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ENTROPIC UNCERATAINTY RELATION OF A SINGLE-TRAPPED ION INTERACTING WITH LASER BEAMS

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In this communication we analyze the quantum motion of an ion isotropically confined in the radial plane of a Paul microtrap when it is irradiated by a properly chosen configuration of external laser beams. We present a derivation of an analytical solution, by means of which we study the entropic uncertainty relation. Using parameters corresponding to existing experimental realization, we have studied the variation effect for the initial total number of vibrational quanta and the phase.

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1 Overview

One of the most striking features of quantum mechanics is the property that certain observables cannot simultaneously be assigned arbitrarily precise values. This property does not compromise claims of completeness for the theory, since it may consistently be asserted that such observables cannot simultaneously be measured to an arbitrary accuracy [1]. On the other hand, recent advances in ion cooling and trapping have opened new prospects in nonclassical state generation. An ion confined in an electromagnetic trap can be described approximately as a particle in a harmonic potential. Its center of mass (c.m.) exhibits a simple quantum-mechanical harmonic motion. By driving the ion appropriately with laser fields, its internal and external degrees of freedom can be coupled to the extent that its center-of-mass motion can be manipulated precisely. The entropy squeezing for a single-mode two-level atom [2] and multimode [3] have been studied recently. It has been shown that the effect of Kerr like medium on the entropy is negative and the effect of detuning on the atomic variable squeezing is positive when atomic coherence is taken into account. The emphasis on the fact that the atomic coherence has an effect on the squeezing of the entropy and the system of Janyes-Cumming model with Kerr-like medium can have a potential application in the field of quantum information. Furthermore, the quantum mutual entropy has been used to examine the entanglement degree in the time development of a single two-level ion interacting with laser field and a generation of Bell-type states involving

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light and vibrational motion of a single trapped ion have been proposed [4]. One advantage of the trapped ion system is that decoherence effects are relatively weak due to the extremely weak coupling between the vibrational modes and the environment. It was realized that this advantage of the trapped ion system makes it a promising candidate for constructing quantum logic gates for quantum computation [5] as well as for producing nonclassical states of the center-of-mass motion. In fact, single-mode nonclassical states, such as Fock states, squeezed states and Schrödinger cat states of the ion's vibration mode were investigated [6–8]. Recently, various schemes of producing two-mode nonclassical states of the vibration mode were proposed using ions in a two-dimensional trap [7,9,10].

The scope of this communication is to employ the uncertainty relation to elucidate the squeezing for a single trapped ion resonantly interacting with laser beams to the first red side-band of the center-of-mass mode. We introduce our Hamiltonian model and give exact expression for the density matrix operator, shedding light on the important question of entropy squeezing behavior. In the language of quantum information theory a definition of squeezing is presented for this system. The utility of the definition is illustrated by examining squeezing in the information entropy of the present system. This analysis is applicable to any quantum two-state system subject to entropy squeezing with appropriate coupling constant.

2 The Model

The dynamics of quantum two-level systems has always been at the focus of interest, but recently has attracted increased attention because of the ideas of quantum computing. Several systems have been suggested as physical realizations of quantum bits allowing for the needed controlled manipulations, and for some of them first elementary steps have been demonstrated in experiments (see reviews in Ref. [11]). We consider an ideal two-level ion of mass m constrained to move in a three-dimensional harmonic potential. Taking the principal trap axis to coincide with the direction of propagation of the driving field, two quantum numbers sufficient to label the vibrational states of the trap. The other one are traced out by summing over the corresponding degrees of freedom. The ion's internal and external degrees of freedom are coupled together by a light field periodically modulated at the frequency ν of the ion trap

$$E(\bar{x},\bar{y},t)=\zeta^{(+)}(\bar{x},\bar{y},t)+\zeta^{(-)}(\bar{x},\bar{y},t), \tag{1}$$

where $\zeta^{(+)}(\bar{x},\bar{y},t)$ represents the positive frequency contribution to the resultant electric field acting upon the ion which can be written as

$$\zeta^{(+)}(\bar{x},\bar{y},t)=\varepsilon[\exp\left(i(\omega_L t-k_L\bar{x}+i\phi)\right)-\exp\left(i(\omega_L t-k_L\bar{y}+i\phi)\right)]. \tag{2}$$

The phase ϕ determine the position of the trap potential with respect to the running wave. Let's denote by $\hbar\omega_0$ the energy separation between the ionic excited state $|\uparrow\rangle$ and its ground state $|\downarrow\rangle$ and assume $\omega_0 >> \upsilon$. We assume that the ion is driven by two π-out of phase laser beams, applied along the two orthogonal directions $\bar{x}$ and $\bar{y}$ with an angle of $\pi/4$ relative to the x and y axis respectively. The two lasers have equal frequency $\omega_L=\omega_0-2\upsilon$, intensity ε and wave vector modulus k_L.

