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DISORDERED SYSTEMS AND THE FUNCTIONAL RENORMALIZATION GROUP, A PEDAGOGICAL INTRODUCTION[1]

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In this article, we review basic facts about disordered systems, especially the existence of many metastable states and and the resulting failure of dimensional reduction. Besides techniques based on the Gaussian variational method and replica-symmetry breaking (RSB), the functional renormalization group (FRG) is the only general method capable of attacking strongly disordered systems. We explain the basic ideas of the latter method and why it is difficult to implement. We finally review current progress for elastic manifolds in disorder.

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1 Introduction

Statistical mechanics is by now a rather mature branch of physics. For pure systems like a ferromagnet, it allows to calculate so precise details as the behavior of the specific heat on approaching the Curie point. We know that it diverges as a function of the distance in temperature to the Curie temperature, we know that this divergence has the form of a power law, we can calculate the exponent, and we can do this with at least 3 digits of accuracy. This is a true success story of statistical mechanics. On the other hand, in nature no system is really pure, i.e. without at least some disorder ("dirt"). As experiments (and theory) seem to suggest, a little bit of disorder does not change the behavior much. Otherwise experiments on the specific heat of Helium would not so extraordinarily well confirm theoretical predictions. But what happens for strong disorder? By this I mean that disorder completely dominates over entropy. Then already the question: "What is the ground state?" is no longer simple. This goes hand in hand with the appearance of so-called metastable states. States, which in energy are very close to the ground state, but which in configuration space may be far apart. Any relaxational dynamics will take an enormous time to find the correct ground state, and may fail altogether, as can be seen in computer simulations as well as in experiments. This means that our way of thinking, taught in the treatment of pure systems, has to be adapted to account for disorder. We will see that in contrast to pure systems,

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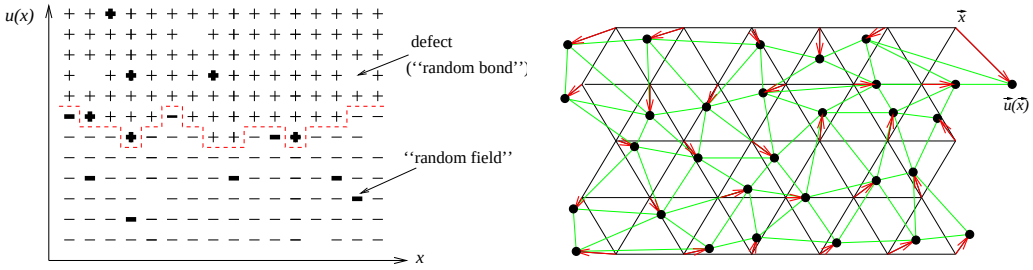


Fig. 1. An Ising magnet at low temperatures (left) has a domain wall described by a function $u(x)$. Without disorder, it is flat. In the presence of disorder it will be deformed. Right: an elastic lattice (e.g. vortex lattice) deformed by disorder. This is described by a vector $\vec{u}(x)$.

whose universal large-scale properties can be described by very few parameters, disordered systems demand the knowledge of the whole disorder-distribution function (in contrast to its first few moments). We show how universality nevertheless emerges.

Experimental realizations of strongly disordered systems are glasses, or more specifically spin glasses, vortex glasses, electron-glasses and structural glasses (not treated here). Furthermore random field magnets, and last not least elastic systems in disorder.

What is our current understanding of disordered systems? It is here that the success story of statistical mechanics, with which I started, comes to an end: Despite 30 years of research, we do not know much: There are a few exact solutions, there are phenomenological methods (like the droplet model), and there is the mean-field approximation, involving a method called replica-symmetry breaking (RSB). This method is correct for infinitely connected systems, e.g. the SK-model (Sherrington-Kirkpatrick model), or for systems with infinitely many components. However it is unclear, how far it applies to real physical systems, in which each degree of freedom is only coupled to a finite number of other degrees of freedom.

In this article, I report recent advances for elastic manifolds in random media. This system has the advantage of being approachable by other (analytic) methods, while still retaining all the rich physics of strongly disordered systems.

