

**TOWARDS A MANIFESTLY GAUGE INVARIANT AND
UNIVERSAL CALCULUS FOR YANG-MILLS THEORY¹****S. Arnone², A. Gatti³, T. R. Morris⁴***Department of Physics and Astronomy, University of Southampton
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A manifestly gauge invariant exact renormalization group for pure $SU(N)$ Yang-Mills theory is proposed, along with the necessary gauge invariant regularisation which implements the effective cutoff. The latter is naturally incorporated by embedding the theory into a spontaneously broken $SU(N|N)$ super-gauge theory, which guarantees finiteness to all orders in perturbation theory. The effective action, from which one extracts the physics, can be computed whilst manifestly preserving gauge invariance at each and every step. As an example, we give an elegant computation of the one-loop $SU(N)$ Yang-Mills beta function, for the first time at finite N without any gauge fixing or ghosts. It is also completely independent of the details put in by hand, *e.g.* the choice of covariantisation and the cutoff profile, and, therefore, guides us to a procedure for streamlined calculations.

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1 ERG and gauge invariance

The basic idea of the exact renormalization group (ERG) is summarised in the diagram below. For a detailed review, and current developments, see for example [1, 2]. In the partition function for the theory, defined in the continuum and in Euclidean space, rather than integrate over all momentum modes in one go, one first integrates out modes between an overall cutoff Λ_0 and the effective Wilsonian cutoff $\Lambda \ll \Lambda_0$. The remaining integral can again be expressed as a partition function, but the bare action, S_{Λ_0} , is replaced by an effective action, S . The new Boltzmann factor, $\exp -S$, is more or less the original partition function, modified by an infrared cutoff Λ [3]. When finally Λ is sent to zero, the full partition function is recovered and all the physics that goes with it (*e.g.* Green functions). In practice however, one does not work with this integral form but rather a differential equation for S , the ERG equation, that expresses how S changes as one lowers Λ .

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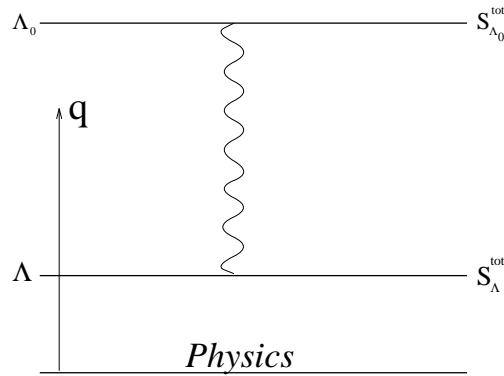


Fig. 1. Integrating out modes.

The application of this technique to quantum field theory brings with it many advantages, because renormalization properties, which are normally subtle and complicated, are here—using Wilson’s insight [4]—straightforward to build in from the beginning. Thus solutions for the effective action may be found directly in terms of renormalized quantities (in fact without specifying a bare action at Λ_0 , which is anyway, by universality, largely arbitrary), and within this framework almost any approximation can be considered (for example truncations [5], derivative expansion [6] etc.) without disturbing this property [1]. The result is that these ideas form a powerful framework for considering non-perturbative analytic approximations in quantum field theory [2].

In particle physics, all the interesting non-perturbative questions also involve gauge theory. However, in order to construct a gauge invariant ERG, we must overcome an obvious conflict: the division of momenta into large and small, according to the effective scale Λ , is not preserved by gauge transformations. (Explicitly, consider a matter field $\phi(x)$. Under a gauge transformation $\phi(x) \rightarrow \Omega(x)\phi(x)$ momentum modes $\phi(p)$ are mapped to a convolution with the modes from Ω .) We only have two choices. Either we break the gauge invariance and try to recover it once the cutoff is removed, by imposing suitable boundary conditions on the ERG equation [7], or we generalise things so that we can write down a gauge invariant ERG equation.

We will go with the second choice [8–10]. Furthermore, we will find that we can continue to keep the gauge invariance manifest at all stages even when we start to compute the effective action. No gauge fixing or ghosts are required. Therefore, any gauge invariant quantities can in principle be evaluated by means of our gauge invariant RG equation.

However, in view of the novelty of the present construction, it is desirable to test the formalism first. We computed the one-loop beta function for $SU(N)$ Yang-Mills theory for a general cutoff profile⁵ and we obtained the usual perturbative result, which is an encouraging confirmation that the expected universality of the continuum limit has been incorporated. The calculation is completely independent of the details put in by hand, *e.g.* the choice of covariantisation and seed action (which will be defined later in Section 3), and therefore guides us to a procedure for

⁵provided some general requirements on normalisation and ultraviolet decay rate are satisfied

streamlined computations. The key ingredient throughout is the use of gauge invariance, whose full power and beauty shines through, as will all become clear in what follows.

This note is organised as follows. In Section 2 we illustrate our regularisation scheme, including a brief description of the novel features of the $SU(N|N)$ gauge group. In Section 3 we state the flow equation in superfield notation, trying to motivate it by considering Polchinski's equation first. We then perform the usual loop expansion and sketch our strategy for computing β_1 . Section 4 is devoted to listing the (un-)broken gauge invariance identities, while Section 5 contains a more detailed description of the simplest part of the calculation, the scalar sector. Finally, in Section 6 we summarise and draw our conclusions.

