CRITICAL BEHAVIOR AT M-AXIAL LIFSHITZ POINTS¹

H. W. Diehl²

Fachbereich Physik, Universität Essen, D-45117 Essen, Federal Republic of Germany

Received 9 June 2002, accepted 18 June 2002

An introduction to the theory of critical behavior at Lifshitz points is given, and the recent progress made in applying the field-theoretic renormalization group (RG) approach to ϕ^4 *n*-vector models representing universality classes of *m*-axial Lifshitz points is surveyed. The origins of the difficulties that had hindered a full two-loop RG analysis near the upper critical dimension for more than 20 years and produced long-standing contradictory ϵ -expansion results are discussed. It is outlined how to cope with them. The pivotal role the considered class of continuum models might play in a systematic investigation of anisotropic scale invariance within the context of thermal equilibrium systems is emphasized. This could shed light on the question of whether anisotropic scale invariance implies an even larger invariance, as recently claimed in the literature.

PACS: 05.20.-y, 11.10.Kk, 64.60.Ak, 64.60.Fr

1 Preliminary remarks

The aim of this article is to give a brief introduction to the field of critical behavior at Lifshitz points [1, 2], and to survey the recent progress that has been made in applying the fieldtheoretic renormalization group (RG) approach [3–8]. As is appropriate for a conference on the topic 'renormalization group', I shall assume some basic familiarity of the reader with RG ideas. However, in view of the mixed background of the participants of the conference, no extensive knowledge of the relevant condensed matter physics is presupposed.

Since the literature on Lifshitz points—or, more generally, on systems with spatially modulated phases—is vast, it is impossible to mention or even cite all relevant papers. The paper is designed to give a reasonably self-contained account of the issues on which we focus, and to serve as a guide to the literature. The choice of the references has been made accordingly. There exist extensive review articles [1] and [2], which survey the literature till 1992 and contain extensive lists of references. The reader may consult these for further information on topics that had to be left out here.

0323-0465/02 © Institute of Physics, SAS, Bratislava, Slovakia

271

¹Plenary lecture at 5th Int. Conf. Renormalization Group 2002, Tatranská Štrba (Slovakia), March 2002

²E-mail address: phy300@theo-phys.uni-essen.de

2 Introduction and background

2.1 Generic phase diagram with a Lifshitz point

The concept of a Lifshitz point was introduced more than 25 years ago [9]. It is a point in the phase diagram at which a disordered phase, a spatially homogeneous ordered phase, and a spatially modular ordered phase meet. A typical phase diagram with a Lifshitz point is depicted in Fig. 1. In the case of a ferromagnet, the disordered and uniform ordered phases are the usual paramagnetic and ferromagnetic phases. The order of the latter corresponds to an infinite 'modulation' wave-length, i.e., a modulation wave-vector $q_0 = 0$. The transition that occurs upon crossing the phase boundary between the disordered and the uniform ordered phases is contin*uous* ('second-order transition'). Thus each point on the line $T = T_{c}(g)$ separating these two phases and emerging from the Lifshitz point is a critical point. The variable g is a second (intensive) thermodynamic variable besides temperature T. What it stands for depends on the type of system considered: In the case of organic crystals like TTF-TCNQ [10, 11], g corresponds to pressure; in the cases of the much studied magnet MnP [12, 13] and the so-called ANNNI model [14], which we will both briefly consider below, g stands for the magnetic field component perpendicular to the order parameter and a ratio of an antiferromagnetic to a ferromagnetic interaction coefficient, respectively. The important point to remember is that g does not couple directly to the order parameter. (If it did, a small change $g \to g + \delta g$, $T \to T_c + \delta T$ along any direction in the gT plane—and specifically along $T_c(g)$ — would destroy the critical behavior; i.e., there would not be a critical line.) For this reason, g is commonly called a *nonordering* thermodynamic field.

