

**EXACT RENORMALIZATION GROUP AS A TOOL
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Basic elements of the exact renormalization group method and recent results within this approach are reviewed. Topics covered are the derivation of equations for the effective action and relations between them, derivative expansion, solutions of fixed point equations and the calculation of the critical exponents, construction of the c -function and a description of the chiral phase transition.

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1 Introduction

To see the role of the exact renormalization group approach let us consider a situation which is rather common in quantum field theory. Suppose that there is a fundamental theory with the action $S_{\Lambda_0}[\phi]$ defined at some energy scale Λ_0 , and suppose that we are interested in physical effects described by such theory at energies of order $\Lambda \ll \Lambda_0$. One way to proceed is to calculate perturbatively the renormalization group (RG) running of coupling constants and anomalous dimensions from Λ_0 down to Λ . However, usually large power and logarithmic corrections develop which make the result unreliable.

Another way is to calculate the low energy effective action $S_\Lambda[\phi]$ relevant at the scale Λ and then perform the perturbative calculations using this action. The formalism to calculate the low energy effective action is the Exact Renormalization Group (ERG). It introduces the notion of the running action $S_\Lambda[\phi]$, i.e. the Wilson effective action at scale Λ , and provides a framework for its calculation starting from the initial action $S_{\Lambda_0}[\phi]$. The basic idea is the following: while lowering the scale the effective action $S_\Lambda[\phi]$ changes in such a way so that to keep the S -matrix unchanged. In practice a stronger condition is imposed. Namely one requires that the Green functions and the generating functional remain unchanged. This guarantees that the physical observables do not depend on the scale at which the effective action is defined.

The ERG is a continuous version of the Wilsonian RG [1] adapted for applications in quantum field theory. The central element of the ERG approach is an equation which determines the

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change of the effective action with the scale. There are quite a few ERG equations known and used in practical calculations. Below we will consider some of the equations and discuss the relation between them.

The goal of the present contribution is to explain main ideas which form the basis of the ERG, outline basic principles of the derivation of ERG equations and list a number of characteristic applications of this method. We included a discussion of some aspects and features of the ERG approach which were not touched or covered sufficiently in previous reviews. Nevertheless, many results are left beyond the scope of this article. The reader is advised to see, for example, Refs. [2–6] for a detailed review and quite complete lists of references to original papers.

The plan of the article is as follows. In Sect. 2 we review the Wegner-Houghton equation as an instructive example, describe the general structure of ERG equations and establish the relation between them. The Polchinski equation will be discussed in some detail. We outline the scheme of calculation of fixed points and critical exponents within the ERG approach and present a few examples in Sect. 3. More non-perturbative results will be reviewed in Sect. 4. These include the construction of the c -function and a study of the chiral phase transition. Sect. 5 contains further discussion of general features of the ERG and some concluding remarks.

2 ERG equations

In this article for the sake of simplicity we consider almost exclusively the case of the 1-component scalar field in the d -dimensional Euclidean space.

It is instructive to start with the Wegner-Houghton equation derived in Ref. [7]. This example illustrates a realization of the basic idea, formulated in the Introduction. Let us consider the partition function

$$\mathcal{Z} = \int \prod_{|p| \leq \Lambda} \mathcal{D}\phi_p e^{-S_\Lambda[\phi]}, \quad (1)$$

where ϕ_p is the field in the momentum representation and the functional integration is performed over the modes with p belonging to the interval $0 \leq |p| \leq \Lambda$. The functional $S_\Lambda[\phi]$ is the effective action defined at scale Λ . All higher modes are supposed to be integrated out. Let Λ' be some lower scale, and let us divide the interval of momenta into low ($0 \leq p \leq \Lambda'$) and high ($\Lambda' < p \leq \Lambda$) momentum frequencies. Then the partition function can be written as

$$\mathcal{Z} = \int \prod_{|p| \leq \Lambda'} \prod_{\Lambda' < |p| \leq \Lambda} \mathcal{D}\phi_p e^{-S_\Lambda[\phi]} = \int \prod_{|p| \leq \Lambda'} \mathcal{D}\phi_p e^{-S_{\Lambda'}[\phi]}, \quad (2)$$

so that $S_{\Lambda'}[\phi]$ includes momentum field modes with p up to the new scale Λ' and is interpreted as the effective action at the scale Λ' . The modes with momenta belonging to the shell $\Lambda' < p \leq \Lambda$ have been integrated out and led to the corresponding modification of the action. This step corresponds to Kadanoff's transformation (called also blocking or coarsening). As we see, there is a well defined boundary between low and high momentum frequencies, and the latter are integrated out completely. In this case it is said that the theory has a sharp momentum cutoff.

