SELF-ORGANIZED CRITICALITY IN SIMPLE MODEL OF EVOLUTION: EXACT DESCRIPTION OF SCALING LAWS[1]

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The the simplest version of the Bak-Sneppen model of self-organized biological evolution with random interaction structure is considered. It's dynamics is described in the framework of master equation. The master equations can be solved exactly both for infinite system and for finite one. The equation for pair correlation function are solved exactly for infinite system. The dynamical regime of self-organized criticality in this model appears to be similar to one of completely integrable systems. Analysis of main characteristics of dynamics take it possible to revive the most essential feature of dynamics.

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1 Introduction

Recently, the dynamics of self-organized criticality (SOC) became a popular object of investigation in the theoretical physics since one believes that the SOC is a manifestation of the most universal self-organization mechanisms in the nature. The most essential feature of the SOC is that the system evolves to a critical state without fine tuning of its parameters. In 1993 P.Bak and K.Sneppen proposed a simple model of biological evolution [1]. It is a dynamical system describing the ecosystem evolution as mutation and natural selection of interacting species. The SOC dynamic in the Bak-Sneppen model (BSM) is in a good agreement with the the specific character of real biological evolution considered in the framework of the Gould-Eldridge "punctuated equilibrium" conception [2]. One can hope that this model represent an important universal type of critical dynamics of avalanche-like processes [3].

The formulation of dynamical rules in the BSM is the following [1]. The state of the ecosystem of N species is characterized by a set $\{x_1, ..., x_N\}$ of N number, $0 \leq x_i \leq 1$. The number x_i represents the barrier of the i-th species toward further evolution. Initially, each x_i is set to a randomly chosen value. At each time step the barrier x_i with minimal value and all the barriers of the neighbors of this "weakest" species are replaced by new random numbers. In the

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random neighbor model (RNM) [4] the $K-1$ replaced non-minimal barriers are chosen at random at each time step. In the local or nearest neighbor model (LM) these are the barriers of the nearest neighbors to the species with minimal barrier. For each species in the LM the nearest neighbors are assumed to be defined. For the LM the most of results are obtained by numericalq experiments or in the framework of mean field approximation. The RNM is more convenient for analytical studies. The master equations obtained in [5] for RNM are very useful for this aim. These equations appeared to be exact solvable. The stationary solution was found in [5]. The time dependent solutions were obtained in [6, 8] (infinite system), [7] (finite system). In [9, 10] it is shown that dynamics of RNN is similar to one of completely integrable system. We present the most important results contained in [7–10] and show the possible way to reach more complete understanding of SOC phenomena in the RNM by means of investigation of correlation functions.

2 Exact solutions of master equations

For description of the BSM behavior one can choose as a characteristic of dynamics the probability $P_n(t)$ that at time point t the barriers of n species are less then λ, where λ is a fixed parameter from the interval [0,1]. For $P_n(t)$in the RNM one can derive the exact master equation [5]. In terms of the generating function

$$q(z,u) \equiv \sum_{t=0}^{\infty} \sum_{n=0}^{N} P_n(t;\lambda) z^n u^t \tag{1}$$

it can be written for the infinite system as follows [8]:

$$q(z,u)[z - u(1+\lambda(z-1))^K] = zq(z,0) + u(z-1)(1+\lambda(z-1))^K q(0,u). \tag{2}$$

Here K is the number of interacting species. For the finite ecosystem with N species and $K=2$ the master equations is presented in the form [7]

$$\frac{1}{u}\left[q(z,u) - q(z,0)\right] = (1-\lambda+\lambda z)^2 \times \\ \times \left\{ \frac{1}{z}\left[1 - \frac{1-z}{N-1}\left(\frac{1}{z} - \frac{\partial}{\partial z}\right)\right]\left[q(z,u) - q(0,u)\right] + q(0,u) \right\}. \tag{3}$$

The function $q(z,0)$ in (2), (3) is defined by initial probability distribution $P_n(0)$ and is assumed to be given.