The term 'thermodynamic field' is probably not too familiar among high-energy physicist, and should by no means be confused with what is meant by a field in field theory. Its usage was suggested in a seminal paper [15] by Griffiths and Wheeler for intensive thermodynamic



Fig. 1. Schematic phase diagram with a Lifshitz point L. The disordered phase is separated from the uniform and modulated ordered phases via a critical line. The crossover exponent φ is defined via the behavior of the critical line near L; the wave-vector exponent β_q describes how the modulation wave-vector q_0 tends to the value $q_{0,L}$ as L is approached along the critical line between the disordered and modulated ordered phases.

variables like *T*, magnetic field, pressure, and chemical potential that take *the same values on both sides of (in general, first-order) bulk phase transition.* Their thermodynamic counterparts are the densities of extensive variables like the magnetization, energy, and entropy densities that normally jump at the transition (though not necessarily all of them) since their values in the respective pure bulk phases differ.

In the modulated ordered phase, the order is characterized by a nonzero modulation wave vector q_0 , which varies with T and g. The transition between the disordered and modulated ordered phase is continuous as well. Hence the Lifshitz point divides the critical line $T_c(g)$ into two sections. The transition between the homogeneous and modulated ordered phases can be of first or second order; for models with a scalar order parameter it is generically discontinuous, for specific models with vector order parameters it is found to be continuous [16].

Let us also note that the modulation wave-vector q_0 does not necessarily have to vanish at the Lifshitz point. To see this, recall that in the case of antiferromagnets, the order parameter which acquires a nonzero value in the uniform ordered phase is *not* the magnetization but the *staggered* magnetization. If we consider lattice models, this is given by the average of the sum of all spins on one (up-spin) sublattice minus the sum of those on the other (down-spin) sublattice. Thus the homogeneous ordered phase does not correspond to a homogeneous magnetization density but to one modulated with the corresponding nonzero wave-vector at the boundary of the Brillouin zone. This value of the modulation wave-vector q_0 applies in particular to the Lifshitz point, so $q_{0,L} \neq 0$. For the sake of simplicity, we shall always use ferromagnetic language in the sequel, taking $q_{0,L}$ to vanish.

2.2 Critical exponents and continuum models

To reach the Lifshitz point, both the temperature T and the nonordering field g must be finetuned. This tells us that g must correspond to a relevant variable in the RG sense. We denote the associated crossover exponent as φ . It describes the behavior of the critical line $T_c(g)$ in the vicinity of L: As indicated in Fig. 1, we have

$$\delta T_{\rm c} \equiv T_c(g) - T_{\rm L} \sim |\delta g|^{1/\varphi} , \quad \delta g \equiv g - g_{\rm L} . \tag{1}$$

In order to describe the behavior of the modulation wave-vector near L, one introduces a wave-vector exponent β_q via

$$\delta q_0 \equiv q_0 - q_{0,\mathrm{L}} \sim |\delta g|^{\beta_q} \,. \tag{2}$$

There are further critical exponents that are needed to characterize the critical behavior at Lifshitz points. We will introduce some of these below.

How a Lifshitz point can occur can be easily understood within Landau theory. Landau theory leads one to consider the following natural generalization of the usual ϕ^4 model with the Hamiltonian (Euclidean action)

$$\mathcal{H} = \int d^d x \left\{ \frac{1}{2} \left(\nabla_\perp \phi \right)^2 + \frac{\rho_0}{2} \left(\nabla_\parallel \phi \right)^2 + \frac{\sigma_0}{2} \left(\triangle_\parallel \phi \right)^2 + \frac{\tau_0}{2} \phi^2 + \frac{u_0}{4!} \left| \phi \right|^4 \right\}.$$
 (3)

Here $\phi = (\phi^{\alpha})$ is an *n*-component order-parameter field. The *d*-dimensional position vector $\boldsymbol{x} = (\boldsymbol{x}_{\parallel}, \boldsymbol{x}_{\perp})$ has an *m*-dimensional 'parallel' component $\boldsymbol{x}_{\parallel}$ and a (d - m)-dimensional

'perpendicular' one x_{\perp} . The coefficients of the squared gradient terms, and those of the Hamiltonian's other interaction terms, generally depend on the thermodynamic variables T and g. We have assumed that the squared gradient terms involve just two distinct coefficients, where the one of $(\nabla_{\perp} \phi)^2$ remains positive and has been transformed to unity by means of an appropriate choice of the amplitude of ϕ . Its parallel counterpart, ρ_0 , is permitted to change sign. To ensure stability when $\rho_0 < 0$, the term $(\Delta_{\parallel} \phi)^2$ with a positive coefficient $\sigma_0 > 0$ has been added. The critical line between the disordered and homogeneously ordered phases is given in Landau theory by $\tau_0 = 0$ with $\rho_0 > 0$. The Lifshitz point is located at $\tau_0 = \rho_0 = 0$ in this classical approximation.

