

**RENORMALIZATION GROUP IN THE STATISTICAL THEORY OF TURBULENCE:  
TWO-LOOP APPROXIMATION<sup>1</sup>**

**L. Ts. Adzhemyan, N. V. Antonov, M. V. Kompaniets, A. N. Vasil'ev**  
*Department of Theoretical Physics, St Petersburg University,  
Uljanovskaja 1, St Petersburg, Petrodvorez, 198 504 Russian Federation*

Received 21 June 2002, in final form 6 August 2002, accepted 15 August 2002

The field theoretic renormalization group is applied to the stochastic Navier–Stokes equation that describes fully developed fluid turbulence. The complete two-loop calculation of the renormalization constant, the beta function and the fixed point is performed. The ultraviolet correction exponent, the Kolmogorov constant and the inertial-range skewness factor are derived to second order of the  $\varepsilon$  expansion.

PACS: 47.27.–i, 05.10.Cc, 47.10.+g

### 1 Introduction

One of the oldest open problems in theoretical physics is that of describing fully developed turbulence on the basis of a microscopic model. The latter is usually taken to be the stochastic Navier–Stokes (NS) equation subject to an external random force which mimics the energy input by the large-scale modes; see, e.g., [1, 2]. The aim of the theory is to verify the basic principles of the celebrated Kolmogorov–Obukhov phenomenological theory, study deviations from this theory, determine the dependence of various correlation functions on the times, distances, external (integral) and internal (viscous) turbulence scales, and derive the corresponding scaling dimensions. Most results of this kind were obtained within the framework of numerous semiphenomenological models which cannot be considered to be the basis for construction of a regular expansion in certain small (at least formal) parameter [1, 2].

An important exception is provided by the renormalization group (RG) method that was earlier successfully applied in the theory of critical behaviour to explain the origin of critical scaling and to calculate universal quantities (critical dimensions and scaling functions) in the form of the  $\varepsilon$  expansions [3].

The RG approach to the stochastic NS equation, pioneered in [4–7], allows one to prove the existence of the infrared (IR) scale invariance with exactly known “Kolmogorov” dimensions and the independence of the correlation functions on the viscous scale (the second Kolmogorov hypothesis), and calculate a number of representative constants in a reasonable agreement with

---

<sup>1</sup>Presented by M. V. K. at 5th Int. Conf. Renormalization Group 2002, Tatranská Štrba (Slovakia), March 2002

experiment; see [8–10] for a review. Much more detailed exposition of the RG theory of turbulence and the bibliography can be found in [11].

In contrast to the standard  $\phi^4$  model of critical behaviour [3], where the critical exponents are known up to the order  $\varepsilon^5$  (five-loop approximation), all the calculations in the RG approach to the stochastic NS equation have been confined with the simplest one-loop approximation. The reason for this distinction is twofold. First, calculation of the multiloop diagrams for this dynamical model appears rather involved due to the presence of frequencies and vector indices in the corresponding integrands. Our experience has shown that the two-loop calculation for the stochastic NS equation is, roughly speaking, as cumbersome as the four-loop calculation for the conventional  $\phi^4$  model. Second, the use of Galilean invariance and functional Schwinger equations allows one to calculate the critical dimensions for the most important physical quantities (velocity and its powers, frequency, energy dissipation rate and so on) exactly [4–7]. The corresponding  $\varepsilon$  series terminate at first-order terms (in this sense, they are given by the one-loop approximation exactly) and the higher-order calculations for them are not needed.

However, the  $\varepsilon$  series for other important quantities do not terminate and the calculation of the higher-order terms for them is of great interest. In this paper, we present the results of the two-loop calculation for a number of such quantities: the ultraviolet (UV) correction exponent  $\omega$ , the Kolmogorov constant  $C_K$  and the inertial-range skewness factor  $\mathcal{S}$ .

## 2 The model, field theoretic formulation and renormalization

Detailed exposition of the RG theory of turbulence and the bibliography can be found in [10, 11]; below we restrict ourselves to only the necessary information.

