CRITICAL BEHAVIOR OF VECTOR MODELS WITH CUBIC SYMMETRY[1]

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We report on some results concerning the effects of cubic anisotropy and quenched uncorrelated impurities on multicomponent spin models. The analysis of the six-loop three-dimensional series provides an accurate description of the renormalization-group flow.

1 Cubic-symmetric models

The magnetic interactions in crystalline solids with cubic symmetry like iron or nickel are usually modeled using the O(3)-symmetric Heisenberg Hamiltonian. However, this is a simplified model, since other interactions are present. Among them, the magnetic anisotropy that is induced by the lattice structure is particularly relevant experimentally. In cubic-symmetric lattices it gives rise to additional single-ion contributions. These terms are usually not considered when the critical behavior of cubic magnets is discussed. However, this is strictly justified only if these nonrotationally invariant interactions, that have the reduced symmetry of the lattice, are irrelevant in the renormalization-group (RG) sense.

This question has been extensively investigated during the past decades [1, 2]. In the field-theoretical context, one considers the ϕ^4 Hamiltonian (ϕ is the M-component order parameter, $M = 3$ for magnets) and adds all cubic-invariant interactions that may be potentially relevant. There are two possible terms: a cubic hopping term $\sum_\mu (\partial_\mu \phi_\mu)^2$ and a cubic-symmetric quartic interaction term $\sum_\mu \phi_\mu^4$. The first term was shown to be irrelevant, although it induces slowly-decaying crossover effects [1]. In order to study the second one, one considers a ϕ^4 theory with two quartic couplings [1]:

$$\mathcal{H}_c = \int d^d x \left\{ \frac{1}{2}(\partial_\mu \phi(x))^2 + \frac{1}{2} r \phi(x)^2 + \frac{1}{4!} v_0 \left[\phi(x)^2 \right]^2 + \frac{1}{4!} w_0 \sum_{i=1}^{M} \phi_i(x)^4 \right\}. \tag{1}$$

The fixed-point (FP) structure of the model (1) has been investigated extensively and there is a general consensus that a critical value M_c exists such that, for $M < M_c$, the stable FP is the O(M)-symmetric one, while for $M > M_c$ criticality is controlled by a new point with cubic symmetry, which is the FP for all RG trajectories starting with $w > 0$. The debated issue is the value of M_c. While old studies indicated $3 < M_c < 4$, recent field-theoretical works find $M_c < 3$ [2]; more precisely, $M_c \approx 2.9$ [3–6]. This result has several important implications for magnets for which $M = 3$. If the system tends to magnetize along the cubic axes—this corresponds to a negative coupling w—then the system undergoes a first-order phase transition, since it is not in the basin of attraction of the cubic FP and therefore the RG flow runs away to infinity. Instead, magnets in which the cubic interaction favors the alignment of the spins along the diagonals of the cube, so that $w_0 > 0$, have a critical behavior with a new set of critical exponents and do not show Goldstone excitations even at the critical point. However, distinguishing the cubic and Heisenberg universality classes is expected to be a hard task in practice. Indeed, the critical exponents differ very little [2, 7]: the cubic exponents are [8] $\nu_c = 0.7109 \pm 0.0006$, $\eta_c = 0.0374 \pm 0.0005$, and $\gamma_c = 1.3955 \pm 0.0012$, and

$$\eta_c - \eta_H = -0.0001 \pm 0.0001 \,,$$
$$\nu_c - \nu_H = -0.0003 \pm 0.0003 \,, \qquad (2)$$
$$\gamma_c - \gamma_H = -0.0005 \pm 0.0007 \,.$$

These differences are much smaller than the typical experimental errors, so distinguishing cubic and O(3) universality class should be very hard.

