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CLASSICAL RELATIVISTIC SYSTEM OF POINTLIKE MASSES WITH LINEARIZED GRAVITATIONAL INTERACTION

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The Hamiltonian formulation of relativistic system of pointlike particles with gravitational field is considered within the linearized theory of gravity. Both the Einstein's theory and the gauge theory of gravity are explored. The gauge-invariant description in the terms of Dirac's observables is obtained. Elimination of physical field variables is performed by means of the Dirac's theory of constraints up to the first order in the gravitational constant. The relation between positional and canonical variables of particles is found.

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1 Introduction

The gravitational interaction of the pointlike masses is described naturally by the field theory [1]. Sometimes, it is desirable to exclude the field degrees of freedom and to reformulate dynamics of a gravitationally bounded many-particle system in the terms of particle variables only. Elimination of the field is based on the substitution of a solution to the field equations into the equations of motion of particles. The complicated form of Einstein's equations forces one to use the linearized theory. Such a theory allows us to find the solution of field equations by means of the Green's function method and to exclude immediately the field variables in an action integral of the system [2, 3]. This leads to the Fokker-type action whose nonlocality is an obstacle in the transition to Hamiltonian picture. Similar problems appear in Hamiltonization of the Wheeler-Feynman electrodynamics [4]. Another way consists in elimination of the field degrees of freedom after the transition to the Hamiltonian description is performed. The gravitational field has been already reduced in the post-Newtonian approximations [5] within the canonical formalism of "particle+field" system by Arnowitt, Deser, and Misner [6]. But we note that the direct substitution of the solution in the terms of canonical particle variables into particle equations of motion and/or the Hamiltonian of the system is not correct from the following point of view. It is known that the covariant particle positions cannot be canonical after the field reduction [7]. Moreover, the direct insertion does not guarantee the preserving of commutation relations in the sense of the Poisson bracket between the components of the energy-momentum and angular momentum

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which form the canonical realization of the Poincaré group. These facts stimulate appearance of new approach [8, 9] based on the Dirac's theory of constraints [10]. The field equations within this approach are treated as constraints which have to be eliminated. In this paper we will perform the reduction of the gravitational field degrees of freedom by using the constraint method. Also we goal to investigate the linearized gauge theory of gravity by Ning Wu [11]. Such a theory has been built on the base of the gauge principle of fundamental interactions with the aim of them unification. Among various theories of the gauge gravity [12] this model is renormalizable.

In the present paper we start from the Hamiltonization of the classical relativistic system of pointlike particles in external gravitational field (Sec. 2). In Sec. 3 we explore the linearized Einstein's theory of gravity. First, we construct the Hamiltonian of the free gravitational field. The next step consists in the construction of Hamiltonian description of the "field+particle" system. On this level the field and particle variables are treated on equal rights. Eliminating gauge degrees of freedom, we reformulate dynamics of our system in the gauge-invariant manner in the terms of Dirac's observables (physical variables). Further, we exclude essentially the physical degrees of freedom of gravitational field in the first-order approximation in the gravitational constant. It is done by three steps [9]: (i) search of the solution to the field equations; (ii) transition to the canonical free-field variables; (iii) fixing free field by imposing additional constraints. These constraints are suppressed by means of the Dirac method. In such a way, we obtain the Hamiltonian formalism in the particle variables. In Sec. 4 we apply step by step our scheme to the linearized gauge theory of gravity by Ning Wu. In the present paper we are interested in comparison of the obtained metrics within the Einstein's theory and the Wu's one.

2 Hamiltonian description of the system of particles in gravitational field

Let us consider a system of N pointlike particles which are described by world lines in the curved space-time[2] $\gamma_a : \tau_a \mapsto x^\mu_a(\tau_a)$. The N-time covariant formalism cannot be used for construction of the Hamiltonian picture. In order to define the Poisson bracket of the point-particle system with the field, we have to use single-time formalism. We introduce the common evolution parameter t and put $\tau_a = t$, $x^0_a(t) = t$. Then the coordinates $\mathbf{x}_a(t) = (x^i_a(t))$ and the velocities $\dot{\mathbf{x}}_a(t) = (\dot{x}^i_a(t))$ are dynamical particle variables. The Lagrangian of the system of pointlike particles in the gravitational field is given by

$$L(t) = -\sum_{a=1}^{N} m_a \sqrt{g_{00}(t,\mathbf{x}_a) + 2g_{0i}(t,\mathbf{x}_a)\dot{x}^i_a(t) + g_{ik}(t,\mathbf{x}_a)\dot{x}^i_a(t)\dot{x}^k_a(t)}. \tag{1}$$

Let us transit by the standard way [6, 13] from the ten fields $g_{\mu\nu}$ to the lapse and shift functions, namely, N, N^i, and 3-dimensional metric γ^{ik} (or γ_{ik}):

$$\gamma_{ik} \equiv -g_{ik}, \qquad \gamma^{ik} = -g^{ik} + \frac{g^{0i}g^{0k}}{g^{00}}, \qquad \gamma_{ik}\gamma^{kj} = \delta^j_i, \tag{2}$$

[2]The Greek indices $\mu, \nu, \ldots$ run from 0 to 3; the Latin indices from the middle of alphabet, $i, j, k, \ldots$ run from 1 to 3 and both types of indices are subject of the summation convention. The Latin indices from the beginning of alphabet, a, b, label the particles and run from 1 to N. The sum over such indices is indicated explicitly. The velocity of light c is equal to unity.

$$N^i \equiv g_{0k}\gamma^{ki} = \frac{g^{0i}}{g^{00}}, \qquad N \equiv \sqrt{g_{00} - g_{ik}N^i N^k} = \sqrt{\frac{1}{g^{00}}}. \tag{3}$$

Using $g_{0i} = -g_{ik}N^k$ and introducing $u_a^i(t) \equiv \dot{x}_a^i(t) - N^i(t, \mathbf{x}_a)$, we have

$$L(t) = -\sum_{a=1}^{N} m_a \sqrt{N^2(t, \mathbf{x}_a) - \gamma_{ik}(t, \mathbf{x}_a)u_a^i(t)u_a^k(t)}. \tag{4}$$

We define the canonical particle momenta as

$$k_{ai}(t) \equiv -\frac{\partial L(t)}{\partial \dot{x}_a^i(t)} = -\frac{m_a \gamma_{ik}(t, \mathbf{x}_a)u_a^k(t)}{\sqrt{N^2(t, \mathbf{x}_a) - \gamma_{ik}(t, \mathbf{x}_a)u_a^i(t)u_a^k(t)}}. \tag{5}$$