Suppose now that the scales are related by $\Lambda' = \Lambda e^{-\delta t}$. The authors of Ref. [7] showed that if $\delta t \ll 1$, then the following relation between the effective actions at these two scales fulfills:

$$\begin{aligned} \frac{S_{\Lambda'}[\phi] - S_{\Lambda}[\phi]}{\delta t} &= \frac{1}{2\delta t} \left\{ \int' dp \left[\ln \frac{\delta^2 S_{\Lambda}}{\delta\phi_p \delta\phi_{-p}} - \frac{\delta S_{\Lambda}}{\delta\phi_p} \frac{\delta S_{\Lambda}}{\delta\phi_{-p}} \left(\frac{\delta^2 S_{\Lambda}}{\delta\phi_p \delta\phi_{-p}} \right)^{-1} \right] \right. \\ &\quad \left. + \text{(rescaling terms)} + \mathcal{O}(\delta t^2) \right\}, \end{aligned} \quad (3)$$

where the prime indicates that the integration is performed over the shell of momenta $\Lambda' < |p| < \Lambda$. The origin of the rescaling terms is the canonical change of the scale of dimensional parameters due to the change of Λ . Here we prefer to omit these details and focus on the general structure of the equation (see Ref. [7] and an explicit example below). In the limit $\delta t \rightarrow 0$ this relation becomes an equation which defines the evolution of the effective action with the scale. Being supplied with an initial condition

$$S_{\Lambda}[\phi]|_{\Lambda=\Lambda_0} = S_0[\phi],$$

the equation determines the evolution of the effective action with the scale. It allows to calculate (in principle) the running effective action $S_{\Lambda}[\phi]$ for a given bare action (“fundamental theory”) $S_0[\phi]$ defined at $\Lambda_0 \gg \Lambda$.

The Polchinski ERG equation is one of the most broadly studied and widely used in applications. It was obtained in Ref. [10] and is formulated in terms of the Wilson effective action with a smooth cutoff function:

$$S[\phi; t] = \frac{1}{2} \int \frac{dp}{(2\pi)^d} \phi_p \cdot P^{-1}(p^2, \Lambda^2) \cdot \phi_{-p} + S_{int}[\phi; t], \quad (4)$$

where

$$P(p^2, \Lambda^2) = \frac{1}{p^2} K \left(\frac{p^2}{\Lambda^2} \right) \quad (5)$$

is a regularized propagator and $K(p^2/\Lambda^2)$ is a (smooth) ultraviolet cutoff profile (regulating function) which has the following properties: (1) $K(0) = 1$; (2) $K(z) \rightarrow 0$ as $z \rightarrow \infty$ fast enough so that all momentum integrals are finite. For a smooth cutoff the boundary between low and high momentum frequencies is blurred, and contributions of high momenta are suppressed rather than integrated out. In concrete calculations the exponential cutoff function $K(z) = \exp(-az^l)$ ($l > 0$, $a > 0$) is often used. The action in the Wegner-Houghton ERG equation is regularized with the sharp cutoff function $K(z) = \theta(1 - z)$, where θ is the Heaviside (step) function. The analysis of the evolution of the effective action with scale becomes more transparent if it is carried out in terms of dimensionless quantities. For this the dimensionless field variable φ_k and dimensionless momentum k are introduced: $\varphi_k = \Lambda^{1+d/2} \phi_p$, $k = p/\Lambda$.

In Ref. [10] the Polchinski ERG equation was derived starting from the invariance of the partition function under the change of the scale (see a detailed discussion of the derivation in Ref. [11]). In terms of φ_k and k it has the form

$$\partial_t S = \mathcal{F}[\varphi_k, S] \quad (6)$$

with the functional \mathcal{F} given by

$$\begin{aligned} \mathcal{F}[\varphi_k, S] &= \int \frac{dk}{(2\pi)^d} K'(k^2) \left[\frac{\partial S}{\partial \varphi_k} \frac{\partial S}{\partial \varphi_{-k}} - \frac{\delta^2 S}{\delta \varphi_k \delta \varphi_{-k}} - \frac{2k^2}{K(k^2)} \varphi_k \frac{\partial S}{\partial \varphi_k} \right] + Sd \\ &+ \int \frac{dk}{(2\pi)^d} \left[\left(1 - \frac{d}{2} - \frac{\eta}{2} \right) \varphi_k \frac{\partial S}{\partial \varphi_k} - \varphi_k k^\mu \frac{\partial'}{\partial k^\mu} \frac{\partial S}{\partial \varphi_k} \right]. \end{aligned} \quad (7)$$

The first line contains terms coming from the ‘‘blocking transformation’’, the terms in the second line are ‘‘rescaling terms’’.

One can check that changing the cutoff $K(k^2)$ can be compensated by a field redefinition [8]. This explains why approximations of equations for different cutoff functions give very close results. Moreover, certain equations can be transformed one into another by field redefinitions [8] (see also [9]). They form a ‘‘universality’’ class of equivalent ERG equations and determine the same low energy effective actions. In particular, one can show that the Polchinski ERG equation is equivalent to the equation for the Legendre effective action [12] or to the equation for the average effective action derived in Ref. [13] (see also [14]).