The equations (2), (3) can be solved exactly [7, 8]. One could easily do it, if the function $q(0,u)$ would be known. Therefore the problem is reduced to finding $q(0,u)$. For infinite system described by master equation (2) it can be done in the following way. In virtue of the usual properties of probabilities, we have:

$$P_n(t) \geq 0, \;\; \sum_{n=0}^{\infty} P_n(t) = 1. \tag{4}$$

Hence, $q(z,t)$ is analytical in z, u for $|z| < 1$ and $|u| < 1$. Let us denote $\alpha(u)$ the analytical in the neighborhood $u = 0$ solution of the algebraic equation for z

$$z - u(1 + \lambda(z-1))^K = 0.$$

It has the form

$$\alpha(u) = u(1-\lambda)^K + Ku^2\lambda(1-\lambda)^{K-1} + \dots . \tag{5}$$

The series for $\alpha(u)$ is convergent and $|\alpha(u)| < 1$, if $|u| < u_0$, and the parameter u_0 is chosen small enough. Assuming $|u| < u_0 < 1$ and substituting in (2) $z = \alpha(u)$ we see that the left hand side vanish because $q(\alpha(u), u)$ is finite for $|u| < u_0$. Thus, the function $q(0,u)$ is expressed in terms of $q(z,0)$ as follows

$$q(0,u) = \frac{\alpha(u)q(\alpha(u),0)}{u(1-\alpha(u))(1+\lambda(\alpha(u)-1))^K} = \frac{q(\alpha(u),0)}{(1-\alpha(u))}. \tag{6}$$

Now, from equations (2) and (6) we obtain the solution for $q(z,u)$

$$q(z,u) = \frac{z(1-\alpha(u))q(z,0) + u(z-1)(1+\lambda(z-1))^K q(\alpha(u),u)}{(1-\alpha(u))[z - u(1+\lambda(z-1))^K]}. \tag{7}$$

The problem to construct the solution $q(u,z)$ of equation (3) appears to be more difficult. Nevertheless, the direct analysis of general solution of differential equation (3) enables one, using the analytical properties of $q(z,u)$, to find the evident expression for the function $q(0,u)$ in terms of $q(z,0)$ [7]:

$$q(0,u) = \frac{\int_{\frac{\lambda-1}{\lambda}}^{1} \exp\left(-R(x,u)\frac{q(x,0)dx}{(1-\lambda+\lambda x)^2(1-x)}\right)}{\int_{\frac{\lambda-1}{\lambda}}^{1} \exp\left(-R(x,u)\frac{(1-u(1-\lambda+\lambda x)^2)dx}{(1-\lambda+\lambda x)^2(1-x)}\right)}. \tag{8}$$

Here,

$$R(z,u) = \frac{N-1}{u}\left[\ln(1-\lambda+\lambda z) - (1-u)\ln(1-z) + \frac{1-\lambda}{\lambda(1-\lambda+\lambda z)}\right].$$

Substituting (8) into (3), we obtain the usual differential equation of the first order for the function $q(z,u)$, satisfying, in virtue of (1), (4), the initial condition $q(1,u) = (1-u)^{-1}$. Its solution is of the form:

$$q(z,u) = z\frac{N-1}{u}e^{R(z,u)}\int_z^{\frac{\lambda-1}{\lambda}} e^{-R(x,u)}\frac{q(x,0)dx}{(1-\lambda+\lambda x)^2(1-x)} +$$

$$+ q(0,u)\left[1 - z\frac{N-1}{u}e^{R(z,u)}\int_z^{\frac{\lambda-1}{\lambda}} e^{-R(x,u)}\frac{(1-u(1-\lambda+\lambda x)^2)dx}{(1-\lambda+\lambda x)^2(1-x)}\right],$$

where $q(0,u)$ is defined by (8).

3 Asymptotical dynamics

Simple characteristic of the RNM dynamics is the first moment $M(t)$ of probability distribution $P_n(t)$:

$$M(t) = \sum_{n=0}^{\infty} P_n(t)n.$$

It has the meaning of the average value of the species number having the barriers less then λ at time point t. For the infinite system the asymptotic of $M(t)$ for large t appears to be of three different kind [8]:

$$M(t) = \left(1 + \frac{\lambda(K-1)}{2(1-K\lambda)}\right) + o(t^{-3/2}) \text{ for } \lambda < \lambda_{cr},$$
$$M(t) = M(0) + t(K\lambda - 1) + o(t^{-3/2}) \text{ for } \lambda > \lambda_{cr},$$
$$M(t) = M(0) + \sqrt{\frac{2(K-1)t}{K\pi}} + O(t^{-1/2}) \text{ for } \lambda = \lambda_{cr},$$

where $\lambda_{cr} \equiv 1/K$.