The particle velocity is expressed in the terms of canonical variables as follows

$$\dot{x}_a^i(t) = N^i(t, \mathbf{x}_a) - \frac{N(t, \mathbf{x}_a)k_a^i(t)}{\sqrt{m_a^2 + \gamma^{ik}(t, \mathbf{x}_a)k_{ai}(t)k_{ak}(t)}}, \quad k_a^i(t) = \gamma^{ik}(t, \mathbf{x}_a)k_{ak}(t). \tag{6}$$

Now we immediately find the canonical Hamiltonian

$$H = -\sum_{a=1}^{N} k_{ai}\dot{x}_a^i - L = \sum_{a=1}^{N} \left[N\sqrt{m_a^2 + \gamma^{ik}k_{ai}k_{ak}} - k_{ai}N^i \right], \tag{7}$$

which generates time evolution in the terms of the Poisson bracket

$$\{x_a^i(t), k_{bj}(t)\} = -\delta_{ab}\delta_j^i. \tag{8}$$

In the case of weak gravitational field the metric can be presented in the form

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \qquad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{9}$$

where $\|\eta_{\mu\nu}\| = \mathrm{diag}(1, -1, -1, -1)$ is the metric of flat space-time. In the linear approximation we have $g^{\mu\lambda}g_{\lambda\nu} = \delta_\nu^\mu$. Then we obtain the Hamiltonian

$$H = \sum_{a=1}^{N} \sqrt{m_a^2 + \mathbf{k}_a^2} + \frac{1}{2}\int J_{\mu\nu}h^{\mu\nu}d^3x. \tag{10}$$

Here the current $J_{\mu\nu}$ is the free-particle energy-momentum tensor and has the following form

$$J_{\mu\nu} = \sum_{a=1}^{N} \frac{k_{a\mu}k_{a\nu}}{\sqrt{m_a^2 + \mathbf{k}_a^2}}\delta^3(\mathbf{x} - \mathbf{x}_a), \tag{11}$$

$$k_{a0} = k_a^0 = \sqrt{m_a^2 + \mathbf{k}_a^2}, \qquad k_{ai} = -k_a^i.$$

Now we have to introduce the Hamiltonian which generates the evolution of the field $h_{\mu\nu}$. In this paper we will consider two theories which define different Hamiltonians of gravitational field. Let us start from the general relativity.

3 General relativity

The exact Lagrangian density for the gravitational field within the Einstein's theory [14] is

$$\mathcal{L} = \frac{1}{16\pi G}\left[-g^{\mu\nu}(\Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\lambda\nu} - \Gamma^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\lambda\sigma})\sqrt{-\det(g_{\mu\nu})} + \partial_\mu\omega^\mu\right], \tag{12}$$

where the divergent term and the connection read

$$\omega^\mu = \sqrt{-\det(g_{\mu\nu})}\partial_\nu g_{\lambda\sigma}(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma}), \tag{13}$$

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}(\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu}). \tag{14}$$

Hereafter we use the notation

$$\partial_\mu \equiv \partial/\partial x^\mu, \qquad \partial_0 f = \partial f/\partial t \equiv \dot{f}.$$

3.1 Lagrangian function of the linearized theory. Hamiltonian formalism

The condition of weakness of the gravitational field with the metric (9) results immediately in the following form of the Lagrangian density

$$\mathcal{L} = -\frac{1}{16\pi G}\Gamma, \qquad \Gamma = \eta^{\mu\nu}(\Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\lambda\nu} - \Gamma^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\lambda\sigma}). \tag{15}$$

In a given approximation the connection takes the following form

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\eta^{\lambda\sigma}(\partial_\nu h_{\sigma\mu} + \partial_\mu h_{\sigma\nu} - \partial_\sigma h_{\mu\nu}). \tag{16}$$

Direct calculations give us

$$\begin{aligned} 4\Gamma \;=\;& -\pi_{ik}\pi^{ik} + \pi^i_i\pi^k_k + 2\partial^i h_{00}(\partial_i h^k_k - \partial_k h^k_i) - 2\partial^i h^j_i\partial_j h^k_k \\ & + \partial^j h^i_i\partial_j h^k_k + 2\partial_j h^i_k\partial^k h^j_i + \partial_j h^k_i\partial_j h^i_k + \Omega, \end{aligned} \tag{17}$$

where we have introduced the abbreviations:

$$\pi_{ik} \equiv \dot{h}_{ik} - \partial_i h_{0k} - \partial_k h_{0i}, \tag{18}$$

$$\Omega \equiv -2\partial_0[(h_{00} - h^k_k)\partial^i h_{0i}] + 2\partial^i[(h_{00} - h^k_k)\dot{h}_{0i}] + 4\partial^i[h_{0k}\partial^k h_{0i} - h_{0i}\partial^k h_{0k}]. \tag{19}$$

Discarding divergent term Ω, we obtain the final expression for the Lagrangian density:

$$\begin{aligned} \mathcal{L} \;=\;& \frac{1}{32\pi G}\left[\frac{1}{2}\pi_{ik}\pi^{ik} - \frac{1}{2}\pi^i_i\pi^k_k - \partial^i h_{00}(\partial_i h^k_k - \partial_k h^k_i) + \partial^i h^j_i\partial_j h^k_k\right. \\ & \left. - \frac{1}{2}\partial_j h^i_i\partial^j h^k_k - \partial_j h^i_k\partial^k h^j_i + \frac{1}{2}\partial_j h^i_k\partial^j h^k_i\right]. \end{aligned} \tag{20}$$

The canonical field momenta are

$$p^{0\mu} \equiv \frac{\partial \mathcal{L}}{\partial \dot{h}_{0\mu}} = 0, \tag{21}$$

$$p^{ik} \equiv \frac{\partial \mathcal{L}}{\partial \dot{h}_{ik}} = \frac{1}{32\pi G}\left(\pi^{ik} - \delta^{ik}\pi^{ll}\right). \tag{22}$$

The Poisson bracket relations are

$$\{h_{\mu\nu}(t,\mathbf{x}), p^{\lambda\sigma}(t,\mathbf{y})\} = I^{\lambda\sigma}_{\mu\nu}\delta^3(\mathbf{x}-\mathbf{y}), \qquad I^{\lambda\sigma}_{\mu\nu} = \frac{1}{2}(\delta^\lambda_\mu\delta^\sigma_\nu + \delta^\lambda_\nu\delta^\sigma_\mu). \tag{23}$$

The equation (21) determines the primary constraints [15]. Accounting (21), one finds the canonical Hamiltonian density (see [15])

$$\begin{aligned}\mathcal{H} \;&=\; 16\pi G\left(p_{ik}p^{ik} - \frac{1}{2}p^i_i p^k_k\right) + 2p^{ik}\partial_i h_{0k} + \frac{1}{32\pi G}\left[\partial^i h_{00}(\partial_i h^k_k - \partial_k h^k_i)\right.\\ &\left. - \partial^i h^j_i \partial_j h^k_k + \frac{1}{2}\partial_j h^i_i \partial^j h^k_k + \partial_j h^i_k \partial^k h^j_i - \frac{1}{2}\partial_j h^k_i \partial^j h^i_k\right].\end{aligned} \tag{24}$$

Having got the canonical formalism for dynamics of the gravitational field, we can build the Hamiltonian description of relativistic system of pointlike masses coupled with the field.