The Hamiltonian for a trapped ion interacting with a bichromatic field can be split in two parts

$$\hat{H} = \hat{H}_0 + \hat{H}_{in}, \tag{3}$$

where

$$\hat{H}_0 = \frac{\hbar\omega_0}{2}(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|) + \hbar\upsilon\left(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2\right), \tag{4}$$

and in the dipole and the rotating wave approximations,

$$\hat{H}_{in} = \wp[\lambda\zeta^{(-)}(\bar{x},\bar{y},t)|\,|\uparrow\rangle\langle\downarrow| + \lambda^*\zeta^{(+)}(\bar{x},\bar{y},t)|\downarrow\rangle\langle\uparrow|]. \tag{5}$$

We denote by $\hat{a}_1(\hat{a}_1^\dagger)$ and $\hat{a}_2(\hat{a}_2^\dagger)$ the annihilation (creation) operators of vibrational quanta relative to the ionic center of mass oscillatory motion along the $\widehat{x}$ and $\widehat{y}$ axes of the radial plane of the trap respectively and $\wp$ the appropriate dipole matrix element. Consider a two-level ion confined in a two dimensional isotropic harmonic potential characterized by the trap frequency. We denote by $\hbar\omega_0$ the energy separation between the ionic excited state $|\uparrow\rangle$ and its ground state $|\downarrow\rangle$.

However it is of rather more use to exhibit the solution explicity for particular initial conditions of relevance to the experiments. With this in mind we will assume that the initial state is prepared to be a particular vibrational quanta + field quanta eigenstate such as $\rho\otimes\varpi$ with the ion prepared in the electronic state:

$$\rho = \xi_1^2\,|\uparrow\rangle\langle\uparrow| + \xi_2^2\,|\downarrow\rangle\langle\downarrow| + \xi_1\xi_2 e^{i\beta}\,|\uparrow\rangle\langle\downarrow| + \xi_1\xi_2^* e^{-i\beta}\,|\downarrow\rangle\langle\uparrow| \in \mathcal{S}(\mathcal{H}_A). \tag{6}$$

Also we suppose that the initial state of the field is given by

$$\begin{aligned} \varpi &= |\theta_1,\theta_2\rangle\langle\theta_1,\theta_2| \in \mathcal{S}(\mathcal{H}_F), \\ |\theta_1,\theta_2\rangle &= \sum_{n=0}^{N}\sqrt{\left(\begin{array}{c} N \\ n \end{array}\right)}p^{n/2}(1-p^2)^{\frac{N-n}{2}}\,|N-n,n\rangle. \end{aligned} \tag{7}$$

The field prepared in a number state $|N-n\rangle$ containing $N-n$ photons, the ion's center of mass motion prepared in a number state $|n\rangle$ containing n quanta, and the ion's internal levels prepared in a mixed state. This particular initial condition is crucial for the generation of entanglement between the states of ionic vibration. The vibrational state, $|\theta_1,\theta_2\rangle$ belong to the class of so-called $SU(2)$ coherent states. Since number states for the 1D motion of an ion in a rf-trap have been already experimentally realized (see [8]), it is evident that the initial state of equations (6) and (7) could be easily prepared experimentally.