The results for the cubic model (1) have implications for other models. First, we should mention the antiferromagnetic three- and four-state Potts models. In fact, as argued in [9, 10], the critical behavior of these models at the high-temperature transition should be described by the cubic Hamiltonian $\mathcal{H}_c$ with $M = 2, 3$ and $w_0 < 0$. The results presented above allow us to make the following predictions. If the three-state model has a critical transition, it should belong to the XY universality class. On the other hand, the four-state model is expected to show a first-order transition.

We can also use the above-presented results to discuss the nature of the bicritical point in models with symmetry O(N_1) $\oplus$ O(N_2). Indeed, they allow to exclude that the bicritical point has enlarged symmetry O($N_1 + N_2$) if $N_1 + N_2 > 2$ [11]. This result has important implications for the SO(5) theory of superconductivity [12]. In the SO(5) theory [12], one considers a model with symmetry O(3)⊕U(1) = O(3)⊕O(2) with two order parameters: one is related to the anti-ferromagnetic order, the other one is associated with d-wave superconductivity. The main issue is whether the SO(5) symmetry can be realized at a bicritical point where two critical lines, with symmetry O(3) and O(2) respectively, meet. In RG terms, this can generally occur if the O(5) FP has only *two* relevant O(3)⊕ O(2)-symmetric perturbations. But, when $N \geq 3$, the instability of the O(N) fixed point with respect to the cubic perturbation shows that at least another relevant perturbation exists. The stable FP is expected to be the tetracritical decoupled FP which can be shown to be stable by nonperturbative arguments [13].

Another important class of systems are uniaxial antiferromagnets in a magnetic field parallel to the field direction [14]. In this case $N_1 = 1$ and $N_2 = 2$. The results presented above show also that the bicritical O(3)-symmetric fixed point is unstable. The multicritical behavior should be controlled by the biconal FP [11], which, however, is expected to be close to the O(3) FP, so
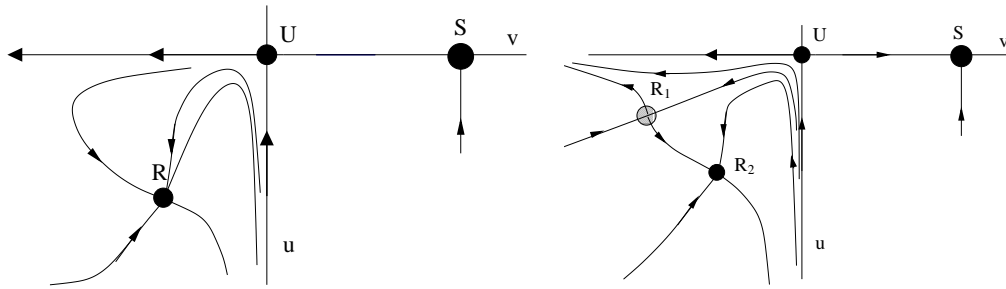
Fig. 1. Softening scenarios for fluctuation-induced first-order transitions.

that critical exponents should be very close to the Heisenberg ones. Thus, differences should be hardly distinguishable in experiments.