3.2 "Field plus particle" system: description in the terms of observables

Using expressions (10) and (24), we arrive to the Hamiltonian of "particle+field" system:

$$H = \sum_{a=1}^{N}\sqrt{m_a^2 + \mathbf{k}_a^2} + \frac{1}{2}\int J^{ik}h_{ik}d^3x + \int\left(\mathcal{H}_{\mathrm{g}} + \frac{1}{2}h_{00}T^0 + h_{0i}T^i\right)d^3x, \tag{25}$$

here

$$\mathcal{H}_{\mathrm{g}} = 16\pi G\left(p_{ik}p^{ik} - \frac{1}{2}p^i_i p^k_k\right) - \frac{1}{64\pi G}h_{kl}\Delta(P^{ik}P^{lj} - P^{kl}P^{ij})h_{ij}, \tag{26}$$

$$T^0 = J^{00} + \frac{1}{16\pi G}\Delta P^{ik}h_{ik}, \quad T^i = J^{0i} - 2\partial_k p^{ik}, \tag{27}$$

$$P^{ik} = \eta^{ik} + \Delta^{-1}\partial^i\partial^k. \tag{28}$$

The inverse operator to the Laplacian $\Delta \equiv -\partial^i\partial_i$ is defined by the equation

$$\Delta^{-1}\delta^3(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|}. \tag{29}$$

The constraints (21) are connected with the gauge invariance of equations of motion. The requirement of preservation of (21) in time produces the secondary constraints. The total set of constraints

$$p^{0\mu} \approx 0, \quad T^\mu \approx 0 \tag{30}$$

belongs to the first class because $\{p^{0\mu}, T^\nu\} = 0$. Hereafter, as usual, the equations of constraints are written with the use of "weak equality". It helps to distinguish constraints from evolution equations of motion.

The field variables canonically conjugated to the momenta $p^{0\mu}$ and the functions T^μ are unphysical and arbitrary. They are not fixed by equations of motion. This ambiguity can be reduced by fixing additional gauge conditions. Another way is the transition to the gauge-invariant description. Such an approach was initiated by Dirac for the electromagnetic field. Here we apply the Dirac's theory to the gauge-invariant reformulation of dynamics of our system.

One can verify that the field variables

$$Q_0 = -8\pi G \Delta^{-1} P_{ik} p^{ik}, \qquad Q_i = -\frac{1}{2}\Delta^{-1}[\partial^k h_{ik} + P_i^l \partial^k h_{lk}] \tag{31}$$

satisfy the following commutation relations in the sense of the Poisson brackets:

$$\{Q_\mu(t,\mathbf{x}), T^\nu(t,\mathbf{y})\} = \delta_\mu^\nu \delta^3(\mathbf{x}-\mathbf{y}), \qquad \{Q_\mu, Q_\nu\} = 0. \tag{32}$$

Then the pairs $(h_{0\mu}, p^{0\mu})$ and (Q_μ, T^μ) constitute the canonical basis of the gauge degrees of freedom. Now let us concentrate on the search of canonical variables which correspond to the gauge-invariant degrees of freedom. First, we need to decouple physical and gauge degrees of freedom by means of the following decomposition [15]:

$$h_{ik} = h_{ik}^\perp + h_{ik}^L, \qquad h_{ik}^\perp = P_{ik}^{lm} h_{lm}, \qquad h_{ik}^L = L_{ik}^{lm} h_{lm}, \tag{33}$$

$$p^{ik} = p_\perp^{ik} + p_L^{ik}, \qquad p_\perp^{ik} = P_{lm}^{ik} p^{lm}, \qquad p_L^{ik} = L_{lm}^{ik} p^{lm}. \tag{34}$$

Here projectors are given by

$$P_{lm}^{ik} = \frac{1}{2}(P_l^i P_m^k + P_m^i P_l^k - P^{ik} P_{lm}), \qquad L_{lm}^{ik} = I_{lm}^{ik} - P_{lm}^{ik}. \tag{35}$$

Then, employing the relations (27) and (31), we obtain

$$h_{ik}^L = -8\pi G P_{ik} \Delta^{-1}(T^0 - J^{00}) + \partial_i Q_k + \partial_k Q_i, \tag{36}$$

$$\begin{aligned} p_L^{ik} \quad = \quad & -\frac{1}{16\pi G} P^{ik} \Delta Q_0 + \frac{1}{2}\partial^k \Delta^{-1}(T^i - J^{0i}) \\ & + \frac{1}{2}\partial^i \Delta^{-1}(T^k - J^{0k}) + \frac{1}{2}\partial^i \partial^k \Delta^{-2} \partial_l (T^l - J^{0l}). \end{aligned} \tag{37}$$

For the transverse (physical) parts of the field variables we have

$$\{h_{ik}^\perp(t,\mathbf{x}), p_\perp^{lm}(t,\mathbf{y})\} = P_{ik}^{lm} \delta^3(\mathbf{x}-\mathbf{y}). \tag{38}$$

The splitting (33)–(34) induces a canonical transformation:

$$((x_a^i, k_{ai}), (h_{ik}, p^{ik})) \mapsto ((y_a^i, q_{ai}), (h_{ik}^\perp, p_\perp^{ik}), (Q_\mu, T^\mu)). \tag{39}$$

The transformation of the particle variables is generated by the functional:

$$F = \int Q_\mu J^{\mu 0} d^3x. \tag{40}$$

Within the framework of the linearized theory we get

$$y_a^i = x_a^i + \{x_a^i, F\}, \qquad q_{ai} = k_{ai} + \{k_{ai}, F\}. \tag{41}$$

Taking into consideration equations (30) and transformation (39), we reduce the Hamiltonian to the form

$$\begin{aligned} H &= \sum_{a=1}^{N} \sqrt{m_a^2 + \mathbf{q}_a^2} + \frac{1}{2} \int J^{ik} \left(h_{ik}^{\perp} - 8\pi G P_{ik} \Delta^{-1} J^{00} \right) d^3x + \int \mathcal{H}_{\mathrm{g}}^{\perp} d^3x \\ &\quad + 8\pi G \int \left(\frac{1}{4} J^{00} \Delta^{-1} J^{00} + J^{0i} \Delta^{-1} P_{ik} J^{0k} + \frac{1}{4} \partial_i J^{0i} \Delta^{-2} \partial_k J^{0k} \right) d^3x. \end{aligned} \tag{42}$$

It does not depend on the gauge variables $h_{0\mu}$ and Q_μ. Here the current and the physical field Hamiltonian are

$$J^{\mu\nu} = \sum_{a=1}^{N} \frac{q_a^\mu q_a^\nu}{\sqrt{m_a^2 + \mathbf{q}_a^2}} \delta^3(\mathbf{x} - \mathbf{y}_a), \qquad \mathcal{H}_{\mathrm{g}}^{\perp} = 16\pi G p_{ik}^{\perp} p_{\perp}^{ik} - \frac{1}{64\pi G} h_{\perp}^{ik} \Delta h_{ik}^{\perp}. \tag{43}$$

Thus, we have done reduction of the gauge degrees of freedom and transition to the description in the terms of Dirac's observables. At this stage the field and particle variables have equal rights. Below we shall carry out elimination of the physical degrees of freedom of the gravitational field.

