ADVECTION OF VECTOR ADMIXTURE BY TURBULENT FLOWS WITH STRONG ANISOTROPY[1]

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Using the field theoretic renormalization group and the operator product expansion the structure of the fluctuations of passively advected magnetic field in a given anisotropic stochastic environment is analyzed. Inertial-range anomalous scaling behaviour is studied, and explicit asymptotic expressions for structure functions are determined. The corresponding anomalous exponents are calculated in the first order in a small parameter of the model as functions of the anisotropy parameters. The negativeness of some exponents indicates a complex multifractal structure of the fluctuations of the passively advected magnetic field in such environment.

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1 Introduction

Instabilities of the hydromechanic motion of fluids leads to generation of complex stochastic processes in these. As a result, the macroscopic dynamics becomes complicated and can cause a nontrivial topological structure of hydrodynamic systems.

Quantum field theory method, including the renormalization group (RG) and the operator product expansion (OPE) approach, has been successfully used for theoretical explanation of various phenomena in stochastic flows (see [1] and refs. therein).

Over the last five years some progress has been achieved in the understanding of the physical origin of multifractality and anomalous scaling in stochastic dynamics [2, 3]. The central role in this was played by the model of the passive advection of a scalar quantity (e.g. temperature, or concentration of the marker) by random Gaussian field in the rapid-change model [4].

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Straightforward generalization of the model of the passive advection of a scalar field are the models of the passively advected vector fields (see e.g. [5–9] and references therein).

The aim of this work is to study of the spacial structure of fluctuations of the passively advected magnetic field in the framework of the kinematic magnetohydrodynamic so-called Kazantzev-Kraichnan model [10]. The fluctuations are generated stochastically by a given anisotropic white in time and self-similar in space Gaussian noise. The goal is the calculation of the anomalous exponents as the functions of the anisotropy parameters. It was found that (from the mathematical point of view) the model is in some aspects equivalent to the model of a passive scalar quantity advected by the Gaussian strongly anisotropic velocity field [11], meaning that the corresponding structure functions of both models exhibit the same anomalous behaviour.

2 Definition of the model

We consider the passive advection of the magnetic field $\mathbf{b} \equiv \mathbf{b}(\mathbf{x},t)$ which is described by the stochastic equation

$$\partial_t \mathbf{b} = \nu_0 \triangle \mathbf{b} - (\mathbf{v}\nabla)\mathbf{b} + (\mathbf{b}\nabla)\mathbf{v} + \mathbf{f}^{\mathbf{A}} + \mathbf{f}, \tag{1}$$

where ν_0 is a "diffusion" coefficient of unrenormalized theory and $\mathbf{v} \equiv \mathbf{v}(\mathbf{x},t)$ is a random solenoidal (owing to the incompressibility) velocity field. The term $\mathbf{f}^{\mathbf{A}} = \chi_0\nu_0(\mathbf{n}\nabla)^2\mathbf{b}$ is related to anisotropy and is needed to have multiplicatively renormalizable model [11]. χ_0 is a new parameter of the model and the unit vector $\mathbf{n}$ specifies the direction of the anisotropy axis. The transverse Gaussian stirring force $\mathbf{f} \equiv \mathbf{f}(\mathbf{x},t)$ with zero mean and a white noise in time pair correlator

$$D^f_{ij} \equiv \langle f_i(\mathbf{x},t) f_j(\mathbf{x}',t')\rangle = \delta(t-t')C_{ij}(\mathbf{r}/L), \quad \mathbf{r} = \mathbf{x} - \mathbf{x}' \tag{2}$$

is the source of the fluctuations of the passive magnetic admixture $\mathbf{b}$. The parameter L is the integral scale related to the stirring and C_{ij} are dimensionless functions finite in the limit $L \to \infty$. In our considerations its precise form is irrelevant.

