

**RENORMALIZATION SCHEME DEPENDENCE IN THE CASE
OF A QCD NON-POWER PERTURBATIVE EXPANSION¹**

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A novel, non-power, expansion of QCD quantities replacing the standard perturbative expansion in powers of the renormalized coupling a has recently been introduced and examined by two of us. Being obtained by analytic continuation in the Borel plane, the new expansion functions $W_n(a)$ share the basic analyticity properties with the expanded quantity. In this note we investigate the renormalization scale dependence of finite order sums of this new expansion for the phenomenologically interesting case of the τ -lepton decay rate.

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1 Renormalization scale and scheme dependence

In the standard perturbation theory the finite order approximations of physical quantities are renormalization scale (μ) and scheme (RS) dependent. The quest for in some sense “optimal” scale and scheme is vital for meaningful applications but has so far no generally accepted solution. There are several recipes [1–3] how to do that. The one proposed in [1] and known as the Principle of Minimal Sensitivity (PMS) selects the scale and scheme by the condition of local scale and scheme invariance. For a physical quantity $\mathcal{R}(Q)$ depending on one external kinematical variable Q and admitting the perturbative expansion in powers of the QCD coupling parameter α_s

$$\mathcal{R}(Q) = a(\mu, \text{RS})(1 + r_1(Q, \mu, \text{RS})a(\mu, \text{RS}) + r_2(Q, \mu, \text{RS})a^2(\mu, \text{RS}) + \dots), \quad a \equiv \alpha_s/\pi, \quad (1)$$

this implies for the sum $R^{(N)}(Q, \mu, \text{RS})$ of first N terms in (1)

$$\left. \frac{\partial \mathcal{R}^{(N)}(Q, \mu, \text{RS})}{\partial \ln \mu} \right|_{opt} = \left. \frac{\partial \mathcal{R}^{(N)}(Q, \mu, \text{RS})}{\partial (\text{RS})} \right|_{opt} = 0. \quad (2)$$

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The PMS thus selects the point where the truncated approximant has locally the property which the all order sum must have globally. In the absence of additional information this choice appears particularly well-motivated. But even if we do not subscribe to PMS it is definitely useful to investigate the scale and scheme dependence of finite order approximants. The scheme can be labeled by the set of free parameters $c_k, k \geq 2$, defining the r.h.s. of the RG equation for the couplant a

$$\frac{\partial a(\mu, \text{RS})}{\partial \ln \mu} = \beta(a) = -ba^2(1 + ca + c_2a^2 + c_3a^3 + \dots), \quad (3)$$

together with some parameter specifying which of the solutions of eq. (3) we have in mind. One way of doing this is by means of the parameter $\tilde{\Lambda}$ defined by the condition $a(\mu = \tilde{\Lambda}) = \infty$. Note that in QCD with n_f massless quarks the first two coefficients in (3), $b = (33 - 2n_f)/6$, $c = (153 - 19n_f)/(66 - 4n_f)$, are universal, and that $\tilde{\Lambda}$ defined above is related to the more commonly used definition of Λ by a simple scale factor close to unity: $\tilde{\Lambda} = \Lambda(2c/b)^{-c/b}$.

At the second order there are two free parameters: the scale μ and $\tilde{\Lambda}$, specifying the scheme, but without loss of generality we can fix the latter and vary the scale only

$$\mathcal{R}^{(2)}(Q, \mu) = a^{(2)}(\mu)[1 + r_1(Q, \mu)a^{(2)}(\mu)], \quad (4)$$

where $a^{(2)}(\mu)$ solves (3) with the first two terms on its r.h.s. only and satisfies

$$b \ln(\mu/\tilde{\Lambda}) = 1/a^{(2)}(\mu) + c \ln[ca^{(2)}(\mu)/(1 + ca^{(2)}(\mu))]. \quad (5)$$

The formal (i.e. to the order considered) scale independence of (4) implies

$$\partial r_1(Q, \mu)/\partial \ln \mu = b, \quad \Rightarrow \quad r_1(Q, \mu) = b \ln(\mu/\tilde{\Lambda}) - \rho_1(Q/\tilde{\Lambda}), \quad (6)$$

where ρ_1 is a scale and scheme invariant depending on Q and the numerical value of $\tilde{\Lambda}$, which can be evaluated using the results in $\overline{\text{MS}}$ RS as $\rho_1 = b \ln(Q/\tilde{\Lambda}_{\overline{\text{MS}}}) - r_1(\mu = Q, \overline{\text{MS}})$.