As the microscopic model of the fully developed, homogeneous, isotropic turbulence of an incompressible viscous fluid one usually takes the stochastic NS equation with a random driving force

$$\nabla_t \varphi_i = \nu_0 \partial^2 \varphi_i - \partial_i P + F_i, \quad \nabla_t \equiv \partial_t + (\varphi \partial). \quad (1)$$

Here  $\varphi_i$  is the transverse (due to the incompressibility) three-dimensional vector velocity field,  $P$  and  $F_i$  are the pressure and the transverse random force per unit mass (all these quantities depend on  $x \equiv \{t, \mathbf{x}\}$ ),  $\nu_0$  is the kinematical viscosity coefficient,  $\partial^2$  is the Laplace operator and  $\nabla_t$  is the Lagrangian derivative. The problem (1) is studied on the entire  $t$  axis and is augmented by the retardation condition and the condition that  $\varphi_i$  vanishes for  $t \rightarrow -\infty$ . We assume for  $F$  a Gaussian distribution with zero mean and correlator

$$\langle F_i(x) F_j(x') \rangle = \delta(t - t') (2\pi)^{-3} \int d\mathbf{k} P_{ij}(\mathbf{k}) d_F(k) \exp[i\mathbf{k}(\mathbf{x} - \mathbf{x}')], \quad (2)$$

where  $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$  is the transverse projector and  $d_F(k)$  is some function of  $k \equiv |\mathbf{k}|$  and model parameters.

The stochastic problem (1), (2) is equivalent to the field theoretic model of the doubled set of fields  $\Phi \equiv \{\varphi, \hat{\varphi}\}$  with action functional

$$S(\Phi) = \hat{\varphi} D_F \hat{\varphi} / 2 + \hat{\varphi} [-\partial_t \varphi + \nu_0 \partial^2 \varphi - (\varphi \partial) \varphi], \quad (3)$$

where  $D_F$  is the random force correlator (2) and the required integrations over  $x = \{t, \mathbf{x}\}$  and summations over the vector indices are understood.

The standard RG technique can be applied to the model (3) if the function  $d_F(k)$  is chosen in the form

$$d_F(k) = D_0 k^{1-2\varepsilon} h(m/k), \quad D_0 > 0, \quad h(0) = 1. \quad (4)$$

Here  $m = 1/L$  is the reciprocal of the integral turbulence scale  $L$  and the function  $h(m/k)$ , rapidly decaying for  $m/k \rightarrow \infty$ , provides the IR regularization. The exponent  $\varepsilon > 0$  plays the part of the RG expansion parameter, similar to that played by  $\varepsilon = 4 - d$  in Wilson's theory of critical phenomena [3]. The real (physical) value of this parameter is  $\varepsilon = 2$ : idealized energy injection by infinitely large eddies corresponds to  $d_F(k) \propto \delta(\mathbf{k})$ , and the function (4) for  $\varepsilon \rightarrow 2$  and the appropriate choice of the amplitude can be considered as a power-law model of the three-dimensional  $\delta$  function.

The model (3) is logarithmic (the coupling constant  $g_0 \equiv D_0/\nu_0^3$  is dimensionless) at  $\varepsilon = 0$ , and the UV divergences have the form of the poles in  $\varepsilon$  in the correlation functions of the fields  $\varphi$  and  $\hat{\varphi}$ . Superficial UV divergences, whose removal requires counterterms, are present only in the 1-irreducible function  $\langle \hat{\varphi}\varphi \rangle$ , and the corresponding counterterm has the form  $\hat{\varphi}\partial^2\varphi$ . Thus for the complete elimination of the UV divergences it is sufficient to perform the multiplicative renormalization of the parameters  $\nu_0$  and  $g_0 = D_0/\nu_0^3$  with the only independent renormalization constant  $Z_\nu$ :

$$\nu_0 = \nu Z_\nu, \quad g_0 = g\mu^{2\varepsilon} Z_g, \quad Z_g = Z_\nu^{-3} \quad (D_0 = g_0\nu_0^3 = g\mu^{2\varepsilon}\nu^3). \quad (5)$$

Here  $\mu$  is the reference mass in the minimal subtraction (MS) scheme, which we always use in what follows,  $g$  and  $\nu$  are renormalized analogues of the bare parameters  $g_0$  and  $\nu_0$ , and  $Z = Z(g, \varepsilon, d)$  are the renormalization constants. In the MS scheme they have the form "1 + only poles in  $\varepsilon$ ," in particular,

$$Z_\nu = 1 + \sum_{k=1}^{\infty} a_k(g)\varepsilon^{-k} = 1 + \sum_{n=1}^{\infty} g^n \sum_{k=1}^n a_{nk}\varepsilon^{-k}, \quad (6)$$

with the one-loop coefficient  $a_{11} = -1/40\pi^2$  [6, 7].