Near the Lifshitz point the interaction coefficients can be expanded about $T = T_{\rm L}$ and $g = g_{\rm L}$. For the coefficients that remain positive, i.e. u_0 and σ_0 , the expansions may be truncated at zeroth order, so that u_0 and σ_0 become independent of T and g. The deviations $\delta \tau_0 \equiv \tau_0 - \tau_{0,\rm L}$ and $\delta \sigma_0 = \sigma_0 - \sigma_{0,\rm L}$ change sign at L (where $\tau_{0,\rm L}$ and $\sigma_{0,\rm L}$ vanish in Landau theory); their expansions must be retained to linear order in $T - T_{\rm L}$ and δg .

3 What is interesting about studying critical behavior at Lifshitz points

One reason for the ongoing interest in critical behavior at Lifshitz points is the wealth of distinct physical systems with such multi-critical points. They range from magnetic ones [12, 13, 17], ferroelectric crystals [18], and charge-transfer salts [10, 11] to liquid crystals [19], systems undergoing structural phase transitions [20] or having domain-wall instabilities [21], and the so-called ANNNI model [14]. Lifshitz points have been discussed recently even in the context of superconductors [22] and polymer blends [23].

From a more general perspective, the problem is interesting because it provides well-defined clear examples of systems exhibiting *anisotropic scale invariance (ASI)*. At conventional critical points, scaling operators \mathcal{O} such as the order parameter ϕ and the energy density transform asymptotically as $\mathcal{O}(\ell x) = \ell^{-\Delta_{\mathcal{O}}} \mathcal{O}(x)$ under scale transformations, with a (in general nontrivial) scaling dimension $\Delta_{\mathcal{O}}$. In the case of ASI, the position coordinates x, or the position and time coordinates (in the case of time-dependent phenomena), divide into two (or more) groups, say $x = (x_{\parallel}, x_{\perp})$, that must be scaled with different powers of the scale factor ℓ to recover $\mathcal{O}(x)$; one has

$$\mathcal{O}(\ell^{\theta} \boldsymbol{x}_{\parallel}, \ell \boldsymbol{x}_{\perp}) = \ell^{-\Delta_{\mathcal{O}}} \mathcal{O}(\boldsymbol{x}) , \qquad (4)$$

with an anisotropy exponent θ different from unity. That ASI applies at Lifshitz points is evident from the Hamiltonian (3): The momentum-space two-point vertex function of the free theory obviously behaves at the Lifshitz point $\tau_0 = \rho_0 = 0$ as

$$\tilde{\Gamma}^{(2)}(\boldsymbol{q}_{\parallel} = \boldsymbol{0}, \boldsymbol{q}_{\perp} \to \boldsymbol{0}) \sim q_{\perp}^{2-\eta_{L2}} \quad \text{and} \quad \tilde{\Gamma}^{(2)}(\boldsymbol{q}_{\parallel} \to \boldsymbol{0}, \boldsymbol{q}_{\perp} = \boldsymbol{0}) \sim q_{\parallel}^{4-\eta_{L4}} ,$$
(5)

where the analogs η_{L2} and η_{L4} of the usual correlation exponent η take their mean-field value zero, but differ from it beyond Landau theory. In terms of these, the anisotropy exponent reads $\theta = (2 - \eta_{L2})/(4 - \eta_{L4})$. The asymptotic validity of ASI means that two distinct correlation lengths ξ_{\perp} and ξ_{\parallel} are needed to characterize the region within which $\phi(\mathbf{x})$ is correlated to $\phi(\mathbf{0})$; these diverge at $g = g_L$ as functions of $\delta T = (T - T_L)/T_L$ like $|\delta T|^{-\nu_{L2}}$ and $|\delta T|^{-\nu_{L4}}$ with different exponents ν_{L2} and $\nu_{L4} = \theta \nu_{L2}$, respectively. A familiar arena of ASI are dynamic critical phenomena near equilibrium [24, 25]. In their case the time t and position x must be scaled differently; the analog of Eq. (4) becomes $\phi(\ell x, \ell^3 t) = \ell^{-\Delta_{\phi}} \phi(x, t)$ for the order parameter, and the role of θ is played by the dynamic exponent 3. Their simplifying feature is that detailed balance and fluctuation-dissipation theorems hold. This entails that the stationary states of the dynamics are guaranteed to be thermal equilibrium ones described by an a priori known Hamiltonian. Hence the problem of the steady-state correlations splits off from the dynamics; all critical exponents of static origin can be determined without dealing with dynamics, only genuine dynamic properties require the analysis of the dynamic field theory.

