonic operators show the generalization of exclusion principle and the interchange of two anyons can not be defined consistently for them. These analysis are very interesting for various fields as condensed matter in which the experimental advances related to the fractional quantum Hall effect have proved that quasi-particles discovered appear to exhibit anyonic behavior. Also, the field of quantum computation which can be constructed from the abstract study of anyonic systems, such that the braiding and fusion of anyonic excitations in quantum Hall electron liquids and 2D magnets are modeled by modular functors opening a new possibility for the realization of quantum computers [25, 26].

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