### 3 Two-loop approximation for the RG functions, fixed point and the UV correction exponent

We have performed calculation of the constant  $Z_\nu$  with the accuracy of  $g^2$  (two-loop approximation). Divergent parts (that is, poles in  $\varepsilon$ ) of the two-loop diagrams can be represented as integrals over three variables: moduli of the two integration momenta and the angle between them. With an appropriate choice of the function  $h(m/k)$  in (4), one modular integration can be got rid of using the differentiation with respect to the IR cut-off  $m$ ; the other two integrations can be performed numerically for any given value of  $d$ , the space dimensionality. Details of this calculation can be found in [12, 13], and below we give only the results for the residues  $a_{22}$  and  $a_{21}$  at the second-order and first-order poles in  $\varepsilon$  in the representation (6) in three dimensions:

$$a_{22}/a_{11}^2 = 1, \quad a_{21}/a_{11}^2 \simeq -1.65. \quad (7)$$

The knowledge of the renormalization constant  $Z_\nu$  to order  $g^2$  allows for the calculation of the RG functions, the anomalous dimension  $\gamma_\nu$  and the beta function  $\beta_g(\varepsilon, g)$ , with the following accuracy:

$$\begin{aligned}\beta(g, \varepsilon) &\equiv \tilde{\mathcal{D}}_\mu g = g(-2\varepsilon + 3\gamma_\nu(g)), \\ \gamma_\nu(g) &\equiv \tilde{\mathcal{D}}_\mu \ln Z_\nu = -2g\partial_g a_1(g) = -2(a_{11}g + 2a_{21}g^2) + O(g^3),\end{aligned}\quad (8)$$

where  $\tilde{\mathcal{D}}_\mu$  is the operation  $\mu\partial/\partial\mu$  at fixed bare parameters. In the MS scheme only the residues at the first-order poles in  $\varepsilon$ , that is, only the coefficients  $a_{k1}$ , contribute to the RG functions owing to the UV finiteness of the latter.

The coordinate of the fixed point is determined by the condition that  $\beta(g_*) = 0$ . From (7) and (8) we thus obtain:

$$g_* = (40\pi^2\varepsilon/3)(1 + \lambda\varepsilon) + O(\varepsilon^3), \quad \lambda \equiv 2a_{21}/3a_{11}^2 \simeq -1.10. \quad (9)$$

Due to the first relation in (8), the value of the anomalous dimension  $\gamma_\nu$  at the fixed point is found exactly,  $\gamma_\nu(g_*) = 2\varepsilon/3$ , and has no corrections of order  $\varepsilon^2$  and higher. On the contrary, the correction exponent  $\omega = \beta'(g_*)$ , determined by the slope of the beta function at the fixed point, is given by an infinite series in  $\varepsilon$ . From (8) and (9) we obtain the first and second terms of its  $\varepsilon$  expansion:

$$\omega = 2\varepsilon(1 - \lambda\varepsilon) + O(\varepsilon^3). \quad (10)$$

#### 4 Two-loop calculation of the Kolmogorov constant

The Kolmogorov constant  $C_K$  can be defined as the (dimensionless) coefficient in the inertial-range expression  $S_2(r) = C_K(\bar{\varepsilon}r)^{2/3}$  for the second-order structure function, predicted by the Kolmogorov–Obukhov theory and confirmed by experiment [1, 2]. Here  $\bar{\varepsilon}$  is the mean energy dissipation rate and the  $n$ -th order (longitudinal, equal-time) structure function is defined as

$$S_n(r) \equiv \langle [\varphi_r(t, \mathbf{x} + \mathbf{r}) - \varphi_r(t, \mathbf{x})]^n \rangle, \quad \varphi_r \equiv (\varphi_i \cdot r_i)/r, \quad r \equiv |\mathbf{r}|. \quad (11)$$

