

**A DEMONSTRATION OF ONE-LOOP SCHEME INDEPENDENCE
IN A GENERALISED SCALAR EXACT RENORMALIZATION GROUP¹****S. Arnone², A. Gatti³, T. R. Morris⁴***Department of Physics and Astronomy, University of Southampton
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Received 21 June 2002, in final form 16 September 2002, accepted 17 September 2002

The standard demand for the quantum partition function to be invariant under the renormalization group transformation results in a general class of exact renormalization group equations, different in the form of the kernel. Physical quantities should not be sensitive to the particular choice of the kernel. Such scheme independence is elegantly illustrated in the scalar case by showing that, even with a general kernel, the one-loop beta function may be expressed only in terms of the effective action vertices, and in this way the universal result is recovered.

PACS: 11.10.Hi

1 Introduction

The non-perturbative meaning of renormalization, as understood by Wilson, is formulated most directly in the continuum in terms of the Exact Renormalization Group (ERG) [1]. Moreover, the fact that solutions S of the corresponding flow equations can then be found directly in terms of renormalized quantities, that all physics (*e.g.* Green functions) can be extracted from this Wilsonian effective action S , and that renormalizability is trivially preserved in almost any approximation [2], turns these ideas into a powerful framework for considering non-perturbative analytic approximations [1–7].

Recently, very general versions of the ERG [8] have been considered, dependent on the choice of a functional Ψ , known as the “kernel” of the ERG [5, 7]. In particular, each ERG is associated with a Ψ , that induces a reparametrisation (field redefinition) along the flow, and acts as a connection between the theory space of actions at different scales Λ . As a result, local to some generic point Λ on the flow, all the ERGs, including these generalised ones, may be shown to be just reparametrisations of each other. When this reparametrisation can be extended globally, the result is an immediate proof of scheme independence for physical observables. Indeed computations of physical quantities then differ only through some field reparametrisation.

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One practical example is an explicit field redefinition that interpolates between results computed using different choices of cutoff functions [7].

Obstructions to full (global) equivalence of ERGs can arise, on the one hand from differences in the global structure of fixed points deduced from the two flows, and on the other from the non-existence at special points of an inverse in an implied change of variables (from S to Λ) [7]. However it is difficult to determine these, given that in practice one has to make approximations in order to solve these theories. Furthermore, computations within a generalised ERG, such as the type being used for gauge theory [5, 9], generate many more terms, whose interpretation seems obscure [5, 9].

This contribution (summarising [10]) addresses these problems within a sufficiently simple and bounded context: the one-loop β function of massless four dimensional one component $\lambda\varphi^4$ scalar field theory. We will see that even for a very general form of Ψ (one involving a general ‘seed’ action \hat{S}), the correct universal result is obtained. To our knowledge, this is the first concrete test of such scheme independence beyond testing for cutoff function independence. The only requirements we have to impose to recover the universal result, are some very weak and general requirements which are necessary in any case to ensure that the Wilsonian action S makes sense. To this level then, all such ERGs are equivalent and merely parametrise changes of scheme.

2 From the Polchinski equation to general ERGs

The Polchinski equation [11] for a self-interacting real scalar field in four Euclidean dimensions takes the form

$$\Lambda\partial_\Lambda S^{int} = -\frac{1}{\Lambda^2} \frac{\delta S^{int}}{\delta\varphi} \cdot c' \cdot \frac{\delta S^{int}}{\delta\varphi} + \frac{1}{\Lambda^2} \frac{\delta}{\delta\varphi} \cdot c' \cdot \frac{\delta S^{int}}{\delta\varphi}, \quad (1)$$

where S^{int} is the interaction part of the effective action. Prime denotes differentiation with respect to the function’s argument (here p^2/Λ^2) and the following shorthand has been introduced: for two functions $f(x)$ and $g(y)$ and a momentum space kernel $W(p^2/\Lambda^2)$, with Λ being the effective cutoff,

$$f \cdot W \cdot g = \iint d^4x d^4y f(x) W_{xy} g(y), \quad W_{xy} = \int \frac{d^4p}{(2\pi)^4} W(p^2/\Lambda^2) e^{ip \cdot (x-y)}. \quad (2)$$

$c(p^2/\Lambda^2) > 0$ is a *smooth*, *i.e.* infinitely differentiable, ultra-violet cutoff profile, which modifies propagators $1/p^2$ to c/p^2 . It satisfies $c(0) = 1$ so that low energies are unaltered, and $c(p^2/\Lambda^2) \rightarrow 0$ as $p^2/\Lambda^2 \rightarrow \infty$ sufficiently fast that all Feynman diagrams are ultra-violet regulated.

