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DYNAMIC PHASE TRANSITIONS IN DIFFUSION-LIMITED REACTIONS[1]

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Many non-equilibrium systems display dynamic phase transitions from active to absorbing states, where fluctuations cease entirely. Based on a field theory representation of the master equation, the critical behavior can be analyzed by means of the renormalization group. The resulting universality classes for single-species systems are reviewed here. Generically, the critical exponents are those of directed percolation (Reggeon field theory), with critical dimension $d_c = 4$. Yet local particle number parity conservation in even-offspring branching and annihilating random walks implies an inactive phase (emerging below $d_c' \approx 4/3$) that is characterized by the power laws of the pair annihilation reaction, and leads to different critical exponents at the transition. For local processes without memory, the pair contact process with diffusion represents the only other non-trivial universality class. The consistent treatment of restricted site occupations and quenched random reaction rates are important open issues.

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1 Introduction: Active to absorbing state phase transitions

Among the prevalent goals in current statistical mechanics is the understanding and characterization of non-equilibrium steady states. This program is hindered by the general absence of an effective free-energy function that would allow a straightforward classification in terms of symmetries and interactions. One might hope, however, that the task becomes more feasible near continuous phase transitions separating different non-equilibrium steady states. For in analogy with equilibrium critical points, one would expect the properties near non-equilibrium phase transitions as well to be *universal*, i.e., independent of the detailed microscopic dynamical rules and the initial conditions. Rather, the emerging power laws and scaling functions describing the long-wavelength, long-time limit should hopefully be characterized by not too many distinct universality classes. Obviously, provided we can cast the problem at hand into a form amenable to field-theoretic methods, the renormalization group (RG) provides a very powerful tool for such investigations. Indeed, during the past twenty-five years or so it has been successfully applied to

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a variety of non-equilibrium processes. Among the lessons we have learned is that critical phenomena or generic scale invariance far from thermal equilibrium, where the detailed-balance constraints do not apply, are considerably richer than equilibrium statics, or even near-equilibrium dynamics. In fact, intuitions drawn from the latter may often be quite deceptive.

A special class of genuine non-equilibrium phase transitions separate *'active'* from *'inactive, absorbing'* stationary states where any stochastic fluctuations cease entirely [1, 2]. These occur in a large variety of systems, e.g., in chemical reactions involving an inert state $\emptyset$ that does not release the reactants A anymore. We may also consider stochastic population dynamics, combining, say, diffusive migration with asexual reproduction $A \to 2A$ (with rate σ), spontaneous death $A \to \emptyset$ (rate μ), and lethal competition $2A \to A$ (rate λ). In the inactive state, where no population members A are left, all processes terminate. Similar effective dynamics may be used to model non-equilibrium physical systems, such as the domain-wall kinetics in Ising chains with competing Glauber (spin flip) and Kawasaki (spin exchange) dynamics [3]. Here, spin flips $\uparrow\uparrow\downarrow\downarrow \to \uparrow\uparrow\uparrow\downarrow$ and $\uparrow\uparrow\downarrow\uparrow \to \uparrow\uparrow\uparrow\uparrow$ may be viewed as domain wall (A) hopping and pair annihilation $2A \to \emptyset$, respectively, whereas a spin exchange $\uparrow\uparrow\downarrow\downarrow \to \uparrow\downarrow\uparrow\downarrow$ represents a branching process $A \to 3A$ in domain wall language. Notice that the paramagnetic and ferromagnetically ordered phases map onto the active and inactive 'particle' states, the latter rendered absorbing if the spin flip rates are computed at zero temperature, allowing no energy increase.

