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RENORMALIZATION GROUP APPROACH AND UNIVERSALITY CLASSES FOR 3D SPIN GLASS MODELS[1]

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A Renormalization Group approach to disordered spin systems is presented, based on a real space coarse graining of the overlap distribution. Universality classes are defined through generalized disorder distribution, thus including a large set of models such as Ising ferromagnet, Gaussian and Z_2 spin glasses, fully frustrated models. The relations between Z_2 gauge models and spin glasses find a natural framework within this context. The approach is supported by Monte Carlo renormalization group computations in three dimensions. Precise estimates of the critical temperature and indexes in the difficult 3D case are also obtained with moderate computer time.

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1 The spin glass model and the overlap field

The present work is based on paper [1]. Let σ_x be Ising spins located at the sites of a d-dimensional cubic lattice Λ, with L points on each side ($x \in \Lambda \subset \mathbb{Z}^d$). The typical spin glass model of Edwards and Anderson [2] is defined by the following Hamiltonian

$$H(J,\, \sigma) = -\sum_{\langle x,\, y\rangle} J_{xy}\sigma_x\sigma_y; \tag{1}$$

Where the sum is over couples of neighboring sites, and the quenched disordered interactions J_{xy} are random variables. We will denote by E the expectation on J variables, and define $E(J_{xy}) = 0,\ E(J_{xy}^2) = 1$. We will usually assume periodic boundary conditions. The Boltzmann–Gibbs measure on the spin variables will be denoted by angular brackets $\langle\cdot\rangle$. Expectation on the disorder is taken only after Boltzmann averages are calculated, and the thermodynamic limit for the appropriate quantities is eventually taken afterwords. Define the *overlap field*, $q_x^{(a,b)} = \sigma_x^{(a)}\sigma_x^{(b)} \in \mathbb{Z}_2$, where the replicas $\sigma^{(a)}$, $a = 1, \ldots, s$ are independently and identically distributed according the Boltzmann measure previously defined. The probability

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distribution of the overlap fields, μ, can be implicitly defined through the overlap expectations, involving both the thermal average and the average E over disorder. For any smooth function F define

$$\langle\!\langle F\left(q_x^{(12)}, q_y^{(23)}, \dots\right)\rangle\!\rangle = E\left[\langle F\left(\sigma_x^{(1)}\sigma_x^{(2)},\, \sigma_y^{(2)}\sigma_y^{(3)}, \dots\right)\rangle\right], \tag{2}$$

where expectation with respect to the μ distribution is denoted by $\langle\!\langle\cdot\rangle\!\rangle$. All physical observables can be expressed in terms of overlap observables, so that the full physical meaning of these models is contained in the overlap probability measure. Let us introduce the total overlap (or simply "overlap"), $q^{(a,b)} = \sum_x q_x^{(a,b)}/|\Lambda|$. Another interesting observable is the two points, connected correlation function of the overlap field, $C(r) = \sum_x \langle\!\langle q_x^{(1,2)} q_{x+r}^{(1,2)}\rangle\!\rangle_c/|\Lambda|$.

2 A renormalization group for spin glass models

As is well known, the total overlap is the order parameter of the model. In the high temperature phase, including the critical point, it should be zero in probability in the infinite volume limit, while in the low temperature phase it is expected to fluctuate [3, 4]. Therefore, the critical point can be characterized by a divergence of the correlation length associated to $C(r)$. For this reason, it is natural to define the Renormalization Group transformation [5, 6] on the overlap field. In the following we will consider only two replicas, and omit the replica indexes on the overlap variables. Let $B_y^n \subset \Lambda$ be the cube of side n, centered on y, and $\Lambda_n \subset \mathbb{Z}^d$ a cubic lattice of side L/n. Introduce on Λ_n the new random field

$$q_x^n = (R_{\rho,n} q)_x = n^{-\rho d/2} \sum_{x' \in B_{nx}^n} q_{x'}, \tag{3}$$

where $1 \le \rho < 2$. The probability distribution of the coarse-grained overlap field is, thus, given by

$$\mu_{\rho,n}(q^n) = (\mathcal{R}_{\rho,n}\mu)(q^n) = \sum_{\{q\}} \mu(q) \prod_{x \in \Lambda_n} \delta(q_x^n,\, (R_{\rho,n} q)_x), \tag{4}$$