The ERG equations were derived and extensively studied for scalar (see Sect. 4) and spinor theories [15]. A non-invariant formalism for gauge theories was developed and applied in a number of papers (see for example Refs. [14, 16, 17]). There the gauge invariance, not preserved by the equation, was restored at the end by imposing certain conditions on the RG flow. A gauge invariant formalism has been recently proposed in Refs. [18] (see Refs. [19, 20] for recent developments).

3 Fixed points and critical exponents

Let us first discuss the type of problems which can be naturally addressed within the ERG approach and then review some results.

An ERG equation can be written in general form (6) with some functional \mathcal{F} (see, for example, Eq. (7)). Search for fixed point (FP) solutions $S^*[\phi]$ is one of the immediate applications of the ERG. A FP is described by the action satisfying the condition $\partial_t S^* = 0$ or, equivalently, the equation

$$\mathcal{F}[\varphi, S^*] = 0. \quad (8)$$

FP solutions do not contain any scale and represent continuum limits of the theory.

Suppose that we have found a FP solution $S^*[\phi]$. The next problem to address is to study a linearized theory around the FP [21]. For this one writes the effective action as $S[\varphi; t] = S^*[\varphi] + \Delta S$ and expands Eq. (6):

$$\partial_t \Delta S = \mathcal{F}[\varphi, S^* + \Delta S] = \mathcal{L} \cdot \Delta S + \mathcal{O}(\Delta S^2), \quad (9)$$

where $\mathcal{L} \cdot \Delta S$ is the part linear in ΔS . The operator \mathcal{L} defines a set of eigenoperators $\Phi_n[\varphi]$ and critical (or scaling) exponents λ_n [22]:

$$\mathcal{L} \Phi_n[\varphi] = \lambda_n \Phi_n[\varphi].$$

As a result the effective action in the vicinity of the FP S^* is equal to

$$S[\varphi; t] = S^*[\varphi] + \sum_n \mu_n e^{\lambda_n t} \Phi_n[\varphi].$$

If $\lambda_n > 0$, $\lambda_n < 0$ or $\lambda_n = 0$ the associated operator $\Phi_n[\varphi]$ and its coupling $\mu_n e^{\lambda_n t}$ are called relevant, irrelevant or marginal, respectively. Relevant operators have couplings which grow along the flow, and they alone are responsible for scaling effects near the FP. The couplings of the irrelevant operators decrease along the flow, such operators contribute to subleading corrections. To determine the behavior of a marginal operator higher order terms in the expansion of the action around the FP must be taken into account. The critical exponents are physical observables and can be measured experimentally.

The ERG equations are quite complicated equations in functional derivatives with respect to the field variable φ and in ordinary derivatives with respect to the flow parameter t . The formal exactness of the ERG by itself does not constitute any calculational progress compared to the perturbation theory. It is the existence of non-perturbative approximation schemes which opens possibilities for exploration of non-perturbative physics and makes, therefore, the approach quite valuable.

The most widely used scheme is the derivative expansion [23]. The main idea is to expand the action of interaction in powers of space-time derivatives of the field:

$$S_{int} = \int d^d x \left[V(\phi(x); t) + \frac{1}{2} (\partial_\mu \phi)^2 Z(\phi(x); t) + \dots \right], \quad (10)$$

where potentials $V(\phi; t)$, $Z(\phi; t)$, etc. do not contain derivatives. Note that here the action is written in terms of the fields in the coordinate representation. The leading-order (LO) approximation is obtained by retaining only the first term in (10) and neglecting all derivatives in the effective action [24–26]. It is called the local potential approximation (LPA). Substituting expansion (10) into the Polchinski ERG equation, Eqs. (6), (7), we obtain a system of coupled partial differential equations for $V(\phi; t)$, $Z(\phi; t)$, etc.

As an example let us consider the next-to-leading order (NLO) approximation. Denote the field variable by z and introduce the function $f(z; t) \equiv \partial V(z; t)/\partial z$. The system of equations becomes [27]

$$\partial_t f = -A f'' - 2B Z' + 2K'(0) f f' + \left(1 + \frac{d}{2} - \frac{\eta}{2}\right) f + \left(1 - \frac{d}{2} - \frac{\eta}{2}\right) z f, \quad (11)$$

$$\begin{aligned} \partial_t Z &= -A Z'' + 2K''(0) (f')^2 + 4K'(0) Z f' + 2K'(0) f Z' \\ &+ \left(1 - \frac{d}{2} - \frac{\eta}{2}\right) z Z' - \eta Z - \frac{\eta}{2}. \end{aligned} \quad (12)$$

Here the prime denotes the derivative with respect to the field variable $z = \varphi$ and A, B stand for the following integrals:

$$A = \int \frac{dk}{(2\pi)^d} K'(k^2), \quad B = \int \frac{dk}{(2\pi)^d} k^2 K'(k^2).$$