The simplest mathematical description for such processes uses a kinetic rate equation in terms of the time-dependent average 'particle' density $n(t)$, which for the above systems reads

$$\partial_t\, n(t) = (\sigma - \mu)\, n(t) - \lambda\, n(t)^2 \,. \tag{1}$$

Obviously, this yields both an inactive and an active phase, as for $\sigma < \mu$ we have $n(t \to \infty) \to 0$, whereas for $\sigma > \mu$ the particle density saturates at $n_s = (\sigma - \mu)/\lambda$. The explicit solution $n(t) = n_0 n_s / \left[n_0 + (n_s - n_0) e^{(\mu-\sigma)t} \right]$ shows that either stationary state is approached exponentially in time. The two phases are separated by a continuous dynamic transition at the critical point $\sigma = \mu$, where the temporal decay becomes algebraic, $n(t) = n_0/(1 + n_0 \lambda t)$. However, eq. (1) entails a mean-field type of approximation, as we have neglected particle correlations on the right-hand side, and effectively factored a two-point correlation function, namely the joint probability of finding two particles at the same position. A more detailed treatment therefore requires a systematic incorporation of spatio-temporal fluctuations, and the ensuing particle correlations.

2 Langevin description, Reggeon field theory, and directed percolation (DP)

We may try a phenomenological incorporation of fluctuation effects using a Langevin-type approach. To this end, we assume diffusive particle transport, and model our 'chemical' system through a non-linear stochastic differential equation with reaction functional $r[n]$,

$$\partial_t\, n(\mathbf{x}, t) = D\, \nabla^2\, n(\mathbf{x}, t) - r[n](\mathbf{x}, t) + \zeta(\mathbf{x}, t) \,. \tag{2}$$

The stochastic variable $\zeta(\mathbf{x}, t)$ incorporates the reaction noise, which we take to have zero mean, and to be local in space-time ('white' in Fourier space), with a density-dependent correlator $c[n]$:

$$\langle \zeta \rangle = 0 \,, \quad \langle \zeta(\mathbf{x}, t)\, \zeta(\mathbf{x}', t') \rangle = c[n]\, \delta(\mathbf{x} - \mathbf{x}')\, \delta(t - t') \,. \tag{3}$$

DP exponents	$d = 1$	$d = 2$	$d = 4 - \epsilon$, $O(\epsilon)$
$n_s \sim \|r\|^{\beta}$	$\beta \approx 0.2765$	$\beta \approx 0.584$	$\beta = 1 - \epsilon/6$
$\xi \sim \|r\|^{-\nu}$	$\nu \approx 1.100$	$\nu \approx 0.735$	$\nu = 1/2 + \epsilon/16$
$t_c \sim \xi^z \sim \|r\|^{-z\nu}$	$z \approx 1.576$	$z \approx 1.73$	$z = 2 - \epsilon/12$
$n_c(t) \sim t^{-\alpha}$	$\alpha \approx 0.160$	$\alpha \approx 0.46$	$\alpha = 1 - \epsilon/4$

Tab. 1. Critical exponents for the saturation density (order parameter) n_s, correlation length ξ, characteristic time scale t_c, and critical density decay $n_c(t)$ for the universality class of directed percolation (DP).

Here, the last expression means that whenever a trajectory average is taken, $c[n]$ needs to be factored in. In the spirit of Landau theory, we may expand the functionals $r[n]$ and $c[n]$ near the inactive phase ($n \ll 1$). In the absence of spontaneous particle production, both must vanish at $n = 0$, which is the condition for an absorbing state. Consequently, an active to absorbing state phase transition should be *generically* described by eqs. (2), (3) with $r[n] \approx Dr\, n + \lambda\, n^2$ and $c[n] \approx 2\sigma\, n$ [4]. Upon neglecting fluctuations, one then recovers eq. (1) with $r = (\mu - \sigma)/D$.