The dissipation rate in the system can be characterized by the following quantity:

$$\delta S(t) = \frac{\delta M(t)}{M(t)} = \frac{\delta \ln M(t)}{\delta P(0)} \delta P(0).$$

It shows the dependence of $M(t)$ from initial distribution $P_n(0)$ and could be called the "memory" of the system. For $\lambda > \lambda_{cr}$, $\delta S(t)$ is decreasing as t^{-1} and the system quickly "forgets" the initial distribution. For $\lambda = \lambda_{cr}$, $\delta S(t)$ goes down as $t^{-1/2}$, and for $\lambda < \lambda_{cr}$ $\delta S(t)$ is decreasing exponentially. The slowest rate of "forgetting" the initial distribution of barriers occurs at critical value of λ.

4 Simplest description of dynamics

The variables $P_n(t)$ have simple meaning, but the description of dynamics of infinite system by master equation (2) is rather complicated. One can choose variables to obtain the simplest form of the master equation. Let us consider the function $d(y,u)$:

$$d(y,u) = \frac{q(\alpha(y),u)}{1-\alpha(y)}, \tag{9}$$

where $\alpha(y)$ was defined in (5). It is analytical in z and u in the neighborhood of $y = 0$ and $u = 0$:

$$d(y,u) = \sum_{n,t=0}^{\infty} C_n(t) y^n u^t. \tag{10}$$

Let us denote $\beta(z)$ the inverse in respect to $\alpha(y)$ function: $\alpha(\beta(z)) = z, \; \beta(\alpha(y)) = y$. Obviously,

$$\beta(z) = \frac{z}{(1-\lambda+\lambda z)^K}. \tag{11}$$

Substituting in (9) $y = \beta(z)$ we obtain

$$q(z,u) = (1-z)d(\beta(z),u). \tag{12}$$

Presenting in (2) the function $q(z,t)$ in the form (12) and setting $z = \alpha(y)$, we obtain the following equation

$$(y-u)d(y,u) = yd(y,0) - ud(0,u). \tag{13}$$

It is the master equation for generating function of variables $C_n(t)$. It looks in terms of the functions $C_n(t)$ as

$$C_n(t+1) = C_{n+1}(t) \ \text{ for } \ n \geq 0, \ t \geq 0. \tag{14}$$

The initial conditions for this equation

$$C_n(0) = c_n \tag{15}$$

are defined by $q(z,0)$:

$$d(y,0) = \sum_{n=0}^{\infty} c_n y^n = \frac{q(\alpha(y),0)}{1-\alpha(y)}. \tag{16}$$

The equations (14) has the simple solution:

$$C_n(t) = c_{n+t}. \tag{17}$$

Transforming the variables from $P_n(t)$ to $C_n(t)$ we have obtained the simplest of possible descriptions of the SOC dynamics in the RNM: the master equations (13), (14) with initial conditions (15), (16). Setting in (13) $u = y$ we see that $d(y,0) = d(0,y)$ and

$$d(y,u) = \frac{yd(y,0) - ud(u,0)}{y-u}. \tag{18}$$

Substituting $y = \beta(z)$ in (18) and using (9), (11), (12), we obtain the solution (7) of the master equation (2).

We have obtained the description of the system evolution in terms of variables $C_n(t)$. It is essentially different from original one defined by the master equation (2). The equation (14) for $C_n(t)$ does not contain the parameter λ controlling the dynamics of $P_n(t)$ according to (2). The restriction on the possible values of initial conditions $P_n(0)$ are the usual general restrictions for probabilities (4) being obviously independent on λ.

The restriction on the possible initial values of the variables $C_n(t)$ are defined by equation (16). Since the function $\alpha(u)$ depends on λ, the initial dates c_n appear to be dependent on it. Thus, in original formulation λ defines the dynamics, and in the terms of variables $C_n(t)$ it specifies the the initial dates.