At the third order, the coefficients r_2 in (1) and c_2 in (3) come into play. As a consequence, both r_2 and the couplant a depend beside μ and RS also on c_2 . We refer to [4] for details and mention only the expression for r_2 which will be used in the following

$$r_2 = \rho_2 - c_2 + (r_1 + c/2)^2, \quad (7)$$

where ρ_2 is another scale and scheme invariant, which unlike ρ_1 , is a pure number. Although at the third order c_2 is a free parameter, we shall not exploit the associated freedom, but will work in the RS where $c_2 = 0$ at all orders. We prefer this choice of the RS to the conventional $\overline{\text{MS}}$ RS since in this case the coupling $a(\mu)$ is well defined and the same at all orders, and any manifestation of the divergence of perturbation expansion concerns exclusively the coefficients of the expansion (1).

2 Non-power expansions

In [6–8] a method was proposed that replaces perturbative expansions of observables in powers of the QCD couplant a by expansions in the set of functions $W_n(a)$ encompassing the available

knowledge of the large order behaviour of standard perturbative expansions. As an example we consider the phenomenologically interesting observable

$$R_\tau = \frac{\Gamma(\tau \rightarrow \nu_\tau + \text{hadrons})}{\Gamma(\tau \rightarrow \nu_\tau + e^- \bar{\nu}_e)} = 3(1 + \delta_{\text{EW}})(1 + \mathcal{R}_\tau), \tag{8}$$

where δ_{EW} is an electroweak correction and the QCD contribution \mathcal{R}_τ is of the form (1). As shown in [10], the term \mathcal{R}_τ can be formally written in the form of the Borel transform

$$\mathcal{R}_\tau(M_\tau) = \int_C e^{-u/a(\mu)} B(\mu, u) F(bu/2) du \tag{9}$$

involving the functions ³

$$F(u) = \frac{-12 \sin(\pi u)}{\pi u(u-1)(u-3)(u-4)}, \quad B(\mu, u) = \sum_{n=0}^{\infty} \frac{D_{n+1}(M_\tau, \mu)}{n!} u^n. \tag{10}$$

In (9) we have written explicitly the dependence on the arbitrary scale μ but suppressed that on M_τ . The coefficients D_n come from the perturbative expansion of the Adler function $D_\tau(s)$ in the Euclidean region $s < 0$

$$D_\tau(s) = D_\tau^{(0)} (1 + D_1(\kappa)a(\kappa\sqrt{-s}) + D_2(\kappa)a^2(\kappa\sqrt{-s}) + \dots), \quad D_\tau^{(0)} = 3(1 + \delta_{\text{EW}}), \tag{11}$$

where the scale ambiguity is now parameterized via the parameter κ relating μ to s : $\mu = \kappa\sqrt{-s}$. The contour \mathcal{C} runs from 0 to ∞ , circumventing the singularities of $B(u)$, which create non-uniqueness of the integral (9). We choose the principal value prescription.