The continuous map $\mathcal{E}_t^*$ describing the time evolution between the ion and the field for this process is defined by the unitary operator generated by the total Hamiltonian H such that

$$\begin{aligned} \mathcal{E}_t^* &: \mathcal{S}(\mathcal{H}_A) \longrightarrow \mathcal{S}(\mathcal{H}_A\otimes\mathcal{H}_F), \\ \mathcal{E}_t^*\rho &= U_t\,(\rho\otimes\varpi)\,U_t^*, \quad U_t \equiv e^{-itH/\hbar}. \end{aligned} \tag{8}$$

It has been shown that, by irradiating the trapped ion with an appropriate configuration of laser beams, the physical system under scrutiny can be studied, in the Lamb-Dicke regime, using

equation (5) it is not difficult to show that, in the Lamb-Dicke limit, the effective interaction Hamiltonian in the interaction picture can be written as [12,13]

$$\hat{H}_{in} = \wp\varepsilon \exp\left(-\frac{\eta^2}{2}\right)\left(\hat{a}_1\hat{a}_2|\uparrow\rangle\langle\downarrow| + \hat{a}_1^{\dagger}\hat{a}_2^{\dagger}|\downarrow\rangle\langle\uparrow|\right), \tag{9}$$

where $\eta = k_L\sqrt{\hbar/(2\upsilon M)}$, is the so-called Lamb-Dicke parameter. For realizing this Hamiltonian model a three level electronic system should be used. Such a system is similar to the one currently used in the experiments performed at NIST [8]. In this case the two-photon Raman transitions between the states of interest are formally equivalent to narrow single photon transitions.

The unitary operator U_t can be written in this picture as

$$U(t) = \sum_{n=0}^{N}\left(e^{-itE_n}|\Phi_+\rangle\langle\Phi_+| + e^{itE_n}|\Phi_-\rangle\langle\Phi_-|\right), \tag{10}$$

where

$$E_n = \hbar g\sqrt{(N-n+1)(n+1)}, \tag{11}$$

and $\pm E_n$ are the eigenvalues with $g = \wp\varepsilon\exp\left(-\eta^2/2\right)$. The eigenvectors associated with the eigenvalues $\pm E_n$ are given by

$$|\Phi_{\pm}\rangle = \frac{1}{\sqrt{2}}\left[|N-n,n;\uparrow\rangle \pm |N-n+1,n+1;\downarrow\rangle\right]. \tag{12}$$

It is interesting to point out here performing a unitary operator yields different cases under the rotating wave approximation, such as equation (9) and by adjusting the laser or cavity frequency, we can obtain different time evolutions of states in the system. The Hamiltonian (9) can be used for constructing quantum computing gates

$$\begin{aligned}
|N-n\rangle\otimes|n\rangle\otimes|\uparrow\rangle \quad &\Longrightarrow \cos E_n t|N-n\rangle\otimes|n\rangle\otimes|\uparrow\rangle \\
&\quad -i\sin E_n t|N-n+1\rangle\otimes|n+1\rangle\otimes|\downarrow\rangle, \\
|N-n\rangle\otimes|n\rangle\otimes|\downarrow\rangle \quad &\Longrightarrow \cos E_{n-1} t|N-n\rangle\otimes|n\rangle\otimes|\downarrow\rangle \\
&\quad -i\sin E_{n-1} t|N-n-1\rangle\otimes|n-1\rangle\otimes|\uparrow\rangle.
\end{aligned} \tag{13}$$

With above time evolutions of states with those obtained from the other cases, we can construct Hadamard and CNOT gates in the system, which are basic elements of a general quantum computing [14].

The time-dependent analytical solution for the final state of the system is given by

$$\begin{aligned}
\mathcal{E}_t^*\rho \;=\; & \sum_{n=0}^{N}\sum_{m=0}^{N}\Big\{\xi_1^2\sum_{i=1}^{2}\sum_{j=1}^{2}\Im_i(n)\Im_j^*(m)|\Upsilon_i(n)\rangle\langle\Upsilon_j(m)| \\
& +\xi_2^2\sum_{i=1}^{2}\sum_{j=1}^{2}\Im_i(n-1)\Im_j^*(m-1)|\Upsilon_{3-i}(n-1)\rangle\langle\Upsilon_{3-j}(m-1)|
\end{aligned}$$

$$+\xi_1\xi_2 e^{i\beta}\sum_{i=1}^{2}\sum_{j=1}^{2}\Im_i(n)\Im_j^*(m-1)|\Upsilon_i(n)\rangle\langle\Upsilon_{3-j}(m-1)|$$
$$+\xi_1\xi_2^* e^{-i\beta}\sum_{i=1}^{2}\sum_{j=1}^{2}\Im_i(n-1)\Im_j^*(m)|\Upsilon_{3-i}(n-1)\rangle\langle\Upsilon_j(m)|\Big\}, \tag{14}$$

where $\Im_i(N-n)$ and $|\Upsilon_i(n)\rangle$ are given by

$$\begin{aligned}\Im_1(n)&=\sqrt{\tfrac{N!}{n!(N-n)!}}\tfrac{\cos(E_n t)}{\sqrt{2^N}}, \qquad \Im_2(n)=-i\sqrt{\tfrac{N!}{n!(N-n)!}}\tfrac{\sin(E_n t)}{\sqrt{2^N}},\\ |\Upsilon_1(n)\rangle&=|N-n,n,e\rangle, \qquad |\Upsilon_2(n)\rangle=|N-n+1,n+1,g\rangle.\end{aligned} \tag{15}$$