2 Physical realizations, model and observables

Before developing the theory to treat elastic systems in a disordered environment, let us give some physical realizations. The simplest one is an Ising magnet. Imposing boundary conditions with all spins up at the upper and all spins down at the lower boundary (see Fig. 1), at low temperatures, a domain wall separates a region with spin up from a region with spin down. In a pure system at temperature $T = 0$, this domain wall is completely flat. Disorder can deform the domain wall, making it eventually rough again. Two types of disorder are common: random bond (which on a course-grained level also represents missing spins) and random field (coupling of the spins to an external random magnetic field). Figure 1 shows, how the domain wall is described by a displacement field $u(x)$. Another example is the contact line of water (or liquid Helium), wetting a rough substrate. (The elasticity is long range). A realization with a 2-parameter displacement field $\vec{u}(\vec{x})$ is the deformation of a vortex lattice: the position of each

vortex is deformed from $\vec{x}$ to $\vec{x} + \vec{u}(\vec{x})$. A 3-dimensional example are charge density waves.

All these models have in common, that they are described by a displacement field

$$x \in \mathbb{R}^d \;\longrightarrow\; \vec{u}(x) \in \mathbb{R}^N \,. \tag{1}$$

For simplicity, we set $N = 1$ in the following. After some initial coarse-graining, the energy $\mathcal{H} = \mathcal{H}_{\rm el} + \mathcal{H}_{\rm DO}$ consists out of two parts; the elastic energy

$$\mathcal{H}_{\rm el}[u] = \int \mathrm{d}^d x \, \frac{1}{2} \left(\nabla u(x)\right)^2 \tag{2}$$

and the disorder

$$\mathcal{H}_{\rm DO}[u] = \int \mathrm{d}^d x \, V(x, u(x)) \,. \tag{3}$$

In order to proceed, we need to specify the correlations of disorder. Suppose that fluctuations u in the transversal direction scale as

$$\overline{\left(u(x) - u(y)\right)^2} \sim |x - y|^{2\zeta} \tag{4}$$

with a roughness-exponent $\zeta < 1$. Starting from a disorder correlator

$$\overline{V(u, x) V(u', x')} = f(x - x') R(u - u') \tag{5}$$

and performing one step in the RG-procedure, one has to rescale more in the x-direction than in the u-direction. This will eventually reduce $f(x - x')$ to a δ-distribution, whereas the structure of $R(u - u')$ remains visible. We therefore choose as our starting-model

$$\overline{V(u, x) V(u', x')} := \delta^d(x - x') R(u - u') \,. \tag{6}$$

There are a couple of useful observables. We already mentioned the roughness-exponent ζ. The second is the renormalized (effective) disorder. It will turn out that we actually have to keep the whole disorder distribution function $R(u)$, in contrast to keeping a few moments. Other observables are higher correlation functions or the free energy.

3 Treatment of disorder

Having defined our model, we can now turn to the treatment of disorder. The problem is to average not the partition-function, but the free energy over disorder: $\overline{\mathcal{F}} = \overline{\ln Z}$. This can be achieved by the beautiful *replica-trick*. The idea is to write

$$\ln \mathcal{Z} = \lim_{n \to 0} \frac{1}{n} \left(\mathrm{e}^{n \ln \mathcal{Z}} - 1\right) = \lim_{n \to 0} \frac{1}{n} \left(\mathcal{Z}^n - 1\right) \tag{7}$$

and to interpret $\mathcal{Z}^n$ as the partition-function of an n times replicated system. Averaging $\mathrm{e}^{-\sum_{a=1}^n \mathcal{H}_a}$ over disorder then leads to the *replica-Hamiltonian*

$$\mathcal{H}[u] = \frac{1}{T} \sum_{a=1}^{n} \int \mathrm{d}^d x \, \frac{1}{2} \left(\nabla u_a(x)\right)^2 - \frac{1}{2T^2} \sum_{a,b=1}^{n} \int \mathrm{d}^d x \, R(u_a(x) - u_b(x)) \,. \tag{8}$$

Let us stress that one could equivalently pursue a dynamic formulation. We therefore should not encounter, and in fact do not encounter, problems associated with the use of the replica-trick.

