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AGING IN FERROMAGNETIC SYSTEMS AT CRITICALITY NEAR FOUR DIMENSIONS[1]

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We report on some results concerning the off-equilibrium response and correlation functions and the corresponding fluctuation-dissipation ratio for a purely dissipative relaxation of an $O(N)$ symmetric vector model (Model A) in d dimensions below the upper critical dimension. The scaling behaviour of these quantities is analyzed and the associated universal functions are given at first order in $\epsilon = 4 - d$ in the high-temperature phase and at criticality. A non trivial limit of the fluctuation-dissipation ratio is found at criticality.

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1 Introduction

It is well known that complex systems such as glasses, spin glasses and generally disordered systems with quenched disorder show very interesting dynamical behaviours depending on temperature and time–scale ranges [1]. One of the most striking is that of *aging*, i. e. physical properties of the system depend on its thermal history. This is mainly due to the fact that the system, say, for example, a spin glass at low temperature, does not reach thermal equilibrium even after a "macroscopic" time has elapsed since the last perturbation on the system. But this kind of behaviour is not only restricted to disordered systems [2]. Indeed consider a ferromagnetic model in a disordered state for the initial time $t = 0$, and quench it to a given temperature $T \geq T_c$ [4]. During the relaxation a small external field h is applied at $\mathbf{x} = 0$ after a waiting time s. At time t the order parameter response to h is given by the response function $R_{\mathbf{x}}(t, s) = \delta\langle\phi_{\mathbf{x}}(t)\rangle/\delta h(s)$, where ϕ is the order parameter and $\langle\cdot\rangle$ stands for the mean over the stochastic dynamics. The two-time correlation function of order parameter fluctuations is another quantity of interest, given by $C_{\mathbf{x}}(t, s) = \langle\phi_{\mathbf{x}}(t)\phi_{\mathbf{0}}(s)\rangle$. The time evolution of the system may be characterized by two different regimes: a transient behaviour with off-equilibrium evolution, for $t < t_R$, and a stationary

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[4] We are not interested here in the problem of phase ordering dynamics, so we will not discuss the effects of a quench to $T < T_c$.

Model		$T < T_c$	$T = T_c$	$T > T_c$
Random Walk[a]	[2]	—	$1/2$	—
Free Gaussian Field[a]	[2]	—	$1/2$	1
d-dim. Spherical Model[a]	[8]	0	$(d-2)/d^\dagger$	1
Ising–Glauber Chain[a]	[7]	—	$1/2$	1
2-dim. Ising Model[b]	[8]		$0.26(1)$	
3-dim. Ising Model[b]	[8]		0.40	

Tab. 1. Values of X^∞ in some models. [a] Exact solution, [b] Monte Carlo simulations. † $2 < d < 4$.

equilibrium evolution for $t > t_R$, where t_R is the relaxation time. In the former a dependence of the behaviour of the system on initial conditions is expected, while in the latter homogeneity of time and time reversal symmetry (at least in absence of external fields) are recovered; as a consequence we expect that for $t_R \ll s, t$, $R_{\mathbf{x}}(t,s) = R^{eq}_{\mathbf{x}}(t-s)$, $C_{\mathbf{x}}(t,s) = C^{eq}_{\mathbf{x}}(t-s)$ where R^{eq} and C^{eq} are determined by the "equilibrium" dynamics of the system, with a characteristic time scale diverging at the critical point (critical slowing down). Moreover the fluctuation-dissipation theorem states that

$$R^{eq}_{\mathbf{x}}(\tau) = -\frac{1}{T}\frac{\mathrm{d}C^{eq}_{\mathbf{x}}(\tau)}{\mathrm{d}\tau} \quad . \tag{1}$$

When the system does not reach the equilibrium all the previous functions will depend both on s (the "age" of the system) and t.