The case of 1-component scalar field was studied in detail in a number of papers, see for example Refs. [26–28]. The FP solutions are given by system (11), (12) with f and Z independent of the flow parameter t , i.e. $\partial_t f = 0$, $\partial_t Z = 0$ in the l.h.s. Let us consider the theory

with Z_2 -symmetry under the reflection $\phi \rightarrow -\phi$. The boundary conditions fixing a solution are usually chosen at $z = 0$ in the form

$$f(0) = 0, \quad f'(0) = \gamma, \quad Z(0) = 0, \quad Z'(0) = 0.$$

The first and the last conditions follow directly from the Z_2 -symmetry (recall that $f(z)$ is the derivative of the potential). The third relation is a normalization condition, meaning that the second term in (10) does not contribute to the massless kinetic term in (4).

In the LPA the Polchinski ERG equation reduces to the following equation for $f(z)$:

$$Af'' = 2K'(0)ff' + \left(1 + \frac{d}{2} - \frac{\eta}{2}\right)f + \left(1 - \frac{d}{2} - \frac{\eta}{2}\right)zf. \quad (13)$$

which follows from Eq. (11). The consistency of approximation requires $\eta = 0$. Eq. (13) is stiff, and for a general value of γ the solution is singular at some finite value $z_0(\gamma)$ of the field variable. Of course, such solution does not give sensible potential and, therefore, is not physical. By fine tuning the parameter γ to a value $\gamma = \gamma_*$ such that $z_0(\gamma_*) = \infty$ one obtains a physical FP solution. By spanning all values of γ one obtains all FP solutions accessible in the LPA [29, 30].

A more general study which gives a deeper insight into the structure of the space of solutions of Eq. (13) was carried out in [31, 32] (also for the case of N -component scalar theory). It was shown that if arbitrary η are allowed, then regular solutions form a discrete set of families corresponding to curves $\eta_n(\gamma)$ in the (γ, η) -plane, where $n = 1, 2, \dots$ labels the curves. They are described by the formula

$$\eta_n(\gamma) = 2 + d\alpha_n\left(\frac{\gamma}{d}\right), \quad (14)$$

where $\alpha_n(z)$ are universal functions which do not depend on the space-time dimension and can be calculated from Eq. (13). Their properties were studied in Ref. [32]. In particular, $\alpha_n(0) = -n/(n+1)$ and $\alpha_n \rightarrow -1$ for $z \rightarrow -\infty$ so that the curves accumulate at $\eta = 2 - d$.

The curves of regular solutions for various d follow the universal pattern given by Eq. (14). When we pass from one number of dimensions to another the pattern shifts vertically and the curves scale in the γ -direction. The physical FPs correspond to the values of $\gamma = \gamma_*$ at which $\eta_n(\gamma)$ cross the γ -axis. This explains correctly the number of FPs in various dimensions $d > 2$. For $d = 2$ the curves approach asymptotically the line $\eta = 0$, this is an indication of the existence of infinite number of FPs in this case.

FP solutions were studied using the ERG first in Ref. [26] for the sharp cutoff Wegner-Houghton equation, and later for the other ERG equations in the LO and NLO approximations, see for example Refs. [27, 33–38]. In all these and other similar studies consistent results were obtained:

- 1) In $d = 4$ dimensions the only FP found is the trivial Gaussian FP.
- 2) In $d = 3$ there are just two FPs: the Gaussian FP and a non-trivial FP. Calculation of the critical exponents confirm that the latter corresponds to the universality class of the Wilson-Fischer FP (Ising model universality class).
- 3) In $d = 2$ the FP corresponding to the $p(p+1)$ conformal models were found [33, 38].

The critical exponents to the LO and NLO of the derivative expansion were calculated in a number of papers. In Table 1 we present a few examples of numerical results for the Wilson-Fischer FP in $d = 3$. These include the anomalous dimension η , the correlation length critical

Approach and approximation	η	ν	w
Wegner-Houghton eq., $\mathcal{O}(\partial^0)$ [26]	0	0.687	0.595
Polchinski eq. [27]			
$\mathcal{O}(\partial^0)$	0	0.649	0.66
$\mathcal{O}(\partial^2)$	0.019-0.056	0.616-0.637	0.70-0.85
Legendre effective action eq. [28, 33–35]			
$\mathcal{O}(\partial^0)$	0	0.660	0.628
$\mathcal{O}(\partial^2)$	0.054	0.618	0.897
World best estimates	0.035(3)	0.631(2)	0.80(4)

Tab. 1. Results of calculations of the anomalous dimension η and critical exponents ν and w for the Wilson-Fischer FP in $d = 3$. $\mathcal{O}(\partial^0)$ and $\mathcal{O}(\partial^2)$ denote the LPA and NLO approximation of the derivative expansion respectively. The entries of the last row (taken from the article by Morris [34]) were obtained by averaging the world best estimates [39].

exponent $\nu = 1/\lambda_1$, associated to the relevant operator, and $w = -\lambda_2$ for the least irrelevant operator. Note that $\eta = 0$ in the LPA. In the case when the exponent under consideration depends on the choice of the cutoff function K in Eq. (5) the intervals of values corresponding to certain ranges of the regulator parameter are indicated. This is analogous to the scheme dependence in the perturbative RG. One can see that the LO and NLO of the derivative expansion give fairly good results. It turns out that the values obtained within the Polchinski ERG equation (see the Table) coincide with those for the optimal regulator within the average effective approach [40].