4 Dimensional reduction

There is a beautiful and rather mind-boggling theorem relating disordered systems to pure systems (i.e. without disorder), which applies to a large class of systems, e.g. random field systems and elastic manifolds in disorder. It is called dimensional reduction and reads as follows [1]:
Theorem: *A d-dimensional disordered system at zero temperature is equivalent to all orders in perturbation theory to a pure system in $d-2$ dimensions at finite temperature.* Moreover the temperature is (up to a constant) nothing but the width of the disorder distribution. A simple example is the 3-dimensional random field Ising model at zero temperature; according to the theorem it should be equivalent to the pure 1-dimensional Ising-model at finite temperature. But it has been shown rigorously, that the former has an ordered phase, whereas we have all solved the latter and we know that there is no such phase at finite temperature. So what went wrong? Let me stress that there are no missing diagrams or any such thing, but that the problem is more fundamental: As we will see later, the proof makes assumptions, which are not satisfied. Nevertheless, the above theorem remains important since it has a devastating consequence for all perturbative calculations in the disorder: However clever a procedure we invent, as long as we do a perturbative expansion, expanding the disorder in its moments, all our efforts are futile: dimensional reduction tells us that we get a trivial and unphysical result. Before we try to understand why this is so and how to overcome it, let me give one more example. Dimensional reduction allows to calculate the roughness-exponent ζ defined in equation (4). We know (this can be inferred from power counting) that the width u of a d-dimensional manifold at finite temperature in the absence of disorder scales as $u \sim x^{(2-d)/2}$. Making the dimensional shift implied by dimensional reduction leads to

$$\overline{\left(u(x)-u(0)\right)^2} \sim x^{4-d} \equiv x^{2\zeta} \quad \text{i.e.} \quad \zeta = \frac{4-d}{2} \,. \tag{9}$$

5 The Larkin-length

To understand the failure of dimensional reduction, let us turn to an interesting argument given by Larkin [2]. He considers a piece of an elastic manifold of size L. If the disorder has correlation length r, and characteristic force $\bar{f}$, this piece will typically see a force of strength

$$F_{\rm DO} = \bar{f}\left(\frac{L}{r}\right)^{\frac{d}{2}} \,. \tag{10}$$

On the other hand, there is an elastic force, which scales like

$$F_{\rm el} = c\, L^{d-2} \,. \tag{11}$$

These forces are balanced at the *Larkin-length* $L = L_c$ with

$$L_c = \left(\frac{c^2}{\bar{f}^2} r^d\right)^{\frac{1}{4-d}} \,. \tag{12}$$

More important than this value is the observation that in all physically interesting dimensions $d < 4$, and at scales $L > L_c$, the membrane is pinned by disorder; whereas on small scales elastic

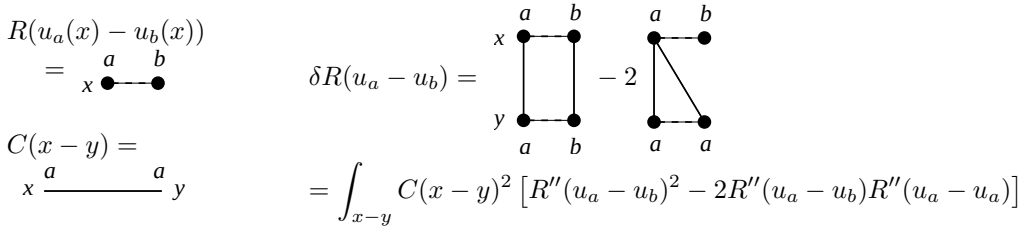


Fig. 2. The disorder vertex $R(u_a(x) - u_b(x))$ and the correlation-function $C(x-y)$, with Fourier-transform $\tilde{C}(k) = \frac{1}{k^2}$, which is diagonal in replica-space (left). Contracting two disorder-vertices with two correlation-functions leads to the two 1-loop contributions δR to the disorder-correlator R (right). The integral $\int_{x-y} C(x-y)^2 = \frac{L^\epsilon}{\epsilon}$, where L is some IR-cutoff.

energy dominates. Since the disorder has a lot of minima which are far apart in configurational space but close in energy (metastability), the manifold can be in either of these minimas, and the ground state is no longer unique. However exactly this is assumed in e.g. the proof of dimensional reduction; as is is most easily seen in its supersymmetric formulation [3].