where $\delta(a,\, b)$ is 1 for $a = b$ and zero otherwise. The sum on the r.h.s. runs over all the $2^{|\Lambda|}$ configurations of the overlap field. The transformation has the semi-group property $\mathcal{R}_{\rho,n_1}\mathcal{R}_{\rho,n_2} = \mathcal{R}_{\rho,n_1 n_2}$. The renormalization group (RG) transformation is naturally defined on the probability distribution of the overlap field. For the purpose of calculations this distribution is well characterized by the expectations of a set of (translation-invariant) observables. It is the approach that has been followed in this work. However, the question may be raised of how to express such a distribution in terms of physically meaningful parameters. It is clear that the traditional exponential form, $\exp(-H)$, has no physical relevance, due to the involved definition of the overlap field distribution in the original model. To state the question in a different way: which additional interactions between the microscopic variables are raised as a result of the RG transformation ? The following parameterization was fruitfully used in [7]. The Hamiltonian, given by (1), is kept fixed, while the disorder is distributed according to a general $\mathbb{Z}_2$ gauge action: $\rho_K(J) = \exp\left(\sum_i K_i W_i(J)\right)$, where the $K_i \in \mathbb{R}$ are parameters and the W_i's are Wilson's

loops, *i.e.*, products of J's along closed paths. Then, the overlap field distribution, $\mu(K)$, is defined, depending on the set of parameters K's. We will discuss it in more detail in section 3 of this paper.

To carry out useful RG calculations an approximation scheme must be chosen. In this work we have used the Monte Carlo approach, for which the main approximation is the use of a finite lattice. The basic effect is that of neglecting the influence of distant regions of the lattice on each other in the calculation of $\mu_{\rho,n}(q^n)$ from $\mu(q)$. Bell and Wilson [8] underline that "the validity of the approximation rests on the same foundation as the renormalization group as a whole". The main source of systematic errors comes from the fact that on the small lattice Λ_n only a few correlation functions are defined, *i.e.*, only fairly simple forms are possible for $\mu_{\rho,n}$. This is known as the truncation error.

An important characteristic of the linear RG transformation defined in (3–4) is the dependence on the parameter ρ. The full RG equations lead to a fixed point of the transformation only for a certain value of this parameter, but in the finite-lattice approximation the value is not unique, and the critical exponents will depend on the particular choice of the parameter [8]. The traditional recipe for MCRG calculations on Ising-like models consists in a non-linear transformation with majority rule (see for instance [9, 10]). Because of the $\mathbb{Z}_2$ symmetry of the overlap field, one can apply it to spin glasses. In the present work, however, we have adopted a different strategy [7, 11]. Consider the linear transformation (3–4). Because of the $\mathbb{Z}_2$ symmetry, some properties of the renormalized distribution $\mu_{\rho,n}$ do not depend on ρ. The overlap field may be decomposed as a product of two random fields, $q_x^n = s_x^n \xi_x^n$, where $s_x^n \in \mathbb{Z}_2$, and $\xi_x^n \in \mathbb{R}$ is non-negative. Let the corresponding distribution be $\hat{\mu}(s, \xi) = \mu(q)$. Upon integration of the ξ field, one obtains the following distribution for the field s:

$$\psi_n(s) = \int (\textstyle\prod_x d\xi_x)\ \hat{\mu}_{\rho,n}(s, \xi) = \sum_{\{q\}} \mu_{\rho,n}(q) \prod_{x\in\Lambda_n} \delta(s_x^n, \operatorname{sign}(q_x)). \tag{5}$$

This defines the linear application Θ of the distributions of real random fields on the distributions of $\mathbb{Z}_2$ random fields, *i.e.*, $\psi = \Theta\mu$. Clearly, ψ_n will not depend on the renormalization parameter ρ, as a rescaling of the overlap field would affect only the ξ variables. Then, looking at the distribution ψ_n we may observe the RG flow of the linear transformation (3–4).

Notice that ψ_n is very different from the distribution obtained through a non-linear transformation with majority rule. Indeed, for k iterations, we would have $\tilde{\mathcal{R}}_m = (\Theta\mathcal{R}_r)^k$ (where $m = r^k$), while $\psi_n = \Theta\mathcal{R}_n$. Moreover, the fact that the distribution ψ_n is defined for any integer n, and not only for integer powers of r, will be a great advantage in the study of spin glasses, as Monte Carlo computations are feasible only for very small lattice sizes. One may question whether this coarse-grained characterization of the RG flow, and of the fixed point, keeps sufficient details to discern physical informations. In the cases presented hereafter the answer is positive, whithin statistical errors. The basin of attraction of a fixed point is defined as the set of overlap field distributions μ^c such that $\psi_n^c \equiv \Theta\mathcal{R}_n\mu^c \to \psi^*$ for $n \to \infty$. The critical exponents may be computed, with the traditional techniques, from the dependence of ψ_n on the initial distribution μ in the vicinity of the critical surface.