Using the exact relation  $S_3(r) = -4\bar{\varepsilon}r/5$  that follows from the energy balance equation [1, 2], the constant  $C_K$  can be related to the inertial-range skewness factor:

$$\mathcal{S} \equiv S_3/S_2^{3/2} = -(4/5)C_K^{-3/2}.$$

Many studies have been devoted to the derivation of  $C_K$  within the framework of the RG approach; see [14–22]. In order to obtain  $C_K$ , it is necessary to augment the solution of the RG equation for  $S_2$  by some formula that relates the amplitude  $D_0$  in the random force correlator (4) to the physical parameter  $\bar{\varepsilon}$ . In particular, in [14–16] the first-order term of the  $\varepsilon$  expansion for the pair correlator was combined with the so-called eddy damped quasi-normal Markovian approximation for the energy transfer function, taken at  $\varepsilon = 2$ . A more elementary derivation, based on the exact relation between  $\bar{\varepsilon}$  and the function  $d_F(k)$  from (4) was given in [17]; see also [10, 11]. In spite of the reasonable agreement with the experiment, such derivations are not

immaculate from the theoretical viewpoints. Their common flaw is that the relation between  $\bar{\varepsilon}$  and  $D_0$  is unambiguous only in the limit  $\varepsilon \rightarrow 2$ :

$$\lim_{\varepsilon \rightarrow 2} \frac{D_0}{4\pi^2(2-\varepsilon)} = \bar{\varepsilon}, \quad (12)$$

so that the coefficients of the corresponding  $\varepsilon$  expansions appear in fact arbitrary; see the discussion in [21] and Sec. 2.10 of [11]. The ambiguity is a consequence of the fact that the notion of the Kolmogorov constant has no definite extension to the nonphysical range  $\varepsilon < 2$ .

The experience on the RG theory of critical behaviour shows that unambiguous  $\varepsilon$  expansions can be written for universal quantities, such as critical exponents, normalized scaling functions and ratios of amplitudes in scaling laws [3]. The constant  $C_K$  extended to the range  $\varepsilon < 2$  as in [14–16] or [17] involves a bare parameter,  $D_0$ , and hence is not universal.

To circumvent this difficulty, we propose below an alternative derivation that relates  $C_K$  to an universal quantity and thus does not involve any relation between  $D_0$  and  $\bar{\varepsilon}$ , and calculate  $C_K$  to second order of the expansion in  $\varepsilon$  (previous attempts have been confined to first order). Consider the ratio

$$Q(\varepsilon) \equiv r\partial_r S_2(r)/|S_3(r)|^{2/3} = r\partial_r S_2(r)/(-S_3(r))^{2/3}. \quad (13)$$

The operation  $r\partial_r \equiv r\partial/\partial r$  kills the constant contribution  $\langle \varphi^2 \rangle$  in  $S_2$  that diverges as  $\Lambda \rightarrow \infty$  for  $\varepsilon < 3/2$ ; in  $S_3$  such constant contributions are absent.

Solving the RG equations for the quantities in  $Q(\varepsilon)$  in the IR range ( $\Lambda r \gg 1$ ) for general  $0 < \varepsilon \leq 2$  gives

$$S_3(r) = D_0 r^{-3\Delta_\varphi} f_3(\varepsilon), \quad r\partial_r S_2(r) = D_0^{2/3} r^{-2\Delta_\varphi/3} f_2(\varepsilon), \quad \Delta_\varphi = 1 - 2\varepsilon/3; \quad (14)$$

see e.g. [10, 11]. Thus the quantity  $Q(\varepsilon) = f_2/(-f_3)^{2/3}$  in (13) does not depend on  $D_0$  and can be calculated in the form of a regular  $\varepsilon$  expansion. We calculated the scaling functions  $f_{2,3}$  to the second order of the  $\varepsilon$  expansion, which corresponds to the two-loop approximation in (9), (10), and obtained:

$$Q(\varepsilon) = (1/3)(20\varepsilon)^{1/3} [1 + 0.525\varepsilon + O(\varepsilon^2)]. \quad (15)$$