3.3 Elimination of the physical degrees of freedom of the field in the first-order approximation in the gravitational constant

In this section we aim to reformulate our system in the terms of particle variables only. This reformulation is especially effective, when the free radiation is not essential. We do it by three steps. (i) We shall solve the Hamiltonian field equations in the linear approximation in the gravitational constant with the help of the Green's function method. In a given approximation the advanced, retarded and symmetric solutions coincide. Here we use the symmetric Green's function. (ii) By means of suitable transformation we shall get canonical free-field variables which appear as the solution to homogeneous field equations. (iii) We shall equate the free field to zero. The constrained variables are excluded from the dynamics with the help of Dirac bracket. Also, the use of the Dirac bracket allows us to eliminate the field from the Hamiltonian.

The Hamiltonian equations of motion for $h_{ik}^{\perp}$ and $p_{\perp}^{ik}$ read

$$\dot{h}_{ik}^{\perp} = 32\pi G p_{ik}^{\perp}, \qquad \dot{p}_{\perp}^{ik} = -\frac{1}{2} P_{lm}^{ik} J^{lm} + \frac{1}{32\pi G} \Delta h_{\perp}^{ik}. \tag{44}$$

Let us rewrite these equations in the following form:

$$\ddot{h}_{ik}^{\perp} - \Delta h_{ik}^{\perp} = -16\pi G P_{ik}^{lm} J_{lm}, \qquad p_{\perp}^{ik} = \frac{1}{32\pi G} \dot{h}_{\perp}^{ik}. \tag{45}$$

The general solution to equations (45) is presented as

$$h_{ik}^{\perp} = \phi_{ik}^{\perp} + \hat{h}_{ik}^{\perp}, \qquad p_{\perp}^{ik} = \chi_{\perp}^{ik} + \hat{p}_{\perp}^{ik} \tag{46}$$

Variables $\phi_{ik}^{\perp}$ and $\chi_{\perp}^{ik}$ are the general solutions to the corresponding homogeneous equations:

$$\ddot{\phi}_{ik}^{\perp} - \Delta\phi_{ik}^{\perp} = 0, \qquad \chi_{\perp}^{ik} = \frac{1}{32\pi G}\dot{\phi}_{\perp}^{ik}. \tag{47}$$

The functions $\hat{h}_{ik}^{\perp}$ depend on particle variables and satisfy inhomogeneous field equation with the pointlike sources. They are determined by

$$\begin{aligned} \hat{h}_{ik}^{\perp} &= P_{ik}^{lm}\hat{h}_{lm}, \\ \hat{h}_{\mu\nu} &= -16\pi G\sum_{a=1}^{N}\int G_{\mu\nu}^{\lambda\sigma}\left[(t-t')^2 - (\mathbf{x}-\mathbf{y}_a(t'))^2\right]\frac{q_{a\lambda}(t')q_{a\sigma}(t')}{\sqrt{m_a^2+\mathbf{q}_a^2(t')}}dt', \end{aligned} \tag{48}$$

where

$$G_{\mu\nu}^{\lambda\sigma}\left[(x^0)^2 - \mathbf{x}^2\right] = \frac{1}{2}\left(\delta_\mu^\lambda\delta_\nu^\sigma + \delta_\nu^\lambda\delta_\mu^\sigma - \eta^{\lambda\sigma}\eta_{\mu\nu}\right)G\left[(x^0)^2 - \mathbf{x}^2\right]. \tag{49}$$

In our problem the Green's function $G\left[(x^0)^2 - \mathbf{x}^2\right]$ of the d'Alambertian is symmetric. Using free-particle equations, when the particle momenta are conserved, an integration yields

$$\hat{h}_{\mu\nu} = -4G\sum_{a=1}^{N}\frac{q_{a\mu}q_{a\nu} - \frac{1}{2}\eta_{\mu\nu}m_a^2}{\sqrt{[\mathbf{q}_a(\mathbf{x}-\mathbf{y}_a)]^2 + m_a^2(\mathbf{x}-\mathbf{y}_a)^2}}. \tag{50}$$

Then, as remembering definition (22), one finds that

$$\hat{p}_{\perp}^{ik} = P_{lm}^{ik}\hat{p}^{lm}, \quad \hat{p}_{ik} = \frac{1}{32\pi G}\left[D_t\hat{h}_{ik} - \partial_i\hat{h}_{0k} - \partial_k\hat{h}_{0i} + \delta_{ik}(D_t\hat{h}_l^l - 2\partial_l\hat{h}_0^l)\right], \tag{51}$$

here

$$D_t = \sum_{a=1}^{N}\frac{q_a^i}{\sqrt{m_a^2+\mathbf{q}_a^2}}\frac{\partial}{\partial y_a^i} \tag{52}$$

is the time derivative.

It is easy to check that the obtained solutions in the terms of particle variables satisfy the equations of constraints

$$J^{00} + \frac{1}{16\pi G}\Delta P^{ik}\hat{h}_{ik} \equiv 0, \qquad J^{0i} - 2\partial_k\hat{p}^{ik} \equiv 0 \tag{53}$$

in the linear approximation in the gravitational constant.

The next step of our procedure of the field reduction consists in the canonical transformation of the field variables to $\phi^{\perp}_{ik}$, $\chi^{ik}_{\perp}$ in accordance with relations (46). This transformation changes the particle variables:

$$y^i_a = z^i_a + \int \left[\left(\phi^{\perp}_{lm} + \frac{1}{2}\hat{h}^{\perp}_{lm} \right) \frac{\partial \hat{p}^{lm}_{\perp}}{\partial \pi_{ai}} - \left(\chi^{lm}_{\perp} + \frac{1}{2}\hat{p}^{lm}_{\perp} \right) \frac{\partial \hat{h}^{\perp}_{lm}}{\partial \pi_{ai}} \right] d^3x, \tag{54}$$

$$q_{ai} = \pi_{ai} - \int \left[\left(\phi^{\perp}_{lm} + \frac{1}{2}\hat{h}^{\perp}_{lm} \right) \frac{\partial \hat{p}^{lm}_{\perp}}{\partial z^i_a} - \left(\chi^{lm}_{\perp} + \frac{1}{2}\hat{p}^{lm}_{\perp} \right) \frac{\partial \hat{h}^{\perp}_{lm}}{\partial z^i_a} \right] d^3x. \tag{55}$$

In the terms of new variables nonvanishing Poisson brackets are

$$\{z^i_a(t), \pi_{bj}(t)\} = -\delta_{ab}\delta^i_j, \qquad \{\phi^{\perp}_{ik}(t,\mathbf{x}), \chi^{lm}_{\perp}(t,\mathbf{x})\} = P^{lm}_{ik}\delta^3(\mathbf{x}-\mathbf{y}). \tag{56}$$

In order to finish our procedure, it remains to put $\phi^{\perp}_{ik} \approx 0$, $\chi^{ik}_{\perp} \approx 0$. These constraints reflect the absence of the free gravitational radiation.

