acta physica slovaca vol. 52 No. 6, 597 – 602 December 2002

SELF-ORGANIZATION OF THE CRITICAL STATE IN PHYSICAL SYSTEMS DESCRIBED BY DIFFERENTIAL EQUATIONS[1]

S. L. Ginzburg[2], N. E. Savitskaya[3]
Petersburg Nuclear Physics Institute, 188300, Gatchina, Russian Federation

Received 21 June 2002, in final form 4 September 2002, accepted 3 October 2002

A general form of the system of differential equations simulating the self-organized criticality is presented. Three physically important cases of this system are studied in detail. It is shown that the critical states of the systems under consideration are really self-organized.

PACS: 64.60.Lx 74.50.+r

1 Introduction

The concept of self-organized criticality (SOC) proposed by P. Bak et al. [1] was proved to be useful for the explanation of the behaviour of the giant dissipative dynamical systems. According to the main principles of this concept, such systems naturally evolve into a critical state that is self-reproduced in further dynamics. This critical state is an ensemble of metastable states. During the evolution process the critical system migrates from one metastable state to another by means of so-called “avalanches”. Such a critical state is called a self-organized one, and the mathematical criterion of self-organization is the power-law distribution of avalanche sizes.

The mathematical model for such type of behaviour was also introduced in [1] and it was referred to as a “sand pile model”. This model is described by a cellular automaton that is the system with discrete phase space.

The self-organized behaviour detecting itself as a power-law distribution of the system characteristics is observed for a wide range of dynamical systems and phenomena. Nevertheless the sand pile model and its modifications still remain the main objects for the theoretical study of self-organization [1, 2]. Therefore, the problem of finding a useful physical system with self-organization, available for experimental investigations of SOC, remains very actual. However this problem is a difficult one, since the majority of physical systems is described by differential equations in contrast with the classical models of SOC that are formulated in terms of cellular automaton.

For the first time the realization of self-organized critical state in physical system described by differential equations was demonstrated in [3–5]. These papers were devoted to study of critical state in a granular superconductor.

[1]Presented by N. E. S. at 5th Int. Conf. Renormalization Group 2002, Tatranská Štrba (Slovakia), March 2002
[2]E-mail address: ginzburg@thd.pnpi.spb.ru
[3]E-mail address: savitska@thd.pnpi.spb.ru

In the present work we consider two important cases of the system of differential equations for the granular superconductors. First of them describes the physical system with asymmetrical potential, so-called "ratchet". The second one is related to the extensively investigated "multilayer" system with varying shapes of interlayer boundaries. We show that these systems are self-organized. Using these results we formulate general mathematical requirements for differential equations to describe the self-organized system.

2 Self-organization of the critical state in multijunction SQUID

It is known [1] that the self-organized system consists of a large number of the interacting threshold elements. The simplest physical example of such an element is single-junction SQUID described as the hollow superconducting cylinder with Josephson junction inserted in [6]. Dynamics of a SQUID is characterized by the gauge-invariant phase difference φ that is proportional to the internal magnetic flux Φ:

$$\varphi = \frac{2\pi}{\phi_0}\Phi, \tag{1}$$

where ϕ_0 is a magnetic flux quantum.

System under consideration is described by the following differential equation [6]:

$$V\sin\varphi + \tau\frac{\partial\varphi}{\partial t} = -\varphi + 2\pi F_{ext}\,, \tag{2}$$

where $F_{ext} = \Phi_{ext}/\phi_0$ describes the external perturbations, $\tau = 4\pi lS/\rho$, ρ is the resistivity of the junction, $V = 8\pi lSj_c/\phi_0$ is the main SQUID parameter, where j_c is the critical current density, l is junction size, and S is the area of the SQUID ring.

The properties of SQUID strongly depend on SQUID parameter V. When the parameter V is large ($V \gg 1$), the energy of the system has a large number of metastable states [6]. In this case the SQUID demonstrates a threshold behaviour [6] and we can approximate the phase difference by the step-wise function [3]

$$\varphi \approx 2\pi p + \pi/2, \tag{3}$$

where p is a integer number. It means that the internal magnetic flux Φ can change only by integer number of magnetic flux quanta and the SQUID has a discrete phase space (1). Thus the single-junction SQUID with $V \gg 1$ can serve as an element of self-organizing system.