By means of standard techniques, a stochastic differential equation of the form (2) with noise correlation (3) can be represented as a functional integral [5]. Essentially, one starts with the Gaussian noise probability distribution $W[\zeta] \propto \exp\left[-\int (\zeta^2/2c) d^dx\, dt\right]$, multiplies with $1 = \int D[n] \prod_{\mathbf{x},t} \delta(\partial_t n - D\nabla^2 n + r - \zeta) = \int D[i\tilde{n}] D[n] \exp\left[-\int \tilde{n}(\partial_t n - D\nabla^2 n + r - \zeta) d^dx\, dt\right]$, thus introducing the auxiliary fields $\tilde{n}$, and then integrates out the stochastic noise ζ. Upon applying a forward discretization, the ensuing functional determinant vanishes, and one arrives at the probability distribution $P[n] \propto \int D[i\tilde{n}] \exp(-S[\tilde{n}, n])$, with the response functional $S[\tilde{n}, n] = \int \tilde{n} \left(\partial_t n - D\nabla^2 n + r[n] - c[n]\tilde{n}/2\right) d^dx\, dt$. After rescaling $n = (\sigma/\lambda)^{1/2}\phi$, $\tilde{n} = (\lambda/\sigma)^{1/2}\tilde{\phi}$, the expanded Langevin equation (2), (3) maps to *'Reggeon' field theory* [6]

$$S[\tilde{\phi}, \phi] = \int \left[\tilde{\phi}\left[\partial_t + D(r - \nabla^2)\right]\phi + u\left(\tilde{\phi}\phi^2 - \tilde{\phi}^2\phi\right)\right] d^dx\, dt\,, \tag{4}$$

with $u = (\sigma\lambda/2)^{1/2}$. Notice the invariance of this action with respect to 'rapidity inversion' $\phi(\mathbf{x}, t) \to -\tilde{\phi}(\mathbf{x}, -t)$, $\tilde{\phi}(\mathbf{x}, t) \to -\phi(\mathbf{x}, -t)$.

The field theory (4) should capture the generic critical behavior for non-equilibrium phase transitions between active and absorbing states, occurring at $r = 0$. Quite remarkably, the very same action is obtained for the threshold pair correlation function [7] in the purely geometric problem of *directed percolation* (DP) [8]. Power counting reveals $d_c = 4$ as the upper critical dimension; hence, the critical exponents as predicted by mean-field theory acquire logarithmic corrections at d_c, and are shifted to different values by the infrared-singular fluctuations in $d < 4$ dimensions. By means of the standard perturbational loop expansion in terms of the diffusion propagator and the vertices $\propto u$, and the application of the RG, the critical exponents can be computed systematically and in a controlled manner in a dimensional expansion with respect to $\epsilon = 4 - d$. The one-loop results, to first order in ϵ, as well as reliable values from Monte Carlo simulations in one and two dimensions [2] are listed in Tab. 1. Moreover, as a consequence of rapidity invariance there are only three independent scaling exponents, namely the anomalous field dimension η, the correlation length exponent ν, and the dynamic critical exponent z. All other exponents are then fixed by scaling relations, such as $\beta = \nu(d + z - 2 + \eta)/2 = z\nu\alpha$ for the order parameter exponent β and the critical density decay exponent α, respectively.

3 From the master equation to stochastic field theory

The above phenomenological approach constitutes a natural extension of the mean-field rate equations to a stochastic partial differential equation. However, it does presume (i) that such a Langevin-type representation is in fact possible, and (ii) it is fundamentally based on conjectures on the noise correlator. Away from thermal equilibrium, there is no analog of the Einstein relation to constrain the form and structure of the noise correlations. Indeed, it has emerged that the latter may quite profoundly affect the scaling behavior of non-equilibrium systems, rendering point (ii) an entirely non-trivial issue. Moreover, as we shall see, even assertion (i) turns out to be relevant in certain important model systems. It is therefore of fundamental importance to be able to construct a long-wavelength or field theory representation of stochastic processes that starts directly from their microscopic definition in terms of a classical master equation, without recourse to any serious additional assumptions or approximations.