In the dynamics we reduce the fixed field by means of Dirac bracket which is constructed as follows. Since $\phi^L_{ik} \equiv 0$ and $\chi^{ik}_L \equiv 0$ for the free field, we can restore canonical variables ϕ_{ik} and χ^{ik}, so that $\phi^{\perp}_{ik} = P^{lm}_{ik}\phi_{lm}$, $\chi^{ik}_{\perp} = P^{ik}_{lm}\chi^{lm}$. Then the Dirac bracket

$$\{F, G\}^* = \{F, G\} - \int \{F, \phi_{ik}(t,\mathbf{x})\} I^{ik}_{lm} \{\chi^{lm}(t,\mathbf{x}), G\} d^3x, \tag{57}$$

coincides with the particle Poisson bracket. Discarding the free field in the Hamiltonian, we get

$$\begin{aligned} H &= \sum_{a=1}^{N} \sqrt{m_a^2 + \boldsymbol{\pi}_a^2} + \frac{1}{4}\int J^{ik}\left(\hat{h}^{\perp}_{ik} - 16\pi G P_{ik}\Delta^{-1}J^{00}\right) d^3x \\ &\quad + 8\pi G \int \left(\frac{1}{4} J^{00}\Delta^{-1}J^{00} + J^{0i}\Delta^{-1}P_{ik}J^{0k} + \frac{1}{4}\partial_i J^{0i}\Delta^{-2}\partial_k J^{0k} \right) d^3x. \end{aligned} \tag{58}$$

We see that the field cancelation leads to compensation of half of the interaction term by the field part of Hamiltonian.

Here the current depending on new particle variables is

$$J^{\mu\nu} = \sum_{a=1}^{N} \frac{\pi^{\mu}_a \pi^{\nu}_a}{\sqrt{m_a^2 + \boldsymbol{\pi}_a^2}} \delta^3(\mathbf{x} - \mathbf{z}_a). \tag{59}$$

The fields $\hat{h}_{ik}$ and $\hat{p}^{ik}$ in the terms of new particle variables have the same form defined by expressions (50), (51).

If we add the following expression

$$-\frac{1}{4} D_t \int \left[16\pi G J^{00}\Delta^{-1}P_{ik}\hat{p}^{ik} + J^{0i}\Delta^{-1}\left(\partial^k \hat{h}_{ik} + P^l_i \partial^k \hat{h}_{lk} \right) \right] d^3x \tag{60}$$

to the Hamiltonian, we finally find

$$H = \sum_{a=1}^{N} \sqrt{m_a^2 + \boldsymbol{\pi}_a^2} + \frac{1}{4}\int J^{\mu\nu}\hat{h}_{\mu\nu} d^3x. \tag{61}$$

After exclusion of the free radiation the metric up to the first order in G becomes as follows

$$g_{\mu\nu} = \eta_{\mu\nu} - 4G \sum_{a=1}^{N} \frac{\pi_{a\mu}\pi_{a\nu} - \frac{1}{2}\eta_{\mu\nu} m_a^2}{\sqrt{[\boldsymbol{\pi}_a(\mathbf{x} - \mathbf{z}_a)]^2 + m_a^2(\mathbf{x} - \mathbf{z}_a)^2}}, \tag{62}$$

where $\pi_{a0} = \sqrt{m_a^2 + \boldsymbol{\pi}_a^2}$.

The combination of the equations (41) and (54) yields the relation between the covariant particle positions and the canonical variables:

$$x_a^i = z_a^i + \frac{1}{2} \int \left(\hat{h}_{lm} \frac{\partial \hat{p}^{lm}}{\partial \pi_{ai}} - \hat{p}^{lm} \frac{\partial \hat{h}_{lm}}{\partial \pi_{ai}} \right) d^3x. \tag{63}$$

The Dirac brackets between particle positions,

$$\{x_a^i, x_b^j\}^* = \int \left(\frac{\partial \hat{h}_{lm}}{\partial \pi_{bj}} \frac{\partial \hat{p}^{lm}}{\partial \pi_{ai}} - \frac{\partial \hat{p}^{lm}}{\partial \pi_{bj}} \frac{\partial \hat{h}_{lm}}{\partial \pi_{ai}} \right) d^3x \not\equiv 0, \tag{64}$$

show that x_a^i cannot be the canonical variable. It is in accordance with no-interaction theorem [7].

Poincaré invariance of the particle system with gravitational interaction causes existence of the ten conserved quantities which in the terms of canonical variables constitute the canonical realization of the Poincaré group. The Poincaré generators depending on particle variables result from the generators of "particle+field" system by means of field reduction. For four-momentum and angular momentum we find the expressions:

$$\begin{aligned}
P^0 &= \sum_{a=1}^{N} \sqrt{m_a^2 + \boldsymbol{\pi}_a^2} + \frac{1}{4} \int J^{\mu\nu} \hat{h}_{\mu\nu} d^3x, \\
P^i &= \sum_{a=1}^{N} \pi_a^i, \qquad M^{ik} = \sum_{a=1}^{N} (z_a^i \pi_a^k - z_a^k \pi_a^i), \\
M^{k0} &= \sum_{a=1}^{N} z_a^k \sqrt{m_a^2 + \boldsymbol{\pi}_a^2} + \frac{1}{4} \int x^k J^{\mu\nu} \hat{h}_{\mu\nu} d^3x - tP^k.
\end{aligned} \tag{65}$$

Self-action singular terms are excluded with the help of procedure based on Riesz potential [16]. It is easy to check that the generators satisfy the commutation relations of the Poincaré group in the first-order approximation:

$$\{P^\mu, P^\nu\}^* = 0, \qquad \{P^\mu, M^{\nu\lambda}\}^* = \eta^{\mu\nu} P^\lambda - \eta^{\mu\lambda} P^\nu,$$

$$\{M^{\mu\nu}, M^{\lambda\sigma}\}^* = -\eta^{\mu\lambda} M^{\nu\sigma} + \eta^{\nu\lambda} M^{\mu\sigma} - \eta^{\nu\sigma} M^{\mu\lambda} + \eta^{\mu\sigma} M^{\nu\lambda}. \tag{66}$$

It turns out that

$$\{x_a^i, M^{k0}\}^* = x_a^k \{x_a^i, H\}^* - \delta^{ik} t, \tag{67}$$

i.e. the particle position satisfies world line condition.