The ERG equations were also used for the calculation of the RG flows of the effective action [35, 36, 41, 42] and the Ising model equation of state [35, 42, 43], as well as for the analysis of phase transitions [44] (see also [6]). These studies show high efficiency and accuracy of the ERG approach. Thus, as it was observed in Ref. [3], the comparison with the results obtained within the perturbation theory and within the ϵ -expansion shows that while the perturbative methods are more accurate for low order couplings, the derivative expansion eventually overtakes for higher orders.

4 More non-perturbative results

4.1 c -function

Another class of non-perturbative results obtained within the ERG is the construction of the c -function. The c -function appears in the context of the c -theorem first proved by Zamolodchikov [45] for 2-dimensional theories. A proof of the theorem in four dimensional AdS space was given in Ref. [46].

The c -theorem states that for Poincaré invariant, renormalizable and unitary theories there exists a non-negative function of the flow parameter t , which was called the c -function, such that: (1) it decreases along RG trajectories, i.e. $dc/dt < 0$; (2) it is stationary at a FP: $dc/dt|_{F.P.} = 0$.

Considered with respect to a local (in the space of all possible interactions) basis of operators and corresponding couplings $\{g^i\}$, the c -function depends on t only through $g^i(t)$: $c = c(g^i(t))$. The total derivative is equal to $dc/dt = -\beta^i \partial c / \partial g^i$, where β^i are β -functions. The c -theorem expresses the irreversibility of RG flows, which is a manifestation of the decoupling of massive

states as the system flows towards the infrared limit and provides a valuable instrument to relate realizations of the same quantum system at different scales.

Moreover, the RG flow is gradient, i.e. for a chosen basis of operators

$$\beta^i \equiv -\frac{dg^i}{dt} = \sum_j \mathcal{G}^{ij}(\{g\}) \frac{\partial c}{\partial g^j},$$

where \mathcal{G}^{ij} is a positive definite metric (known as the Zamolodchikov metric) in the space of couplings. This property of the c -function means that only FPs are allowed in the space of couplings, and limit cycles or more complicated behaviors are excluded.

One of the first studies of the c -function within the ERG approach was carried out in Ref. [36] within the Wegner-Houghton formulation. The c -function within the Polchinski ERG formalism in the LO of the derivative expansion was studied in Ref. [47]. It is based on the observation that LPA equation (13) (with $\eta = 0$) can be cast into the form [48]

$$aG(\phi) \frac{\partial \rho}{\partial t} = -\frac{\delta \mathcal{D}[\rho]}{\delta \rho},$$

where

$$G(\phi) = e^{-(d-2)\phi^2/4}, \quad \rho(\phi; t) = e^{-V(\phi; t)},$$

$$\mathcal{D}[\rho] = a \int d\phi G(\varphi) \left[\frac{1}{2} \left(\frac{d\rho}{d\phi} \right)^2 + \frac{d}{4} \rho^2 (1 - 2 \ln \rho) \right],$$

and a is a normalization factor. It was found that in terms of G and \mathcal{D} the c -function and the metric can be written as follows [47]:

$$c(\{g\}) = \frac{1}{A} \ln \left(\frac{4\mathcal{D}}{d} \right), \quad \mathcal{G}_{ij} = a\mathcal{D} \ln A \int d\phi G \rho^2 \mathcal{O}_i(\phi) \mathcal{O}_j(\phi),$$

where $O_i(\phi) = \partial V / \partial g^i$ and A is another normalization factor.

4.2 Quark-meson transition

Systems with many degrees of freedom are often described by different relevant excitations (fields) at different scales. Thus, in the theory of strong interactions at $\Lambda^{-1} \ll 1$ fm the relevant degrees of freedom are quarks and gluons and the adequate theory is the QCD. At $\Lambda^{-1} \gg 1$ fm the observed particles are mesons and hadrons, and their interaction is described by the chiral model (nonlinear σ -model).