Having obtained the time-dependent analytical solution for the final state of the system, we are therefore in a position to study any phenomena related to the system. In the following section and by employing the results obtained here we shall be able to discuss the entropy squeezing which has received a little attention in the literature.

3 Entropy squeezing and uncertainty

According to Shannon's information theory, entropy is the only rigorous quantitative measure of the uncertainty or lack of information associated to a random variable. Entropic uncertainty relations thus provide a rigorous mathematical formulation for the uncertainty principle of quantum mechanics, unlike the standard Heisenberg uncertainty relation. Entropic uncertainty relations also provide a natural link between the uncertainty principle and the branches of quantum physics that make use of information theory, such as quantum communication, quantum computation and quantum cryptography. There has been considerable recent interest in the information-theoretic (or entropic) uncertainty relations. In the information-theoretical formulation of uncertainty relation, for expressing the uncertainty of an observable one uses the Shannon entropy of its probability distribution instead of its variance (see e.g. Ref. [15]).

The uncertainty relation for a two-level atom characterized by the Pauli operators S_x, S_y and S_z, is given by

$$\Delta S_x \Delta S_y \geq \frac{1}{2}|\langle S_z\rangle|, \tag{16}$$

where $\Delta S_\alpha = \sqrt{\langle S_\alpha^2\rangle - \langle S_\alpha\rangle^2}$. Fluctuations in the component S_α of the atomic dipole is said to be squeezed if S_α satisfies the condition

$$V(S_\alpha) = (\Delta S_\alpha - \sqrt{\frac{|\langle S_z\rangle|}{2}} < 0, \qquad \alpha = x \quad \text{or} \quad y. \tag{17}$$

Recently in an even N-dimensional Hilbert space, the optimal entropic uncertainty relation for sets of $N+1$ complementary observables with the non-degenerate eigenvalues has been investigated [16]. This can be described by the inequality

$$\sum_{k=1}^{N+1} H(S_k) \geq \ln\left[\frac{N}{2}\right]^{\frac{N}{2}} + \ln\left[1+\frac{N}{2}\right]^{1+\frac{N}{2}}, \tag{18}$$

where $H(S_k)$ represents the information entropy of the variable S_k. On the other hand, for an arbitrary quantum state the probability distribution for N possible outcomes of measurements of the operator S_α is

$$P_i(S_\alpha) = \langle \Psi_{\alpha i} \mid \rho \mid \Psi_{\alpha i} \rangle, \quad \alpha = x, y, z \quad i = 1, 2, ..., N, \tag{19}$$

where $\mid \Psi_{\alpha i}\rangle$ is an eigenvector of the operator S_α such that $S_\alpha|\Psi_{\alpha i}\rangle = \lambda_{_{\alpha i}}|\Psi_{\alpha i}\rangle$. The corresponding information entropies are defined as

$$H(S_\alpha) = -\sum_{i=1}^{N} P_i(S_\alpha) \ln P_i(S_\alpha), \quad \alpha = x, y, z. \tag{20}$$

Thus, to obtain the information entropies of the atomic operators $\widehat{S}_x$, $\widehat{S}_y$ and $\widehat{S}_z$ for a two-level system, with $N = 2$, one can use the reduced atomic density operator $\widehat{\rho}(t)$. For the present case we find that,

$$H(S_x) = -\ln\left(\left[\frac{1}{2} + Re\{\rho_{\uparrow\downarrow}(t)\}\right]^{\left[\frac{1}{2}+Re\{\rho_{\uparrow\downarrow}(t)\}\right]} \times \left[\frac{1}{2} - Re\{\rho_{\uparrow\downarrow}(t)\}\right]^{\left[\frac{1}{2}-Re\{\rho_{\uparrow\downarrow}(t)\}\right]}\right),$$