To characterize the distance from equilibrium of an aging system, evolving at a fixed temperature T, the fluctuation-dissipation ratio (FDR) is usually introduced [2]:

$$X_{\mathbf{x}}(t,s) = \frac{T\, R_{\mathbf{x}}(t,s)}{\partial_s C_{\mathbf{x}}(t,s)} \ , \tag{2}$$

In recent years, several works [1–5] have been devoted to the study of the FDR for systems exhibiting domain growth [6], or for aging systems such as glasses and spin glasses, showing that in the low-temperature phase $X(t,s)$ turns out to be a non-trivial function of its two arguments. In particular analytical and numerical studies indicate that the limit

$$X^\infty_{\mathbf{x}=0} = \lim_{s\to\infty}\lim_{t\to\infty} X_{\mathbf{x}=0}(t,s), \tag{3}$$

vanishes throughout the low-temperature phase both for glasses and simple ferromagnetic systems [3].

Only recently [2,7–10] attention has been paid to the FDR, for non-equilibrium, non-disordered, and unfrustrated systems at criticality. It has been argued that the FDR (3) is a novel universal quantity of non-equilibrium critical dynamics. The value of $X^\infty_{\mathbf{x}=0}$ has been determined for the models reported in Tab. 1. In all cases $X^\infty_{\mathbf{x}=0}$ has values ranging between 0 and $\frac{1}{2}$ while for some urn models a different range has been found [11]. Also the scaling form for $R_{\mathbf{x}=\mathbf{0}}(t,s)$ was rigorously established using conformal invariance [12].

It is easy to realize from the above mentioned works and from models listed in Tab. 1 that only exact solutions have been analytically determined so far for the FDR, with the drawbacks that only simple systems have already been dealt with. In this work we report on our investigation of the non-equilibrium correlation and response functions and the associated FDR for the $O(N)$ ferromagnetic model with purely dissipative relaxation dynamics (Model A) both at the critical point and in the high-temperature phase, using a field-theoretical approach. A more detailed discussion may be found in Ref. [13].

Let us consider the purely dissipative relaxation dynamics of a N-component field $\varphi(\mathbf{x},t)$ described by the stochastic Langevin equation

$$\partial_t\varphi(\mathbf{x},t) = -\Omega\frac{\delta\mathcal{H}[\varphi]}{\delta\varphi(\mathbf{x},t)} + \xi(\mathbf{x},t)\ , \tag{4}$$

where $\mathcal{H}[\varphi]$ is the Landau-Ginzburg Hamiltonian

$$\mathcal{H}[\varphi] = \int \mathrm{d}^d x \left[\frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}r_0\varphi^2 + \frac{1}{4!}g_0\varphi^4\right], \tag{5}$$

Ω the kinetic coefficient, and $\xi(\mathbf{x},t)$ a zero-mean stochastic Gaussian noise with

$$\langle\xi_i(\mathbf{x},t)\xi_j(\mathbf{x}',t')\rangle = 2\Omega\,\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')\delta_{ij}. \tag{6}$$

The equilibrium correlation functions, generated by the Langevin equation (4) and averaged over the noise ξ, can be obtained by means of the field-theoretical action [14]

$$S[\varphi,\tilde\varphi] = \int \mathrm{d}t \int \mathrm{d}^d x \left[\tilde\varphi\frac{\partial\varphi}{\partial t} + \Omega\tilde\varphi\frac{\delta\mathcal{H}[\varphi]}{\delta\varphi} - \tilde\varphi\Omega\tilde\varphi\right]. \tag{7}$$

where $\tilde\varphi(\mathbf{x},t)$ is the response field.

In Ref. [15] this formalism was extended in order to incorporate a macroscopic initial condition into Eq. (7): one has also to average over the initial configuration $\varphi_0(\mathbf{x}) = \varphi(\mathbf{x}, t=0)$ with a weight $e^{-H_0[\varphi_0]}$ given by

$$H_0[\varphi_0] = \int \mathrm{d}^d x\,\frac{\tau_0}{2}(\varphi_0(\mathbf{x}) - a(\mathbf{x}))^2. \tag{8}$$

This specifies an initial state $a(\mathbf{x})$ with correlations proportional to τ_0^{-1}. In this way all response and correlation functions may be obtained, following standard methods [14], by a perturbative expansion of the functional weight $e^{-(S[\varphi,\tilde\varphi]+H_0[\varphi_0])}$.