2 Random impurities and softening: General considerations

The critical behavior of systems with quenched disorder is of considerable interest. Experimentally, dilute systems can be obtained by mixing an (anti)-ferromagnetic material with a nonmagnetic one or by considering a fluid in a porous material, for instance in Vycor. A practical tool in the study of the effect of randomness on second-order phase transitions is provided by the Harris criterion [15]. It states that the addition of impurities to a system which undergoes a second-order phase transition does not change the critical behavior if the specific-heat critical exponent α_p of the pure system is negative. If α_p is positive, the transition is altered. Moreover, even if the pure-system FP remains stable, disorder may still have physical consequences. It may change the attraction domain of the pure stable FP so that some pure systems undergoing a first-order transition in the absence of disorder may show a critical behavior for some dilution. Softening of the phase transition may also occur if new stable FP's are generated by disorder. Two possible scenarios are illustrated in Fig. 1. The pure system corresponds to $u = 0$ and has two FP's: one, labelled S, with $v > 0$ is stable, while the second one, labelled U, with $v = 0$ is unstable. Therefore, pure systems with $v > 0$ show a critical behavior controlled by S, while systems with $v < 0$ undergo a first-order phase transition. Then, we introduce randomness in the system, which corresponds to considering strictly negative values of u. The pure FP is stable against disorder and indeed RG trajectories with $v > 0$ still flow towards S: disorder does not change the critical behavior. On the other hand, disorder is relevant for $v < 0$, if new FP's outside the basin of attraction of S appear, as illustrated in Fig. 1. If the scenario on the left occurs, systems corresponding to $v < 0$ show now a critical transition that belongs to a new universality class controlled by the new random FP. If the scenario on the right of Fig. 1 applies, the behavior is more complex. The transition remains first order for low enough impurity concentration, then for a given concentration becomes continuous and in the universality class of the unstable (tricritical) FP, and finally, for larger concentrations (but still under the percolation threshold) it is in the attraction domain of the stable FP. In Fig. 1 we have assumed that the attraction domain does not change, but it possible that the boundary of the basin of attraction is not the line $v = 0$. Therefore, it could also be possible that some systems with $v > 0$ do not have a critical behavior

controlled by S in the presence of disorder, or the opposite case, i.e. that systems with $v < 0$ have S criticality in the presence of disorder. Another exotic possibility was found by Cardy in two dimensions [16]. In this case, the pure stable FP is marginally unstable against disorder, but the RG trajectories are closed paths starting and finishing in the pure FP, so that the critical behavior is unchanged. This peculiar FP occurs only when disorder is marginally unstable.

The occurence of softening has been studied carefully for the two-dimensional case. It was argued in Ref. [17], and later put on a rigorous basis [18], that in two dimensions thermal first-order transitions become continuous in the presence of quenched disorder coupled to the local energy density. Such a conclusion was confirmed by Cardy that showed for a very specific model that softening persists in $2+\varepsilon$ dimensions [16]. However, in three dimensions the analysis of [17] shows that the occurrence of softening may depend on nonuniversal features.

3   Randomly dilute M-vector cubic models

The critical properties of dilute M-vector models are often described in terms of an $O(M)$-symmetric ϕ^4 Hamiltonian; the presence of uncorrelated random impurities is taken into account by coupling a random field with the local energy density. Using the Harris criterion, one sees that for $M \geq 2$ the pure FP is stable against disorder since $\alpha_p < 0$. On the other hand, in the Ising case the specific-heat exponent is positive and thus disorder is relevant: the random Ising model (RIM) shows a new type of critical behavior as confirmed experimentally and theoretically, see, e.g., Refs. [2, 4, 6, 19] and references therein.

The Harris criterion also allows to determine the stability properties of the stable FP of the cubic-symmetric model. Since $\alpha_p < 0$ in all cases, the stable FP is unchanged. Of course, as discussed in the previous section, this does not exclude the presence of new stable FP's due to disorder—they indeed do appear in two dimensions [16]—and therefore the softening of the first-order transition observed in pure systems with $w < 0$.

In order to investigate this possibility, we consider the Hamiltonian [20]

$$\mathcal{H}_e = \int d^d x \left\{ \sum_{i,a} \frac{1}{2} \left[(\partial_\mu \phi_{a,i})^2 + r \phi_{a,i}^2 \right] + \sum_{ij,ab} \frac{1}{4!} \left(u_0 + v_0 \delta_{ij} + w_0 \delta_{ij} \delta_{ab} \right) \phi_{a,i}^2 \phi_{b,j}^2 \right\}, \quad (3)$$

where $a, b = 1, ...M$ and $i, j = 1, ...N$. Using the standard replica trick, one may show that dilute cubic-symmetric systems are recovered in the limit $N \to 0$. The coupling u_0 is negative, being proportional to minus the variance of the quenched disorder.

