INFLUENCE OF ANISOTROPY ON THE SCALING REGIMES IN FULLY DEVELOPED TURBULENCE[1]

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Fully developed turbulence with anisotropy is investigated by means of the renormalization group approach and double expansion regularization for dimensions $d \geq 2$. Modification of the standard minimal subtraction scheme has been used to analyze the restoration of the stability of the Kolmogorov scaling regime under a transition from $d = 2$ to 3. The results are in qualitative agreement with results obtained in the framework of a typical analytical regularization scheme.

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1 Introduction

A traditional approach to the description of fully developed turbulence is based on the stochastic Navier-Stokes equation [1]. An exact solution of the Navier-Stokes equation does not exist, and so one is forced to find some convenient methods to treat the problem at least step by step.

A suitable and also powerful tool in the theory of developed turbulence is the so-called quantum-field renormalization group (RG) method [2, 3]. In early papers, the RG approach was applied only to isotropic models of developed turbulence. However, the method can also be used in the theory of anisotropically developed turbulence. A crucial question immediately arises here: whether the scaling regime remains stable under transition from the isotropic to the

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anisotropic case. In other words, do the stable fixed points of the RG equations remain stable under the influence of anisotropy?

In Refs. [4, 5], d-dimensional models with $d > 2$ were investigated for two cases, weak anisotropy [4] and strong anisotropy [5], and it was shown that the stability of the isotropic fixed point is lost for dimensions $d < d_c = 2.68$. It was also shown that the stability of the fixed point, even for dimension $d = 3$, takes place only for sufficiently weak anisotropy. The only problem in these investigations is that it is impossible to use them in the case $d = 2$, because new ultraviolet (UV) divergences appear in the Green functions, when one considers $d = 2$. Correct treatment of the two-dimensional isotropic turbulence was given in Ref. [6]. The correctness in the renormalization procedure was reached by introducing a new local term into the model, which allows one to remove additional UV divergences in accordance with the basic principles of renormalization [7, 8].

In Ref. [9] the double-expansion procedure introduced in Ref. [6] and the minimal subtraction (MS) scheme for an investigation of developed turbulence with weak anisotropy for $d = 2$ were used. The main result of the paper was the conclusion that the two-dimensional fixed point is not stable under weak anisotropy. They also tried to restore the stability of the fixed point using analytical continuation from $d = 2$ to the three-dimensional turbulence. From analysis made in Ref. [9], it follows that it is impossible to restore the stable regime by transition from dimension $d = 2$ to 3 in the framework of the standard MS scheme.

In Ref. [10,11] a modified MS scheme was introduced and applied in which the d-dependence of the UV divergences of graphs were kept. It was shown that the using of such modification in the procedure of renormalization allows to restore the stability of the fixed point of fully developed turbulence with weak anisotropy by transition from $d = 2$ to 3.

In the present paper we follow the idea of Ref. [10] but we allow the parameters of the anisotropy to have arbitrary possible values (strong anisotropy). Our aim is to analyze the influence of the strong anisotropy on the stability of the scaling regime in the case with $d \geq 2$ and also to study the restoration of the stability by transition from $d = 2$ to 3 and to find the dependence of the borderline dimension d_c on the parameters of anisotropy. We also compare our results with that obtained in the framework of a typical analytical regularization scheme (so called ϵ-expansion) [5].

2 Description of the model

We work with fully developed turbulence, and assume a strong anisotropy of the system. In the statistical theory of anisotropically developed turbulence, the turbulent flow can be described by a random velocity field $\mathbf{v}(\mathbf{x}, t)$, and its evolution is given by the randomly forced Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} - \nu_0 \Delta \mathbf{v} - \mathbf{f}^{\mathbf{A}} = \mathbf{f}, \tag{1}$$

where we assume incompressibility of the fluid, which is given mathematically by the well-known conditions $(\nabla \mathbf{v}) = 0$ and $(\nabla \mathbf{f}) = 0$. In eq. (1) the parameter ν_0 (subscript 0 denotes bare parameters) is the kinematic viscosity, the term $\mathbf{f}^{\mathbf{A}}$ is related to anisotropy, and its form is dictated by the condition to have a multiplicatively renormalizable model. In our case it has the following form

$$\mathbf{f^A} = \nu_0 \left[\chi_{10}(\mathbf{n}\nabla)^2\mathbf{v} + \chi_{20}\mathbf{n}\nabla^2(\mathbf{nv}) + \chi_{30}\mathbf{n}(\mathbf{n}\nabla)^2(\mathbf{nv})\right] \,. \tag{2}$$