2 Regularisation via $SU(N|N)$

2.1 General idea

As a necessary first step, we need a gauge invariant implementation of the non-perturbative continuum effective cutoff Λ . The standard ERG cutoff is implemented by inserting $c^{-1}(p^2/\Lambda^2)$ into the kinetic term of the action. c is a smooth ultraviolet cutoff profile with $c(0) = 1$, decaying sufficiently rapidly as $p/\Lambda \rightarrow \infty$ that all quantum corrections are regularised. To restore the gauge invariance we covariantise so that the regularised bare action takes the form:

$$\frac{1}{2g^2} \text{tr} \int d^Dx F_{\mu\nu} c^{-1}(-D^2/\Lambda^2) \cdot F^{\mu\nu}. \quad (1)$$

Here $F_{\mu\nu} = i[D_\mu, D_\nu]$ is the standard field strength, built from the covariant derivative $D_\mu = \partial_\mu - iA_\mu$. We scale out the coupling g for good reason: since gauge invariance will be exactly preserved, the form of the covariant derivative is protected [11], which in this parametrisation simply means that A suffers no wavefunction renormalization. Eq. (1) is nothing but covariant higher derivative regularisation and is known to fail at one-loop [12]. Slavnov solved this problem by introducing gauge invariant Pauli-Villars fields [13]. These appear bilinearly so that their one-loop determinants cancel the remaining divergences. We cannot use these ideas directly since the bilinearity property cannot be preserved by the ERG flow [8,10]. Instead, we discovered a novel and elegant solution: we embed (1) in a spontaneously broken $SU(N|N)$ super-gauge theory [14]. We will see that the result has similar characteristics to Slavnov's scheme but sits much more naturally in the effective action framework. Indeed the regularising properties will follow from the supersymmetry in the fibres of the high energy unbroken supergroup. We will then design an ERG in which the spontaneous breaking scale and higher derivative scale are identified and flow together as we lower Λ .

2.2 The $SU(N|N)$ super group

The graded Lie algebra of $SU(N|M)$ in the $(N+M)$ -dimensional representation is given by

$$\mathcal{H} = \begin{pmatrix} H_N & \theta \\ \theta^\dagger & H_M \end{pmatrix}. \quad (2)$$

$H_N (H_M)$ is an $N \times N (M \times M)$ Hermitian matrix with complex bosonic elements and θ is an $M \times N$ matrix composed of complex Grassmann numbers. \mathcal{H} is required to be supertraceless, *i.e.*

$$\text{str}(\mathcal{H}) = \text{tr}(\sigma_3 \mathcal{H}) = \text{tr}(H_N) - \text{tr}(H_M) = 0 \tag{3}$$

(where $\sigma_3 = \text{diag}(\mathbb{1}_N, -\mathbb{1}_M)$ is the obvious generalisation of the Pauli matrix to this context). The traceless parts of H_N and H_M correspond to $SU(N)$ and $SU(M)$ respectively and the traceful part gives rise to a $U(1)$, so we see that the bosonic sector of the $SU(N|M)$ algebra forms a $SU(N) \times SU(M) \times U(1)$ sub-algebra.

Specialising to $M = N$, we see that the $U(1)$ generator becomes just $\mathbb{1}_{2N}$ and thus commutes with all the other generators. We cannot simply drop it however because it is generated by other elements of the algebra (*e.g.* $\{\sigma_1, \sigma_1\} = 2\mathbb{1}_{2N}$). Bars suggested removing it by redefining the Lie bracket to project out traceful parts [15]: $[\ , \]_{\pm} \mapsto [\ , \]_{\pm} - \frac{\mathbb{1}}{2N} \text{tr}[\ , \]_{\pm}$. We can use this idea but only on the gauge fields: the matter fields require the full commutator because invariance of the Lagrangian in this sector requires the bracket to be Leibnitz [14]. A simpler and equivalent solution [14] is to keep the $\mathbb{1}_{2N}$ and note that the corresponding gauge field, \mathcal{A}^0 , which we have seen is needed to absorb gauge transformations produced in the $\mathbb{1}_{2N}$ direction, does not however appear in the Lagrangian at all! (Its absence is then protected by a no- \mathcal{A}^0 shift-symmetry: $\delta \mathcal{A}_{\mu}^0 = \Lambda_{\mu}$.)