The random velocity field $\mathbf{v}$ obeys the Gaussian statistics with zero mean and the noise

$$D^v_{ij}(\mathbf{x},t) \equiv \langle v_i(\mathbf{x},t) v_j(0,0)\rangle = \frac{D_0\delta(t)}{(2\pi)^d}\int d^d\mathbf{k} \frac{e^{i\mathbf{k}\mathbf{x}}\, T_{ij}(\mathbf{k})}{(k^2 + r_c^{-2})^{d/2+\epsilon/2}}, \tag{3}$$

where r_c is the correlation length, d is the dimensionality of the coordinate space, $D_0 > 0$ is an amplitude factor related to the coupling constant g_0 of the model by relation $D_0/\nu_0 \equiv g_0$ and $0 < \epsilon < 2$ is a free parameter. Its "physical" value $\epsilon = 4/3$ mimics the Kolmogorov statistics of the velocity field in developed turbulence. The tensor quantity $T_{ij}(\mathbf{k})$ characterizes the space vector structure of $\mathbf{v}$. In case of incompressible fluid, it is represented by a transverse structure with uniaxial anisotropy, where the distinguished direction of the anisotropy axis is defined by the unit vector $\mathbf{n}$ [11]

$$T_{ij}(\mathbf{k}) \equiv \left(1 + \alpha_1 \frac{(\mathbf{n}\mathbf{k})^2}{k^2}\right) P_{ij}(\mathbf{k}) + \alpha_2 n_s n_l P_{is}(\mathbf{k}) P_{jl}(\mathbf{k}); \quad P_{ij}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{k^2}. \tag{4}$$

The anisotropy parameters $\alpha_{1,2} > -1$, and $P_{ij}(\mathbf{k})$ being the ordinary transverse projection operator.

The main object of the study is the asymptotic behaviour of the equal-time structure functions $S_N(r)$, which represent the equal-time correlations of the N-th powers of differences of the projection of the field $\mathbf{b}$ onto the direction along two separate space coordinates $\mathbf{x}$ and $\mathbf{x}'$

$$S_N(r) \equiv \langle [b_r(\mathbf{x},t) - b_r(\mathbf{x}',t)]^N \rangle, \qquad b_r \equiv \mathbf{b}\mathbf{r}/r, \qquad r \equiv |\mathbf{x}-\mathbf{x}'|. \tag{5}$$

In the quantum field theory language, they are defined as

$$S_N(r) \equiv \int D\mathbf{b}' D\mathbf{b} D\mathbf{v} [b_r(\mathbf{x},t) - b_r(\mathbf{x}',t)]^N e^{S(\mathbf{b},\mathbf{b}',\mathbf{v})} \tag{6}$$

with the Janssen-Dominicis action functional [12, 13] for the set of three fields $\mathbf{b}', \mathbf{b}, \mathbf{v}$

$$\begin{aligned} S(\mathbf{b},\mathbf{b}',\mathbf{v}) \quad \equiv \quad & \mathbf{b}' D^f \mathbf{b}'/2 + \mathbf{b}'[-\partial_t - (\mathbf{v}\nabla) + \nu_0\Delta + \chi_0\nu_0(\mathbf{n}\nabla)^2]\mathbf{b} + \\ & \mathbf{b}'(\mathbf{b}\nabla)\mathbf{v} - \mathbf{v}(D^v)^{-1}\mathbf{v}/2, \end{aligned} \tag{7}$$

where $\mathbf{b}'$ is an auxiliary field. All necessary integrations over space-time coordinates and summations over the vector indices are implied.

3 Analysis of statistical properties of the passively advected magnetic field

Renormalization group analysis of S_N in the non-dissipative range of scales $r \gg r_d$, where r_d is the Kolmogorov dissipative (inner) length leads to the equal-time structure functions in the form:

$$S_N(r) = D_0^{-N/2} r^{N(1-\epsilon/2)} R_N(r/r_c) \tag{8}$$

with some still unknown scaling functions $R_N(r/r_c)$ (for details see [11]).

Physically interesting range of scales is the so-called inertial range, specified by inequalities $r_d \ll r \ll r_c$, where the behaviour of the functions $R_N(r/r_c)$ can be studied by the operator product expansion (OPE) technique [1, 14].