Note that the regularised kinetic term in the effective action may be written as $\frac{1}{2} \partial_\mu \varphi \cdot c^{-1} \cdot \partial_\mu \varphi$. This will be referred to as the seed action and denoted by \hat{S} . In terms of the total effective action, $S = \hat{S} + S^{int}$, and $\Sigma \doteq S - 2\hat{S}$, the ERG equation reads (up to a vacuum energy term that was discarded in [11])

$$\Lambda\partial_\Lambda S = -\frac{1}{\Lambda^2} \frac{\delta S}{\delta\varphi} \cdot c' \cdot \frac{\delta S}{\delta\varphi} + \frac{1}{\Lambda^2} \frac{\delta}{\delta\varphi} \cdot c' \cdot \frac{\delta S}{\delta\varphi} \Rightarrow \Lambda\partial_\Lambda e^{-S} = -\frac{1}{\Lambda^2} \frac{\delta}{\delta\varphi} \cdot c' \cdot \left(\frac{\delta \Sigma}{\delta\varphi} e^{-S} \right), \quad (3)$$

i.e. the infinitesimal RG transformation results in a change in the integrand which is a total functional derivative.

Incidentally, this establishes a rather counterintuitive result, that integrating out degrees of freedom is just equivalent to redefining the fields in the theory [7, 9]. In the present case, the change in the partition function may be shown to correspond to the change of variables $\varphi \rightarrow \varphi + \delta\Lambda \Psi$, with the “kernel” $\Psi = -\frac{1}{\Lambda^3} c' \frac{\delta\Sigma}{\delta\varphi}$ that appears in eq. (3) [7].

Different forms of ERG equations correspond to choosing different kernels Ψ . There is a tremendous amount of freedom in this choice, just as there is a great deal of freedom in choosing the form of a blocking transformation in the condensed matter or lattice realisation of the Wilsonian RG [1]. The flow equation (1) is distinguished only by its relative simplicity (related to incorporating the cutoff only in the kinetic term). Nevertheless, physical quantities should turn out to be universal *i.e.* independent of these choices.

As shown above, Polchinski’s equation comes from setting the seed action equal to the effective kinetic term in the Wilsonian effective action. If we are to reproduce that very term at the classical level, the bilinear term in \hat{S} must continue to be equal to $\frac{1}{2} \partial_\mu \varphi \cdot c^{-1} \cdot \partial_\mu \varphi$. Furthermore, we choose to leave the $\varphi \leftrightarrow -\varphi$ symmetry alone, by requiring that \hat{S} is even under this symmetry. We are left with a generalised ERG parametrised by the infinite set of seed action n -point vertices, $n = 4, 6, 8, \dots$. We will leave each of these vertices as completely unspecified functions of their momenta except for the requirement that *the vertices be infinitely differentiable and lead to convergent momentum integrals*. (The first condition ensures that no spurious infrared singularities are introduced and that all effective vertices can be Taylor expanded in their momenta to any order [2, 5]. The second is necessary for the flow equation to make sense at the quantum level and also ensures the flow actually corresponds to integrating out modes [7, 9].)

We are therefore incorporating in the momentum dependence of *each* of the seed action n -point vertices, $n = 2, 4, 6, \dots$, an infinite number of parameters. Remarkably, however, we can still compute the one-loop β function. Moreover, as we will see in the next section, we can invert the flow equation by expressing \hat{S} vertices in terms of S , and in this way demonstrate explicitly that the result is universal — *viz.* independent of the choice of c and \hat{S} .

3 One-loop beta function with general seed action

As usual, universal results are obtained only after the imposition of appropriate renormalization conditions which allow us to define what we mean by the physical (more generally renormalized) coupling and field. (The renormalized mass must also be defined and is here set to zero implicitly by ensuring that the only scale that appears is Λ .)

We write the vertices of S as $S^{(2n)}(\vec{p}; \Lambda) \equiv S^{(2n)}(p_1, p_2, \dots, p_{2n}; \Lambda)$ (and similarly for the vertices of \hat{S}). In common with earlier works [4, 11], we define the renormalized four point coupling λ by the effective action’s four-point vertex evaluated at zero momenta: $\lambda(\Lambda) = S^{(4)}(\vec{0}; \Lambda)$. This makes sense once we express quantities in terms of the renormalized field, defined (as usual) to bring the kinetic term into canonical form.

We will also rescale the field $\varphi \mapsto \frac{1}{\sqrt{\lambda}} \varphi$, so as to put the coupling constant in front of the action. This ensures the expansion in the coupling constant coincides with the one in \hbar , which is more elegant, although it will introduce a ‘fake’ contribution to the anomalous dimension γ in these variables. It is also analogous to the treatment pursued for gauge theory in refs. [5, 9]

(where gauge invariance introduces further simplifications in particular forcing $\gamma = 0$ for the new gauge field). The following analysis thus furnishes a demonstration that these ideas also work within scalar field theory.