1.1 Scaling forms

When a ferromagnetic system is quenched from a disordered initial state to its critical point, the correlation length grows as $t^{1/z}$, where z is the dynamical critical exponents. So in momentum space, applying standard scaling arguments, all the universal functions depend only on the two products $q^z\,t$ and $q^z\,s$ [5]. In particular we expect the scaling forms [8, 12, 15]

$$R_q(t,s) = q^{-2+\eta+z}\left(\frac{t}{s}\right)^{\theta} F_R(\Omega q^z(t-s), t/s)\ , \tag{9}$$

$$C_q(t,s) = q^{-2+\eta}\left(\frac{t}{s}\right)^{\theta} F_C(\Omega q^z(t-s), t/s)\ , \tag{10}$$

[5] When finite volume effects and a nonvanishing initial magnetization are taken into account, new time scales emerge in this picture. For an exhaustive analysis see Ref. [16].

where θ is the initial-slip exponent of response function [15]. The functions $F_R(y,x)$ and $F_C(y,x)$ are universal apart from the normalizations for small arguments.

For future reference, let us introduce the ratio

$$\mathcal{X}_{\mathbf{q}} = \frac{\Omega R_{\mathbf{q}}(t,s)}{\partial_s C_{\mathbf{q}}(t,s)} \quad \text{and} \quad \mathcal{X}^{\infty}_{\mathbf{q}=0} = \lim_{s\to\infty}\lim_{t\to\infty}\mathcal{X}_{\mathbf{q}=0}(t,s)\ , \tag{11}$$

whose relation with $X_{\mathbf{x}}$ is discussed in Sec. 3.

1.2 Gaussian FDR

Response and correlation functions are eactly known for the Gaussian Model [15], so that the FDR can be determined (in [2] the related quantity $X_{\mathbf{x}}$ has been considered, see Sec. 3), finding the tree-level expression

$$\mathcal{X}_q(t,s)|_{t.l.} = \left(1 + e^{-2\Omega(q^2+r_0)s} + \Omega q^2 \tau_0^{-1} e^{-2\Omega(q^2+r_0)s}\right)^{-1} . \tag{12}$$

If the theory is off-critical ($r_0 \neq 0$) the limit of this ratio for $s\to\infty$ is 1 for all values of q, in agreement with the idea that in the high-temperature phase all modes have a finite equilibration time, so that equilibrium is recovered and as a consequence the fluctuation-dissipation theorem applies. For the critical theory, i.e. $r_0 \propto T - T_c = 0$, if $q \neq 0$ the limit ratio is again equal to one, whereas for $q=0$ we have $\mathcal{X}^0_{q=0}(t,s) = 1/2$. This analysis clearly shows that the only mode characterized by aging, i.e. that "does not relax" to the equilibrium, is the zero mode in the critical limit.

2 One-loop FDR

We report here the results for the non-equilibrium response and correlation functions of the model described in Sec. 1 at one-loop order in an $\epsilon = 4 - d$ expansion. They have been determined by using the method of renormalized field theory in the minimal subtraction scheme. All details may be found in Ref. [13]. Introducing

$$f(v) = 2\left[\int_0^v \mathrm{d}\xi\ \ln\xi\, e^{\xi} + (1-e^{v})\ln v\right] \quad \text{and} \quad P_N = \frac{N+2}{N+8}\ , \tag{13}$$

we get for the critical theory $r_0 = 0$, at the IR fixed point for the renormalized coupling constant,

$$F_R(y,x) = e^{-y} + O(\epsilon^2)\ , \tag{14}$$

$$F_C(y,x) = e^{-y} - \left[1 + \epsilon\frac{P_N}{4} f\left(\frac{2y}{x-1}\right)\right] e^{-y\frac{x+1}{x-1}} + O(\epsilon^2)\ . \tag{15}$$