4 Gauge theory of gravity

Now let us explore an alternative theory of gravity. In this section we apply the developed procedures of reduction to the gauge theory of gravity by Ning Wu [11].

Within the Wu's theory the gravitation is treated as physical interaction in flat space-time with the Minkowski metric $\|\eta_{\mu\nu}\|$. The author of Ref. [11] replaces the equivalence principle of general relativity by the gauge principle of fundamental interactions. Then the gravitational interaction are completely determined by a local gravitational gauge invariance. Description and properties of the used gauge group can be found in original papers [11]. Here we are concentrated on examination of the results of such a theory which looks interesting in the context of the unification of fundamental interactions.

In Ref. [11] the gravitational field is described by the gauge potential $C^{\bar{\mu}}_{\nu}$. Indices μ, ν are ordinary Lorentz indices. On the other hand, the group (barred) indices $\bar{\mu}$, $\bar{\nu}$ running from 0 to 3 move by the Riemanian metric

$$g^{\bar{\mu}\bar{\nu}} = \eta^{\lambda\sigma} G^{\bar{\mu}}_{\lambda} G^{\bar{\nu}}_{\sigma}, \qquad G^{\bar{\mu}}_{\nu} = \delta^{\bar{\mu}}_{\nu} - C^{\bar{\mu}}_{\nu}. \tag{68}$$

This metric characterizes the curved space-time in which the particles move. In the case of weak gravitational field the expression (68) takes the form

$$g^{\bar{\mu}\bar{\nu}} = \eta^{\bar{\mu}\bar{\nu}} - \eta^{\bar{\mu}\lambda} C^{\bar{\nu}}_{\lambda} - \eta^{\bar{\nu}\lambda} C^{\bar{\mu}}_{\lambda}, \qquad g_{\bar{\mu}\bar{\nu}} = \eta_{\bar{\mu}\bar{\nu}} + \eta_{\bar{\mu}\lambda} C^{\lambda}_{\bar{\nu}} + \eta_{\bar{\nu}\lambda} C^{\lambda}_{\bar{\mu}}. \tag{69}$$

In a given approximation the Lagrangian density of such a theory is written as

$$\mathcal{L} = \frac{1}{16\pi G} \eta^{\mu\rho} \eta^{\nu\sigma} \eta_{\bar{\mu}\bar{\nu}} F^{\bar{\mu}}_{\mu\nu} F^{\bar{\nu}}_{\rho\sigma}, \tag{70}$$

where $F^{\bar{\lambda}}_{\mu\nu} = \partial_\mu C^{\bar{\lambda}}_{\nu} - \partial_\nu C^{\bar{\lambda}}_{\mu}$ is the field strength; $\partial_\mu = \delta^{\bar{\nu}}_{\mu} \partial_{\bar{\nu}}$.

From (70) the canonical momenta follow:

$$E^0_{\bar{\mu}} \equiv \frac{\partial \mathcal{L}}{\partial \dot{C}^{\bar{\mu}}_0} = 0, \tag{71}$$

$$E^i_{\bar{\mu}} \equiv \frac{\partial \mathcal{L}}{\partial \dot{C}^{\bar{\mu}}_i} = \frac{1}{4\pi G} \eta_{\bar{\mu}\bar{\nu}} \eta^{ij} F^{\bar{\nu}}_{0j}. \tag{72}$$

The field Poisson brackets are

$$\{C^{\bar{\mu}}_{\lambda}(t, \mathbf{x}), E^{\sigma}_{\bar{\nu}}\} = \delta^{\bar{\mu}}_{\bar{\nu}} \delta^{\sigma}_{\lambda} \delta^3(\mathbf{x} - \mathbf{y}). \tag{73}$$

Taking into account the constraints (71), we obtain a density of the canonical Hamiltonian:

$$\mathcal{H} = -\frac{1}{16\pi G} \eta_{\bar{\mu}\bar{\nu}} \eta^{ij} \eta^{kl} F^{\bar{\mu}}_{ik} F^{\bar{\nu}}_{jl} + 2\pi G \eta^{\bar{\mu}\bar{\nu}} \eta_{ij} E^i_{\bar{\mu}} E^j_{\bar{\nu}} + E^i_{\bar{\mu}} \partial_i C^{\bar{\mu}}_0. \tag{74}$$

Now we can construct the Hamiltonian of the "field+particle" system. One finds

$$H = \sum_{a=1}^{N} \sqrt{m_a^2 + \mathbf{k}_a^2} + \int J^i_{\bar{\mu}} C^{\bar{\mu}}_i d^3x + \int \mathcal{H}_{\mathrm{g}} d^3x + \int C^{\bar{\mu}}_0 T^0_{\bar{\mu}} d^3x. \tag{75}$$

Here we have introduced the notations

$$\mathcal{H}_{\rm g} = -\frac{1}{16\pi G}\eta_{\bar\mu\bar\nu}\eta^{ij}\eta^{kl}F^{\bar\mu}_{ik}F^{\bar\nu}_{jl} + 2\pi G\eta^{\bar\mu\bar\nu}\eta_{ij}E^i_{\bar\mu}E^j_{\bar\nu}, \qquad T^0_{\bar\mu} = J^0_{\bar\mu} - \partial_i E^i_{\bar\mu}, \tag{76}$$

$$J^\mu_{\bar\nu} = \sum_{a=1}^{N}\frac{k^\mu_a k_{a\nu}}{\sqrt{m_a^2 + \mathbf{k}_a^2}}\delta^3(\mathbf{x} - \mathbf{x}_a). \tag{77}$$

The set of the first class constraints of the theory is

$$E^0_{\bar\mu} \approx 0, \qquad T^0_{\bar\mu} \approx 0. \tag{78}$$

Let us isolate physical and gauge degrees of freedom by means of decomposition:

$$E^i_{\bar\mu} = \overset{\perp}{E}{}^i_{\bar\mu} + \overset{L}{E}{}^i_{\bar\mu}, \qquad \overset{\perp}{E}{}^i_{\bar\mu} = P^i_k E^k_{\bar\mu}, \qquad \overset{L}{E}{}^i_{\bar\mu} = \partial^i\Delta^{-1}(T^0_{\bar\mu} - J^0_{\bar\mu}). \tag{79}$$