The simplest system of interacting single-junction SQUIDs is one-dimensional multijunction SQUID. It is described as two superconducting layers that are infinite in y dimension and are connected by Josephson junctions of size l [6]. These junctions are situated along x axis and the distance between two neighbour junctions is a random value b_i. Following to [5], we assume that the system is perturbed by means of current injection. The density of injected current is j_i^1 for i-th contact. System under consideration is described by the generalization of Maxwell equations for discrete lattice [6].

$$V\Psi(\varphi_i) + \tau\frac{d\varphi_i}{dt} = J_i(\varphi_{i+1} - \varphi_i) + J_{i-1}(\varphi_{i-1} - \varphi_i) + 2\pi F_i\,, \qquad i \neq 1, N \tag{4}$$
$$V\Psi(\varphi_1) + \tau\frac{d\varphi_1}{dt} = J_1(\varphi_{i+1} - \varphi_i) + 2\pi F_1\,,$$
$$V\Psi(\varphi_N) + \tau\frac{d\varphi_N}{dt} = J_{N-1}(\varphi_{N-1} - \varphi_N) + 2\pi F_N\,,$$
$$V = \frac{16\pi^2 al\lambda_L j_c}{\phi_0}, \qquad \tau = \frac{8\pi al\lambda_L}{\rho}, \qquad J_i = \frac{a}{b_i}, \qquad F_i = \frac{8\pi\lambda_L al}{\phi_0} j_i^1\,,$$

where $a =< b_i >$, $\Psi(\varphi_i) = \sin\varphi_i$, and N is a number of junction.

In the present paper we consider the perturbation method proposed in [2] for one-dimensional sand pile model. For our system, this method is equivalent to increase of magnetic field in a randomly chosen cell by $\delta = 1/2$ [5], that leads to induction of positive current into the junction situated on the right side of chosen cell and the same but negative current is injected into the junction situated on the left side of the chosen cell:

$$F_i \to F_i - 1/2 \qquad F_{i+1} \to F_{i+1} + 1/2\,. \tag{5}$$

We study our multijunction SQUID and other systems described below by computer simulations with $N = 129$, $V = 40$, $\tau = 1$ and $J_i = 1$ for all i using the Euler integration scheme with $\delta t = 0.01$. Starting from the stable state (all $\varphi_i = 0$) we perturb the system according to rules (5), this perturbation launches the dynamical process (avalanche), during this process F_i are not changed for any i. When the avalanche stops and the system reaches the metastable state (all $\partial\varphi_i/\partial t < 10^{-7}$), we perturb the system again and so on.

Finally, SQUID under consideration comes into the critical state that is an ensemble of metastable states. The distribution of the "currents" $z_i^{st} = V/2\pi\sin\varphi_i$ and the "magnetic field profile" $h_i = \frac{1}{2\pi}(\varphi_{i+1} - \varphi_i)$ for one of the metastable state are shown in Fig. 1a, b. Comparing our system with the sand pile model we see that the "magnetic field" plays the role of pile height and "current" is an analog of pile slope.

To verify whether the arising critical state is self-organized one, we define the quantity which is analog of the avalanche size. This is the integral of the voltage over the avalanche time [3]. We take into account only "positive" part of system that is similar to the classical sand pile Fig. 1b.

$$u_k = \frac{\phi_0}{2\pi M}\sum_{i=1}^{M}(\varphi_i(t_{ek})) - \varphi_i(t_{bk})), \tag{6}$$

where k is the avalanche number, $M = 64$, t_{bk}, and t_{ek} are the starting and ending moments of the k-th avalanche, respectively.

From Fig. 2a we see that the probability density of voltage $\rho(u)$ demonstrates the power-law behaviour. Hence the critical state of our system is the self-organized.