2.3 Higher derivative $SU(N|N)$ super-gauge theory

We promote the gauge field to a connection for $SU(N|N)$:

$$\mathcal{A}_{\mu} = \mathcal{A}_{\mu}^0 \mathbb{1} + \begin{pmatrix} A_{\mu}^1 & B_{\mu} \\ \bar{B}_{\mu} & A_{\mu}^2 \end{pmatrix}, \tag{4}$$

where the A_{μ}^i are the two bosonic gauge fields for $SU(N) \times SU(N)$, and B_{μ} is a fermionic gauge field. The field strength $\mathcal{F}_{\mu\nu}$ is now a commutator of the super-covariant derivative $\nabla_{\mu} = \partial_{\mu} - i\mathcal{A}_{\mu}$. The super-gauge field part of the Lagrangian is then

$$\mathcal{L}_{\mathcal{A}} = \frac{1}{2g^2} \mathcal{F}_{\mu\nu} \{c^{-1}\} \mathcal{F}^{\mu\nu}. \tag{5}$$

Here we take the opportunity to be more sophisticated about the covariantisation of the cutoff. For any momentum space kernel $W(p^2/\Lambda^2)$, there are infinitely many covariantisations. The form used in (1) is just one of them. Another way would be to use Wilson lines [8, 11]. In general, the covariantisation results in a new set of vertices (infinite in number if W is not a polynomial):

$$\mathbf{u}\{W\}\mathbf{v} = \sum_{n,m=0} \int_{x,y} \int_{x_i y_j} W_{\mu_1 \dots \mu_n, \nu_1 \dots \nu_m}(x_i; y_j; x, y) \text{str}[\mathbf{u}(x) \mathcal{A}_{\mu_1}(x_1) \dots \mathcal{A}_{\mu_n}(x_n) \mathbf{v}(y) \mathcal{A}_{\nu_1}(y_1) \dots \mathcal{A}_{\nu_m}(y_m)]. \tag{6}$$

($\mathbf{u}(x)$ and $\mathbf{v}(y)$ are any two supermatrix representations.) These can be graphically represented as in Fig. 2.

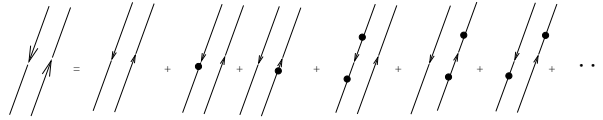


Fig. 2. Expansion of the covariantisation in terms of super-gauge fields.

2.4 Spontaneous breaking in fermionic directions

Now we add a super-scalar field, $\mathcal{L} = \mathcal{L}_A + \mathcal{L}_C$,

$$C = \begin{pmatrix} C^1 & D \\ \bar{D} & C^2 \end{pmatrix} \in U(N|N), \tag{7}$$

with a Lagrangian that encourages spontaneous symmetry breaking:

$$\mathcal{L}_C = \frac{1}{2} \nabla_\mu \cdot C \{ \tilde{c}^{-1} \} \nabla^\mu \cdot C + \frac{\lambda}{4} \text{str} \int (C^2 - \Lambda^2)^2. \tag{8}$$

Choosing the classical vacuum expectation value $\Lambda \sigma_3$ breaks all and only the fermionic directions, and expanding about this, by $C \mapsto C + \Lambda \sigma_3$, gives

$$\begin{aligned} \mathcal{L}_C = & \frac{1}{2} \nabla_\mu \cdot C \{ \tilde{c}^{-1} \} \nabla^\mu \cdot C - i \Lambda [A_\mu, \sigma_3] \{ \tilde{c}^{-1} \} \nabla^\mu \cdot C \\ & - \frac{1}{2} \Lambda^2 [A_\mu, \sigma_3] \{ \tilde{c}^{-1} \} [A^\mu, \sigma_3] + \frac{\lambda}{4} \text{str} \int (\Lambda \{ \sigma_3, C \} + C^2)^2. \end{aligned} \tag{9}$$

Since (fermionic) bosonic parts (anti)commute with σ_3 , we see in the second line that B gains a mass $\sqrt{2}\Lambda$ (B eats D), and C^1 and C^2 gain masses $\sqrt{2\lambda}\Lambda$. These heavy fields play the rôle of Slavnov’s gauge invariant Pauli-Villars fields.

2.5 Proof of regularisation

A proof that this all adds up to a regularisation of four dimensional $SU(N)$ Yang-Mills theory has been given in [14]. We only have room to summarise the conclusions.

If c^{-1} and \tilde{c}^{-1} are chosen to be polynomials of rank r , \tilde{r} , we require $r > \tilde{r} - 1$ and $\tilde{r} > -1$ simply to ensure that at high momentum the propagators go over to those of the unbroken $SU(N|N)$ theory. The stronger constraints $\tilde{r} > 1$ and $r - \tilde{r} > 1$ then ensure finiteness in all perturbative diagrams except pure \mathcal{A} one-loop graphs with up to 4 external legs. This maximises the regularising power of the covariant higher derivatives and is ensured simply by power counting. The remaining diagrams can be shown to be finite within spontaneously broken $SU(N|N)$ gauge theory as follows. One-loop diagrams with 2 or 3 external \mathcal{A} legs are finite because of supersymmetric cancellations in group theory factors: $\text{str} A_\mu = \text{str} \mathbb{1} = 0$. Transverse parts of such diagrams with four external legs are finite by power counting, whilst the longitudinal parts are finite once gauge invariance properties are taken into account [14].