6 The functional renormalization group (FRG)

Let us now discuss a way out of the dilemma: Larkin's argument suggests that 4 is the upper critical dimension. So we would like to make an $\epsilon = 4 - d$ expansion. On the other hand, dimensional reduction tells us that the roughness is $\zeta = \frac{4-d}{2}$ (see (9)). Even though this is systematically wrong below four dimensions, it tells us correctly that at the critical dimension $d = 4$, where disorder is marginally relevant, the field u is dimensionless. This means that having identified any relevant or marginal perturbation (as the disorder), we find immediately another such perturbation by adding more powers of the field. We can thus not restrict ourselves to keeping solely the first moments of the disorder, but have to keep the whole disorder-distribution function $R(u)$. Thus we need a *functional renormalization group* treatment (FRG). This was first proposed in 1986 by D. Fisher [4]. Performing an infinitesimal renormalization, i.e. integrating over a momentum shell à la Wilson, leads to the flow $\partial_\ell R(u)$, with ($\epsilon = 4 - d$)

$$\partial_\ell R(u) = (\epsilon - 4\zeta)\, R(u) + \zeta u R'(u) + \frac{1}{2} R''(u)^2 - R''(u)R''(0)\,. \tag{13}$$

The first two terms come from the rescaling of R and u respectively. The last two terms are the result of the 1-loop calculations, which are sketched in Fig. 2.

More important than the form of this equation is it actual solution, sketched in Fig. 3.

After some finite renormalization, the second derivative of the disorder $R''(u)$ acquires a cusp at $u = 0$; the length at which this happens is the Larkin-length. How does this overcome dimensional reduction? To understand this, it is interesting to study the flow of the second and forth moment. Taking a derivative of (13) w.r.t. u and setting u to 0, we obtain

$$\partial_\ell R''(0) = (\epsilon - 2\zeta)\, R''(0) + R'''(0)^2 \;\longrightarrow\; (\epsilon - 2\zeta)\, R''(0) \tag{14}$$

$$\partial_\ell R''''(0) = \epsilon R''''(0) + 3R''''(0)^2 + 4R'''(0)R''''(0) \;\longrightarrow\; \epsilon R''''(0) + 3R''''(0)^2\,. \tag{15}$$

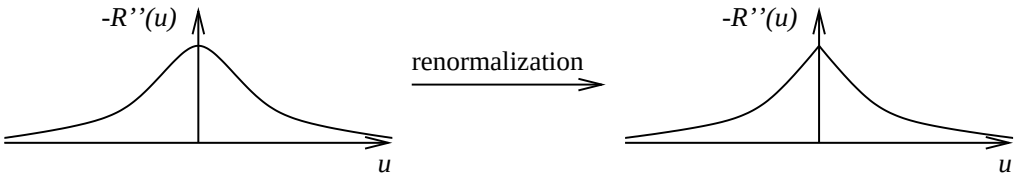


Fig. 3. Change of $-R''(u)$ under renormalization and formation of the cusp.