Fortunately, for reaction-diffusion systems there exists indeed a standard route from the master equation to an effective 'Hamiltonian' (more precisely, the Liouville time evolution operator), and therefrom immediately to a field theory action [9]. The key point is that all possible configurations here can be labeled by specifying the occupation numbers n_i of, say, the sites of a d-dimensional lattice; we shall henceforth assume that there are no site occupation restrictions, i.e., $n_i = 0, 1, 2, \ldots$ The master equation then addresses the time evolution of the configurational probability $P(\{n_i\}; t)$. For example, the corresponding contribution from the binary coagulation process $2A \to A$ at site i reads $\partial_t P(n_i; t)|_\lambda = \lambda \left[(n_i + 1) n_i P(n_i + 1; t) - n_i (n_i - 1) P(n_i; t) \right]$. This sole dependence on the integer variables $\{n_i\}$ calls for a second-quantized bosonic operator representation with the standard commutation relations $[a_i, a_j^\dagger] = \delta_{ij}$ and the empty state $|0\rangle$ such that $a_i|0\rangle = 0$. We then define the Fock states via $|\{n_i\}\rangle = \prod_i (a_i^\dagger)^{n_i} |0\rangle$ (notice that the normalization is different from standard many-particle quantum mechanics), and thence construct the formal state vector $|\Phi(t)\rangle = \sum_{\{n_i\}} P(\{n_i\}; t)\, |\{n_i\}\rangle$. The master equation now imposes a linear time evolution that can be written as an imaginary-time 'Schrödinger' equation $\partial_t |\Phi(t)\rangle = -H\, |\Phi(t)\rangle$, with a generally non-Hermitian stochastic 'Hamiltonian' $H(\{a_i^\dagger\}, \{a_i\})$; e.g., for the on-site coagulation reaction one finds explicitly $H_{\lambda\, i} = -\lambda (1 - a_i^\dagger) a_i^\dagger a_i^2$.

Our goal is to evaluate time-dependent statistical averages for observables F, necessarily mere functions of the occupation numbers as well, whence $\langle F(t) \rangle = \sum_{\{n_i\}} F(\{n_i\}) P(\{n_i\}; t)$. Straightforward algebra utilizing the identity $[e^a, a^\dagger] = e^a$ shows that this average can be cast into a 'matrix element' $\langle F(t) \rangle = \langle \mathcal{P} | F(\{a_i\}) | \Phi(t) \rangle = \langle \mathcal{P} | F(\{a_i\})\, e^{-Ht} | \Phi(0) \rangle$ with the state vector $|\Phi(t)\rangle$ and the projector state $\langle \mathcal{P}| = \langle 0| \prod_i e^{a_i}$, with $\langle \mathcal{P}|0\rangle = 1$. For example, probability conservation implies $1 = \langle \mathcal{P}| e^{-Ht} |\Phi(0)\rangle$, i.e., for infinitesimal times $\langle \mathcal{P}|\Phi(0)\rangle = 1$ and $\langle \mathcal{P}| H = 0$, which is satisfied if $H(\{1\}, \{a_i\}) = 0$. Notice furthermore that commuting the factor $e^{\sum_i a_i}$ through all the other operators has the effect of shifting $a_i^\dagger \to 1 + a_i^\dagger$ everywhere.

As a final step, we employ the coherent-state path integrals familiar from quantum many-particle systems [10], and perform the continuum limit to arrive at the desired field theory. For our earlier population dynamics example with diffusive motion and the reactions $A \to 2A$, $A \to \emptyset$, and $2A \to A$, this results in the action (omitting contributions from the initial state)

$$S[\hat{\psi}, \psi] = \int \left[\hat{\psi} \left(\partial_t - D\, \nabla^2 \right) \psi + \sigma (1 - \hat{\psi}) \hat{\psi} \psi - \mu (1 - \hat{\psi}) \psi - \lambda (1 - \hat{\psi}) \hat{\psi} \psi^2 \right] d^d x\, dt\,. \quad (5)$$