ASI is abundant in *non-equilibrium* systems such as driven diffusive systems [26] and surface growth processes [27], which have attracted considerable attention during the past decade. In their case detailed balance is not normally valid. Finding the steady-state solutions therefore is a nontrivial task, requiring the investigation of the long-time limit of the dynamics. This must be solved before the ASI of the steady-state correlations can be investigated.

Two features make critical behavior at *m*-axial Lifshitz points a very attractive stage for the study of ASI: (i) One can stay entirely within the realm of equilibrium statistical physics, and (ii) the class of models (3) involves a parameter, m, that can be varied. Some time ago Henkel [28,29] argued that ASI should imply additional invariances, just as scale and rotational invariance in local field theories lead to the larger symmetry group of conformal transformations [30–32]. He made predictions for the form of the scaling function of the pair correlation function at criticality, which are in conformity with analytic results for certain spherical models [33]. Recent Monte Carlo results for an equilibrium [34] and non-equilibrium systems [35] appear to support these claims. Yet the general validity of such additional invariances has not been shown, nor is there a good understanding of their origin or the conditions under which they hold. For example, in Ref. [28] the assumption that the anisotropy exponent takes the rational values $\theta = 2/\wp$, $\wp \in \mathbb{N}$, is needed to ensure that the considered sub-algebra closes. However, in the case of the uniaxial Lifshitz point in d = 3 dimensions studied by means of simulations [34], this condition is unlikely to be fulfilled since the ϵ expansion [5,6] reveals that θ differs from 1/2 at order ϵ^2 , although the d = 3 estimate $\theta \simeq 0.487$ it yields for m = n = 1 is pretty close to it. Building on the field-theoretic analysis described in Refs. [5] and [6], and outlined below, one should be able to clarify whether the suggested 'generalized conformal invariance' and the predicted form of two-point scaling functions hold indeed.

Before turning to the RG analysis of the models (3), let us discuss two examples of systems with a Lifshitz point.

4 Two examples of systems with Lifshitz points: the ANNNI model and MnP

We begin with the axial-next-nearest-neighbor Ising (ANNNI) model, defined through the Hamiltonian

$$\mathcal{H}_{\text{ANNNI}} = -\frac{J_0}{k_{\text{B}}T} \sum_{i,\delta_{\perp}} s_i \, s_{i+\delta_{\perp}} - \frac{J_1}{k_{\text{B}}T} \sum_{i,\delta_{\parallel}} s_i \, s_{i+\delta_{\parallel}} - \frac{J_2}{k_{\text{B}}T} \sum_{i,\delta_{\parallel}'} s_i \, s_{i+\delta_{\parallel}'} \,, \tag{6}$$

where $s_i \pm 1$ are Ising spins residing on the sites *i* of the cubic lattice illustrated in Fig. 2. Here δ_{\parallel} and δ'_{\parallel} are nearest-neighbor (nn) and next-nearest-neighbor (nnn) displacements along one



Fig. 2. ANNNI model. Along one axis the spins are coupled via ferromagnetic nearest-neighbor bonds of strength J_1 and antiferromagnetic axial next-nearest neighbor bonds of strength $|J_2| \equiv -J_2$. Along the other directions there are only ferromagnetic nearest-neighbor interactions of strength J_0 .