Since $R(u)$ is an even function, $R'''(0)$ and $R''''(0)$ are 0 and the above equations for $R''(0)$ and $R''''(0)$ are in fact closed. Equation (14) tells us that the flow of $R''(0)$ is trivial and that $\zeta = \epsilon/2 \equiv \frac{4-d}{2}$. This is exactly the result predicted by dimensional reduction. The appearance of the cusp can be inferred from equation (15). Its solution is

$$R''''(0)\big|_\ell = \frac{c\,\mathrm{e}^{\epsilon\ell}}{1 - 3\,c\,(\mathrm{e}^{\epsilon\ell} - 1)\,/\epsilon}\,, \qquad c := R''''(0)\big|_{\ell=0} \tag{16}$$

Thus after a finite renormalization $R''''(0)$ becomes infinite: The cusp appears. By analyzing the solution of the flow-equation (13), one also finds that beyond the Larkin-length $R''(0)$ is no longer given by (14) with $R'''(0)^2 = 0$. The correct interpretation of (14), which remains valid after the cusp-formation, is (for details see below)

$$\partial_\ell R''(0) = (\epsilon - 2\zeta)\,R''(0) + R'''(0^+)^2\,. \tag{17}$$

Renormalization of the whole function thus overcomes dimensional reduction. The appearance of the cusp also explains why dimensional reduction breaks down. The simplest way to see this is by redoing the proof for elastic manifolds in disorder, which in the absence of disorder is a simple Gaussian theory. Terms contributing to the 2-point function involve $R''(0)$, $TR''''(0)$ and higher derivatives of $R(u)$ at $u = 0$, which all come with higher powers of T. To obtain the limit of $T \to 0$, one sets $T = 0$, and only $R''(0)$ remains. This is the dimensional reduction result. However we just saw that $R''''(0)$ becomes infinite. Thus $R''''(0)T$ may also contribute, and the proof fails.

7 Why is a cusp necessary?

The appearance of a cusp might suggest that our approach is fatally ill. Let me present a simple argument, why a cusp *is a physical necessity and not an artifact.* To this aim, consider a toy model with only one Fourier-mode $u = u_q$

$$\mathcal{H}[u] = \frac{1}{2}q^2u^2 + \sqrt{\epsilon}\,\tilde{V}(u)\,. \tag{18}$$

Since equation (13) has a fixed point of order $R(u) \sim \epsilon$ for all $\epsilon > 0$, $V(u)$ scales like $\sqrt{\epsilon}$ for ϵ small and we have made this dependence explicit in (18) by using $V(u) = \sqrt{\epsilon}\tilde{V}(u)$. The only further input comes from the physics: For $L < L_c$, i.e. before we reach the Larkin length, there

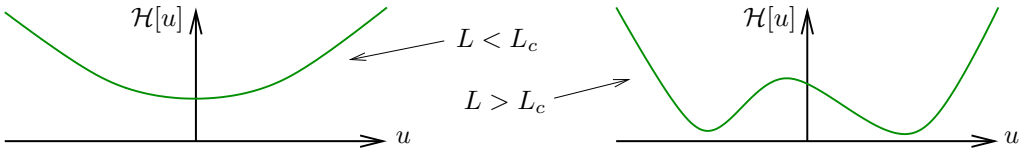


Fig. 4. The toy model (18) before (left) and after (right) the Larkin-scale.

is only one minimum, as depicted in Fig. 4. On the other hand, for $L > L_c$, there are several minima. Thus there is at least one point for which

$$\frac{\mathrm{d}^2}{\mathrm{d}u^2}\mathcal{H}[u] = q^2 + \sqrt{\epsilon}\,\tilde{V}''(u) < 0\,. \tag{19}$$

In the limit of $\epsilon \to 0$, this is possible if and only if $\frac{1}{\epsilon}R''''(0)$, which a priori should be finite for $\epsilon \to 0$, becomes infinite:

$$\frac{1}{\epsilon}R''''(0) = \overline{V''(u)V''(u')}\Big|_{u=u'} = \infty\,. \tag{20}$$

This argument shows that a cusp is indeed a physical necessity.

8 Beyond 1 loop?

Functional renormalization has successfully been applied to a bunch of problems at 1-loop order. From a field theory, we however demand more. Namely that it

- allows for systematic corrections beyond 1-loop order
- be renormalizable
- and thus allows to make universal predictions.