3 Self-organization of the critical state in asymmetrical system

Prescinding from the physical meaning of differential equations for SQUID (4) we can consider them as equations describing the system with symmetrical periodical potential. However we

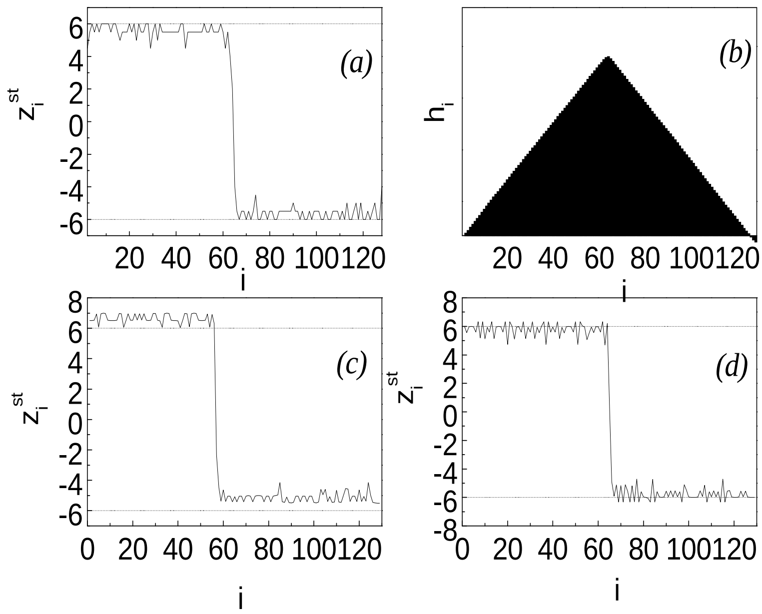

Fig. 1. The profile of a) dimensionless "currents" b) "magnetic field" for the one of metastable states of one-dimensional multijunction SQUID. The profile of z_i^{st} c) for asymmetric system, d) for "multilayer" system.

often deal with situation when the system potential is periodical but an asymmetrical one. The most popular example of such a system is the extensively investigated system with "ratchet" potential [7]. The main feature of "ratchets" is presence of nonzero flux in the system under the action of the zero-mean noise.

To construct such a system we replace the symmetrical function $\Psi(\varphi) = \sin\varphi$ in (4) by asymmetric one that is usually used for description of ratchets $\Psi(\varphi) = (\cos\varphi + 1/8\cos 2\varphi)$. In this case the variable φ is not a "phase" but, for example, a spatial variable.

Then the equation for single element of the system (2) has the following form:

$$\tau\frac{\partial\varphi}{\partial t} = -\varphi + 2\pi F - V(\cos\varphi + \frac{1}{8}\cos 2\varphi) = -\frac{\partial U}{\partial\varphi}\,. \tag{7}$$

For $V \gg 1$ such element also demonstrates the threshold behaviour and its energy U has a large number of metastable states.

We study a system of N interacting elements (7) using the regime described above with perturbation rules (5).

Every change of F_i leads to avalanche process and after the transition time the system reaches the critical state that is an ensemble of metastable states. We see that the structure of these metastable states (Fig. 1c, where z_i^{st} is an analog of the "current" and calculated as $z_i^{st} = (V/2\pi)\Psi(\varphi_i)$) differs from that observed earlier (Fig. 1a). The asymmetrical profile is conditioned by the asymmetry of function $\Psi(\varphi)$.

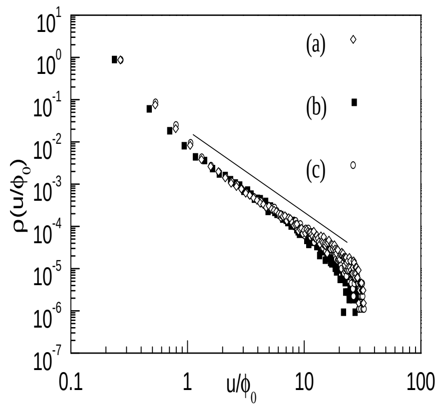

Fig. 2. Probability density of a) voltage for multijunction SQUID, b) the quantity (6) for asymmetric system, c) quantity (6) for "multilayer" system. The fitting line has a slope $\alpha = -1.73$.

For the every avalanche we consider the quantity that is analog of voltage and calculated from eq. (6) with $M = 57$. The value for M is chosen so that only "positive" subsystem is taken into account. The probability density for quantity (6) is shown on the Fig. 2b. From this figure we see that it demonstrates the power-law behaviour then the system under consideration is self-organized.