Finally, we need to show that at energies much lower than the cutoff, the theory we are supposed to be regularising is recovered, namely $SU(N)$ Yang-Mills. (In the ERG context the

cutoff in question is $\Lambda_0 \rightarrow \infty$ where explicitly or implicitly, the partition function is defined.) There is a case to answer because the massless sector that remains, contains the second gauge field, A^2 . In fact this gauge field is unphysical because the supertrace in (3) gives it a wrong sign action, as can be seen from eq. (5),⁶ leading to negative norms in its Fock space [14]. Fortunately, the Appelquist-Carazzone theorem saves the day: since the A^1 and A^2 live in disjoint groups, the lowest dimension interaction between A^1 and A^2 is proportional to $\text{tr} (F_{\mu\nu}^1)^2 \text{tr} (F_{\mu\nu}^2)^2$. Since this is irrelevant, the A^2 sector decouples in the limit that $\Lambda_0 \rightarrow \infty$.

This completes the proof of finiteness to all orders of perturbation theory, in four (or less) dimensions. In the limit $N = \infty$, the scheme can be shown to regularise in any dimension [14].

3 Manifestly gauge invariant flow equation and its loop expansion

3.1 Polchinski's equation

We are ready to write a gauge invariant flow equation. To motivate it consider Polchinski's version of Wilson's ERG [16]. We can cast it in the form [20]

$$\Lambda \partial_\Lambda S = -\frac{1}{\Lambda^2} \frac{\delta S}{\delta \varphi} \cdot c' \cdot \frac{\delta \Sigma}{\delta \varphi} + \frac{1}{\Lambda^2} \frac{\delta}{\delta \varphi} \cdot c' \cdot \frac{\delta \Sigma}{\delta \varphi}. \quad (10)$$

Here φ is for example a single scalar field. Σ is the combination $S - 2\hat{S}$, where \hat{S} is the regularised kinetic term $\hat{S} = \frac{1}{2} \partial_\mu \varphi \cdot c^{-1} \cdot \partial_\mu \varphi$. In this form it is clear that the ERG leaves the partition function invariant because the Boltzmann measure factor flows into a total functional derivative:

$$\Lambda \partial_\Lambda \exp(-S) = -\frac{1}{\Lambda^2} \frac{\delta}{\delta \varphi} \cdot c' \cdot \left(\frac{\delta \Sigma}{\delta \varphi} \exp(-S) \right). \quad (11)$$

At this stage we recognize that there is nothing particularly special about the Polchinski / Wilson version. There are infinitely many other ERG flow equations with this property [17], the continuum analogue of the infinitely many possible blockings on the lattice. All we have to do is to choose a gauge invariant one by making a gauge covariant replacement for $\Psi = c' \cdot \delta \Sigma / \delta \varphi$.

3.2 $SU(N)$ gauge invariant ERG

Writing $\varphi \mapsto A_\mu$, this can be done simply by replacing $\cdot c' \cdot$ with $\{c'\}$ and replacing \hat{S} with a gauge invariant generalisation. Thus:

$$\Lambda \partial_\Lambda S = -\frac{1}{\Lambda^2} \frac{\delta S}{\delta A_\mu} \{c'\} \frac{\delta \Sigma_g}{\delta A_\mu} + \frac{1}{\Lambda^2} \frac{\delta}{\delta A_\mu} \{c'\} \frac{\delta \Sigma_g}{\delta A_\mu} + \dots, \quad (12)$$

where $\hat{S} = \frac{1}{4} F_{\mu\nu} \{c^{-1}\} F_{\mu\nu} + \dots$. Recall that the coupling g was scaled out, cf. eq. (1). It must reappear somewhere in the flow equation and some thought shows that the appropriate place is in the combination $\Sigma_g = g^2 S - 2\hat{S}$. We have added the ellipsis in the recognition that further regularisation will be needed over and above the gauge invariant higher derivatives.

⁶The supertrace is a necessity since it is this, not the trace, that leads to invariants when supergroups are used [14, 15].

3.3 $SU(N|N)$ gauge invariant ERG

We get the remaining regularisation by promoting A to \mathcal{A} , adding the super-scalar sector, and then shifting \mathcal{C} to the fermionic symmetry breaking vacuum expectation value. We want to ensure that under the ERG flow, this vacuum expectation value flows with the effective cutoff, *i.e.* as $\langle \mathcal{C} \rangle = \Lambda \sigma_3$. One can show that this follows at the classical level if we work instead with a dimensionless superscalar, $\mathcal{C} \mapsto \mathcal{C}\Lambda$, so that the shift becomes $\mathcal{C} \rightarrow \mathcal{C} + \sigma_3$. It is technically very convenient if the ERG equation allows for the classical two-point vertices to be equal to those coming from \hat{S} [10], as is true (10) and (12) above. To keep this property in the spontaneously broken phase we need different kernels for B and D which we can make by adding \mathcal{C} commutator terms. Constructing the appropriate Ψ , we thus obtain in the symmetric phase, a fully manifestly $SU(N|N)$ gauge invariant flow equation:

$$\Lambda \partial_\Lambda S = -a_0[S, \Sigma_g] + a_1[\Sigma_g], \quad (13)$$

where

$$\begin{aligned} a_0[S, \Sigma_g] &= \frac{1}{2\Lambda^2} \left(\frac{\delta S}{\delta \mathcal{A}_\mu} \{c'\} \frac{\delta \Sigma_g}{\delta \mathcal{A}_\mu} - \frac{1}{4} [\mathcal{C}, \frac{\delta S}{\delta \mathcal{A}_\mu}] \{M\} [\mathcal{C}, \frac{\delta \Sigma_g}{\delta \mathcal{A}_\mu}] \right) \\ &\quad + \frac{1}{2\Lambda^4} \left(\frac{\delta S}{\delta \mathcal{C}} \{H\} \frac{\delta \Sigma_g}{\delta \mathcal{C}} - \frac{1}{4} [\mathcal{C}, \frac{\delta S}{\delta \mathcal{C}}] \{L\} [\mathcal{C}, \frac{\delta \Sigma_g}{\delta \mathcal{C}}] \right), \\ a_1[\Sigma_g] &= \frac{1}{2\Lambda^2} \left(\frac{\delta}{\delta \mathcal{A}_\mu} \{c'\} \frac{\delta \Sigma_g}{\delta \mathcal{A}_\mu} - \frac{1}{4} [\mathcal{C}, \frac{\delta}{\delta \mathcal{A}_\mu}] \{M\} [\mathcal{C}, \frac{\delta \Sigma_g}{\delta \mathcal{A}_\mu}] \right) \\ &\quad + \frac{1}{2\Lambda^4} \left(\frac{\delta}{\delta \mathcal{C}} \{H\} \frac{\delta \Sigma_g}{\delta \mathcal{C}} - \frac{1}{4} [\mathcal{C}, \frac{\delta}{\delta \mathcal{C}}] \{L\} [\mathcal{C}, \frac{\delta \Sigma_g}{\delta \mathcal{C}}] \right). \end{aligned} \quad (14)$$

In here, we can take \hat{S} , hereafter referred to as the seed action, to be simply $\int d^4x (\mathcal{L}_\mathcal{A} + \mathcal{L}_\mathcal{C})$, although there is considerable flexibility over the exact choice as there is with the covariantisation, and recognising this, we were able to turn this to our advantage [18–20]. The kernels M , H and L are then determined in terms of c , \tilde{c} and other parameters in \hat{S} (here λ) by the requirement that the classical solution S can have the same two-point vertices as \hat{S} . They are found to be

$$M(x) = - \left(\frac{2c^2}{x\tilde{c} + 2c} \right)', \quad xH(x) = \left(\frac{2x^2\tilde{c}}{x + 2\lambda\tilde{c}} \right)', \quad xL(x) = \left(\frac{x^2\tilde{c}(\lambda\tilde{c}^2 - c)}{(x + 2\lambda\tilde{c})(x\tilde{c} + 2c)} \right)', \quad (15)$$

where prime denotes differentiation with respect to x and c, \tilde{c} are meant to be functions of x .

Although g appears explicitly as a parameter in these flow equations, it is not yet defined as the running Yang-Mills coupling. As usual, this is done via a renormalization condition: for the pure A^1 part we require

$$S = \frac{1}{2g^2(\Lambda)} \text{tr} \int d^4x (F_{\mu\nu}^1)^2 + O(D^3). \quad (16)$$

At first sight, it appears that we have specialised the kernels for the gauge fields so that no longitudinal terms appear. In fact, any longitudinal term $\sim \nabla_\mu \cdot \delta S / \delta \mathcal{A}_\mu$ may be converted to \mathcal{C} commutator terms, $\mathcal{C} \cdot \delta S / \delta \mathcal{C}$, *i.e.* L type terms, via $SU(N|N)$ gauge invariance.

$$\Lambda \partial_\Lambda \text{ (circle with } S \text{)} = -\frac{1}{\Lambda^2} \sum_f \left(\text{diagram 1} - \text{diagram 2} \right)$$

Fig. 3. Graphical representation of the flow equation.

The supermatrix functional derivatives are most easily computed by noting that they have a very simple effect on supertraces. Either we have ‘supersowing’, $\text{str} A \frac{\delta}{\delta X} \text{str} X B = \text{str} A B$, or ‘supersplitting’, $\text{str} \frac{\delta}{\delta X} A X B = \text{str} A \text{str} B$. Drawing single supertraces as closed curves, and using Fig. 2, we get a useful diagrammatic interpretation which counts supertraces, analogous to the ’t Hooft double-line notation [21], which will be widely used in what follows (cf. Section 5).

Supersowing and supersplitting follow from the completeness relations for the supergenerators. Just as in the analogous formulae for $SU(N)$, generically there are $1/N$ corrections, but they involve ordinary traces (or equivalently σ_3) which would violate $SU(N|N)$. In the case of this $SU(N|N)$ gauge theory, they must all cancel out and they do [14, 18], so the double line notation is exact – even at finite N [18].

The equations for the effective action vertices may be derived much more easily if superfields are split into their diagonal and off-diagonal components, *i.e.* $A_\mu = A_\mu + B_\mu$ and $C = C + D$ [18]. This is, of course, direct consequence of the symmetry structure, with σ_3 (anti)commuting with (fermionic) bosonic parts. It also resembles what is usually done in the context of the standard model, namely to deal with the mass eigenstates rather than with the fields themselves. The kernels and their expansion in powers of gauge fields may be split accordingly. (See Figs. 4 and 5 for some of those actually used in the calculation.)