Fig. 3. Phase diagram of the three-dimensional ANNNI model, taken from Ref. [2] (with permission kindly granted by its author); to illustrate the ordering in the (2,2) phase the inset with the arrows has been added. L is a uniaxial Lifshitz point.

axis, while δ_{\perp} denote nn displacements along any of the other (perpendicular) directions. The nn couplings are ferromagnetic, $J_0 > 0$, $J_1 > 0$; the nnn bond is antiferromagnetic, $J_2 < 0$.

The phase diagram of the three-dimensional ANNNI model is depicted in Fig. 3. Obviously, the nonordering field g here translates into the ratio $\kappa = -J_2/J_1$. The phase labeled (2,2), also called $\langle 2 \rangle$ structure, corresponds to an antiferromagnetic ordering of the kind indicated by the arrows in the inset. The region marked 'modulated' actually has considerably more structure: From the multi-phase point $k_B T/J_1 = 0$, $\kappa = 1/2$, commensurate phases of type $\langle 3 \rangle$ and $\langle 2^p 3 \rangle$, $p = 1, 2, ..., \infty$, split off. Here $\langle 3 \rangle$ denotes a periodic layer sequence of 3-bands, while $\langle 2^p 3 \rangle$ signifies a periodic layer sequence of p 2-bands followed by a 3-band, where a k-band means that the magnetization has the same sign in k successive layers. Though interesting, these details cannot be expounded here because they are off our main topic. They can be found in the literature [2, 14]. The essential point for us to note is that L is a *uniaxial* (m = 1) Lifshitz point.



Fig. 4. Phase diagram of MnP for a magnetic field parallel to the **b** axis, according to Refs. [12] and [13].

It has been repeatedly studied via Monte Carlo simulations to determine the values of the critical exponents for the case m = n = 1, d = 3 [34, 36–38].

An experimentally much studied system is the orthorhombic metallic compound MnP [12, 13, 39]. Its phase diagram differs, depending on whether the magnetic field H is directed along the a, b, or c crystal axis. Both for H || a and H || b a uniaxial Lifshitz point is found, but not for H || c. The phase diagram of Ref. [12] for the case H || b is reproduced in Fig. 4 (with permission kindly granted by the first author, Y. Shapira). The magnetic field component $H \cdot b$ plays the role of the thermodynamic field g. In the phase labeled 'para' the magnetic moments ('spins') s(x) are aligned along the $H \propto b$ direction. In the 'ferro' phase the magnetization $m = \langle s \rangle$ has a component along the c axis and hence is tilted with respect to b. The component $c \cdot s$ plays the role of the order parameter; since a is a very hard axis, $s \cdot a$ can be ignored [39]. In the fan phase the spins rotate in the bc plane (but make no full turn as in the screw phase 'SCR' not considered here); there is modulated order with a modulation wave-vector $q_0 \propto a$. The ferro-fan transition is first order. The meeting point of the para, ferro, and fan phases is an m = n = 1 Lifshitz point.

5 Dimensionality expansions and renormalization group analysis

Setting $\rho_0 = \tau_0 = 0$ in the Hamiltonian (3), we see that q_{\perp} scales as q_{\parallel}^2 at the Lifshitz point of the Gaussian ($u_0=0$) theory. By dimensional analysis we have $[x_{\perp}] = \mu^{-1}$, $[x_{\parallel}] = \sigma_0^{1/4} \mu^{-1/2}$, $[\tau_0] = \mu^2$, $[\rho_0] = \sigma_0^{1/2} \mu$, and $[u_0] = \sigma_0^{m/4} \mu^{\epsilon}$ with $\epsilon \equiv d^*(m) - d$, where

$$d^*(m) = 4 + \frac{m}{2}, \quad m \le 8.$$
 (7)

From the given engineering dimensions we can read off how the interaction coefficients transform under the scale transformation $\mu \to \mu \ell$. The coupling constant u_0 becomes marginal at $d = d^*(m)$, the upper critical dimension (UCD). Note that m = 8 implies $d^* = m = 8$; i.e., this



Fig. 5. md plane with the line of upper critical dimensions $d^*(m)$. For further explanations see main text.

is the case of the isotropic Lifshitz point in which only the parallel components of position and momentum remain.