Following [6–9] we expand $B(u)$ in powers of a special function $w(u)$ that maps the holomorphy domain of $B(u)$ (or its known part) onto a unit circle. For D_τ and \mathcal{R}_τ , $w(u)$ has the form

$$w(u) = \frac{\sqrt{1+u} - \sqrt{1-u/2}}{\sqrt{1+u} + \sqrt{1-u/2}}, \tag{12}$$

which enters the definition of the functions $W_n(a)$

$$W_n(a) \equiv \frac{1}{n!} \left(\frac{8}{3}\right)^n \left(\frac{2}{b}\right)^n \frac{2}{ab} \int_C e^{-2u/(ab)} F(u) w^n(u) du \tag{13}$$

relevant for \mathcal{R}_τ . For D_τ the $W_n(a)$ are also given by (13), but with $F(u) = 1$. The $W_n(a)$ take into account the positions $u = -2/b$ and $u = 4/b$ of the two leading singularities of $B(u)$. But we also know that these singularities have the form $(1 + ub/2)^{\gamma_1}$ and $(1 - ub/4)^{\gamma_2}$ with $\gamma_1 = -2.589$ and $\gamma_2 = -2.58$. To use this knowledge we define

$$\widetilde{W}_n(a) = \frac{1}{n!} \left(\frac{8}{3}\right)^n \left(\frac{2}{b}\right)^n \frac{2}{ab} \int_C e^{-2u/(ab)} (1+u)^{\gamma_1} (1-u/2)^{\gamma_2} F(u) w^n(u) du \tag{14}$$

³Eqs. (9,10) were derived in [10] using the one-loop expression for the analytic continuation of $a(-s)$ from Euclidean to Minkowskian region in the formula relating $D(s)$ to R_τ , thus, setting $c = 0$. Using the NLO expression (5) for $a(-s)$ would lead to a more complicated relation between $B(u)$ and R_τ . However, as we use $F(u)$ merely to define our expansion functions, we can use the expression derived in [10], still retaining a consistent expansion of R_τ in terms of our functions to all orders.

and expand R_τ in terms of them. We explore the scale dependence of both expansions, in $W_n(a)$ and in $\widetilde{W}_n(a)$. The incorporation of the nature of a singularity turns out to be quite important. For details about the functions W_n and \widetilde{W}_n see [6–8].

As was shown in [7], expansions in terms of the functions W_n or \widetilde{W}_n are convergent under rather loose conditions on the coefficients. On the other hand, the functions themselves are singular at $a = 0$ [8], the series

$$W_n(a) \sim \sum_{j \geq n} c_{nj} a^j, \quad \widetilde{W}_n(a) \sim \sum_{j \geq n} \tilde{c}_{nj} a^j \quad (15)$$

being asymptotic. We choose the normalization such that $c_{nn} = 1, \tilde{c}_{nn} = 1, n \geq 1$.

3 Renormalization scale dependence for non-power expansions

The scale and scheme dependence of \mathcal{R}_τ in the standard perturbation theory was discussed in [4, 10]. In terms of the functions $W_n(a)$ or $\widetilde{W}_n(a)$ we can rewrite \mathcal{R}_τ as

$$\mathcal{R}_\tau = \mathcal{W}_1(a) + \bar{r}_1 \mathcal{W}_2(a) + \bar{r}_2 \mathcal{W}_3(a) + \dots, \quad \mathcal{W}_n = W_n \text{ or } \widetilde{W}_n, \quad (16)$$

where the coefficients $\bar{r}_k(M_\tau, \mu)$ are related to the $r_k(M_\tau, \mu)$ of (1) as follows

$$\bar{r}_1 = r_1 - c_{12}, \quad \bar{r}_2 = r_2 - \bar{r}_1 c_{23} - c_{13}, \quad \text{etc.} \quad (17)$$

The finite sums $\mathcal{R}_W^{(N)}$ of the first N terms in the expansion (16) have the same property of formal scale independence as the conventional finite sums in powers of the coupling a , i.e. their derivatives with respect to $\ln \mu$ start at the order $N + 1$

$$\frac{\partial \mathcal{R}_W^{(N)}(\mu)}{\partial \ln \mu} = \sum_{k=N+1}^{\infty} s_k W_k(a), \quad (18)$$

where s_k are some numbers, which is a generalization of the analogous relation in the conventional perturbation theory. In our numerical studies we set $Q = M_\tau = 1.8$ GeV in the expression for the invariant $\rho_1(Q)$, and took $b = 4.5, c = 1.8, \rho_2 = -6.27$, corresponding to $n_f = 3$ [4]. In the NLO we work in standard $\overline{\text{MS}}$ scheme, in the NNLO in the scheme where $c_2 = 0$. We did not resort to the conventional practice of expanding the solution of eq. (5) in inverse powers of $\ln(\mu/\Lambda)$, but solved this equation numerically.