4 Self-organization of the critical state in the "multilayer" system

Considering the variable φ as a spatial one we can interpret the periodic part of the potential $\Psi(\varphi)$ as a potential of multilayer system. All examples of $\Psi(\varphi)$ discussed earlier allow us to vary only the period of the function. However one often encounters the systems with varying shapes of interlayer boundaries such as domain systems. To describe such objects we need the "two-scale" function, for example, the Jacobi function

$$\Psi(\varphi) = sn\left(\frac{2K(m)\varphi}{\pi}, m\right), \qquad K(m) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - m\sin^2(\varphi)}}, \tag{8}$$

where $K(m)$ is the complete elliptic integral of the first kind. The parameter m is responsible for the shape of layers.

Replacing $\sin\varphi$ in (2) by $sn(\frac{2K(m)\varphi}{\pi})$ we have the following equation:

$$\tau\frac{\partial\varphi}{\partial t} = -\varphi + 2\pi F - V sn(\frac{2K(m)\varphi}{\pi}) = -\frac{\partial U}{\partial\varphi}\,. \tag{9}$$

For $V \gg 1$ this also demonstrates the threshold behaviour and its energy U corresponds to large number of metastable states.

We consider the system consisting of N elements described by eq. (4) with $\Psi(\varphi) = sn(\frac{2K(m)\varphi}{2\pi})$, and $m = 0.99$ using the regime mentioned above and perturbation rules (5). After the transition process, the system reaches the critical state consisting of the large number of the metastable states. The structure of these states is equivalent to that observed in SQUID (see Fig. 1d where $z_i^{st} = (V/2\pi)\Psi(\varphi_i)$).

For the every avalanche in the critical state we calculate the quantity (6). The probability density for this quantity is presented in Fig. 2c. It is seen that it demonstrates the power-low behaviour. It means that the critical state of our system is self-organized.

5 Conclusions

In our recent papers on study of critical state in granular superconductors [3–5] we demonstrated that the self-organized criticality can arise in the physical system described by differential equations. In the present paper we used these results to study in detail two important modifications of the system of differential equations (4). The first one describes the system with asymmetrical potential, for example, the ratchet system. The second modification includes the two-scale periodic function describing the potential of multilayer system. We have shown that the critical states in these systems are self-organized.

Taking into account these results we conclude that the system of differential equations simulating the physical system with SOC can be written in form (4), where φ is a dynamical variable, $\Psi(\varphi) = \Psi(\varphi + 2\pi)$ is periodical limited function, and F describes the external perturbations. Coefficients J_i characterize the interactions between the elements. The main requirement for the system to be self-organized is the large value of the parameter V $(V \gg 1)$.

Obtained result gives the opportunity to find real physical systems where the self-organized critical state can be observed.

Acknowledgement: This work was supported by the RFBR (projects Nos. 02-02-16979, 02-02-06687), the Scientific Council of the "Superconductivity" direction (project No. 96021 "Profile"), the State programs "Investigations of collective and quantum effects in condensed matter" and "Quantum Macrophysics". N. S. would like to thank "Science Support Foundation Grant for Talented Young Researches".

References

[1] P. Bak, C. Tang, K. Wiesenfeld: *Phys. Rev. Lett.* **59** (1987) 381

[2] L. Kadanoff, S. R. Nagel, L. Wu, S.-m. Zhou: *Phys. Rev.* **A 39** (1989) 6524

[3] S. L. Ginzburg: *JETP* **79** (1994) 334

[4] S. L. Ginzburg, M. A. Pustovoit, N. E. Savitskaya: *Phys. Rev.* **E 57** (1998) 1319

[5] S. L. Ginzburg, N. E. Savitskaya: *JETP* **90** (2000) 202

[6] K. K. Likharev *Dynamics of Josephson Junction and Circuits*, Gordon and Breach, New York (1986)

[7] C. R. Doering, W. Horsthemke, J. Riordan: *Phys. Rev. Lett.* **72** (1994) 2984;
R. D. Astumian, F. Moss: *Chaos* **8** (1998) 533