Expanding the action, the beta function $\beta(\Lambda) = \Lambda\partial_\Lambda\lambda$ and the anomalous dimension in powers of the coupling constant yields the loopwise expansion of the flow equation

$$\Lambda\partial_\Lambda S_0 = -\frac{1}{\Lambda^2} \frac{\delta S_0}{\delta\varphi} \cdot c' \cdot \frac{\delta(S_0 - 2\hat{S})}{\delta\varphi}, \quad (4)$$

$$\Lambda\partial_\Lambda S_1 = -\beta_1 S_0 - \frac{\gamma_1}{2} \varphi \cdot \frac{\delta S_0}{\delta\varphi} - \frac{2}{\Lambda^2} \frac{\delta S_1}{\delta\varphi} \cdot c' \cdot \frac{\delta(S_0 - \hat{S})}{\delta\varphi} + \frac{1}{\Lambda^2} \frac{\delta}{\delta\varphi} \cdot c' \cdot \frac{\delta(S_0 - 2\hat{S})}{\delta\varphi}, \quad (5)$$

etc., where S_0 (S_1) is the classical (one-loop) effective action and β_1 and γ_1 are the one-loop contributions to β and γ . These latter may now be extracted directly from eq. (5), as specialised to the two-point and four-point effective couplings, $S^{(2)}(\vec{p}; \Lambda)$ and $S^{(4)}(\vec{p}; \Lambda)$ respectively, once the renormalization conditions have been taken into account. In these variables they read

$$S^{(2)}(p, -p; \Lambda) = S^{(2)}(0, 0; \Lambda) + p^2 + O(p^4/\Lambda^2), \quad S^{(4)}(\vec{0}; \Lambda) = 1. \quad (6)$$

As usual, both conditions are saturated at tree level. Hence there must be no quantum corrections to the four-point vertex at $\vec{p} = \vec{0}$, or to the $O(p^2)$ part of the two-point vertex.

The flow equations for these special parts of the quantum corrections thus greatly simplify, reducing to algebraic equations which then determine the β_i and γ_i . At one loop:

$$\beta_1 + 2\gamma_1 = \frac{8c'_0}{\Lambda^2} \left[1 - \hat{S}^{(4)}(\vec{0}) \right] S_1^{(2)}(0) - \frac{1}{\Lambda^2} \int_q c' \left(\frac{q^2}{\Lambda^2} \right) \left[S_0^{(6)} - 2\hat{S}^{(6)} \right] (\vec{0}, q, -q), \quad (7)$$

$$\beta_1 + \gamma_1 = -\frac{1}{\Lambda^2} \int_q c' \left(\frac{q^2}{\Lambda^2} \right) \left[S_0^{(4)} - 2\hat{S}^{(4)} \right] (p, -p, q, -q) \Big|_{p^2}, \quad (8)$$

where $c'_0 = c'(0)$, $\int_q \doteq \int \frac{d^4 q}{(2\pi)^4}$, and the notation $|_{p^2}$ means that one should take the coefficient of p^2 in the series expansion in p .

In order to evaluate eq. (7), we need to calculate $S_1^{(2)}(0)$ and $S_0^{(6)}(\vec{0}, q, -q)$. We would also need $\hat{S}^{(4)}(\vec{0})$ and $\hat{S}^{(6)}(\vec{0}, q, -q)$, but we can avoid using explicit expressions for them, and thus keep \hat{S} general, by using the flow equations for the corresponding vertices of S .

Our general strategy (see also [12]) is to use the flow equation to express any hatted vertices in terms of the effective ones. This will cause almost all the non-universal terms, those depending on the details of the cutoff function and/or on the explicit form of the seed action, to cancel out. The remaining ones will disappear once γ_1 is substituted using eq. (8). The simplest way to appreciate that non-universal terms cancel is to recognise they can be paired up into total Λ -derivatives, which can be taken outside the momentum integral as the integrand is regulated both in the ultra-violet and the infrared. Furthermore, as those terms contribute to β_1 , they must be dimensionless and thus cannot depend upon Λ after the momentum integral has been carried out, hence the result vanishes identically!