Bare parameters χ_{10}, χ_{20} and χ_{30} characterize the weight of the individual structures in eq. (2). The large-scale random force per unit mass $\mathbf{f}$ is assumed to have Gaussian statistics defined by the averages

$$\langle f_i\rangle = 0,\ \langle f_i(\mathbf{x_1},t)f_j(\mathbf{x_2},t)\rangle = D_{ij}(\mathbf{x_1}-\mathbf{x_2},t_1-t_2). \tag{3}$$

In our case, the two-point correlation matrix

$$D_{ij}(\mathbf{x},t) = \delta(t)\int \frac{d^d\mathbf{k}}{(2\pi)^d}\tilde{D}_{ij}(\mathbf{k})\exp(i\mathbf{kx}) \tag{4}$$

is convenient to parameterize as [10]:

$$\begin{aligned}\tilde{D}_{ij}(\mathbf{k}) &= g_{10}\nu_0^3k^{2-2\delta-2\epsilon}[(1+\alpha_1\xi_k^2)P_{ij}(\mathbf{k}) + \alpha_2R_{ij}(\mathbf{k})] \\ &+ g_{20}\nu_0^3k^2[(1+\alpha_{30}\xi_k^2)P_{ij}(\mathbf{k}) + (\alpha_{40}+\alpha_{50}\xi_k^2)R_{ij}(\mathbf{k})]\,,\end{aligned} \tag{5}$$

where a vector $\mathbf{k}$ is the wave vector, d is the dimension of the space (in our case: $2 \le d$), $\epsilon \ge 0$ and $\delta = (d-2)/2$ are dimensionless parameters of the model (the physical value of ϵ is $\epsilon = 2$). The values $\epsilon = 0$ and $\delta = 0$ correspond to a logarithmic perturbation theory for calculation of Green functions when g_{10} and g_{20}, which play the role of bare coupling constants of the model, become dimensionless. The second line in eq. (5) is needed if one want to include into consideration the two-dimensional case (see below). In the case $d > 2$ it is enough to work with the first line in eq. (5). The problem of continuation from $\epsilon = 0$ to physical values has been discussed in [12]. The $(d \times d)$-matrices P_{ij} and R_{ij} are the transverse projection operators and in the wave-number space are defined by the relations

$$P_{ij}(\mathbf{k}) = \delta_{ij} - \frac{k_ik_j}{k^2}\,,\ \ R_{ij}(\mathbf{k}) = \left(n_i - \xi_k\frac{k_i}{k}\right)\left(n_j - \xi_k\frac{k_j}{k}\right)\,, \tag{6}$$

where ξ_k is given by the equation $\xi_k = (\mathbf{kn})/k$. In eq. (6), the unit vector $\mathbf{n}$ specifies the direction of the anisotropy axis. The tensor $\tilde{D}_{ij}$, given by eq. (5), is the most general form with respect to the condition of incompressibility of the system under consideration, and contains two dimensionless free parameters α_1 and α_2. From the positiveness of the correlation tensor D_{ij} one immediately gets restrictions on the above parameters, namely $\alpha_1 \ge -1$ and $\alpha_2 \ge -1$. Coupling constant g_{20} and the parameters α_{30}, α_{40} and α_{50} are consequence of $d = 2$ of the model (see [10]).

3 UV-divergences, RG analysis and stability of the fixed point

Using the well-known Martin–Siggia-Rose formalism [13], one can transform the stochastic problem (1) with the correlator (4) into the field-theoretic model of fields $\mathbf{v}$ and $\mathbf{v}'$. Here $\mathbf{v}'$ is independent of the $\mathbf{v}$ auxiliary incompressible field, which we have to introduce when transforming the stochastic problem into a functional form.