A formalism to tackle the change of relevant degrees of freedom within the ERG approach was developed and used for the analysis of phenomena like the chiral phase transition in Refs. [49, 50] (see also [6]). Here we just outline the main elements of this formalism. The idea is to use the ERG flow equation for a model of quarks and gluons for scales $\Lambda > \Lambda_\sigma$, and a correspondingly modified ERG equation for quarks and mesons for $\Lambda < \Lambda_\sigma$, where Λ_σ is the transition scale ($\sim 600 - 700$ MeV). At $\Lambda = 1.5$ GeV the system was described by a QCD inspired model which symbolically can be written as

$$\Gamma_\Lambda[\psi, \bar{\psi}] = \int \frac{dq}{(2\pi)^4} \bar{\psi}_q \hat{q} \psi_{-q} + \frac{1}{2} \int \prod_{i=4}^4 \frac{dq_i}{(2\pi)^4} \gamma_\Lambda(q_i) (2\pi)^4 \delta(\sum q_i) \mu(q_i) (\bar{\psi}\psi) (\bar{\psi}\psi), \quad (15)$$

where $\psi, \bar{\psi}$ denote the quark fields and $(\bar{\psi}\psi)$ ($\bar{\psi}\psi$) stands for different 4-quark interactions allowed by vector and axial vector symmetries. The gluons are supposed to be integrated out, and the form of the 4-quark couplings is motivated by the gluon exchange and a confining potential. It turns out that, as the integration of modes is performed and the scale lowers down, the effective action develops a pole-like structure of a bound state: $\gamma_\Lambda \sim -g(q_1, q_2)\tilde{G}(s)g(q_3, q_4)$, where $g(q_1, q_2)$ corresponds to the Bethe-Salpeter wave function and $\tilde{G}(s)$ is the bound-state propagator with a pole-like dependence on s . At the scale $\Lambda = \Lambda_\sigma = 0.63 \text{ GeV}$ another effective action was introduced which symbolically could be written as

$$\Gamma_\Lambda[\psi, \bar{\psi}, \sigma] = \Gamma_\Lambda[\psi, \bar{\psi}] + \frac{1}{2}O^+\tilde{G}O - \sigma^+O + \frac{1}{2}\sigma^+\tilde{G}^{-1}\sigma, \quad (16)$$

where O, O^+ are composite operators quadratic in $\psi, \bar{\psi}$. They are appropriately defined so that the pole-like structure cancels. σ, σ^+ are collective fields playing the role of mesons of the linear σ -model with Yukawa couplings. Action (16) is used as the initial condition for the running effective action at lower scales. Further integration of the ERG equation gives the meson potential which develops a non-trivial minimum and thus describes the spontaneous breaking of chiral symmetry at $0 \leq \Lambda < \Lambda_\phi$. From this the authors of Ref. [49] calculated the vacuum expectation value $\langle \sigma \rangle = 0.18 \text{ MeV}$ and the chiral condensate $\langle \bar{\psi}\psi \rangle \approx (175 \text{ MeV})^3$. A substantial discrepancy between the calculated value of the pion decay constant and its experimental value was explained by the crudeness of the approximation [49]. A more systematic study of the quark-meson system within the average effective action approach with gluon effects taken into account is presented in [50].

5 General Remarks and Conclusions

From the numerous studies it can be concluded that the ERG is a powerful method in quantum field theory. Its effectiveness and reliability in practical calculations has been confirmed in numerous applications in various models. In cases when the results can be compared with other methods or with experimental data it was observed that the ERG gives a fairly good precision already at the LO or NLO of the derivative expansion. We would like to mention that even at the level of the LPA the effective potential $V(\phi; t)$ is given by quite a non-trivial expression which does not rely on expansion in powers of the field or any small parameter and includes Feynman diagrams of all orders and topologies [12].

In the present article we do not discuss the relation between the ERG and the perturbative RG. We restrict ourselves to a few comments. In principle, perturbative calculations can be carried out within the ERG approach and reproduce known results. For this one performs the perturbative expansion in the Polchinski ERG equation, Eqs. (6), (7), and resolves the resulting system of equations, for example, by iterations. This amounts to calculation with a smooth momentum cutoff regulator $K(p^2/\Lambda^2)$ in Eq. (5). Needless to say that this regularization technique for performing perturbative computations is not among the efficient ones. A detailed discussion of the relation between the ERG approximations, e.g. the derivative expansion, and the perturbation theory expansion, issues of the scheme dependence, renormalization conditions, etc. can be found, for example, in Refs. [4, 51, 52].

The advantage of the ERG is that it allows approximation schemes which are not based on a small parameter expansion. Perhaps the most successful of them is the derivative expansion dis-

cussed in detail in Sect. 3. A weak point of this technique is the absence of an obvious parameter or a practical criteria controlling its convergence. A complete and consistent study of its convergence and accuracy is still missing. In a number of papers the polynomial approximation of the effective potential in powers of the field was used [36, 53]. Though in certain cases such truncations give rather good numerical results, the procedure is not convergent and generates spurious solutions [30]. Development of other non-perturbative approximations is highly necessary for addressing a wider class of physical problems, especially those which require different degrees of freedom at different scales. We feel that the ERG approach, being exact by construction, allows for such approximations.