However, this has been a puzzle since 1986, and it has even been suggested that the theory is not renormalizable due to the appearances of terms of order $\epsilon^{\frac{3}{2}}$. Why is the next order so complicated? The reason is that it involves terms proportional to $R'''(0)$. A look at Fig. 3 explains the puzzle. Shall we use the symmetry of $R(u)$ to conclude that $R'''(0)$ is 0? Or shall we take the left-hand or right-hand derivatives, related by

$$R'''(0^+) := \lim_{\substack{u>0\\u\to 0}} R'''(u) = -\lim_{\substack{u<0\\u\to 0}} R'''(u) =: -R'''(0^-). \tag{21}$$

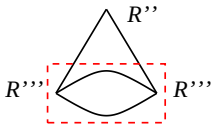
In the following, I will present the solution of this puzzle. First at 2-loop order [5] and then at large N [7]. The latter approach allows for another independent control-parameter, and sheds further light on the cusp-formation.

9 Results at 2-loop order

For the flow-equation at 2-loop order, we find [5]

$$\partial_\ell R(u) = (\epsilon - 4\zeta)\, R(u) + \zeta u R'(u) + \frac{1}{2} R''(u)^2 - R''(u)R''(0) \\ + \frac{1}{2}\left(R''(u) - R''(0)\right) R'''(u)^2 - \frac{1}{2} R'''(0^+)^2 R''(u)\,. \tag{22}$$

The first line is the result at 1-loop order, already given in (13). The second line is new. The most interesting term is the last one, which involves $R'''(0^+)^2$ and which we therefore call *anomalous*. The hard task is to fix the prefactor $(-\frac{1}{2})$. We have up to now invented six algorithms to fix it; one leads to inconsistencies and shall not be reported here. The other five algorithms are consistent with each other: The sloop-algorithm, recursive construction, reparametrization invariance, renormalizability, and potentiality. For lack of space, we restrain our discussion to the last two ones. At 2-loop order the following diagram appears



$$\longrightarrow \frac{1}{2}\left(R''(u) - R''(0)\right) R'''(u)^2 - \frac{1}{2} R''(u) R'''(0^+)^2 \tag{23}$$

leading to the anomalous term. The integral (not written here) contains a subdivergence, which is indicated by the box. Renormalizability demands that its leading divergence (which is of order $1/\epsilon^2$) be canceled by a 1-loop counter-term. The latter is unique thus fixing the prefactor of the anomalous term. (The idea is to take the 1-loop correction δR in Fig. 2 and replace one of the R'' in it by $\delta R''$ itself, which the reader can check to leading to the terms given in (23) plus terms which only involve even derivatives.)

Another very physical demand is that the problem remain potential, i.e. that forces still derive from a potential. The force-force correlation function being $-R''(u)$, this means that the flow of $R'(0)$ has to be strictly 0. (The simplest way to see this is to study a periodic potential.) From (9) one can check that this does not remain true if one changes the prefactor of the last term in (9); thus fixing it.

Let us give some results for cases of physical interest. First of all, in the case of a periodic potential, which is relevant for charge-density waves, the fixed-point function can be calculated analytically as (we choose period 1, the following is for $u \in [0,1]$)

$$R^*(u) = -\left(\frac{\epsilon}{72} + \frac{\epsilon^2}{108} + O(\epsilon^3)\right) u^2(1-u)^2 + \text{const.} \tag{24}$$

This leads to a universal amplitude. In the case of random field disorder (short-ranged force-force correlation function) $\zeta = \frac{\epsilon}{3}$. For random bond disorder (short-ranged potential-potential correlation function) we find numerically $\zeta = 0.20829804\epsilon + 0.006858\epsilon^2$. This compares well with numerical simulations, see Tab. 1.

10 Large N

In the last section, we have discussed renormalization in a loop expansion, i.e. expansion in ϵ. In order to independently check consistency it is good to have another non-perturbative approach.

ζ	one loop	two loop	estimate	simulation and exact
$d=3$	0.208	0.215	0.215 ± 0.01	0.22 ± 0.01 [8]
$d=2$	0.417	0.444	0.42 ± 0.02	0.41 ± 0.01 [8]
$d=1$	0.625	0.687	0.67 ± 0.02	$2/3$

Tab. 1. Results for ζ in the random bond case.