It is worth noting that the  $\varepsilon$  expansion for  $f_3$  can be obtained not only from the direct perturbative calculation, but also from the exact expression

$$S_3(r) = -\frac{6\Gamma(2-\varepsilon)}{2^{2\varepsilon}\pi^{3/2}\Gamma(3/2+\varepsilon)} D_0 r^{-3\Delta_\varphi} \quad (16)$$

that follows from the energy balance equation. In the limit  $\varepsilon \rightarrow 2$  equation (16) along with the formula (12) reproduce the correct coefficient  $-4/5$  (see above).

The value of  $Q(\varepsilon)$  at  $\varepsilon = 2$  determines the Kolmogorov constant and skewness factor through the exact relations  $C_K = 6 \cdot 10^{-2/3} Q(2)$ ,  $\mathcal{S} = -[1.5 \cdot Q(2)]^{-3/2}$ , which follow from the definitions and the identity  $r\partial_r r^\lambda = \lambda r^\lambda$  for any  $\lambda$ . Substituting the value of (15) we obtain  $C_K = 3.02$ ,  $\mathcal{S} = -0.15$ . If we retained only the first-order term in (15) we would have obtained  $C_K = 1.47$  and  $\mathcal{S} = -0.45$ . We also recall the experimental estimates recommended in [1]:  $C_K \approx 1.9$  and  $\mathcal{S} \approx -0.28$ .

Over the last four decades, deviations from the Kolmogorov theory due to intermittency and the issue of anomalous scaling have been much debated [1, 2]. Within the framework of the RG, such deviations can be related to Galilean invariant composite fields (operators) with *negative* critical dimensions [10, 11, 17]. For the so-called Kraichnan's rapid-change model (see [23] for a review and references) this idea allows one to construct regular  $\varepsilon$  expansions for the anomalous exponents that measure the deviations of the scaling laws from the predictions of the Kolmogorov theory [24, 25]. However, for the model (3), (4) no such operator has been presented in any study we know of. Although the problem should be considered open, this fact suggests that Galilean invariant objects may have a finite limit at  $m \rightarrow 0$  in agreement with the first Kolmogorov hypothesis. Above we calculated the constant  $C_K$  within the assumption that the second-order structure function  $S_2$  in (11) is finite at  $m = 0$ . Such an assumption can be justified by the real experiments, which indicate that the corresponding exponent is hardly distinguishable from the Kolmogorov value [1], or by the example of the Kraichnan model, in which the second-order function is not anomalous [23]. It should also be stressed that such a derivation is internally consistent because no negative dimensions in model (3) for asymptotically small  $\varepsilon$  [10, 11].

## 5 Conclusion

We have accomplished the complete two-loop calculation of the renormalization constant and RG functions for the stochastic problem (1)–(4) and derived the coordinate of the fixed point, the UV correction exponent  $\omega$ , the Kolmogorov constant  $C_K$  and the inertial-range skewness factor  $\mathcal{S}$  to second order of the corresponding  $\varepsilon$  expansions. The new point is not only the inclusion of the second-order correction, but also the derivation of  $C_K$  through an universal (in the sense of the theory of critical behaviour) quantity.

Of course, one should have not expected that the second-order terms of the  $\varepsilon$  expansions would be small in comparison to the first-order terms. The experience from the RG theory of critical behaviour shows that such corrections are not small for dynamical models (in contrast to static ones) and for amplitudes (in contrast to exponents); see [3]. It is thus rather surprising that in our case the account of the two-loop contributions leads to reasonable changes in the results.

Although the  $\varepsilon^2$  correction to  $\omega$  in (10) is rather large, it does not change its sign and hence does not destroy the IR stability of the fixed point.

The first-order approximation  $C_K = 1.47$  underestimates, and the second-order approximation  $C_K = 3.02$  overestimates the conventional experimental value of the Kolmogorov constant  $C_K \approx 1.9$  [1]. Thus the experimental value of  $C_K$  (and hence for  $\mathcal{S}$ ) lies in between the two consecutive approximations. A similar situation is encountered for the well-known Heisenberg model [1], where the analogue of the Kolmogorov constant is known exactly and lies between the first-order and second-order approximations given by the corresponding  $\varepsilon$  expansion [26]. If we assume, by the analogy with the Heisenberg model, that the (unknown) exact predictions for  $C_K$  and  $\mathcal{S}$  lie between the first two approximations, we may conclude that our calculation has given a satisfactory estimate for these quantities.