3 Spin models with quenched gauge-field interactions

Consider the E-A spin Hamiltonian (1), and assume the quenched interactions are distributed according to a general, gauge-invariant distribution function. Expectation with respect to this distribution will be denoted by E_K. A fairly large set of models may be defined in this way, including, for instance, the Gaussian and the $\mathbb{Z}_2$ E.-A. models. For the sake of simplicity consider the following disorder distribution:

$$\rho_K(J) = C_K \, e^{K_3 \sum_\alpha \Box_\alpha}, \quad J_{xy} = \pm 1, \tag{6}$$

where $C_K \in \mathbb{R}$ is a normalization constant, and the symbol $\Box$ denotes the plaquette terms of the kind $J_{x,y}J_{y,z}J_{z,w}J_{w,x}$. Considering the disorder variables only, this is the well known pure gauge $\mathbb{Z}_2$ model [12]. Let us denote by w_γ the product of the J's along a closed path γ, $w_\gamma = \prod_\gamma J_{xy} = \prod_{\alpha \in S} \Box_\alpha$, where S is a surface bounded by γ. Two canonical asymptotic regimes may be distinguished: a *weak decrease* regime, characterized by $E(w_\gamma) \approx e^{-L}$; a *strong decrease* regime, where $E(w_\gamma) \approx e^{-A}$. L and A denote the perimeter of γ and the area of the surface S. In various models a transition from the strong to the weak decrease is observed as the average plaquette increases. The model (6), in 3 dimensions, exhibits a continuous phase transition at $K_3 = K_3^c \simeq 0.7613$. The corresponding order parameter is the square Polyakov loop, $p^2 = (p_1^2 + p_2^2 + p_3^2)/3$, where p_i is the average Wilson loop along a path that closes itself exploiting the periodicity of the b.c.'s in direction i. The quantity $E(w_\gamma)$ is relevant for the spin system associated to the gauge field [13], as it is related to the average frustration f on the range γ, according to $f_\gamma = (1 - E(w_\gamma))/2$. The qualitative behaviour of the associated spin system is expected to be as follows [7]. In the limit $K_3 \to +(-)\infty$ the Ising (fully frustrated) model is obtained, provided $p_1 = p_2 = p_3 = 1$. For $-K^c < K_3 < K^c$ the disorder is in the confined phase, characterized by strong decrease regime. The average plaquette is approximated by $E_{K_3}(\Box) \approx K_3$, except in the region very close to the transition. However, frustration decreases rapidly to 1/2 for increasing range. Spin glass behaviour is expected. In the range $K^c < K_3 < +\infty$ the disorder is in the deconfined phase and weak decrease regime applies: frustration effects are very small locally ($E_{K_3}(\Box) \approx 1$) and increase weakly at long range. The spin system is expected to be ferromagnetic. These different qualitative regimes should correspond to different universality classes of the spin system, that will be characterized in terms of different fixed points.

4 Monte Carlo Renormalization Group

We have exploited the RG approach introduced so far to analyze the critical point of the generalized $\mathbb{Z}_2$ spin glass in dimension $d = 3$, at $K_3 = 0,\, 0.3,\, 0.8$. We have performed dynamical Monte Carlo computations, employing Parallel Tempering algorithm (MC-PT) and the technique of multi-spin coding. We have chosen to analyze the system down to a temperature $T \approx 0.9 T_c$, in order to distinguish also from an eventual merging of the curves characterizing a Kosterlitz-Thouless critical point. Due to the very long computer time required to reach ergodicity at these temperatures, and the ordinary computational resources employed (mainly 8 PCs, with AMD K7 cpu at 550MHz), we have been limited to lattice sizes $\leq 16^3$. This strongly affects the precision of the results. Nevertheless, the estimates obtained for T_c and ν are competitive with

K_3	L_2 *vs.* L_1	T_c	ν
0	12 *vs.* 8	1.225 (10)	1.89 (9)
	16 *vs.* 8	1.221 (12)	1.96 (12)
	16 *vs.* 12	1.215 (25)	2.11 (34)
0.3	12 *vs.* 8	1.693 (35)	1.86 (20)
	16 *vs.* 8	1.655 (24)	1.95 (15)
	16 *vs.* 12	1.61 (8)	2.2 (8)
0.8	12 *vs.* 8	4.483 (1)	0.659 (1)
	16 *vs.* 8	4.481 (1)	0.661 (2)
	16 *vs.* 12	4.479 (3)	0.663 (3)

Tab. 1. Estimates of the critical temperature (T_c) and the critical index ν.

those obtained with different techniques based on finite size scaling on much larger lattices and employing dedicated machines or supercomputers [14, 15] Thermalization and ergodicity were tested for each J sample by two traditional criteria [16]: the distribution of the total overlap must be even, and the time spent by the MC-PT run at each temperature must be approximately the same [1]. This last criteria was fulfilled within a factor 2 by the 66% of the runs, and within a factor 4 by the 97%. We have chosen to simulate lattices of side $L = 8,\ 12,\ 16$, performing the largest possible RG step, $n = 4,\ 6,\ 8$, respectively. Thus, we considered a small renormalized lattice of constant side $L' = 2$. This has been done in order to minimize the transient effects due to irrelevant couplings, and to enhance the flow in the relevant direction. At $K_3 = 0$ we ran 36096, 14016 and 3296 samples, respectively; at $K_3 = 0.3$ the number of samples is 4824, 2944, 1664, while at $K_3 = 0.8$ we ran 4000, 2368 and 320 samples.