This is achieved by the large-N limit, which can be solved analytically and to which we turn now. We start from

$$\mathcal{H}[\vec{u},\vec{j}] = \frac{1}{2T}\sum_{a=1}^{n}\int_x \vec{u}_a(x)\left(-\nabla^2+m^2\right)\vec{u}_a(x) - \sum_{a=1}^{n}\int_x \vec{j}_a(x)\vec{u}_a(x) \\ -\frac{1}{2T^2}\sum_{a,b=1}^{n}\int_x B\left((\vec{u}_a(x)-\vec{u}_b(x))^2\right) \ . \tag{25}$$

where in contrast to (8), we use an N-component field $\vec{u}$. For $N=1$, we identify $B(u^2)=R(u)$. We also have added a mass m to regularize the theory in the infra-red and a source $\vec{j}$ to calculate the effective action $\Gamma(\vec{u})$ via a Legendre transform. For large N the saddle point equation reads

$$\tilde{B}'(u_{ab}^2) = B'\left(u_{ab}^2 + 2TI_1 + 4I_2[\tilde{B}'(u_{ab}^2)-\tilde{B}'(0)]\right) \tag{26}$$

This equation gives the derivative of the effective (renormalized) disorder $\tilde{B}$ as a function of the (constant) background field $u_{ab}^2 = (u_a - u_b)^2$ in terms of: the derivative of the microscopic (bare) disorder B, the temperature T and the integrals $I_n := \int_k \frac{1}{(k^2+m^2)^n}$.

The saddle-point equation can again be turned into a closed functional renormalization group equation for $\tilde{B}$ by taking the derivative w.r.t. m:

$$\partial_l \tilde{B}(x) \equiv -\frac{m\partial}{\partial m}\tilde{B}(x) = (\epsilon-4\zeta)\tilde{B}(x) + 2\zeta x\tilde{B}'(x) + \frac{1}{2}\tilde{B}'(x)^2 - \tilde{B}'(x)\tilde{B}'(0) + \frac{\epsilon T\tilde{B}'(x)}{\epsilon+\tilde{B}''(0)} \tag{27}$$

This is a complicated nonlinear partial differential equation. It is therefore surprising, that one can find an analytic solution. (The trick is to write down the flow-equation for the inverse function of $\tilde{B}'(x)$, which is linear.) Let us only give the results of this analytic solution: First of all, for long-range correlated disorder of the form $\tilde{B}'(x) \sim x^{-\gamma}$, the exponent ζ can be calculated analytically as $\zeta = \frac{\epsilon}{2(1+\gamma)}$. It agrees with the replica-treatment in [9] and the 1-loop treatment in [10]. Second, it demonstrates that before the Larkin-length, $\tilde{B}(x)$ is analytic and thus dimensional reduction holds. Beyond the Larkin length, $\tilde{B}''(0)=\infty$, a cusp appears and dimensional reduction is incorrect. This shows again that the cusp is not an artifact of the perturbative expansion, but an important property even of the exact solution of the problem (here in the limit of large N).

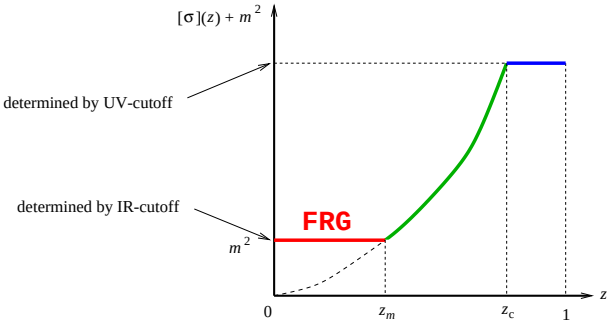


Fig. 5. The function $[\sigma]\,(u) + m^2$ as given in [9].