One can analytically continue the parallel and perpendicular momentum integrals in their respective dimensions m and d - m, and hence take those as continuously varying. Ideally one would like to determine the expansions in m and d - m of the critical exponents and other universal quantities about any point of the line $d^*(m)$. The situation is illustrated in Fig. 5. The point (m, d) = (0, 4) marks the UCD of the standard (isotropic) $|\phi|^4$ model; analyzing it via the conventional $\epsilon = 4 - d$ expansion means to move away from this point along the path indicated by \Downarrow . More generally, ϵ expansions at fixed m correspond to paths parallel to the d-axis. How to reach the physically interesting point (1, 3) along such a path is indicated. In the case of the isotropic (m=d) Lifshitz point, things are different: The conventional expansion in $\epsilon_8 \equiv 8 - d$ is along the diagonal m = d [8,9].

We must also keep in mind an obvious condition points (m, d) to which extrapolations make sense must satisfy: d must be larger than $d_*(m, n)$, the lower critical dimension (LCD) below which a Lifshitz—or, if m = 0, a critical—point cannot occur. Since the LCD of the Ising model with short-range interactions is $d_*(0, 1) = 1$, we clearly must have d > 1. In the n > 1 case with the continuous O(n) symmetry, familiar spin-wave arguments and an analysis based on an adequate nonlinear sigma model [40] show that the ordered phase becomes thermally unstable at any T > 0 if $d \le d_*^{O(n)}(m) \equiv 2 + \frac{m}{2}$ (cf. Fig. 5).

The existence of a Lifshitz point with 0 < m < 8 also requires that the modulated ordered phase be thermally stable. In the critical region of this phase, fluctuations of the Fourier components ϕ_q of the order parameter with wave-vectors $q \simeq \pm q_0$ are dominant. As pointed out in Ref. [41], one therefore expects that an *n*-component system with a helical structure behaves critically as a 2*n*-component ϕ^4 model whose ϕ^4 terms are O(n) but not O(2n) symmetric. Specifically for n = 1, one arrives at an anisotropic two-component model. It has (besides others) an O(2) symmetric fixed point that is believed to be stable [41]. This suggests that the long-range order (LRO) of the modulated phase should be destroyed in $d \le 2$ dimensions by thermal fluctuations at any temperature T > 0. Furthermore, if the presumed isotropy of the Hamiltonian (3) in the parallel subspace can indeed be taken for granted, one can exploit the invariance under arbitrary rotations along the lines of Mermin and Wagner [42] to show the absence of a helical phase with orientational LRO for $m \le d \le m + 1$ [43,44].³

The goal of expanding in m and d about a general point on the line $d^*(m)$ was envisaged already in Ref. [9]. Yet its realization turned out to be extremely difficult. Computing the ϵ expansions of the critical exponents for general values of m to first order in ϵ is easy. Their $O(\epsilon)$ coefficients are independent of m, and essentially determined by combinatorial factors. (The operator product expansion is structurally similar to the one that applies to the standard $|\phi|^4$ theory. Upon generalizing Cardy's analysis in Ref. [45] to the m > 0 case, one can determine these coefficients without having to work out the Feynman diagrams in detail.) The real challenges start at order ϵ^2 .

The first results to order ϵ^2 for general m were given in 1977 by Mukamel [46]. The ϵ^2 term of η_{L2} he determined by means of Wilson's momentum-shell integration method was independent of m, that of η_{L4} differed from it by a simple m-dependent factor. He also computed β_q to $O(\epsilon^2)$ for m < 6. For m = 1, these results were confirmed by Hornreich and Bruce [47]. Utilizing also Wilson's technique, Sak and Grest [48] did an independent calculation; because of the severe technical difficulties they encountered for general m they confined themselves to m = 2 and m = 6. Their results are at variance with Mukamel's; in particular, the ϵ^2 coefficients of $\eta_{L2}(m, n)$ for m = 2 and m = 6 are not equal, and hence *not* m-independent.

The application of modern field-theory RG approaches to the problem began in 1998. Mergulhão and Carneiro formulated normalization conditions and derived RG equations for the renormalized theory [3]. In a subsequent paper [4], they reproduced Sak and Grest's results for η_{L2} and η_{L4} with m = 2 and m = 6, and performed a two-loop calculation for these two values of m. They fixed the perpendicular dimension d - m at $d^*(2) - 2 = 3$ and $d^*(6) - 6 = 1$, taking the parallel one as $2 - \epsilon_{\parallel}$ and $6 - \epsilon_{\parallel}$, respectively. The respective paths starting from the points (m, d) = (2, 5) and (6, 7) are indicated in Fig. 5.