Vector $\overset{\perp}{E}{}^i_{\bar\mu}$, whose components are subject of relation $\partial_i \overset{\perp}{E}{}^i_{\bar\mu} \equiv 0$, can be expressed in the terms of independent variables as follows (see [17])

$$\overset{\perp}{E}{}^i_{\bar\mu} = \Pi^i_\alpha e^\alpha_{\bar\mu}, \qquad e^\alpha_{\bar\mu} = E^\alpha_{\bar\mu}, \qquad \alpha = 1, 2. \tag{80}$$

Here we use the projector

$$\Pi^i{}_\alpha \equiv \delta^i_\alpha - \delta^i_3\frac{\partial_\alpha}{\partial_3}, \qquad \alpha = 1, 2. \tag{81}$$

The inverse operator to derivative ∂_3 is defined so that

$$\frac{1}{\partial_3}\delta^3(\mathbf{x}) \equiv \left(\frac{\partial}{\partial x^3}\right)^{-1}\delta^3(\mathbf{x}) = \frac{1}{2}\delta(x^1)\delta(x^2)\mathrm{sign}(x^3). \tag{82}$$

In order to decouple the gauge and gauge-invariant variables, we perform the canonical transformation

$$((x^i_a, k_{ai}), (C^{\bar\mu}_i, E^i_{\bar\mu})) \mapsto ((y^i_a, q_{ai}), (c^{\bar\mu}_\alpha, b^\alpha_{\bar\mu}), (Q^{\bar\mu}_0, T^0_{\bar\mu})) \tag{83}$$

determined by the generating functional

$$F = -\int C^{\bar\mu}_i\left[\Pi^i_\alpha e^\alpha_{\bar\mu} + \partial^i\Delta^{-1}(T^0_{\bar\mu} - J^0_{\bar\mu})\right]d^3x. \tag{84}$$

With the help of (84) we derive the conjugated variables

$$Q^{\bar\mu}_0 = -\frac{\delta F}{\delta T^0_{\bar\mu}} = -\Delta^{-1}\partial^i C^{\bar\mu}_i, \qquad c^{\bar\mu}_\alpha = -\frac{\delta F}{\delta e^\alpha_{\bar\mu}} = \Pi^i_\alpha C^{\bar\mu}_i. \tag{85}$$

In a given approximation we have

$$y^i_a = x^i_a + \{x^i_a, F\}, \qquad q_{ai} = k_{ai} + \{k_{ai}, F\}. \tag{86}$$

From equations (85) we obtain

$$C_i^{\bar\mu} = \overset{\perp}{C}{}_i^{\bar\mu} + \partial_i Q_0^{\bar\mu}, \qquad \overset{\perp}{C}{}_i^{\bar\mu} = P_i^{\alpha} c_{\alpha}^{\bar\mu}. \tag{87}$$

Therefore the transverse parts of the potential and the corresponding momenta are parameterized by means of the canonical variables $(c_{\alpha}^{\bar\mu},\, e_{\bar\mu}^{\alpha})$ which are not noncovariant. Hereafter it is convenient to describe the field by the functions $\overset{\perp}{C}{}_i^{\bar\mu}$, $\overset{\perp}{E}{}_{\bar\mu}^{i}$ which give us

$$\{\overset{\perp}{C}{}_i^{\bar\mu}(t,\mathbf{x}), \overset{\perp}{E}{}_{\bar\nu}^{k}(t,\mathbf{y})\} = \delta_{\bar\nu}^{\bar\mu} P_i^k \delta^3(\mathbf{x}-\mathbf{y}). \tag{88}$$

Accounting constraints, we see that the Hamiltonian depends on observables only:

$$H = \sum_{a=1}^{N} \sqrt{m_a^2 + \mathbf{q}_a^2} + \int J_{\bar\mu}^i \overset{\perp}{C}{}_i^{\bar\mu}\, d^3x + \int \mathcal{H}_{\rm g}^{\perp} d^3x + 2\pi G \int \eta^{\mu\nu} J_\mu^0 \Delta^{-1} J_\nu^0 d^3x. \tag{89}$$

Here

$$\mathcal{H}_{\rm g}^{\perp} = -\frac{1}{8\pi G}\eta_{\bar\mu\bar\nu}\eta^{ij}\, \overset{\perp}{C}{}_i^{\bar\mu}\, \Delta\, \overset{\perp}{C}{}_j^{\bar\nu} + 2\pi G \eta^{\bar\mu\bar\nu}\eta_{ij}\, \overset{\perp}{E}{}_{\bar\mu}^{i} \overset{\perp}{E}{}_{\bar\nu}^{j}\, . \tag{90}$$

is the physical Hamiltonian of the gravitational field.

The Hamiltonian (89) generates evolution of the field variables:

$$\dot{\overset{\perp}{C}}{}_i^{\bar\mu} = 4\pi G \eta^{\bar\mu\bar\nu}\eta_{ij}\, \overset{\perp}{E}{}_{\bar\nu}^{j}, \qquad \dot{\overset{\perp}{E}}{}_{\bar\mu}^{i} = -P_k^i J_{\bar\mu}^k + \frac{1}{4\pi G}\eta_{\bar\mu\bar\nu}\eta^{ij}\Delta\, \overset{\perp}{C}{}_j^{\bar\nu}\, . \tag{91}$$

Let us rewrite equations (91) in the following form

$$\ddot{\overset{\perp}{C}}{}_i^{\bar\mu} - \Delta\, \overset{\perp}{C}{}_i^{\bar\mu} = -4\pi G P_i^k J_k^{\bar\mu}, \qquad \overset{\perp}{E}{}_{\bar\mu}^{i} = \frac{1}{4\pi G}\eta_{\bar\mu\bar\nu}\eta^{ij}\, \dot{\overset{\perp}{C}}{}_j^{\bar\nu}\, . \tag{92}$$

They are ordinary inhomogeneous differential equations which can be solved by means of the Green's function method. The general solution to the field equations is presented by

$$\overset{\perp}{C}{}_i^{\bar\mu} = \overset{\perp}{\phi}{}_i^{\bar\mu} + \overset{\perp}{X}{}_i^{\bar\mu}, \qquad \overset{\perp}{E}{}_{\bar\mu}^{i} = \overset{\perp}{\chi}{}_{\bar\mu}^{i} + \overset{\perp}{Y}{}_{\bar\mu}^{i}\, . \tag{93}$$