Let us mention another important feature of the ERG. In the space of theories one may introduce locally, i.e. in a vicinity of a given FP, a basis of operators \mathcal{O}_i and a set of corresponding coupling constants $\{g^i(\Lambda)\}$ which play the role of local coordinates (masses are also treated as couplings). Some of the couplings will be relevant or marginal (we denote them by $\{\tilde{g}^i(\Lambda)\}$), others will be irrelevant (see Sect. 3). Using the ERG equation one can show that the running effective action can be expressed as a self-similar flow of a finite number of relevant couplings:

$$S_\Lambda[\phi] = S[\phi; \tilde{g}(\Lambda)] \left(1 + \mathcal{O} \left(\frac{\Lambda}{\Lambda_0} \right) \right). \quad (17)$$

The irrelevant couplings bring only corrections of order Λ/Λ_0 [2, 11]. The limit $\Lambda_0 \rightarrow \infty$ is often called the continuum limit in the literature. Property (17) suggests that there should be certain similarity between the ERG and the RG approach based on the exact functional equation by Bogoliubov and Shirkov [54] (see [55] for a review). Namely, in both cases the underlying symmetry is the functional self-similarity, i.e. the invariance of the solution (running effective action) under the choice of the initial condition (bare action S_0). Studying this relation in more detail and sharing techniques of the functional RG equation approach [56] and the ERG could be mutually beneficial. If one considers a system with an external classical field H then all reasonable interactions of the quantum field ϕ with H can be incorporated into certain couplings of the effective action, i.e. we pass from a self-similar flow $S[\varphi; \tilde{g}(\Lambda)]$ to $S[\phi; \tilde{g}(\Lambda, H)]$. This observation is in full accordance with the result obtained in Ref. [57] in the framework of the perturbative RG.

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References

- [1] L. Kadanoff: *Physica* **2** (1966) 263;
K. Wilson: *Phys. Rev.* **B 4** (1971) 3174,3184;
K. G. Wilson, J. Kogut: *Phys. Rep.* **C 12** (1974) 77
- [2] T. R. Morris: *Yukawa Intern. Seminar '97, Prog. Theor. Phys. Suppl* **131** (1998) 395
- [3] T. R. Morris: In *New Developments in Quantum Field Theory, NATO ASI Series 366*, Plenum Press, NY (1998)
- [4] Yu. Kubyshin: *Int. J. Mod. Phys.* **B 12** (1998) 1321
- [5] C. Bagnuls, C. Bervillier: *Phys. Rep.* **348** (2001) 91
- [6] J. Berger, N. Tetradis, C. Wetterich: *Phys. Rep.* **363** (2002) 223
- [7] F. Wegner, A. Houghton: *Phys. Rev.* **A 8** (1973) 401
- [8] J. I. Latorre, T. R. Morris: *J. High En. Phys.* **0011** (2000) 004
- [9] F. J. Wegner: *J. Phys.* **C 7** (1974) 2098
- [10] J. Polchinski: *Nucl. Phys.* **B 231** (1984) 269
- [11] R. Ball, R. Thorne: *Ann. Phys.* **236** (1994) 117
- [12] T. R. Morris: *Int. J. Mod. Phys.* **A 9** (1994) 2411
- [13] C. Wetterich: *Phys. Lett.* **B301** (1993) 90
- [14] J. Pawłowski: *Acta Physica Slovaca* **52** (2002) 475
- [15] J. Comellas, Yu. Kubyshin, E. Moreno: *Nucl. Phys.* **B 490** (1997) 653
- [16] M. Reuter, C. Wetterich: *Nucl. Phys.* **B 391** (1993) 147; **B 408** (1993) 91; **B 417** (1994) 181; **B 427** (1994) 291; *Phys. Rev.* **D 56** (1997) 7893
- [17] C. Becchi: *hep-th/9607188*;
M. Bonini, M. D'Attanasio, G. Marchesini: *Nucl. Phys.* **B 418** (1994) 81; **B 421** (1994) 429;
U. Ellwanger: *Phys. Lett* **B 335** (1994) 364;
U. Ellwanger, M. Hirsch, A. Weber: *Z. Phys.* **C 69** (1996) 687
- [18] T. R. Morris: *hep-th/9810104*; *Nucl. Phys.* **B 573** (2000) 97; *J. High En. Phys.* **0012** (2000) 012
- [19] S. Arnone, Yu. Kubyshin, T. R. Morris, J. Tighe: *hep-th/0106258*
- [20] S. Arnone, A. Gatti, T. Morris: *Acta Physica Slovaca* **52** (2002) 621
- [21] F. J. Wegner: *Phys. Rev.* **B 5** (1972) 4529; In *Phase Transitions and Critical Phenomena*, Vol. VI, Eds. C. Domb and M.S. Green, Acad. Press (1976) p. 7
- [22] G. R. Golner, E. K. Riedel: *Phys. Lett.* **A 58** (1976) 11;
K. E. Newman, E. K. Riedel: *Phys. Rev.* **B 30** (1984) 6615;
E. K. Riedel, G. R. Golner, K. E. Newman: *Ann. Phys.* **161** (1985) 178
- [23] R. J. Myerson: *Phys. Rev.* **B 12** (1975) 2789;
G. R. Golner: *Phys. Rev.* **B 33** (1986) 7863
- [24] J. F. Nicoll, T. S. Chang, H. E. Stanley: *Phys. Rev. Lett.* **33** (1974) 540,1525;
Phys. Rev. A **13** (1976) 1251
- [25] V. I. Tokar: *Phys. Lett.* **A 104** (1984) 135
- [26] A. Hasenfratz, P. Hasenfratz: *Nucl. Phys.* **B 270** (1986) 687
- [27] R. Ball, P. Haagensen, J. I. Latorre, E. Moreno: *Phys. Lett.* **B 347** (1995) 80
- [28] T. R. Morris: *Phys. Lett.* **B 329** (1994) 241
- [29] G. Felder: *Com. Math. Phys.* **111** (1987) 101;
A. E. Filipov, S. A. Breus: *Phys. Lett* **A 158** (1991) 300