11 Relation to Replica Symmetry Breaking (RSB)

There is another treatment of the limit of large N given by Mézard and Parisi [9]. They start from (25) but *without* a source-term j. In the limit of large N, a Gaussian variational ansatz of the form

$$\mathcal{H}_{\rm g}[\vec{u}] = \frac{1}{2T}\sum_{a=1}^{n}\int_x \vec{u}_a(x)\left(-\nabla^2+m^2\right)\vec{u}_a(x) - \frac{1}{2T^2}\sum_{a,b=1}^{n}\sigma_{ab}\,\vec{u}_a(x)\vec{u}_b(x) \tag{28}$$

becomes exact. The art is to make an appropriate ansatz for σ_{ab}. The simplest possibility, $\sigma_{ab} = \sigma$ for all $a \neq b$ reproduces the dimensional reduction result, which breaks down at the Larkin length. Beyond that scale, a replica symmetry broken (RSB) ansatz for σ_{ab} is suggestive. To this aim, one can break σ_{ab} into four blocks of equal size, choose one value for the both outer diagonal blocks, and then iterate the procedure on the diagonal blocks. One finds that the more often one iterates, the better the results. In fact, one has to repeat this procedure infinite many times. This seems like a hopeless endeavor, but Parisi has shown that the infinitely often replica symmetry broken matrix can be parameterized by a function $[\sigma](z)$ with $z \in [0,1]$. In the SK-model, z has the interpretation of an overlap between replicas. While there is no such simple interpretation for the model (28), we retain that $z = 0$ describes distant states, whereas $z = 1$ describes nearby states. The solution of the large-N saddle-point equations leads to the curve depicted in Fig. 5. Knowing it, the 2-point function is given by $\langle u_k u_{-k}\rangle = \frac{1}{k^2}\left(1 + \int_0^1 \frac{{\rm d}z}{z^2}\,\frac{[\sigma](z)+m^2}{k^2+[\sigma](z)+m^2}\right)$. The important question is: What is the relation between the two approaches, which both pretend to calculate the same 2-point function? Comparing the analytical solutions, we find that the 2-point function given by FRG is the same as that of RSB, if in the latter expression we only take into account the contribution from the most distant states, i.e. those for z between 0 and z_m (see Fig. 5). To understand why this is so, we have to remember that the two calculations were done under quite different assumptions: In contrast to the RSB-calculation, the FRG-approach calculated the partition function in presence of an external field j, which was then used to give via a Legendre transformation the effective action. Even if the field j is finally turned to 0, the system might remember its preparation, as is the case for a magnet: Preparing the system in presence of a magnetic field will result in a magnetization which aligns with this field. The magnetization will

remain, even if finally the field is turned off. The same phenomena happens here: By explicitly breaking the replica-symmetry through an applied field, all replicas will settle in distant states, and the close states from the Parisi-function $[\sigma](z) + m^2$ (which describes *spontaneous* RSB) will not contribute. However the full RSB-result can be reconstructed by remarking that the part of the curve between z_m and z_c is independent of the infrared cutoff m, and then integrating over m [7]. We also note that a similar effective action has been proposed in [11]. While it agrees qualitatively, it does not reproduce the correct FRG 2-point function, as it should.

12 Perspectives

More interesting problems have been treated by the above methods, and much more has to be done. Besides equilibrium problems, driven systems are also studied experimentally. An example is the domain wall in a random field magnet, driven through the system by an applied magnetic field. This was treated in [12, 13], and it was concluded that for non-periodic disorder, there is only one fixed point, describing both random bond and random field disorder. Our 2-loop calculations [5, 6] present the first consistent field theory, capable to distinguish between statics and dynamics. They also show that a conjecture by [13] that $\zeta = \frac{\epsilon}{3}$ be exact to all orders is violated at second order. We have applied the same methods to the statics at 3-loop order and to the random field problem. An expansion in $1/N$, (by now we have obtained the effective action), should allow to finally describe such notorious problems as the strong-coupling phase of the Kardar-Parisi-Zhang equation. Finally, it is still open of whether FRG can also be applied to spin glasses as e.g. the SK-model. We leave this problem as a challenge to the reader.

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