The main characteristics of the avalanche-like processes in the SOC dynamics of RNM are defined by asymptotic of $P_n(t)$ for large t. Since $\alpha(0) = 0$, it follows from (9) that $q(0,u) = d(0,u)$, i.e $P_0(t) = C_0(t)$. In virtue of (17), $C_0(t) = c_t$. Hence, in terms of $C_n(t)$ the SOC phenomena manifest itself in the the asympotical behavior of initial dates c_n for large n. To make the picture of the dynamics in terms of $C_n(t)$ more clear let us call the state of the system the ordered one, if all barriers of the species are more then λ. Than $P_0(t) = C_0(t) = c_t$ is the probability for the system to be ordered at the time point t and $1 - c_t$ is the probability of disturbance of the order at this moment. The last situation could be called the "disordering catastrophe" or (in accordance with usual terminology of the SOC theory) the avalanche. Thus, the set of initial dates c_n gives a rough description of the system evolution in terms of "oder-avalanche". The asymptotic c_n for $n \to \infty$ forecasts the specific of the avalanche-like processes, which will be happened in the system in the future. It is of the form: [10]. For $\lambda = \lambda_{cr}$

$$c_n = \sqrt{\frac{K-1}{2\pi K n}} \left\{ 1 - \frac{K}{2n(K-1)} \left[M_2 + \frac{K^2 - K + 1}{6K^2} \right] + O\left(\frac{1}{n^2}\right) \right\},$$

and for $\lambda \neq \lambda_{cr}$

$$c_n = \theta(\lambda_{cr} - \lambda)(1 - K\lambda) + \frac{z_0 \sqrt{K} e^{-n\gamma(z_0)}}{\sqrt{2\pi(K-1)n^3}} \{\phi_0 -$$

$$- \frac{z_0^2 K}{2n(K-1)} \left[\phi_2 + \frac{2(K+1)}{z_0 K} \phi_1 + \frac{K^2 + 11K + 1}{6 z_0^2 K^2} \phi_0 \right] + O\left(\frac{1}{n^2}\right) \}$$

Here the following notations are used: $\gamma(z) \equiv \ln(\beta(z))$, $z_0 = (1 - \lambda)/[\lambda(K-1)]$,

$$M_2 \equiv \sum_{n=0}^{\infty} P_n(0) n^2, \quad \phi_n = \frac{\partial^{n+1}}{\partial z^{n+1}} \left(\frac{q(z,0)}{1-z} \right) \Big|_{z=z_0}.$$

As for large time asymptotic of $M(t)$, we see that in the interval [0,1] of the possible values of parameter λ the point $\lambda = \lambda_{cr}$ plays a special role. For $\lambda < \lambda_{cr}$ and $\lambda > \lambda_{cr}$ the asymptotic of initial dates c_n for large n appear to be of essentially different types: if $\lambda < \lambda_{cr}$, $\lim_{n\to\infty} c_n = 1 - \lambda/\lambda_{cr} \neq 0$ and if $\lambda > \lambda_{cr}$, $\lim_{n\to\infty} c_n = 0$. Therefore it is naturally to consider λ_{cr} as an inherent characteristic of the self-organization processes in the systems. For $\lambda = \lambda_{cr}$ the leading term of asymptotic of $M(t)$ and c_n are the power functions with universal amplitude and the asymptotical dynamics of "critical" avalanches appears to be scale invariant in accordance with usual properties of the SOC processes.

5 Pair correlation function

For description of dynamical correlations in the BSM one can choose as its characteristics the probability $P_{n_1,n_2}(t)$ that at time point t in the system n_1 species have the barriers from the interval $[0, \lambda_1]$, and n_2 species have barriers from the interval $[\lambda_1, \lambda_2]$. It is assumed that $0 < \lambda_1 < \lambda_2 < 1$. It is convenient for analysis of equation for $P_{n_1,n_2}(t)$ to use the generating function $P(z_1, z_2; u)$ defined as follows:

$$P(z_1, z_2; u) = \sum_{n_1=0, n_2=0, t=0}^{\infty} P_{n_1 n_2}(t) z_1^{n_1} z_2^{n_2} u^t .$$