The work was supported by the Nordic Grant for Network Cooperation with the Baltic Countries and Northwest Russia No. FIN-18/2001, the program "Universities of Russia" and the GRACENAS Grant No. E00-3-24. N.V.A. was also supported by the Academy of Finland (Grant No. 79781).

## References

- [1] A. S. Monin, A. M. Yaglom: *Statistical Fluid Mechanics, Vol.2*, MIT Press, Cambridge (1975)
- [2] U. Frisch: *Turbulence: The Legacy of A.N. Kolmogorov*, Cambridge University Press, Cambridge (1995)
- [3] J. Zinn-Justin: *Quantum Field Theory and Critical Phenomena*, Clarendon, Oxford (1989)
- [4] D. Forster, D. R. Nelson, M. J. Stephen: *Phys. Rev. A* **16** (1977) 732
- [5] C. De Dominicis, P. C. Martin: *Phys. Rev. A* **19** (1979) 419
- [6] J. D. Fournier, U. Frisch: *Phys. Rev. A* **28** (1983) 1000
- [7] L. Ts. Adzhemyan, A. N. Vasil'ev, Yu. M. Pis'mak: *Theor. Math. Phys.* **57** (1983) 1131
- [8] P. L. Sulem, J. D. Fournier, U. Frisch: *Lecture Notes in Physics* **104** (1979) 321
- [9] E. V. Teodorovich: *Izv. Akad. Nauk., Fiz. Atmosf. Okeana* **29** (1993) 149
- [10] L. Ts. Adzhemyan, N. V. Antonov, A. N. Vasil'ev: *Phys. Usp.* **39** (1996) 1193
- [11] L. Ts. Adzhemyan, N. V. Antonov, A. N. Vasiliev: *Field theoretic renormalization group in fully developed turbulence*, Gordon and Breach Publishing (1998)
- [12] L. Ts. Adzhemyan, Yu. S. Kabrits, M. V. Kompaniets, A. N. Vasil'ev: *Vestnik SPbU, Ser. 4, issue 1* (No 4), (2000) 3
- [13] L. Ts. Adzhemyan, N. V. Antonov, M. V. Kompaniets, A. N. Vasil'ev: *E-print LANL nlin.CD/0207007*, submitted to *Phys. Rev. E*
- [14] V. Yakhot, S. A. Orszag: *Phys. Rev. Lett.* **57** (1986) 1722
- [15] V. Yakhot, S. A. Orszag: *Journ. Sci. Comp.* **1** (1986) 3
- [16] W. P. Dannevik, V. Yakhot, S. A. Orszag: *Phys. Fluids* **30** (1987) 2021
- [17] L. Ts. Adzhemyan, N. V. Antonov, A. N. Vasil'ev: *Sov. Phys. JETP* **68** (1989) 733
- [18] Y. Zhou, G. Vahala, M. Hossain: *Phys. Rev. A* **40** (1989) 5865
- [19] D. Carati: *Phys. Rev. A* **41** (1990) 3129
- [20] D. Carati: *Phys. Fluids A* **2** (1990) 1854
- [21] S. H. Lam: *Phys. Fluids A* **4** (1992) 1007
- [22] M.J. Giles: *Phys. Fluids A* **6** (1994) 595
- [23] G. Falkovich, K. Gawędzki, M. Vergassola: *Rev. Mod. Phys.* **73** (2001) 913
- [24] L. Ts. Adzhemyan, N. V. Antonov, A. N. Vasil'ev: *Phys. Rev. E* **58** (1998) 1823
- [25] L. Ts. Adzhemyan, N. V. Antonov, A. N. Vasil'ev: *Acta Physica Slovaca* **52** (2002) 541
- [26] L. Ts. Adzhemyan, N. V. Antonov: *Theor. Math. Phys.* **115** (1998) 562