In Figs. 1a-c we compare the scale dependence of the conventional perturbation expansions of \mathcal{R}_τ at the LO, NLO and NNLO with the corresponding expansions in the functions W_n and \widetilde{W}_n for $\widetilde{\Lambda}_{\overline{\text{MS}}}^{(3)} = 0.31$ GeV. The local maxima of the curves in Fig. 1a define the PMS choices, the intersections of the NLO and NNLO curves with the LO one correspond to the “effective charges”(EC) approach of [2]. Conventionally the scale μ is identified with M_τ , but this seemingly natural choice has a serious drawback as the resulting finite order approximations depend on the choice of the scheme ⁴.

⁴The point is that in different schemes the same choice $\mu = M_\tau$ leads to different results for $\mathcal{R}_\tau^{(N)}$. Conventionally one works in the $\overline{\text{MS}}$ RS, but there is no compelling theoretical argument for this choice. Had we worked, for instance, in MS or MOM schemes instead, the same choice $\mu = M_\tau$ would correspond in Fig. 1a to the points $\mu_{\text{MS}} = 0.68$ GeV and $\mu_{\text{MOM}} = 3.9$ GeV respectively and thus yield significantly different values of $\mathcal{R}_\tau^{(N)}$. On the other hand, the scale fixings based on the PMS and EC criteria lead to the same value of $\mathcal{R}_\tau^{(N)}$ in any scheme.