In the limit $r/r_c \to 0$ these scaling functions take the following asymptotic forms

$$R_N(r/r_c) = \sum_F A_F\,(r/r_c)^{\Delta_F}, \tag{9}$$

where summation over all possible renormalized composite operators F is implied (see below), Δ_F are their critical dimensions and A_F are the Wilson coefficients regular in r/r_c.

The specific feature of the models describing turbulence is the existence of the so-called "dangerous" composite operators with negative critical dimensions (see [1, 15] and references therein). Their contribution into the OPE leads to singular behaviour of the scaling functions for $r/r_c \to 0$. The leading singular contribution is given by the operator with minimal Δ_F. As a result, S_N have singular powerlike behaviour as $r/r_c \to 0$:

$$S_N(r) = D_0^{-N/2} r^{N(1-\epsilon/2)} \sum_F A_F\,(r/r_c)^{\Delta_F} \propto r^{N(1-\epsilon/2)} (r/r_c)^{\Delta_N}, \tag{10}$$

with the most singular exponent Δ_N.

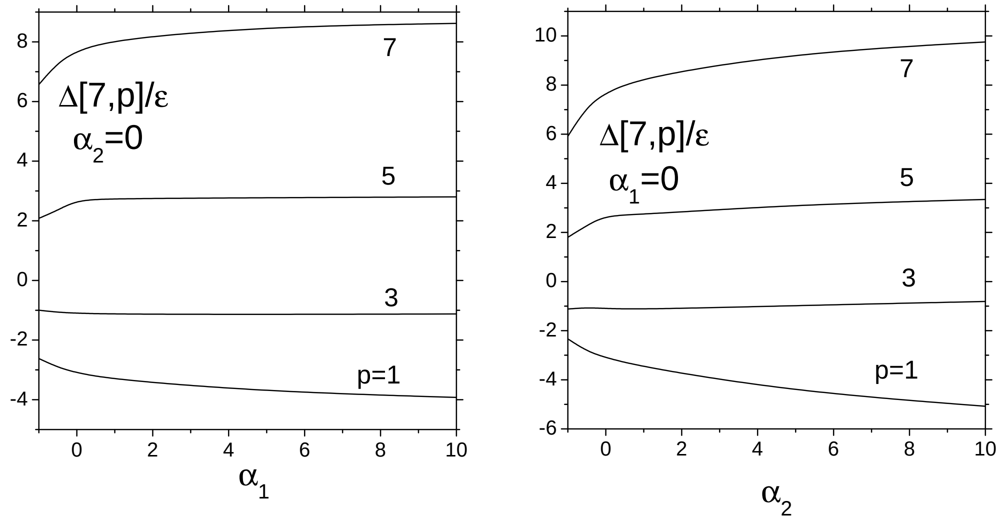

Fig. 1. Behaviour of the critical dimensions $\Delta[7,p]/\epsilon$ for $d=3$ as functions of the anisotropy parameters α_1 and α_2.

4 Critical dimensions of composite operators

A composite operator is any monomial or polynomial constructed of primary fields and their derivatives at the same space-time point $x \equiv (\mathbf{x}, t)$. In our case, anomalous exponents Δ_F are critical dimensions of the scalar composite operators constructed solely of the fields $\mathbf{b}$ without derivatives:

$$F[N,p](x) \equiv [\mathbf{nb}(x)]^p [b_i(x) b_i(x)]^l, \tag{11}$$

with $N \equiv 2l + p$, giving the leading singular contributions to the sum (9). The operators (11) mix only with each other in renormalization

$$F[N,p] = \sum_{N',p'} Z_{[N,p][N',p']} F^R[N',p'] \tag{12}$$

and the corresponding infinite renormalization matrix $Z_{[N,p][N',p']}$ takes the block-triangular form, i.e., $Z_{[N,p][N',p']} = 0$ for $N' > N$.

The elimination of ultraviolet (small scale) singularities of correlation functions containing such operators leads to the non-trivial values of the matrix of critical dimensions $\Delta[N,p]$ expressed via matrix $Z_{[N,p][N',p']}$. The critical dimensions Δ_F are the eigenvalues of the matrix $\Delta[N,p]$.