It is important to emphasize again that the single approximation used here was the continuum

limit; specifically, no assumptions whatsoever on the stochastic noise were invoked. The mean-field rate equation (1) is recovered by inserting the solution $\hat{\psi} = 1$ to the stationarity condition $\delta S/\delta\psi = 0$ into $\delta S/\delta\hat{\psi} = 0$, with $n(t) = 2\langle\psi(\mathbf{x},t)\rangle$. However, higher moments of the field ψ cannot be directly identified with the corresponding density correlations. Very remarkably, though, after shifting $\hat{\psi}(x,t) = 1 + \tilde{\psi}(x,t)$, such that $\langle\tilde{\psi}\rangle = 0$, and appropriate field rescaling one arrives again at the field theory (4) with $r = (\mu - \sigma)/D$ and $u = (2\sigma\lambda)^{1/2}$, with the additional vertex $\lambda\,\tilde{\phi}^2\phi^2$. Yet, the coupling λ has scaling dimension κ^{2-d}, where κ denotes a momentum scale, whereas u becomes marginal at $d_c = 4$. Thus, at least in the vicinity of the upper critical dimension, λ constitutes an irrelevant variable in the RG sense, and in general the ratio $\lambda/u \sim \kappa^{-d/2}$ is expected to scale to zero asymptotically. Consequently, for processes in the directed percolation universality class the microscopic master equation and the coarse-grained Langevin description both lead to the effective action of Reggeon field theory.

4 Diffusion-limited annihilation

Let us now investigate the k-th order *annihilation* reaction $kA \to \emptyset$ [11]. The corresponding mean-field rate equation reads $\partial_t\, n(t) = -\lambda\, n(t)^k$. For simple radioactive decay ($k = 1$), it is solved by the familiar exponential $n(t) = n_0\, e^{-\lambda_1 t}$, whereas one obtains power laws for $k \geq 2$, namely $n(t) = \left[n_0^{1-k} + (k-1)\lambda\, t\right]^{-1/(k-1)}$. In order to consistently include fluctuations in the latter case, we start out from the master equation once again, which leads to the action

$$S[\hat{\psi},\psi] = \int \left[\hat{\psi}\left(\partial_t - D\nabla^2\right)\psi - \lambda(1-\hat{\psi}^k)\psi^k\right] d^dx\, dt\ . \tag{6}$$

After shifting $\hat{\psi}(x,t) = 1 + \tilde{\psi}(x,t)$, it becomes clear that this field theory actually has no simple Langevin representation. For in order to interpret $\tilde{\psi}$ as the corresponding noise auxiliary field, it should appear quadratically in the action only, and with negative prefactor. Thus, the Langevin equation derived from the action (6) for the pair annihilation process would imply unphysical 'imaginary' noise with $c[n] = -2\lambda n^2$.

Analyzing the field theory (6) further, we see that the diffusion propagator does not become renormalized at all, implying that $\eta = 0$ and $z = 2$ to all orders in the perturbation expansion. The critical dimension of the annihilation vertex is found to be $d_c(k) = 2/(k-1)$, allowing for the possibility of non-trivial scaling behavior in low physical dimensions only for the pair and triplet processes. The simple structure of the action permits summing the entire perturbation series for the vertex renormalization by means of a Bethe-Salpeter equation that in Fourier space reduces to a geometric series [11]. For pair annihilation ($k = 2$), this yields the following asymptotic behavior for the particle density: $n(t) \propto t^{-1}$ for $d > 2$, $n(t) \propto t^{-1}\ln t$ at $d_c = 2$, and $n(t) \propto t^{-d/2}$ for $d < 2$. The slower decay for $d \leq 2$ originates in the fast mutual annihilation of any close-by reactants; after some time has elapsed, this leaves only well-separated particles. The annihilation dynamics thus produces *anti*-correlations, mimicking effective repulsion (which is also the physical interpretation of the negative sign in $c[n]$). Similarly, for triplet annihilation ($k = 3$) the density decays as $n(t) \propto t^{-1/2}$ for $d > 1$, with mere logarithmic corrections $n(t) \propto \left(t^{-1}\ln t\right)^{1/2}$ at $d_c = 1$.