The action of the fields $\mathbf{v}$ and $\mathbf{v}'$ is given in the form

$$
\begin{aligned}
S &= \frac{1}{2}\int d^d\mathbf{x_1}dt_1 d^d\mathbf{x_2}dt_2\left[v_i'(\mathbf{x_1},t_1)D_{ij}(\mathbf{x_1}-\mathbf{x_2},t_1-t_2)v_j'(\mathbf{x_2},t_2)\right] \\
&+ \int d^d\mathbf{x}dt\left\{\mathbf{v}'(\mathbf{x},t)\left[-\partial_t\mathbf{v}-(\mathbf{v}\nabla)\mathbf{v}+\nu_0\nabla^2\mathbf{v}+\mathbf{f^A}\right](\mathbf{x},t)\right\}\ . \qquad (7)
\end{aligned}
$$

The functional formulation gives the possibility to use the field-theoretic methods, including the RG technique to solve the problem. By means of the RG approach it is possible to extract large-scale asymptotic behavior of the correlation functions after an appropriate renormalization procedure which is needed to remove UV-divergences.

In our case ($d = 2$), the UV divergences are present in the 1-irreducible Green functions $\langle\mathbf{v}'\mathbf{v}\rangle$ and $\langle\mathbf{v}'\mathbf{v}'\rangle$. The last one is finite when $d > 2$, and to remove the divergences correctly in the specific case $d = 2$ it is necessary to introduce and work with additional terms (the second line in eq. (5)), which are local contrary to the standard terms (first line in eq. (5)) [6, 10]. Thus, our model is multiplicatively renormalizable. In [6,14] a double-expansion method with a simultaneous deviation $2\delta = d-2$ from the spatial dimension $d = 2$ and also a deviation ϵ from the k^2 form of the forcing pair correlation function proportional to $k^{2-2\delta-2\epsilon}$ was proposed. In the present paper we follow the formulation founded on the two-expansion parameters (details see in [10]).

Using the standard field-theoretic analysis with standard renormalization procedure and modified MS-scheme we are left with eight-charge model [10] and in the one loop approximation we have the following system of β-functions

$$
\begin{aligned}
\beta_{g_1} &= g_1(-2\epsilon+3A(g_1d_1+g_2e_1))\,, \\
\beta_{g_2} &= g_2\left[2\delta+3A(g_1d_1+g_2e_1)+\frac{A}{2}\left(\frac{g_1^2}{g_2}a_1+g_1b_1+g_2c_1\right)\right]\,, \\
\beta_{\chi_i} &= -A\left[(g_1d_{i+1}+g_2e_{i+1})-\chi_i(g_1d_1+g_2e_1)\right]\,, \\
\beta_{\alpha_{i+2}} &= -\frac{A}{2}\left[-\left(\frac{g_1^2}{g_2}a_{i+1}+g_1b_{i+1}+g_2c_{i+1}\right)+\alpha_{i+2}\left(\frac{g_1^2}{g_2}a_1+g_1b_1+g_2c_1\right)\right]\,, \\
i &= 1,2,3\,, \qquad (8)
\end{aligned}
$$

where $A = S_{d-1}/(2\pi)^{d-1}(d^2-1)$, $S_{d-1} = 2\pi^{(d-1)/2}/\Gamma((d-1)/2)$ ($d-1$ dimensional sphere), and functions $Y_i = a_i, b_i, c_i, d_i, e_i$ are nontrivial integrals of the following form

$$
Y_i = \int_{-1}^{1} dx\frac{P_{Y_i}(\alpha,\chi,d,x^2)}{x_1^{j_1}x_2^{j_2}x_3^{j_3}}(1-x^2)^{\frac{d-3}{2}}\,, \qquad (9)
$$

where $x_1 = 1+\chi_1x^2$, $x_2 = 1+\chi_1x^2+(\chi_2+\chi_3x^2)(1-x^2)$, $x_3 = x_1+x_2$, $j_1, j_2, j_3 = 2, 3$, and P_{Y_i} are huge polynomial functions of $\alpha = \alpha_i$, $i = 1, ..., 5$, $\chi = \chi_j$, $j = 1, ..., 3$, d and x^2. The scale dependent effective variables are governed by the set of differential equations ($s = k/\Lambda$ is rescaled wave number)

$$
s\frac{d\bar{C}}{ds} = \beta_C(\bar{C};\alpha_{1,2},d), \quad s\in[0,1] \qquad (10)
$$