To characterize the distribution ψ_n, we have measured the following observables:

$$\begin{array}{ll} A_1 = \frac{1}{3|\Lambda'|}\sum_{x,l:|l|=1}\langle\!\langle s_x s_{x+l}\rangle\!\rangle & A_4 = \langle\!\langle S^2\rangle\!\rangle \\ A_2 = \frac{1}{3|\Lambda'|}\sum_{x,l:|l|=\sqrt{2}}\langle\!\langle s_x s_{x+l}\rangle\!\rangle & A_5 = \langle\!\langle S^4\rangle\!\rangle \\ A_3 = \frac{1}{3|\Lambda'|}\sum'_{x,y,z,w}\langle\!\langle s_x s_y s_z s_w\rangle\!\rangle & A_6 = \langle\!\langle (0.4 + S^2)^{-1}\rangle\!\rangle \end{array} \tag{7}$$

where $S = |\Lambda'|^{-1}\sum_x s_x$, and the sum in A_3 runs over the 4-uples of sites located on plaquettes. Notice that observable A_6 is sensible to the small values of S, on the opposite of A_4 and A_5. Moreover, we have explicitly measured the β derivative of each observable, calculating the connected correlation function with the Hamiltonian.

The critical temperature is estimated as the temperature where the values do not depend on the lattice size, *i.e.*, do not depend on the RG parameter n. Unfortunately, a 2^3 lattice is far too small to measure the effect of irrelevant perturbations and estimate the matrix of the linearized RG transformation. Thus, the critical index ν has been estimated with the simple formula: $\nu^{-1} = (\ln L_1/L_2)^{-1}\ln(\partial_\beta A(T_c; L_1)/\partial_\beta A(T_c; L_2))$. The results are shown in Tab. 1.

The statistical errors were computed applying the Jackknife method. We notice that the spin models corresponding to $K_3 = 0$ and $K_3 = 0.3$ exhibit the same critical index ν, though the T_c turns out to be quite different, because of the different values of the average frustration ($f = 0.5$ and 0.35, respectively). On the other side, the model at $K_3 = 0.8$ reveals a very different critical behaviour. The RG approach to criticality provides a direct method to check universality

K_3	T	A_1	A_2	A_3	A_4	A_5	A_6
0	$T_c(12,\ 8)$	0.647(6)	0.580(8)	0.520(7)	0.653(7)	0.570(8)	1.143(10)
	$T_c(16,\ 8)$	0.650(7)	0.583(9)	0.523(8)	0.656(8)	0.573(9)	1.139(11)
	$T_c(16,\ 12)$	0.656(18)	0.589(23)	0.528(20)	0.660(19)	0.578(21)	1.133(27)
0.3	$T_c(12,\ 8)$	0.670(18)	0.605(22)	0.542(21)	0.675(18)	0.594(20)	1.112(26)
	$T_c(16,\ 8)$	0.688(14)	0.625(17)	0.565(15)	0.691(13)	0.513(15)	1.091(17)
	$T_c(16,\ 12)$	0.72(5)	0.66(6)	0.61(6)	0.72(5)	0.65(6)	1.05(7)
0.8	$T_c(12,\ 8)$	0.3429(14)	0.2920(15)	0.1840(11)	0.3969(13)	0.2805(13)	1.5103(21)
	$T_c(16,\ 8)$	0.3505(12)	0.2990(13)	0.1924(11)	0.4032(11)	0.2877(11)	1.5012(19)
	$T_c(16,\ 12)$	0.369(5)	0.316(5)	0.213(4)	0.419(4)	0.305(5)	1.479(7)

Tab. 2. Characterization of the fixed point. The observables $A_1, \ldots, A_6$ were computed at the critical temperature $T_c(L_1, L_2)$, corresponding to the point where the measures on lattices L_1 and L_2 match.

of the critical point. We measured the values of observables $A_1, \ldots, A_6$ at the critical point. That gives a characterization of the fixed point distribution. Therefore, we can observe directly the universality classes to which the models belong, without relying on the phenomenological comparison of the critical indexes. The results, listed in Tab. 2, confirm the expected picture.

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