- [30] T. R. Morris: *Phys. Lett.* **B 334** (1994) 355
- [31] Yu. Kubyshin, R. Neves, R. Potting: *Int. J. Mod. Phys.* **A 16** (2001) 2065
- [32] Yu. Kubyshin, R. Neves, R. Potting: *hep-th/0202199*
- [33] T. R. Morris: *Phys. Lett.* **B 345** (1995) 139
- [34] T. R. Morris: *Nucl. Phys.* **B 458** (1996) 477
- [35] T. R. Morris: *Nucl. Phys.* **B 495** (1997) 477
- [36] P. Haagensen, Yu. Kubyshin, J. I. Latorre, E. Moreno: *Phys. Lett.* **B 323** (1994) 330
- [37] N. Tetradis, C. Wetterich: *Nucl. Phys.* **B 422** (1994) 541;
N. Tetradis, D. F. Litim: *Nucl. Phys.* **B 464** (1996) 492
- [38] Yu. Kubyshin, R. Neves, R. Potting: *The Exact Renormalization Group*, Eds. A. Krasnitz et al., World Scientific (1999) p. 159
- [39] J. Zinn-Justin: *QFT and Critical Phenomena*, C.P., Oxford (1993)
- [40] D. Litim: *Acta Physica Slovaca* **52** (2002) 635
- [41] C. Wetterich: *Nucl. Phys.* **B 352** (1991) 529;
N. Tetradis, C. Wetterich: *Nucl. Phys.* **B 398** (1993) 659;
J. Adams et al.: *Mod. Phys. Lett.* **A 10** (1995) 2367
- [42] J. Berges, N. Tetradis, C. Wetterich: *Phys. Rev. Lett.* **77** (1996) 873
- [43] J. Berges, C. Wetterich: *Nucl. Phys.* **B 487** (1997) 675
- [44] S. Seide, C. Wetterich: *Nucl. Phys.* **B 562** (1999) 524
- [45] A. Zamolodchikov: *JETP Lett.* **43** (1986) 730
- [46] S. Forte, J. I. Latorre: *Nucl. Phys.* **B 535** (1998) 709
- [47] J. Generowicz, C. Harvey-Fros, T. R. Morris: *Phys. Lett.* **B 407** (1997) 27
- [48] G. Zumbach: *Nucl. Phys.* **B 413** (1994) 754;
G. Zumbach: *Phys. Lett.* **A 190** (1994) 225
- [49] U. Ellwanger, C. Wetterich: *Nucl. Phys.* **B 423** (1994) 137
- [50] D.-U. Jungnickel, C. Wetterich: *Phys. Rev.* **D 53** (1996) 5142;
J. Berges, D.-U. Jungnickel, C. Wetterich: *Phys. Rev.* **D 59** (1999) 034010
- [51] T. Morris, J. Tighe: *J. High En. Phys.* **9908** (1999) 007
- [52] S. Arnone, A. Gatti, T. Morris: *Acta Physica Slovaca* **52** (2002) 615
- [53] A. Margaritis, G. Ódor, A. Patkós: *Z. Phys.* **C 39** (1988) 109
- [54] N. N. Bogoliubov, D. V. Shirkov: *Dokl. Acad. Nauk* **103** (1955) 203; *Nuov. Cim.* **3** (1956) 845
- [55] D. V. Shirkov: *hep-th/9602024*
- [56] V. F. Kovalev, D. V. Shirkov: *Phys. Rep.* **352** (2001) 219
- [57] D. I. Kazakov: *Acta Physica Slovaca* **52** (2002) 181