It can be shown that in virtue of the RNM dynamical rules for infinite system the function $P(z_1, z_2; u)$ obeys the equation:

$$P(z_1, z_2; u)z_2(z_1 - u\phi) + P(0, z_2; u)u\phi(z_2 - z_1) + P(0, 0; u)u\phi z_1(1 - z_2) = \\ = P(z_1, z_2; 0)z_1 z_2 \quad (19)$$

where

$$\phi \equiv (p_1 z_1 + p_2 z_2 + p_3)^K, \ p_1 = \lambda_1, \ p_2 = \lambda_2 - \lambda_1, \ p_3 = 1 - \lambda_1 - \lambda_2.$$

This equation appears to be exact solvable. Let us denote $\alpha(u)$ the analytical in $u = 0$ solution of equation

$$\alpha = u[(p_1 + p_2)\alpha + p_3]^K.$$

Substituting in equation (19) $z_1 = z_2 = \alpha(u),$ we obtain the following relation

$$P(0, 0, u) = \frac{P(\alpha, \alpha; 0)}{1 - \alpha}. \quad (20)$$

Let $\beta(z; u)$ be the analytical in $z = 0$, $u = 0$ solution of equation

$$\beta = u(p_1\beta + p_2 z + p_3)^K.$$

Substituting in (19) $z_1 = \beta(z_2; u)$ we have

$$P(0, z_2, u) = \frac{P(\beta, z_2; 0)z_2(1 - \alpha) - P(\alpha, \alpha; 0)\beta(1 - z_2)}{(1 - \alpha)(z_2 - \beta)}. \quad (21)$$

Now, we can substitute in (19) instead of $P(0, 0, u)$, $P(0, z_2, u)$ the right parts of equalities (20), (21) and obtain the linear equation for $P(z_1, z_2; u)$, having the solution of the form

$$P(z_1, z_2; u) = \frac{P(\alpha, \alpha; 0)u\phi(\beta - z_1)(1 - z_2)}{(z_1 - u\phi)(1 - \alpha)(z_2 - \beta)} + \frac{P(\beta, z_2; 0)u\phi(z_1 - z_2)}{(z_1 - u\phi)(z_2 - \beta)} + \frac{P(z_1, z_2; 0)z_1}{(z_1 - u\phi)} \quad (22)$$

If the initial distribution of barriers is given, than the generating function $P(z_1, z_2; 0)$ and the right hand site of (22) are known. Using the generating function $P(z_1, z_2; u)$ one can calculate different dynamical characteristic of correlations in the system. We do not give instance of calculations of such a kind and note only that the pair distribution function $P_{n_1,n_2}(t)$ can not be expressed in terms of the the probability $P_n(t)$.

6 Conclusion

We have shown how the master equations for the RNM can be solved. The essential feature of the SOC dynamics in the infinite system is the presence of the critical barrier $\lambda_{cr} = 1/K$ characterizing the types of dynamical pictures considered in the RNN evolution: over-critical (if $\lambda > \lambda_{cr}$) sub-critical (if $\lambda < \lambda_{cr}$) and critical (if $\lambda = \lambda_{cr}$). The form of the master equation depends on the chose of dynamical variables. In terms of a special set of variables $C_n(t)$ it appears to be very simple and the SOC dynamics in RNM looks as renumeration $C_n(t+1) = C_{n+1}(t)$ of variables. They are expressed straightforwardly in terms of infinite set of the constants defined explicitly by the initial conditions. For each time step one of this constant is forgotten, i.e its value does not influence on the further stages of the system evolution. The consequent loose of the information about initial state is all what happens in the self-organization process described by RNM. The system of such a kind could be called completely integrable dissipative system.

The scaling laws of SOC dynamics of RNM can be studied by means of distribution functions of barriers describing the system dynamics. We considered only simplest of them: $P_n(t)$ and $P_{n_1,n_2}(t)$. Using $P_n(t)$ and $P_{n_1,n_2}(t)$ it is possible to calculate different characteristic of critical avalanches in RNM. Now it is not clear, whether the full description of the RNM dynamics can be obtained in this way. If it is not the case, then there is for RNM the problem to construct the full set of independent correlation function, which can be considered as a good task for further investigations.

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