In two recent papers [5, 6] Shpot and myself have been able to perform a full two-loop calculation for general $m \in (0, 8)$ and to determine the ϵ expansions of all critical, crossover, wave-vector, and correction-to-scaling (Wegner) exponents to $O(\epsilon^2)$. The results are analytical except that the two-loop terms of the required renormalization functions and the series expansion coefficients of the exponents' ϵ^2 terms involve four well-defined single integrals $j_{\phi}(m)$, $j_{\sigma}(m)$, $j_{\rho}(m)$, and $J_u(m)$ which for general m we have not been able to evaluate analytically, though for the special values m = 0, 2, 6, and 8. For other values of m theses integrals can be computed numerically, as we did for m = 1, 2, ..., 7.

These results stand a number of nontrivial checks. First of all, they reduce to the well-known results for the standard $|\phi|^4$ model in the limit $m \to 0.^4$ Second, the analytical results they

³In Ref. [44] the fact that the one-loop shift of $T_c(g)$ on the helicoidal section of the critical line diverges in the infrared if $m \le d \le m + 1$ is interpreted as signaling the absence of helical LRO. However, this property alone is not sufficient to rule out LRO: The corresponding shift of T_c of the standard one-component ϕ^4 model is infrared divergent for $d \le 2$. Nevertheless, the d = 2 Ising model has a ferromagnetic low-temperature phase.

⁴More precisely, this applies to quantities retaining their physically significance for m = 0, like $\eta_{L2}(m=0) \equiv \eta$

yield for m = 2 and m = 6 are consistent with and extend those of Sak and Grest [48] and of Mergulhão and Carneiro [4]. [The original $O(\epsilon^2)$ results of Ref. [4] for $\nu_{L2}(m=2,6)$ and $\nu_{L4}(m=2,6)$ disagreed with ours but become identical to those upon elimination of two minor computational errors.] The isotropic case m = d provides a third, highly *nontrivial*, check. To see this, note that the ϵ expansions of the critical exponents $\lambda = \eta_{L4}, \ldots$ is of the form

$$\lambda(n,m,d)|_{d=d^*(m)-\epsilon} = \lambda_0 + \lambda_1(n)\,\epsilon + \lambda_2(n,m)\,\epsilon^2 + \mathcal{O}(\epsilon^3)\,,\tag{8}$$

i.e., the *m*-dependence starts at order ϵ^2 . To compare with the $\epsilon_8 \equiv 8 - d$ expansion for the isotropic case, we set $m = d = 8 - \epsilon_8$ (which gives $\epsilon = \epsilon_8/2$) in Eq. (8). Hence the expansions of the critical exponents λ of the isotropic Lifshitz point to quadratic order in ϵ_8 should result from the right-hand side via the replacements $\epsilon \to \epsilon_8/2$ and $\lambda_2(n,m) \to \lambda_2(n,8-)$. The so-obtained ϵ_8 expansions of the exponents $\lambda = \eta_{L4}(n,d,d), \nu_{L4}(n,d,d), \varphi(n,d,d), \beta_q(n,d,d)$, and the correction-to-scaling exponent $\omega_{L4}(n,d,d)$ have been verified by means of a direct field-theoretic investigation of the m = d case [8]. Yet, there is one group of authors [49] who obtained—and still favor—results at variance with ours in Refs. [5,6]; their findings and criticism have been refuted in Refs. [7, 8].