Computing the derivative with respect to s of the two-time correlation function and taking its ratio with the response function we have

$$\mathcal{X}_q^{-1}(s) = 1 + e^{-2q^2 s} - \frac{P_N\epsilon}{4} e^{-2q^2 s}\left[\frac{e^{2q^2 s}-1}{q^2 s} - f(2q^2 s) + 2f'(2q^2 s)\right] + O(\epsilon^2)\ . \tag{16}$$

The limit of the FDR for $s \to \infty$ is equal to 1 for all $q \neq 0$. Instead for $q = 0$ we have (using Eq. (13))

$$\mathcal{X}^{\infty}_{\mathbf{q}=0} = \frac{1}{2}\left(1 - \frac{\epsilon}{4}\frac{N+2}{N+8}\right) + O(\epsilon^2) \; , \tag{17}$$

which is the result we were interested in. Taking into account the effect of the mass r_0 (deviation from critical temperature), one finds that $\mathcal{X}^{\infty}_q$ is equal to 1 for all q in the high temperature phase [13].

3 Discussion

In this work we reported some results for the FDR, as defined in (11), for the N-vector model with Model A dynamics, both at criticality and in the high-temperature phase. The main result is that the ratio $\mathcal{X}^{\infty}_q$ is always 1 unless at criticality for $q = 0$, when it takes the value given in Eq. (17).

To compare our result with some particular limit considered in the literature [2, 8] we have to relate $\mathcal{X}^{\infty}_q$ to the analog in the real $\mathbf{x}$ space. The following heuristic argument may be useful to realize that the two ratios are exactly equal, i.e. $X^{\infty}_{\mathbf{x}=0} = \mathcal{X}^{\infty}_{\mathbf{q}=0}$. We may rewrite the FDR in real $\mathbf{x}$ space as a mean value of that in momentum space with a weight given by $R_{\mathbf{q}}$:

$$X^{-1}_{\mathbf{x}=0} \equiv \frac{\int \mathrm{d}^d q \, \partial_s C_{\mathbf{q}}(t,s)}{T \int \mathrm{d}^d q \, R_{\mathbf{q}}(t,s)} = \frac{\int \mathrm{d}^d q \, R_{\mathbf{q}}(t,s) \, \frac{\partial_s C_{\mathbf{q}}(t,s)}{T R_{\mathbf{q}}(t,s)}}{\int \mathrm{d}^d q \, R_{\mathbf{q}}(t,s)} = \langle \mathcal{X}^{-1}_{\mathbf{q}} \rangle_{R_{\mathbf{q}}} \; . \tag{18}$$

Now, since we expect $R_{\mathbf{q}} \propto e^{-q^2(t-s)}$, in the limit $s, t \to \infty$ (in the right order) $X^{-1}_{\mathbf{x}=0}$ will take contributions only from the $q = 0$ mode, i.e. apart a normalization, the weight function $R_{\mathbf{q}}$ is a $\delta(\mathbf{q})$.

In the limit $N \to \infty$ Eq. (17) reduces to $X^{\infty} = 1/2 - \epsilon/8 + O(\epsilon^2)$ that is the same as the expansion of the result for the Spherical Model near four dimension (see Tab. 1). Let us now make a comparison with Monte Carlo results listed in Tab. 1. For $N = 1$ and $\epsilon = 2$, $X^{\infty} = 5/12 < 1/2$ which is qualitatively in agreement with what has been found for the Ising model in $d = 2$. Setting, instead, $\epsilon = 1$, one obtains $11/24 \sim 0.46$, which is fairly in good agreement with the result for the Ising Model in $d = 3$. To have a reliable quantitative prediction the knowledge of higher loop contributions is required. A two-loop computation improves the previous estimates [17].