5 Branching and annihilating random walks (BARW)

In order to allow again for a genuine phase transition, we combine the annihilation $kA \to \emptyset$ ($k \geq 2$) with *branching* processes $A \to (m+1)\,A$. The associated rate equation reads $\partial_t\, n(t) = -\lambda\, n(t)^k + \sigma\, n(t)$, with the solution $n(t) = n_s / \left(1 + \left[(n_s/n_0)^{k-1} - 1\right] e^{-(k-1)\sigma t}\right)^{1/(k-1)}$. Mean-field theory thus predicts the density to approach the saturation value $n_s = (\sigma/\lambda)^{1/(k-1)}$ as $t \to \infty$ for any positive branching rate. Above the critical dimension $d_c(k) = 2/(k-1)$ therefore, the system only has an *active* phase; $\sigma_c = 0$ represents a degenerate 'critical' point, with scaling exponents essentially determined by the pure annihilation model: $\alpha = 1/(k-1) = \beta$, $\nu = 1/2$, and $z = 2$. However, Monte Carlo simulations for these *branching and annihilating random walks* (BARW) revealed a much richer picture, in low dimensions clearly distinguishing between the cases of *odd* and *even* number of offspring m [3, 12]: For $k = 2$, $d \leq 2$, and m *odd*, a transition to an inactive, absorbing phase is found, characterized by the DP critical exponents. On the other hand, for *even* offspring number a phase transition in a novel universality class with $\alpha \approx 0.27$, $\beta \approx 0.92$, $\nu \approx 1.6$, and $z \approx 1.75$ emerges in one dimension.

The aforementioned mapping to a stochastic field theory, combined with RG methods, succeeded to elucidate the physics behind those remarkable findings [13]. The action for the most interesting pair annihilation case becomes

$$S[\tilde{\psi}, \psi] = \int \left[\hat{\psi}\left(\partial_t - D\,\nabla^2\right)\psi - \lambda\,(1 - \hat{\psi}^2)\,\psi^2 + \sigma\,(1 - \hat{\psi}^m)\,\hat{\psi}\psi\right] d^d x\, dt\,. \tag{7}$$

Upon combining the reactions $A \to (m+1)A$ and $2A \to \emptyset$, one notices immediately that the loop diagrams generate the lower-order branching processes $A \to (m-1)A$, $A \to (m-3)A \ldots$ Moreover, the one-loop RG eigenvalue $y_\sigma = 2 - m(m+1)/2$ (computed at the annihilation fixed point) shows that the reactions with smallest m are the most relevant. For *odd* m, we see the generic situation is given by $m = 1$, i.e., $A \to 2A$, supplemented with the spontaneous decay $A \to 0$. After a first coarse-graining step, this latter process (with rate μ) must be included in the effective model, which hence becomes identical with action (5). We are thus led to Reggeon field theory (4) describing the DP universality class, *provided* the induced decay processes are sufficiently strong to render $\sigma_c > 0$. Yet for $d > 2$ the renormalized mass term $\sigma_R - \mu_R$ remains positive, which leaves us with merely the active phase captured by mean-field theory. For $d \leq 2$, however, the involved fluctuation integrals are infrared-divergent, thus indeed allowing the induced decays to overcome the branching processes to produce a phase transition. As the DP upper critical dimension is $d_c = 4$, the scaling exponents display an unusual discontinuity at $d = 2$, jumping from the non-trivial two-dimensional DP to the mean-field values (for any $d > 2$) as a consequence of the vanishing critical branching rate [13].