$$H(S_y) = -\ln\left(\left[\frac{1}{2} + Im\{\rho_{\downarrow\uparrow}(t)\}\right]^{\left[\frac{1}{2}+Im\{\rho_{\downarrow\uparrow}(t)\}\right]} \times \left[\frac{1}{2} - Im\{\rho_{\downarrow\uparrow}(t)\}\right]^{\left[\frac{1}{2}-Im\{\rho_{\downarrow\uparrow}(t)\}\right]}\right),$$

$$H(S_z) = -\ln\left(\rho_{\uparrow\uparrow}(t)^{\rho_{\uparrow\uparrow}(t)} \times (1-\rho_{\uparrow\uparrow}(t))^{(1-\rho_{\uparrow\uparrow}(t))}\right), \tag{21}$$

where the quantities $\rho_{ij}(t)$ and $\rho_{ij}(t) = \rho^*_{ji}(t), (i, j =\uparrow, \downarrow)$ are determined from the relations,

$$\rho_{ij}(t) = \langle i|\mathcal{E}^*_t \rho|j\rangle. \tag{22}$$

In the present case, we find that the uncertainty relation of the entropy can be used as a general criterion for the squeezing of an atom. Specifically, we study squeezing in terms of the information entropy of a two-level atom interacting with a two-mode in a nonlinear medium. For a two-level atom, where $N = 2$, we have $0 \leq H(S_\alpha) \leq \ln 2$, and hence, the information entropies of the operators S_x, S_y, S_z will satisfy the inequality

$$H(S_x) + H(S_y) + H(S_z) \geq 2 \ln 2. \tag{23}$$

In other word if we define $\delta H(S_\alpha) = \exp[H(S_\alpha)]$, then the inequality (23) can be written as

$$\delta H(S_x)\delta H(S_y) \geq \frac{4}{\delta H(S_z)}. \tag{24}$$

Now if $\delta H(S_\alpha) = 1$, then the atom will be in a pure state, however when $\delta H(S_\alpha)$ takes the value 2, then the atom will be in a completely mixed state, since the quantities $\delta H(S_x)$ and $\delta H(S_y)$ are only measuring the uncertainties of the atomic polarization components S_x and S_y respectively. We define here the squeezing of the atom using the entropic uncertainty relation, named entropy squeezing. The fluctuation in the component S_α ($\alpha = x$ or y) of the atomic dipole are said to be "squeezed in entropy" if the information entropy $H(S_\alpha)$ of S_α satisfies the condition,

$$E(S_\alpha) = \delta H(S_\alpha) - \frac{2}{\sqrt{\mid \delta H(S_z) \mid}} < 0, \tag{25}$$

where $\alpha = x$ or y. The formula (25) is directly connecting quantum information with quantum fluctuations, and containing statistical moments to all orders; while (18) only contains the statistical variances. Meaningful information can be retrieved from equation (25) in the case of $\langle S_z \rangle = 0$ since the right-hand side of equation (24) is always non-zero whereas the inequality $\Delta S_x \Delta S_y \geq \frac{1}{2}|\langle S_z \rangle|$ is trivially satisfied. Further, when the atom is in an eigenstate of the operator S_x (or S_y) it can be shown that the operator S_x (or S_y) exhibits optimal entropy squeezing where no entropy squeezing occurs in S_x (or S_y). In this case $\langle S_z \rangle = 0$, and no atomic variance squeezing is predicted from equation (18), (see for example Ref. [2,3]).

4 Discussion

On the basis of the analytical solution presented in the previous section, we shall examine the temporal evolution of the entropy squeezing and variance squeezing. In the experiments done so far, it is possible to probe directly which electronic state the ion occupies. In the work of [8] the internal state $|\downarrow\rangle$ is the $2s\ ^2S_{\frac{1}{2}}$ $(F = 2, M_F = 2)$ state of $^9Be^+$, and the state $|\uparrow\rangle$ corresponds to the $2s\ ^2S_{\frac{1}{2}}$ $(F = 1, M_F = 1)$ state. So to test the dependence of Lamb-Dicke parameter, η on the entropy squeezing, we plot in Fig. 1 the time evolution of the squeezing entropy $E(S_x)$ and $E(S_y)$ as well as the variances $V(S_x)$ and $V(S_y)$ versus scaled time when the ion is prepared initially in the ground or excited state, such that $\theta = \pi$ or 0, while the fields are in coherent states. A coherent state of motion of the ion corresponds to a minimum uncertainty wave-packet whose center oscillates classically in the harmonic well and retains its shape.