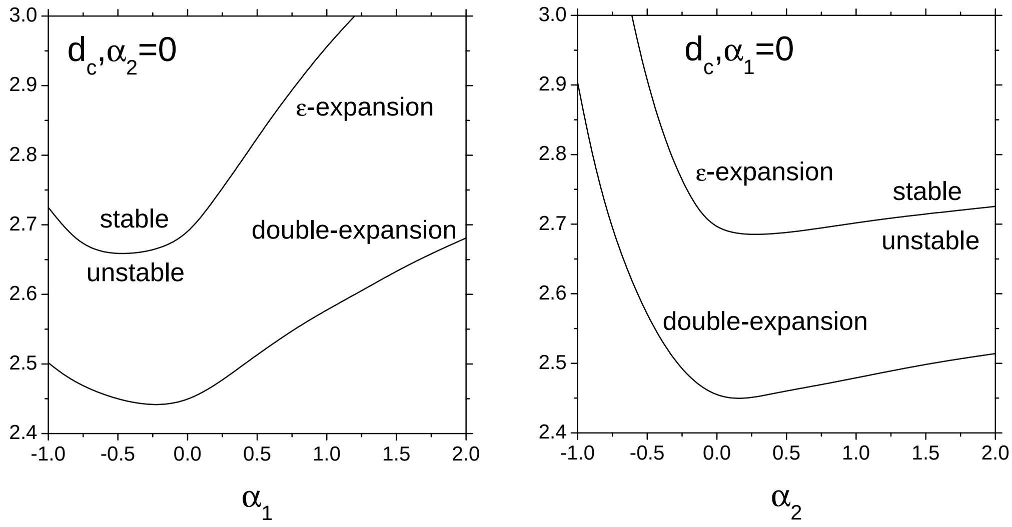

Fig. 1. Dependence of the borderline dimension d_c on the anisotropy parameters $\alpha_{1,2}$ for physical value $\epsilon = 2$ and for both double-expansion and ϵ-expansion technique.

with the initial conditions $\bar{C}|_{s=1} = C$, where we denote $C = \{g_1, g_2, \chi_i, \alpha_{i+2}\}, i = 1, 2, 3$. The large scale limit is described by the fixed point of the RG equations $\bar{C}|_{s=0} = C^*$.

We perform a numerical analysis of this system of differential equations, and our aim is to find and analyze the so-called borderline dimension d_c between stable and unstable regimes as a function of the anisotropy parameters $\alpha_{1,2}$. In Fig. 1 one can see the dependence of d_c on anisotropy parameters for $\epsilon = 2$. We also compare our results (double-expansion technique, where the analytical continuation is taken from $d = 2$ to $d = 3$) with that obtained by using standard ϵ-expansion method (the analytical continuation is performed from $d = 3$ to 2). One can see that results obtained by these two different approaches are qualitatively the same.

4 Conclusion

We have investigated the influence of the strong anisotropy on the fully developed turbulence using the quantum field RG double-expansion method, and a modified minimal subtraction scheme in which the space dimension dependence of the divergent parts of the Feynman diagrams is kept. Such modification of the minimal subtraction scheme is needed when one wants to compute the d dependence of important quantities, and is necessary for restoration of the stability of scaling regimes when one makes transition from dimension $d = 2$ to $d = 3$. We have calculated the dependence of d_c on the anisotropy parameters $\alpha_{1,2}$ for physical value of $\epsilon = 2$. Below this dimension the scaling regime is unstable. In the limit case of infinitesimally small anisotropy ($\alpha_1 \to 0$ and $\alpha_2 \to 0$) and in the so-called energy pumping (physical) regime $\epsilon = 2$, we have found the borderline dimension $d_c = 2.44$ (see also [10]). We have compared our results with the ones obtained by standard ϵ-expansion scheme, where analytical continuation is taken from $d = 3$ to 2 [5], and where the borderline dimension $d_c = 2.68$ was found (in the same infinitesimal case $\alpha_{1,2} \to 0$).

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