Fig. 4. Graphical representation of 0-point kernels. The f -kernel in Fig. 3 stands for any of these. $K \doteq c' + M$.

Fig. 5. Graphical representation of 1-point kernels. The boxed diagram indicates the position of incoming momenta. The σ_3 s coming from the symmetry breaking are represented by stars, while (“) stands for $(p; q, r)$.

$$-4\beta_1 \square_{\mu\nu}(p) + \mathcal{O}(p^3) = \sum_f \left[\text{Diagram 1} + \text{Diagram 2} - \text{Diagram 3} \right]$$

Fig. 6. Graphical representation of eq. (21).

Finally, shifting \mathcal{C} to $\mathcal{C} + \sigma_3$ allows us to perform computations in which not only unbroken $SU(N) \times SU(N)$ gauge invariance, but also broken fermionic gauge invariance, is manifestly preserved at every step. As well as providing yet another beautiful balance in the formalism, one sees very clearly how a massive vector field (B) as created by spontaneous symmetry breaking, and its associated Goldstone mode (D), actually form a single unit, tied together by the underlying gauge invariance [18].

3.4 Loop expansion

Expanding the action and the beta function $\beta(g) = \Lambda \partial_\Lambda g$ in powers of the coupling constant:

$$S = \frac{1}{g^2} S_0 + S_1 + g^2 S_2 + \dots \quad \beta = \Lambda \partial_\Lambda g = \beta_1 g^3 + \beta_2 g^5 + \dots \quad (17)$$

yields the loopwise expansion of the flow equation

$$\Lambda \partial_\Lambda S_0 = -a_0 [S_0, S_0 - 2\hat{S}], \quad (18)$$

$$\Lambda \partial_\Lambda S_1 = 2\beta_1 S_0 - 2a_0 [S_0 - \hat{S}, S_1] + a_1 [S_0 - 2\hat{S}], \quad (19)$$

etc., where S_0 (S_1) is the classical (one-loop) effective action. The one-loop coefficient, β_1 , can be extracted directly from eq. (19) once the renormalization condition, eq. (16), is imposed. (Since gauge invariance already forces the anomalous dimension of the gauge field to vanish [9, 10, 18], we only need to define the renormalized coupling $g(\Lambda)$.)

From eq. (16)

$$S_{\mu\nu}^{AA}(p) + S_{\mu\nu}^{AA\sigma}(p) = \frac{2}{g^2} \square_{\mu\nu}(p) + \mathcal{O}(p^3) = \frac{1}{g^2} S_0^{AA}(p) + \mathcal{O}(p^3), \quad (20)$$

with $\square_{\mu\nu}(p)$ being the transverse combination $(p^2 \delta_{\mu\nu} - p_\mu p_\nu)$. Eq. (20) implies the $\mathcal{O}(p^2)$ component of all the higher loop contributions $S_n^{AA}(p) + S_n^{AA\sigma}(p)$ must vanish. Thus the equation for β_1 becomes algebraic ($\Sigma_0 = S_0 - 2\hat{S}$):

$$-2\beta_1 S_0^{AA}(p) + \mathcal{O}(p^3) = a_1 [\Sigma_0]_{\mu\nu}^{AA}(p). \quad (21)$$

In order to calculate the r.h.s. of eq. (21), we will adopt the following strategy:

- i. introduce the “integrated kernels” in the S_0 part of the first diagram and integrate by parts so as to end up with Λ -derivatives of vertices of the effective action;
- ii. use the flow equations for the effective couplings;

- iii. use the relation between the integrated kernels and their corresponding two-point functions to simplify the diagrams obtained so far;
- iv. repeat the above procedure when any three-point effective coupling is generated.

This simple procedure, which will be described in more detail in Section 5, ensures that any dependence upon n -point vertices of the seed action, $n \geq 3$, will cancel out. This implies that the calculation is actually independent of the choice of \hat{S} , provided it is a covariantisation of its two-point vertices⁷ and these latter vertices are infinitely differentiable and lead to convergent momentum integrals [19, 20]. Moreover, pursuing that strategy will also guarantee that just the kernels' vertices with special momenta remain that by gauge invariance can be expressed as derivatives of their generators (for an example see Section 4), which means independence of the choice of covariantisation.

4 (Un-)Broken gauge invariance

The invariance under the (broken) $SU(N|N)$ gauge symmetry results in the following set of trivial Ward identities

$$\begin{aligned}
 q^\nu U_{a\nu b\cdots}^{XAY\cdots}(\cdots p, q, r, \cdots) &= U_{a\cdots b\cdots}^{XY\cdots}(\cdots p, q+r, \cdots) - U_{a\cdots b\cdots}^{XY\cdots}(\cdots p+q, r, \cdots), \\
 q^\nu U_{a\nu b\cdots}^{XBY\cdots}(\cdots p, q, r, \cdots) &= \pm U_{a\cdots b\cdots}^{X\hat{Y}\cdots}(\cdots p, q+r, \cdots) \mp U_{a\cdots b\cdots}^{\hat{X}Y\cdots}(\cdots p+q, r, \cdots) \\
 &\quad + 2U_{a\cdots b\cdots}^{XD\sigma Y\cdots}(\cdots p, q, r, \cdots),
 \end{aligned} \tag{22}$$

where U is any vertex, a and b are Lorentz indices or null as appropriate and \hat{X}, \hat{Y} are opposite statistics partners of X, Y . The sign of the terms containing \hat{X}, \hat{Y} depends on whether B goes past a σ_3 on its way back and forth.