In fact the initial state of the field equation (9) belongs to the class of the so called $SU(2)$ coherent states defined as

$$|\tau, J\rangle = \frac{1}{(1+|\tau|^2)^J} \sum_{n=0}^{2J} \tau^n \sqrt{\frac{(2J)!}{(2J-n)!n!}}\, |2J-n, n\rangle. \tag{26}$$

where $\tau \in C$, $2J \in N$. The states $|N-n, n\rangle$ appearing in equation (7) are eigenstates of the operator $\left(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2\right)$ all pertaining to the eigenvalue $N = 2J_0$ representing the initial total number of vibrational quanta. In Fig. 1, we plot the squeezing factors as functions of the scaled time gt and assume the initial total number of vibrational quanta $N = 9$, and the initial state of the ion is an excited such as $\xi_1 = 1$. We see from Fig. 1 that there is no squeezing occurring $E(S_y)$ but we have squeezing in $E(S_x)$. In the meantime we observe that there are fluctuations occurring in the variance $V(S_x)$ more than that in the $E(S_x)$ and $V(S_y)$ more than that in the $E(S_y)$. On the other hand it is easy to realize that, there is changing occurred in $E(S_y)$ compared with $E(S_x)$, where we observe decreasing in the number of the fluctuations, see Fig. 1b. The quadrature variances $V(S_y)$ shows shape similar to that of $V(S_x)$ but with a slight decreasing in the value of the fluctuations as the time increases. Exploiting an analytical method based on the analysis of the Rabi frequencies, it has been recently demonstrated in Ref. [17] that, starting from a total vibrational excitation number N, there exist an N-dependent time instant at which the internal and external degrees of freedom of the trapped ion are disentangled ($\langle S_z(t) \rangle = 1$ or $\langle S_z(t) \rangle = -1$) or maximally entangled ($\langle S_z(t) \rangle = 0$).

It has been analytically proved that, if $N >> 1$ is odd, at the time instant $g\tau = \pi(N-1)/4$, one has the atomic inversion $\langle S_z(t) \rangle = 1 (\langle S_z(t) \rangle = -1)$ if $(N-1)/2$ is even (odd).