It is now obvious why the case of *even* offspring number m is fundamentally different: Here, the most relevant branching process is $A \to 3A$, and spontaneous particle death with associated exponential decay is *not* generated under the coarse-grained dynamics, which precludes the above mechanism for producing an inactive phase with exponential particle decay. This important distinction from the odd-m case can be traced to a microscopic local conservation law, for the reactions $2A \to \emptyset$ and $A \to 3A$, $A \to 5A \ldots$ always destroy or produce an even number of reactants, and consequently preserve particle number parity. Formally, this is reflected in the invariance of the action (7) under the combined inversions $\psi \to -\psi$, $\hat{\psi} \to -\hat{\psi}$. As we saw earlier, the branching rate σ certainly constitutes a relevant variable near the critical dimension

$d_c = 2$. Therefore the phase transition can only occur at $\sigma_c = 0$, and for any $\sigma > 0$ there exists only an active phase, described by mean-field theory. In two dimensions one readily predicts the following logarithmic corrections: $\xi(\sigma) \propto \sigma^{-1/2} \ln(1/\sigma)$, and $n(\sigma) \propto \sigma\,[\ln(1/\sigma)]^{-2}$. However, setting $m = 2$ in the one-loop value for the RG eigenvalue y_σ, we notice that the branching vertex becomes irrelevant for $d < 4/3$. More information on the low-dimensional behavior can be gained through a one-loop analysis at *fixed* dimension, albeit uncontrolled [13]. The ensuing RG flow equations for the renormalized, dimensionless branching rate $\sigma_R = \sigma/D\kappa^2$, and annihilation rate $\lambda_R = C_d\lambda/D\kappa^{2-d}$, with $C_d = \Gamma(2-d/2)/2^{d-1}\pi^{d/2}$ read (for $m = 2$):

$$\frac{d\sigma_R}{d\ell} = \sigma_R \left[2 - \frac{3\,\lambda_R}{(1+\sigma_R)^{2-d/2}}\right] , \quad \frac{d\lambda_R}{d\ell} = \lambda_R \left[2 - d - \frac{\lambda_R}{(1+\sigma_R)^{2-d/2}}\right] . \tag{8}$$

The effective coupling is then identified as $g = \lambda_R/(1+\sigma_R)^{2-d/2}$, which approaches the annihilation fixed point $g^* = 2 - d$ as $\sigma_R \to 0$, while for $\sigma_R \to \infty$ the flow tends towards the Gaussian fixed point $g^* = 0$ describing the active state. The separatrix between the two phases is given by the *unstable* RG fixed point $g^* = 4/(10-3d)$, which enters the physical regime below the novel critical dimension $d_c' \approx 4/3$. For $d < d_c'$, this describes a dynamic phase transition with $\sigma_c > 0$; the emerging *inactive* phase is characterized by a vanishing branching rate, and thus by the *algebraic* scaling behavior of the pure pair annihilation model, $n(t) \propto t^{-d/2}$. This very different character of the inactive phase also explains why the transition cannot possibly fall into the DP category. The aforementioned fixed-dimension RG analysis yields the rather crude values $\nu \approx 3/(10-3d)$, $z \approx 2$, and $\beta \approx 4/(10-3d)$ for this *parity-conserving* (PC) universality class. Moreover, the absence of any mean-field ounterpart for this transition precludes a sound derivation of 'hyperscaling' relations such as $\beta = z\nu\alpha$.

Similar arguments suggest for triplet annihilation $3A \to \emptyset$ combined with branching $A \to (m+1)A$ that DP behavior with $\sigma_c > 0$ should ensue for $m \bmod 3 = 1, 2$, as then the processes $A \to \emptyset$, $A \to 2A$, and $2A \to A$ are dynamically generated. For $m = 3, 6, \ldots$, on the other hand, because of $d_c(k = 3) = 1$ one expects logarithmic corrections only in one-dimensional systems.