By specialising (22) to a proper set of momenta, one of which has to be infinitesimal, it is possible to express n -point vertices with one null momentum as derivatives of $(n-1)$ -point's, independently of the choice of covariantisation. As an example, let us consider the three-point pure- A effective vertex at vanishing first momentum, $S_{\mu\nu\rho}^{AAA}(0, k, -k)$. By using (22),

$$\epsilon^\mu S_{\mu\nu\rho}^{AAA}(\epsilon, k, -k - \epsilon) = S_{\nu\rho}^{AA}(k + \epsilon) - S_{\nu\rho}^{AA}(k) = \epsilon^\mu \partial_\mu^k S_{\nu\rho}^{AA}(k) + \mathcal{O}(\epsilon^2). \tag{23}$$

At order ϵ , $S_{\mu\nu\rho}^{AAA}(0, k, -k) = \partial_\mu^k S_{\nu\rho}^{AA}(k)$.

Also $S_{\mu\nu\rho\sigma}^{AAAA}(0, 0, k, -k) = \frac{1}{2} \partial_\mu^k \partial_\nu^k S_{\rho\sigma}^{AA}(k)$.

5 A sample of the calculation: the C sector

In this section the simplest part of the computation will be described, that is the scalar sector. All the steps of the strategy previously outlined will be illustrated by means of diagrams, as the cancellations taking place are evident already at that level. Of course, performing the full and complete calculation yields the same result.

We start by defining the integrated kernel. As for any differentiable function $f(\frac{p^2}{\Lambda^2})$, $\Lambda \partial_\Lambda f(\frac{p^2}{\Lambda^2}) = -2 \frac{p^2}{\Lambda^2} f'(\frac{p^2}{\Lambda^2})$, then

⁷set equal to the effective ones for convenience.

$$\frac{1}{\Lambda^4} H = -\frac{1}{2p^4} \Lambda \partial_\Lambda \left(\frac{2x^2 \bar{c}}{x + 2\lambda \bar{c}} \right) \equiv -\Lambda \partial_\Lambda \Delta^{CC} \quad (24) \quad \left| \begin{array}{c} \circ \\ \circ \end{array} \right| = -\Lambda \partial_\Lambda \left| \begin{array}{c} \circ \\ \circ \end{array} \right|$$

The integrated kernel is introduced via eq. (24) into the S_0 part of the first diagram in Fig. 6. One then integrates by parts, so as to end up with a total Λ -derivative plus the tree-level $\Lambda \partial_\Lambda S_{\mu\nu}^{AA\bar{C}\bar{C}}$ vertex joined by a Δ^{CC} (see Fig. (7) for the diagrammatic representation). The latter will be dealt with, using its flow equation.

The next step consists in using eq. (18) as specialised to $S_{\mu\nu}^{AA\bar{C}\bar{C}}$. Some of the diagrams are shown in Fig. 8.

Already at this level, we note that some of the diagrams either do not contribute at all (cf. Fig. 9) or they give a potentially universal contribution, *i.e.* something depending only on two-point vertices and integrated kernels (cf. Fig. 10).

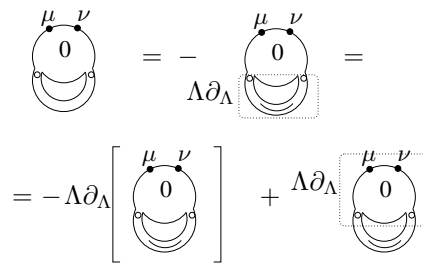


Fig. 7. The integrated kernel trick.

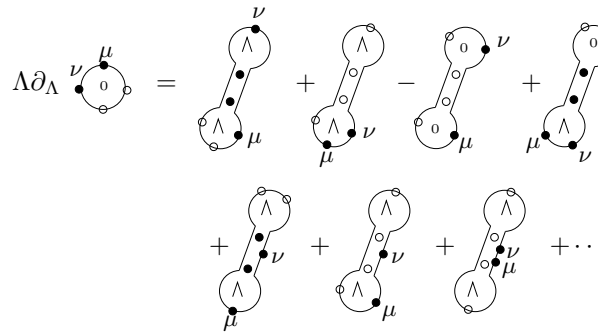
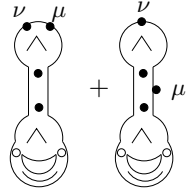


Fig. 8. Eq. (18) as specialised to $S_{\mu\nu}^{AA\bar{C}\bar{C}}$. The ellipsis stands for similar diagrams which have not been drawn.