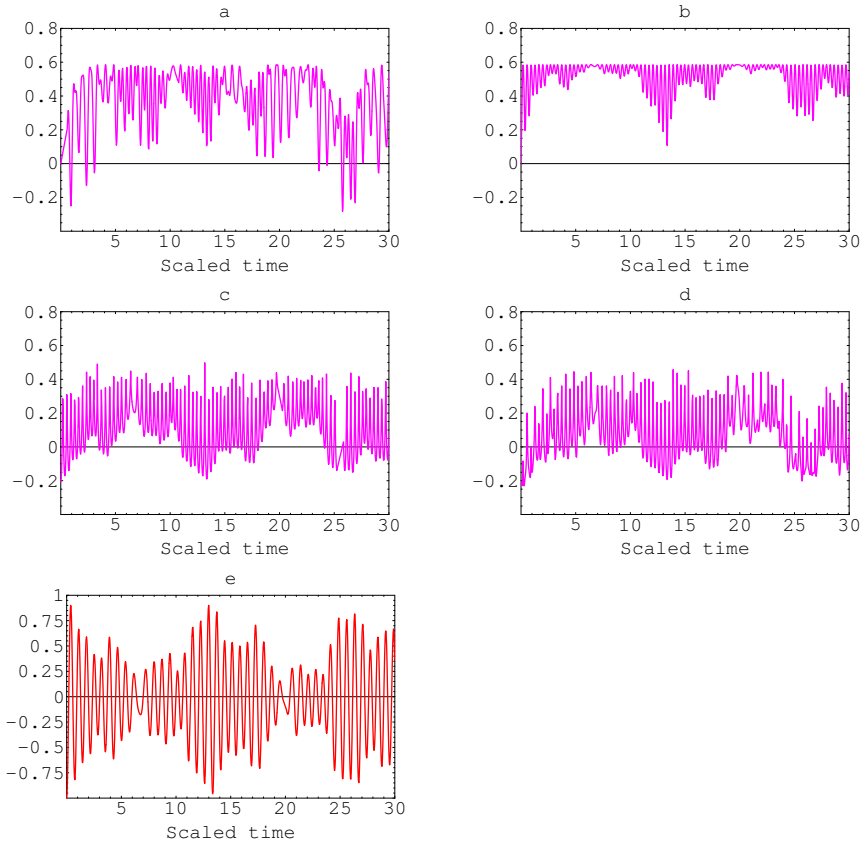

Fig. 1. The squeezing factors of a single trapped ion as a function of gt for $N = 9$, and $\xi_1 = 1$ where (a) The entropy squeezing factor $E(S_x)$; (b) The entropy squeezing factor $E(S_y)$; (c) The variance squeezing factor $V(S_x)$; (d) The variance squeezing factor $V(S_y)$; and (e) The time evolution of the atomic inversion under the same conditions.

This indicates that the internal and external degrees of freedom manifest a marked tendency to disentangle each other. In Fig. 1e we plotted the atomic inversion $\langle S_z(t)\rangle$ against the scaled time gt when the initial total vibrational number N = 9. In Fig. 2, we have considered the same values in Fig. 1 but for different value of the initial total number of vibrational quanta ($N = 10$), in this case we see that the general behavior is almost the same except in $E(S_x)$ we get more squeezing, once the initial total number of vibrational quanta is increased, a new features have been observed in Fig. 3. These figures illustrate in particular the existence of a $N-$dependent time instant at which the system under scrutiny exhibits different quantum behaviors dependent on the parity of N. The interesting physical aspect is given by the peculiar nonclassical sensitivity to the granularity of the initial total number of vibrational quanta N.

Finally it is to be noted that for the eigenstate of the operators S_x or S_y, the entropy squeezing

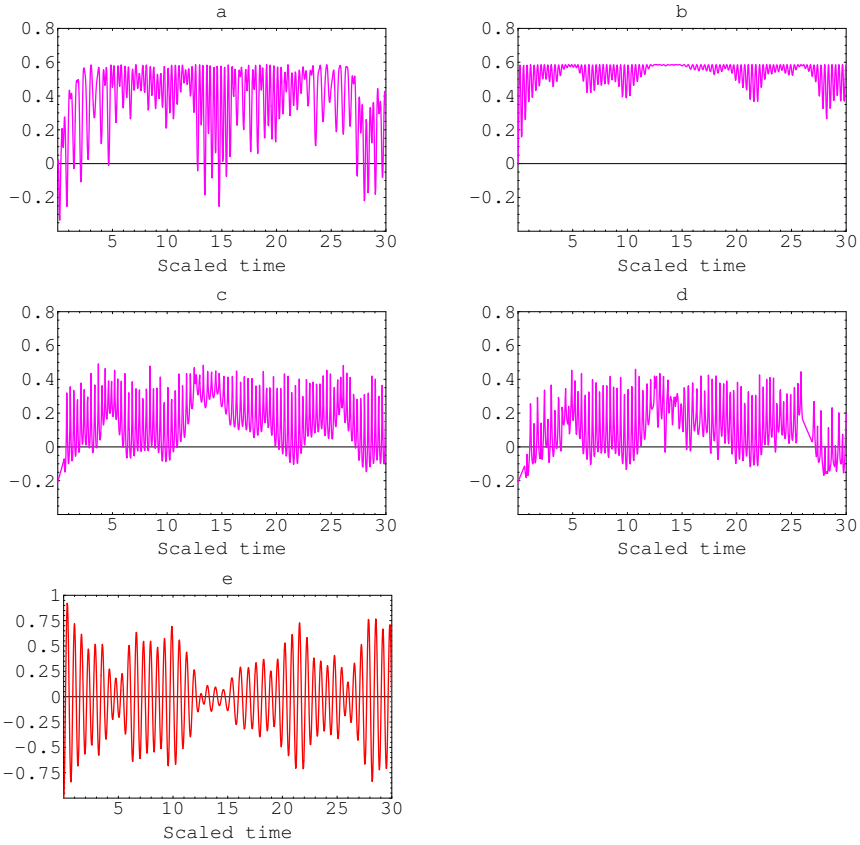

Fig. 2. The squeezing factors of a single trapped ion as a function of gt for $N = 10$, and $\xi_1 = 1$ where (a) The entropy squeezing factor $E(S_x)$; (b) The entropy squeezing factor $E(S_y)$; (c) The variance squeezing factor $V(S_x)$; (d) The variance squeezing factor $V(S_y)$; and (e) The time evolution of the atomic inversion under the same conditions.