In order to see whether new stable FP's with $u < 0$ exist, we shall use the ϵ and the fixed-dimension expansion. First, we analyze the special cases $v = 0$ and $w = 0$. For $v = 0$ the Hamiltonian (3) describes an MN-component model with cubic anisotropy, characterized by the presence of two stable FP's [1, 2]. The one for $u > 0$, $w = 0$ is in the self-avoiding walk (SAW) universality class, but it is irrelevant for our problem, since it is unreachable from the physical region $u < 0$. The other, with $u < 0$, $w > 0$, belongs to the RIM universality class. In the case $w = 0$, the Hamiltonian (3) describes N coupled M-vector models, and it is also called MN model [1]. Again, the flow is characterized by two stable FP's: the SAW and the $O(M)$-symmetric ones. They are both irrelevant for our problem. The SAW FP has $u > 0$, while the $O(M)$-symmetric one has $u = 0$ and it is unstable against w perturbations. For $M = 2$ and

generic N, the Hamiltonian (3) is invariant under a general transformation, cf. Ref. [21]. For $N = 0$, it maps the RIM FP into a new RIM FP belonging to the region with $u < 0$, $v > 0$, $w < 0$.

The above-reported considerations show the presence of only one (for $M = 2$ two) FP that could be possibly stable: the RIM FP with $v = 0$ and, for $M = 2$, the second RIM FP related to the previous one by symmetry. Of course, other FP's may have $u < 0$, $v \neq 0$, $w \neq 0$ and thus a more general analysis is needed in order to have a complete knowledge of the RG flow.

The RG flow can be investigated near four dimensions using the perturbative ϵ expansion. The results are reported in [8]. No new FP's (apart the one predicted by symmetry [21] for $M = 2$) are found in the region of physical interest $u < 0$. Thus, near four dimensions the critical behavior is not changed by the addition of random impurities for any $M \geq 2$. Moreover, there is no softening of the transition for pure systems that are outside the attraction domain of the stable FP.

The ϵ-expansion analysis shows that the random FP that is found in $2 + \epsilon$ dimensions [16] eventually disappears as the dimension is increased. The interesting question is therefore if, for $d = 3$, one observes a qualitative behavior analogous to the two-dimensional or four-dimensional case. Such a question can only be investigated in a strictly three-dimensional scheme. For this reason, we computed the RG functions to six loops in the fixed-dimension expansion [8], and carefully investigated the FP structure of the model.

First of all, we checked the stability properties of the already known FP's. For all values of M we found, in agreement with the Harris criterion, that the pure stable FP remains stable after dilution. Furthermore, the numerical estimate of the crossover exponent agrees with the theoretical prediction $\phi_u = \alpha_{\mathrm{p}}/\nu_{\mathrm{p}}$.

We also studied the stability of the RIM FP in the plane $v = 0$, computing the M-independent crossover exponent ϕ_v at the RIM FP. Our final estimate $\phi_v = 0.04 \pm 0.05$ suggests that the v perturbation is relevant and thus that the RIM FP is unstable, although the relatively large error bar does not allow us to exclude the opposite case. This point deserves further investigations, for example using other resummation methods. Note that, even if the perturbation is relevant, the RG dimension is very small and thus one expects very strong crossover effects.

We searched for the presence of new FP's. For $M = 2$, we only found the FP predicted by symmetry. For larger N, our analysis did not provide evidence for new FP's in the physical region $u < 0$. Therefore, no softening is expected, at least in the region of sufficiently low impurity concentration where the field-theoretical approach is justified.

Finally, we mention that the above results can be used to determine the critical behavior of dilute three- and four-state antiferromagnetic Potts models, which should be described by the dilute cubic model (3) in the two- and three-component cases respectively and for $w_0 < 0$ [8]. They imply that the dilute three-component antiferromagnetic Potts model presents a continuous transition belonging to the XY universality class, as in the pure case. In the four-state case, the weak first order transition expected in the pure case should not be softened by random dilution.

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