In isotropic case ($p=0$), only even correlation functions S_{2N} (5) persist and the contribution to them is determined by the critical dimensions $\Delta_N \equiv \Delta[N,0]$, which have been calculated in

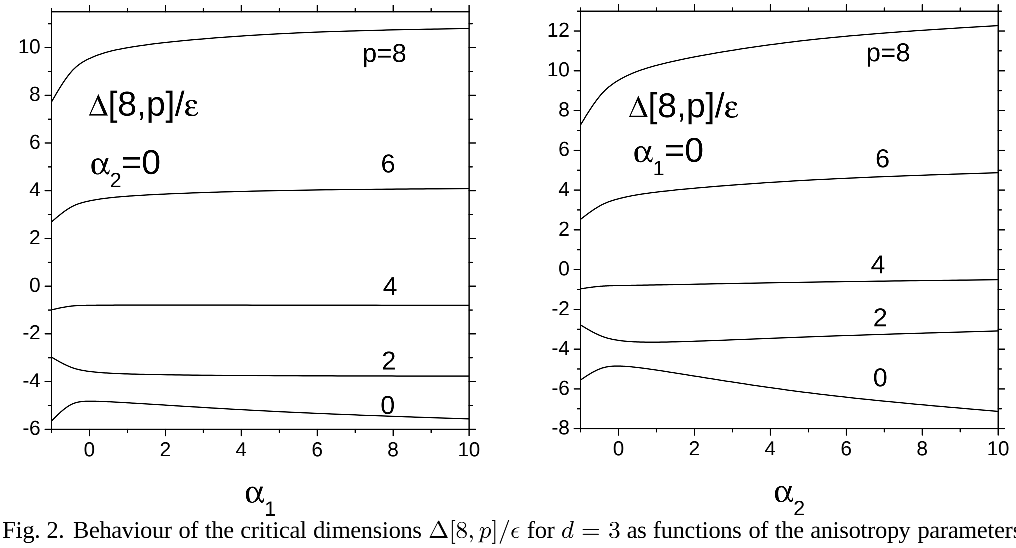

Fig. 2. Behaviour of the critical dimensions $\Delta[8,p]/\epsilon$ for $d=3$ as functions of the anisotropy parameters α_1 and α_2.

the leading order of perturbative series in small ϵ

$$\Delta_N = -\frac{2N(N-1)\epsilon}{d+2}, \qquad (13)$$

and coincide with those obtained for the advection of passive scalar quantity [3, 11].

In anisotropic case an analytical expression has been obtained for the exponents Δ_N to the 1st order in ϵ for $N=2$. For $N>2$ the exponents can be found analytically only within an expansion in small parameters characterizing the intensity of anisotropy. Such expansions have been obtained up to the next to leading order of anisotropy parameters for all exponents with $N \leq 4$. The exponents beyond this expansion have been obtained numerically. The main conclusion is that the exponents Δ_N remain negative in anisotropic case and decrease monotonically as N increases for both odd and even values of N.

Although, at a first sight, the vector model under consideration is more complicated than model of the passively advected scalar quantity, nevertheless, the considering structure functions in both models have the same asymptotic behaviour in the inertial range. Thus, we do not present explicit expressions for the anomalous dimensions here (they can be found in Ref. [11]). Instead we present numerical analysis of the anomalous dimensions for the higher-order structure functions S_N with $N=7,8$. In Fig. 1 and Fig. 2, the eigenvalues $\Delta[N,p]$ for $N=7,8$ of the matrix of anomalous dimensions are presented as functions of the anisotropy parameters α_1, α_2. One can see that the exponents exhibit a hierarchy related to the degree of anisotropy.

5 Conclusion

The field theoretic renormalization group and the operator product expansion have been applied to analyze the kinematic MHD Kazantzev-Kraichnan model with small-scale anisotropy. The anomalous scaling behaviour of the structure functions has been found and the anomalous exponents have been calculated. The negative values obtained for these exponents imply a singular asymptotic behaviour of structure functions S_N for $r/r_c \to 0$. As it is known from various phenomenological multifractal models of stochastic systems [16], this is a signal for complex multifractal structure of passively advected magnetic field fluctuations in turbulent incompressible fluids.

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