is quite a good measure. In Fig. 4e, we have depicted the atomic inversion, from this curve, we can see clearly the collapses and revivals for the electronic state atomic inversion of the trapped ion. Also, In this model, we may prepare the initial state for interaction times t_n, after having detected the ion in the internal state $|\downarrow\rangle$ following Ref. [18], such as

$$|\psi\rangle = -\frac{i}{\sqrt{2}}(|1\rangle \otimes |0\rangle - e^{i\phi}|0\rangle \otimes |1\rangle), \tag{27}$$

where we assumed $N - n = 0$ (the field in the vacuum state) and $n = 1$ (ion vibrational motion in the first excited state). Which is also a Bell-type state involving the quantized cavity field as well as the ion's vibrational motion. If we choose ϕ such that $\phi = \pm\pi/2$, we are able to obtain

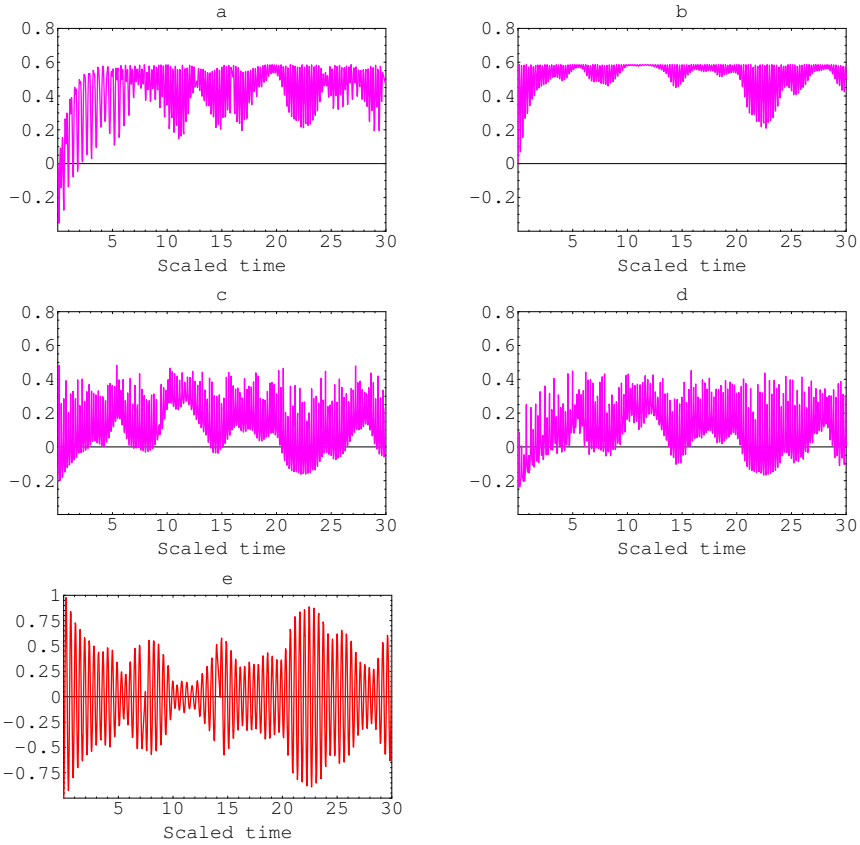

Fig. 3. The squeezing factors of a single trapped ion as a function of gt for $N = 15$, and $\xi_1 = 1$ where (a) The entropy squeezing factor $E(S_x)$; (b) The entropy squeezing factor $E(S_y)$; (c) The variance squeezing factor $V(S_x)$; (d) The variance squeezing factor $V(S_y)$; and (e) The time evolution of the atomic inversion under the same conditions.

the four states constituting the Bell state basis

$$\begin{aligned} |\psi_1\rangle &= \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle), \\ |\psi_2\rangle &= \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle), \\ |\psi_3\rangle &= \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle), \\ |\psi_4\rangle &= \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle). \end{aligned} \tag{28}$$

More general states could also be generated, depending on the initial conditions. In particular,

if cavities can be connected with each other by quantum wires, the system under consideration here would be a workable node of a larger quantum network.

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