6 Concluding remarks and outlook

In this brief overview, I have outlined how non-linear stochastic processes can be represented by field theory actions, allowing for a thorough analysis and classification by means of the renormalization group. Systems with a single 'particle' species displaying a non-equilibrium phase transition from an active to an inactive, absorbing state are generically captured by the DP universality class. The second prominent example, applicable when additional symmetries (degenerate absorbing states) are present, is that of branching and annihilating random walks with even offspring number (PC). In fact, there appears to be only one additional possibility for non-trivial scaling behavior in 'bosonic' systems with multiple site occupation, namely the combination of *binary* reactions $2A \to \emptyset$ and $2A \to (m+2)A$ (annihilation and fission; pair contact process with diffusion, PCPD) [14]. These reactions subsequently generate $2A \to mA, (m-2)A, \ldots$, $2(m+1)A, \ldots$, thus producing *infinitely* many couplings with identical scaling dimensions, which renders the field theory non-renormalizable. Yet, the master equation immediately leads to the exact evolution equation $\partial_t\, n(t) = (\sigma - \lambda)\langle m(m-1)\rangle$, which establishes the existence of a phase transition when the particle production and annihilation rates precisely balance. In the

inactive, absorbing phase, obviously the power laws of the pure annihilation (m even) or coagulation (m odd) models are recovered. In the active phase, the particle density diverges in a finite time, and the transition itself is *discontinuous*. These irregular features are 'cured' by restricting the site occupation numbers, whereupon the phase transition becomes continuous, with critical exponents $\alpha \approx 0.23 \ldots 0.25$, $\beta \approx 0.43 \ldots 0.50$ (apparently depending on the value of the diffusion constant [16]) and $z \approx 1.83$ that belong to neither the DP nor PC universality classes. Yet different exponents were recently found for the triplet reactions $3A \to 2A$ and $3A \to 4A$ [17]. It is however unclear at present how to generally and systematically incorporate such site restrictions into a field theory description in a tractable manner [18]. Another open problem concerns the obviously very relevant issue of quenched disorder in the reaction rates. For example, the RG investigation for DP with random percolation threshold yields run-away flows to diverging couplings, with an as yet unclear interpretation [19].

Lastly, the rich variety of multi-component systems is only partially understood. Only DP processes with arbitrarily many particle species have been fully classified, namely any non-linear coupling leads directly to the DP universality class again [20], whereas novel multi-critical behavior may ensue for hierarchical systems comprising spontaneous transformations $A \to B$, $B \to C \ldots$ [20, 21]. Yet already in the simple binary process $A + B \to \emptyset$, the initial conditions play a vital role. For equal A and B densities, the fact that $n_A - n_B$ remains conserved under the dynamics leads to segregation and reaction-front dominated kinetics, with shifted upper critical dimension $d_c = 4$ and asymptotic decay $n_A \propto n_B \propto t^{-d/4}$ for lower dimensions [22]. On the other hand, whenever there is a majority species, the minority decays to zero exponentially, although this may be masked by very long crossover times. The (bosonic) N-species BARW generalization $A_i + A_i \to \emptyset$, $A_i \to A_i + 2A_j$ turns out to be exactly analyzable, with a degenerate phase transition only at vanishing branching rate, and critical exponents $\alpha = d/2$, $\beta = 1$, $\nu = 1/d$, $z = 2$ [13]. The situation for $N = 1$ is thus qualitatively different from all multi-component cases. Moreover, in certain cases the difference in diffusion constants is a relevant control parameter [23]. When there are 'passive' agents, the corresponding variables are readily integrated out, leaving however interactions of the remaining degrees of freedom that are non-local in time. A classic example for such memory effects is dynamic percolation [24], which at criticality yields stationary isotropic percolation clusters. A recent argument claims this to govern the generic universality class for non-equilibrium phase transitions with infinitely many absorbing states [25]. Many intriguing problems are still open, and RG techniques will be invaluable tools for the further analysis of scale-invariant non-equilibrium systems.

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