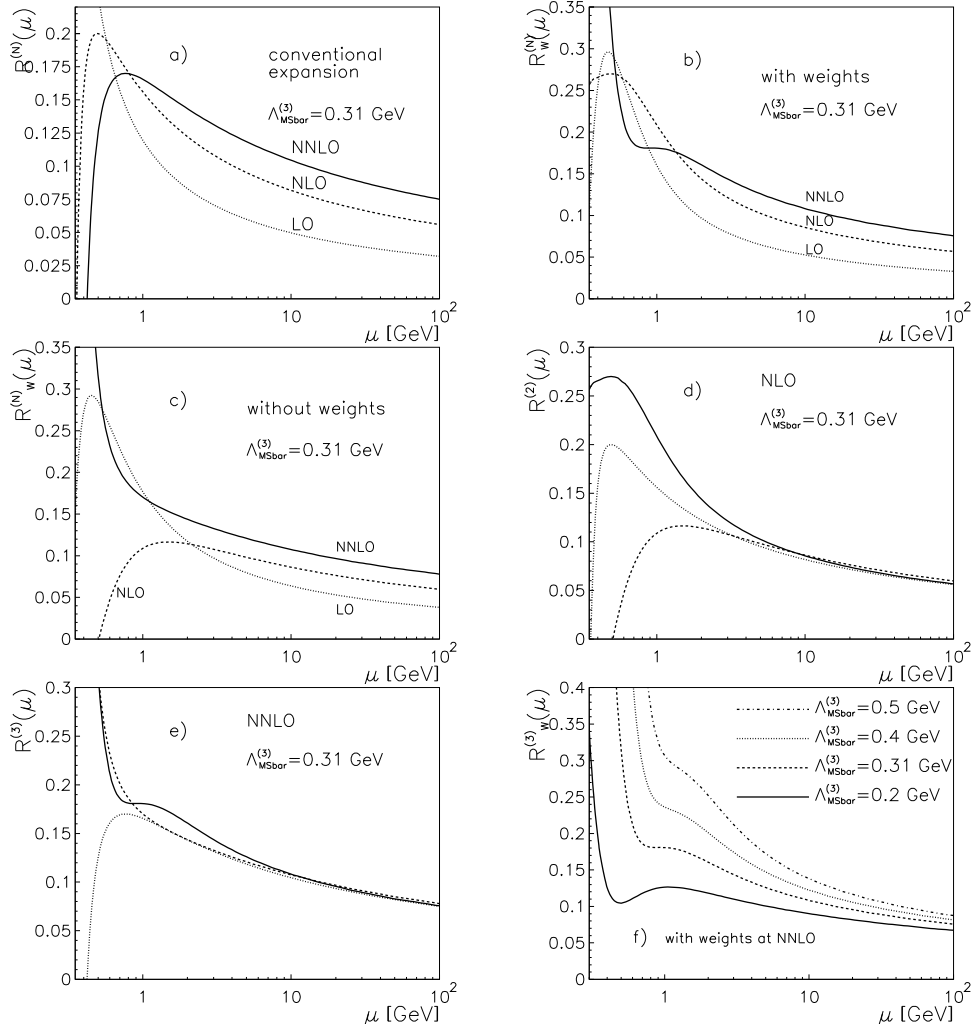


Fig. 1. Scale dependence of $\mathcal{R}_\tau^{(N)}$ in the conventional PQCD (a) as well as for the expansion (16) in the functions \widehat{W}_n (b) and W_n (c). In d) and e) $\mathcal{R}_\tau^{(2)}$ and $\mathcal{R}_\tau^{(3)}$ of the conventional PQCD (dotted curves) are compared with the corresponding approximations using $W_n(a)$ (dashed) and $\widehat{W}_n(a)$ (solid). f) the dependence of $\mathcal{R}_\tau^{(3)}$ obtained with functions $\widehat{W}_n(a)$ on $\Lambda_{\overline{\text{MS}}}^{(3)}$.

In Figs. 1d-e we compare $\mathcal{R}_W^{(N)}$, $N = 2, 3$ in the conventional perturbation theory with the results obtained within non-power expansion (16) for both sets of functions W_n and \widetilde{W}_n . Finally, in Fig. 1f the dependence of $\mathcal{R}_W^{(3)}$ obtained with the functions \widetilde{W}_n on $\widetilde{\Lambda}_{\overline{\text{MS}}}^{(3)}$ is displayed. Several interesting conclusions can be drawn from these figures:

- Scale dependence of the NLO and NNLO approximants $\mathcal{R}_W^{(2)}$ and $\mathcal{R}_W^{(3)}$ differs, for both W_n and \widetilde{W}_n , significantly from that of the conventional perturbation theory.
- There is a striking difference between the scale dependence of the approximants $\mathcal{R}_W^{(2)}$ and $\mathcal{R}_W^{(3)}$, both for W_n and \widetilde{W}_n .
- There is no region of local stability of $\mathcal{R}_W^{(3)}$ obtained with the functions W_n , whereas using the functions \widetilde{W}_n there is a plateau for $\widetilde{\Lambda}_{\overline{\text{MS}}}^{(3)} \lesssim 0.3$ GeV, but even for higher values of $\widetilde{\Lambda}_{\overline{\text{MS}}}^{(3)}$ there is at least a “knee” in $\mathcal{R}_W^{(3)}$.
- The value of $\mathcal{R}_\tau^{(3)}$ obtained with functions \widetilde{W}_n is very close to the PMS optimal point of the conventional NNLO approximation. Remarkably, at this order the approximation obtained with W_n starts to deviate from the conventional NNLO approximation close to just this stationary point.
- The preceding conclusions depend only weakly on the value of $\widetilde{\Lambda}_{\overline{\text{MS}}}^{(3)}$ in the reasonably wide interval (200, 400) MeV.

Similar analyses of the non-power expansions introduced by Shirkov [11] et al. and Cvetic [12] et al. could bring interesting new insights.

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