This work may be easily extended to more realistic models than those previously considered in literature, contributing to the understanding of out-of-equilibrium dynamic phenomena, currently under intensive investigation, by means of the powerful tools of perturbative field theory.

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References

[1] E. Vincent, J. Hammann, M. Ocio, J. P. Bouchaud, L. F. Cugliandolo: in *Complex Behavior of Glassy Systems*, Springer Lecture Notes in Physics **492** (1997) 184
J. P. Bouchaud, L. F. Cugliandolo, J. Kurchan, M. Mézard: in *Spin Glasses and Random Fields*, Directions in Condensed Matter Physics, vol. **12**, edited by A. P. Young; World Scientific, Singapore, 1998

[2] L. F. Cugliandolo, J. Kurchan, G. Parisi: *J. Phys. I (France)* **4** (1994) 1641

[3] L. F. Cugliandolo, J. Kurchan, L. Peliti: *Phys. Rev.* **E 55** (1997) 3898,
L. F. Cugliandolo, J. Kurchan: *J. Phys.* **A 27** (1994) 5749,
L. F. Cugliandolo, D.S. Dean: *J. Phys.* **A 28** (1995) 4213,
W. Zippold, R. Kühn, H. Horner: *Eur. Phys. J.* **B 13** (2000) 531,
S. A. Cannas, D. A. Stariolo, F. A. Tamarit: *Physica* **A 291** (2001) 362

[4] S. Franz, M. Mézard, G. Parisi, L. Peliti: *Phys. Rev. Lett.* **81** (1998) 1758,
J. Stat. Phys. **97** (1999) 459

[5] F. Corberi, E. Lippiello, M. Zannetti: *Phy. Rev.* **E 63** (2001) 061506; ibid. **65** (2002) 046136

[6] A. J. Bray: *Adv. Phys.* **43** (1994) 357,
T. J. Newman, A. J. Bray: *J. Phys.* **A 23** (1990) 4491,
L. Berthier, J.L. Barrat, J. Kurchan: *Eur. Phys. J.* **B 11** (1999) 635,
A. Barrat: *Phys. Rev.* **E 57** (1998) 3629

[7] E. Lippiello, M. Zannetti: *Phys. Rev.* **E 61** (2000) 3369,
C. Godrèche, J.M. Luck: *J. Phys.* **A 33** (2000) 1151

[8] C. Godrèche, J. M. Luck: *J. Phys.* **A 33** (2000) 9141

[9] L. Berthier, P. C. W. Holdsworth, M. Sellitto: *J. Phys.* **A 34** (2001) 1805

[10] C. Godrèche, J. M. Luck: *J. Phys. Cond. Matter* **14** (2002) 1589

[11] C. Godrèche, J. M. Luck: *Eur. Phys. J.* **B 23** (2001) 473,
J. Phys. Cond. Matter. **14** (2002) 1601

[12] M. Henkel, M. Pleimling, C. Godrèche, J.M. Luck: *Phys. Rev. Lett.* **87** (2001) 265701

[13] P. Calabrese, A. Gambassi: *Phys. Rev.* **E 65** (2002) 066120

[14] R. Bausch, H.K. Janssen, H. Wagner: *Z. Phys.* **B 24** (1976) 113

[15] H. K. Janssen, B. Schaub, B. Schmittmann: *Z. Phys.* **B 73** (1989) 539
H. K. Janssen: in *From Phase Transitions to Chaos—Topics in Modern Statistical Physics*, edited by G. Györgyi, I. Condor, L. Saswári, T. Tel; World Scientific, Singapore, 1992

[16] H. W. Diehl, U. Ritschel: *J. Stat. Phys.* **73** (1993) 1,
U. Ritschel, H.W. Diehl: *Phys. Rev.* **E 51** (1995) 5392,
Nucl. Phys. **B 464** [FS] (1996) 512

[17] P. Calabrese, A. Gambassi: in preparation.