Fig. 9. Diagrams not contributing to β_1 .

$$\left(\hat{S}_{\mu\nu\alpha}^{AAA}(p, -p, 0) c'(0) \hat{S}_{\alpha}^{ACC}(0, k, -k) + \hat{S}_{\nu\alpha}^{AA}(p) c'_{\mu}(p; 0, -p) \hat{S}_{\alpha}^{ACC}(0, k, -k) \right) \Delta^{CC}(k) \Big|_{p^2} = 0$$



$$\begin{aligned} & \hat{S}_{\nu\alpha}^{AA}(p) c' \left(\frac{p^2}{\Lambda^2} \right) \hat{S}_{\mu\alpha}^{AACCC}(p, -p, k, -k) \Delta^{CC}(k) \Big|_{p^2} = \\ & \hat{S}_{\nu\alpha}^{AA}(p) c'(0) \hat{S}_{\mu\alpha}^{AACCC}(0, 0, k, -k) \Delta^{CC}(k) = \\ & \frac{1}{2} \hat{S}_{\nu\alpha}^{AA}(p) c'(0) \partial_{\mu}^k \partial_{\alpha}^k \hat{S}^{CC}(k) \Delta^{CC}(k). \end{aligned}$$

Fig. 10. A potentially universal contribution.

Many of the remaining terms in the tree-level equation for $S_{\mu\nu}^{AACCC}$ may be further simplified by making use of the relation between the integrated kernel and the corresponding two-point function. Such a relation may be easily obtained from the tree-level equation for the effective two-point coupling, in the present example S^{CC} . By rewriting it in terms of the inverse coupling, $(S^{CC})^{-1}$, we get $(S^{CC})^{-1} = \Delta^{CC}$, *i.e.* $S^{CC} \Delta^{CC} = 1$. This leads to the simplifications shown in Fig. 11.

The last step concerns how to handle the terms that contain two three-point effective couplings. The procedure is pretty much the same, except that one has to recognise the derivative of the “square of the kernel” (see Fig. 12). At the algebra level, it amounts to writing the second diagram in Fig. 12 as the sum of two equal contributions and, then, to shifting the loop momentum so as to complete the Λ -derivative.

The procedure outlined in the above can be used in the whole calculation: all the hatted vertices cancel out and one is left with potentially universal terms only. The relation between integrated kernels and their corresponding two-point functions, however, is more complicated in the general case. As a matter of fact, it takes the form $S_{IK}(p) \Delta_{KJ}(p) = \delta_{IJ} + R_{IJ}(p)$, where the “remainder” R_{IJ} , absent in the scalar sector, is a (un-)broken gauge transformation. In the A sector, for example, $R_{\mu\nu}(p) = -p_{\mu} p_{\nu} / p^2$ [18].

Once the potentially universal terms have been collected, the momentum integrals should be carried out. We used dimensional regularisation as a preregulator to avoid all the subtleties related to cancelling divergences against each other. (Had we done the calculation in a way that preserves $SU(N|N)$, preregularisation would not have been needed.)

6 Summary and conclusions

A manifestly gauge invariant ERG, together with the necessary non-perturbative gauge invariant regularisation scheme, has been proposed. No gauge fixing is required to define it, nor is it needed

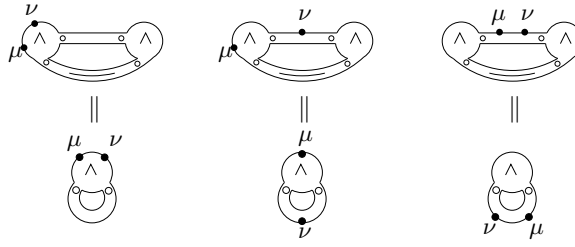


Fig. 11. Simplifications in the four-point effective vertex contribution.

$$\begin{aligned}
 & - \text{Diagram} = \text{Diagram} = \frac{1}{2} \text{Diagram} \\
 & = \frac{1}{2} \Lambda \partial_\Lambda \left[\text{Diagram} \right] - \frac{1}{2} \text{Diagram} - \frac{1}{2} \text{Diagram}
 \end{aligned}$$

Fig. 12. How to handle two joined three-point effective vertices.

to compute the solutions [8–10], thus avoiding the Gribov problem [22]. Although there has been no room for explanation, the ERG, especially the gauge sector (12), may be reinterpreted in terms of Wilson loops, the natural order parameter for gauge theory. The ERG then has an interpretation in the large N limit as quantum mechanics of a single Wilson loop, with close links to the Migdal-Makeenko equation [9, 23].

As a basic test of the formalism, the one-loop $SU(N)$ beta function has been computed and the expected universal result has been obtained. The strategy which has proven to be very efficient consists in eliminating the elements put in by hand by using the flow equations for the effective action vertices, where physics is actually encoded. (See also [19] for the analysis of the scalar case). A diagrammatic technique to represent the various vertices has been sketched, and already at the level of diagrams the big potential of the method comes out.

The calculation is totally independent of the details put in by hand, such as the choice of covariantisation and the cutoff profile, and gauge invariance is no doubt the main ingredient all the way to the final result.

We expect the procedure to be quite general and hope that it may be used to investigate non-perturbative aspects of gauge theories.

For the future, we intend to include matter in the fundamental representation, and turn our attention to non-perturbative approximations and QCD. It also seems a simple matter to incorporate space-time supersymmetry, opening up intriguing possibilities for deeper investigations of Seiberg-Witten methods and the AdS/CFT correspondence [24–26].

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