As an example, we use eq. (4) to express $\hat{S}^{(6)}(\vec{0}, q, -q)$ in terms of the effective action vertices,

$$\hat{S}^{(6)}(\vec{0}, q, -q) = \frac{\Lambda^2}{4q^2} \frac{c_q}{c'_q} \left\{ \Lambda \partial_\Lambda S_0^{(6)}(\vec{0}, q, -q) + \frac{8c'_0}{\Lambda^2} [1 - \hat{S}^{(4)}(\vec{0})] S_0^{(4)}(0, 0, q, -q) - 2c'_0 \frac{c_q}{q^2 c'_q} \Lambda \partial_\Lambda S_0^{(4)}(0, 0, q, -q) - \frac{6}{q^2} S_0^{(4)}(0, 0, q, -q) \Lambda \partial_\Lambda [c_q S_0^{(4)}(0, 0, q, -q)] \right\}, \quad (9)$$

and substitute it back into eq. (7). The first term in eq. (9) and the $S_0^{(6)}$ term in eq. (7) pair up into $\Lambda \partial_\Lambda \int_q \frac{1}{2q^2} c(\frac{q^2}{\Lambda^2}) S_0^{(6)}(\vec{0}, q, -q)$, which vanishes since the result of a convergent dimensionless integral cannot depend upon Λ . Also, the second term in eq. (9) exactly cancels the first term in eq. (7) once the one-loop two-point vertex at null momentum is computed. This latter calculation is carried out exactly the same way, namely by eliminating the four-point seed action vertex in favour of the effective one. The result takes the form

$$S_1^{(2)}(0) = - \int_q \frac{c_q S_0^{(4)}(0, 0, q, -q)}{2q^2}, \quad (10)$$

with no integration constant since for a massless theory, there must be no other explicit scale in the theory apart from the effective cutoff.

Of the two remaining terms in eq. (7), one is cancelled by γ_1 , while the other reads

$$\beta_1 = \frac{3}{2} \int_q \frac{1}{q^4} \Lambda \partial_\Lambda \left\{ c(\frac{q^2}{\Lambda^2}) S_0^{(4)}(0, 0, q, -q) \right\}^2, \quad (11)$$

which is nothing but the standard one-loop diagrammatic result for the β function, written in terms of the regularised propagator and the effective $S_0^{(4)}$. Here the derivative with respect to Λ cannot be taken outside the integral, as the latter would not then be properly regulated in the infrared. Indeed, it yields

$$\beta_1 = -\frac{3}{2} \frac{\Omega_4}{(2\pi)^4} \int_0^\infty dq \partial_q \left\{ c(\frac{q^2}{\Lambda^2}) S_0^{(4)}(0, 0, q, -q) \right\}^2 = \frac{3}{16\pi^2}, \quad (12)$$

which is the standard one-loop result [13].⁵

4 Summary and conclusions

Starting with the generalised ERG flow equation, we computed tree level two, four and six point vertices. At one-loop we computed the effective mass $S_1^{(2)}(0)$ and wavefunction renormalization γ_1 . By combining all these with the flow of the one-loop four-point vertex at zero momenta, we arrived at the expected universal result $\beta_1 = 3/(4\pi)^2$.

The flow equation we used differs from the Polchinski flow equation (1), equivalently eq. (3), because the seed action \hat{S} is no longer set to be just the kinetic term, but is generalised to include all arbitrary even higher-point vertices. (We also scaled out the coupling λ .)

⁵The term in braces depends only on q^2/Λ^2 . Ω_4 is the four dimensional solid angle. The result follows from the convergence of the integral and normalisation conditions $c(0) = 1$ and eq. (6). As far as invariance with respect to the choice of the cutoff function is concerned, this is standard.

We then proceeded to compute the tree and one-loop corrections exploiting the ability, within the ERG, to derive directly the renormalized couplings and vertices (*i.e.* without having to refer back to an overall cutoff and bare action).

We could now argue that we should have expected these results, without the detailed calculation. Nevertheless this is the first specific test of these ideas beyond that of just cutoff function independence, and in the process we found the restrictions on \hat{S} sufficient to ensure scheme independence at this level. They are merely that the seed vertices be infinitely differentiable and lead to convergent momentum integrals, which as we noted are necessary conditions in any case.

It is important to stress that many of our specific choices (what we chose to generalise in Ψ , how we incorporated wavefunction renormalization, organised and solved the perturbative expansion) are not crucial to the calculation. On the contrary there are very many ways to organise the computation; we just chose our favourite one. The crucial step in navigating the generalised corrections, appears to be the recognition that one should eliminate the elements put in by hand, in this case vertices of \hat{S} , in favour of the induced solution, the Wilsonian effective action S , which encodes the actual physics.

For us, this is the most important conclusion of the present paper since it implies a practical prescription for streamlined calculations which can be used even in more involved settings such as in the manifestly gauge invariant framework [5,9,14], where there is no equivalent calculation one can directly compare to.

Acknowledgement: The authors wish to thank the organisers of the fifth international Conference “RG 2002” for providing such a stimulating environment. T.R.M. and S.A. acknowledge financial support from PPARC Rolling Grant PPA/G/O/2000/00464.

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