The fields $\overset{\perp}{\phi}{}_i^{\bar\mu} = P_i^{\alpha}\phi_{\alpha}^{\bar\mu}$, $\overset{\perp}{\chi}{}_{\bar\mu}^{i} = \Pi_{\alpha}^i \chi_{\bar\mu}^{\alpha}$ satisfy homogeneous equations. On the other hand, the fields $\overset{\perp}{X}{}_i^{\bar\mu}$, $\overset{\perp}{Y}{}_{\bar\mu}^{i}$ depend on particle variables. They are written as follows

$$\overset{\perp}{X}{}_i^{\bar\mu} = P_i^k X_k^{\bar\mu}, \qquad X_\nu^{\bar\mu} = -G\sum_{a=1}^{N} \frac{q_a^{\mu} q_{a\nu}}{\sqrt{[\mathbf{q}_a(\mathbf{x}-\mathbf{y}_a)]^2 + m_a^2(\mathbf{x}-\mathbf{y}_a)^2}}, \tag{94}$$

$$\overset{\perp}{Y}{}_{\bar\mu}^{i} = P_k^i Y_{\bar\mu}^i, \qquad Y_{\bar\mu}^i = \frac{1}{4\pi G}\eta_{\bar\mu\bar\nu}\eta^{ij}(D_t X_j^{\bar\nu} - \partial_j X_0^{\bar\nu}). \tag{95}$$

Now we intend to perform the canonical transformation to the new field variables $\overset{\perp}{\phi}{}_i^{\bar{\mu}}$, $\overset{\perp}{\chi}{}_{\bar{\mu}}^i$ (mean $\phi_\alpha^{\bar{\mu}}$ and $\chi_{\bar{\mu}}^\alpha$) in accordance with relations (93). This transformation leads to the new particle variables:

$$y_a^i = z_a^i + \int \left[\left(\overset{\perp}{\phi}{}_l^{\bar{\mu}} + \frac{1}{2} \overset{\perp}{X}{}_l^{\bar{\mu}} \right) \frac{\partial \overset{\perp}{Y}{}_{\bar{\mu}}^l}{\partial \pi_{ai}} - \left(\overset{\perp}{\chi}{}_{\bar{\mu}}^l + \frac{1}{2} \overset{\perp}{Y}{}_{\bar{\mu}}^l \right) \frac{\partial \overset{\perp}{X}{}_l^{\bar{\mu}}}{\partial \pi_{ai}} \right] d^3x, \tag{96}$$

$$q_{ai} = \pi_{ai} - \int \left[\left(\overset{\perp}{\phi}{}_l^{\bar{\mu}} + \frac{1}{2} \overset{\perp}{X}{}_l^{\bar{\mu}} \right) \frac{\partial \overset{\perp}{Y}{}_{\bar{\mu}}^l}{\partial z_a^i} - \left(\overset{\perp}{\chi}{}_{\bar{\mu}}^l + \frac{1}{2} \overset{\perp}{Y}{}_{\bar{\mu}}^l \right) \frac{\partial \overset{\perp}{X}{}_l^{\bar{\mu}}}{\partial z_a^i} \right] d^3x. \tag{97}$$

We reduce field degrees of freedom by the constraints $\phi_\alpha^{\bar{\mu}} \approx 0$, $\chi_{\bar{\mu}}^\alpha \approx 0$. After that the Hamiltonian formalism is formed by the Dirac bracket, which is equal to the particle Poisson bracket, and the Hamiltonian:

$$H = \sum_{a=1}^{N} \sqrt{m_a^2 + \boldsymbol{\pi}_a^2} + \frac{1}{2} \int J_{\bar{\mu}}^i \overset{\perp}{X}{}_i^{\bar{\mu}} \, d^3x + 2\pi G \int \eta^{\mu\nu} J_\mu^0 \Delta^{-1} J_\nu^0 d^3x. \tag{98}$$

Adding the expression

$$-\frac{1}{2} D_t \int J_\mu^0 \Delta^{-1} \partial^l X_l^\mu d^3x \tag{99}$$

to the Hamiltonian that is equivalent to canonical transformation gives us

$$H = \sum_{a=1}^{N} \sqrt{m_a^2 + \boldsymbol{\pi}_a^2} + \frac{1}{2} \int J_\nu^\mu X_\mu^\nu d^3x. \tag{100}$$

Then we find the final form of the metric:

$$g^{\bar{\mu}\bar{\nu}} = \eta^{\bar{\mu}\bar{\nu}} + 2G \sum_{a=1}^{N} \frac{\pi_a^\mu \pi_a^\nu}{\sqrt{[\boldsymbol{\pi}_a(\mathbf{x} - \mathbf{z}_a)]^2 + m_a^2 (\mathbf{x} - \mathbf{z}_a)^2}}. \tag{101}$$

The covariant particle positions and the canonical variables are connected by means of the following relations:

$$x_a^i = z_a^i + \frac{1}{2} \int \left(X_l^\mu \frac{\partial Y_\mu^l}{\partial \pi_{ai}} - Y_\mu^l \frac{\partial X_l^\mu}{\partial \pi_{ai}} \right) d^3x. \tag{102}$$

Then Poisson brackets between particle positions,

$$\{x_a^i, x_b^k\}^* = \int \left(\frac{\partial X_l^\mu}{\partial \pi_{bk}} \frac{\partial Y_\mu^l}{\partial \pi_{ai}} - \frac{\partial Y_\mu^l}{\partial \pi_{bk}} \frac{\partial X_l^\mu}{\partial \pi_{ai}} \right) d^3x \not\equiv 0, \tag{103}$$

show that x_a^i cannot be canonical.

Similarly to (65), the Poincaré generators can be written down on the base of (65) by means of replacement of the corresponding density of interaction term.

5 Conclusions

The gauge-invariant Hamiltonian formulation of the Einstein's and Wu's linearized theories of gravity, when the particle and field variables are treated on equal rights, is obtained. In order to find the description in the terms of particle variables only, we have effectively eliminated the field degrees of freedom with the help of the Dirac's theory of constraints. The used procedure of the field reduction was elaborated in our previous paper [9]. Here we have limited ourselves by the first-order approximation in the gravitational constant.

The field elimination permits us to get the relation between covariant particle positions and canonical variables for gravitationally bounded system of particles. Moreover, such an approach ensures the Poincaré invariance of the theory: the ten canonical Poincaré generators in the terms of particles variables satisfy commutation relations of the Poincaré group in a given approximation.

Discarding gravitational field, we have obtained expression for metric tensor depending on the canonical particle variables for the Einstein's and Wu's theories of gravity (see (62) and (101)). The metric (62) in the general relativity is in agreement with the result of classical work [16] in the terms of configuration variables. On the other hand, the Wu's gravitational field $h^W_{\mu\nu}$ and the Einstein's field $h^E_{\mu\nu}$ are related as follows

$$2h^W_{\mu\nu} = h^E_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^E, \qquad h^E \equiv \eta^{\mu\nu}h^E_{\mu\nu}.$$

Such a difference cannot be eliminated by means of canonical transformation. Thus, in the first-order approximation in the gravitational constant with the explicit accounting of relativistic kinematics the general relativity and the Wu's theory of gravity are not physically equivalent. However, the Wu's metric leads to the Newtonian potential. Also $h^W_{\mu\nu}$ and $h^E_{\mu\nu}$ give us similar expressions in the massless case.

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