The origin of the technical difficulties which had prevented two-loop calculations for so long and caused the mentioned long-standing controversies can be traced back to the difficult form of the free propagator $G(\mathbf{x})$ at the Lifshitz point $\tau_0 = \rho_0 = 0$. In the isotropic cases m = 0 and m = d, $G(\mathbf{x})$ is a simple power of \mathbf{x} . However, for general m (and $\sigma_0 = 1$) it becomes

$$G(\boldsymbol{x}_{\parallel}, \boldsymbol{x}_{\perp}) = \int_{\boldsymbol{q}} \frac{\mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{x}}}{q_{\parallel}^{4} + q_{\perp}^{2}} = x_{\perp}^{-2+\epsilon} \Phi_{m,d} \left(x_{\parallel} \, x_{\perp}^{-1/2} \right) \,, \tag{9}$$

where $\Phi_{m,d}$ are extremely complicated scaling functions of the form

$$\Phi_{m,d}(\upsilon) = C_{m,\epsilon}^{(1)} {}_{1}\mathbf{F}_{2}\left(1 - \frac{\epsilon}{2}; \frac{1}{2}, \frac{2+m}{4}; \frac{\upsilon^{4}}{64}\right) - C_{m,\epsilon}^{(2)} \upsilon^{2} {}_{1}\mathbf{F}_{2}\left(\frac{3}{2} - \frac{\epsilon}{2}; \frac{3}{2}, 1 + \frac{m}{4}; \frac{\upsilon^{4}}{64}\right).$$
(10)

The latter are generalizations of generalized hypergeometric functions known as Fox-Wright $_1\psi_1$ functions [5, 6]. They have asymptotic expansions in powers of v^{-4} . On the line $d^*(m)$ they can be expressed in terms of modified Struve and Bessel functions. The cases m = 2 and m = 6 are special in that remarkable simplifications occur: The asymptotic expansions of $\Phi_{2,5}$ and $\Phi_{6,7}$ truncate at zeroth or first order in v^{-4} , respectively, and these functions reduce to a simple exponential and a similar elementary function, respectively. This is what makes the calculation analytically feasible for m = 2, 6.

To set up the RG for general m and $d = d^*(m) - \epsilon$, one can introduce a renormalized field ϕ_{ren} and renormalized coefficients σ , τ , ρ , and u via

$$\boldsymbol{\phi} = Z_{\phi}^{1/2} \boldsymbol{\phi}_{\text{ren}}, \quad \sigma_0 = Z_{\sigma} \sigma, \quad \delta \tau_0 = \mu^2 Z_{\tau} \tau,$$

$$\delta \rho_0 \sigma_0^{-1/2} = \mu Z_{\rho} \rho, \quad u_0 \sigma_0^{-m/4} = \mu^{\epsilon} Z_u u, \qquad (11)$$

and $\nu_{L2}(m=0) \equiv \nu$. The series expansion coefficients of other exponents that are not required (nor have a known immediate physical meaning) in the m = 0 theory, such as $\eta_{L4}(m)$, may nevertheless have finite $m \to 0$ limits [6, 8].

and determine the renormalization factors by minimal subtraction of poles. The residues of the Zs at two-loop order can be either analytically calculated or expressed in terms of single integrals whose integrands involve scaling functions such as $\Phi_{m,d^*(m)}$.

Let us illustrate this by means of the example of the two-loop graph \bigcirc , which is $\propto G(\boldsymbol{x})^3$ in position space. To determine the Laurent expansion of this distribution, we apply it to a test function $f(\boldsymbol{x}_{\parallel}, \boldsymbol{x}_{\perp})$, substituting Eq. (9) for $G(\boldsymbol{x})$ and making the variable transformation $\boldsymbol{x}_{\parallel} \rightarrow \boldsymbol{v} = \boldsymbol{x}_{\parallel}/\sqrt{x_{\perp}}$. One is led to an integral of the form $\int_0^\infty dx_{\perp} x_{\perp}^{2\epsilon-3} g_{\epsilon,m}(x_{\perp}) = (4\epsilon)^{-1}g''_{0,m}(0) + O(\epsilon^0)$. Upon evaluating this one sees that $G(\boldsymbol{x})^3$ has poles $\propto \Delta_{\parallel}^2 \delta(\boldsymbol{x})/\epsilon$ and $\propto \Delta_{\perp} \delta(\boldsymbol{x})/\epsilon$ whose residues are proportional to

$$\int_{0}^{\infty} dv \, v^{m+3} \, \Phi_{m,d^*}(v)^3 \quad \text{and} \quad \int_{0}^{\infty} dv \, v^{m-1} \, \Phi_{m,d^*}(v)^3 \,, \tag{12}$$

respectively. Up to a simple factor given in Eq. (46) of Ref. [6], these are the integrals $j_{\sigma}(m)$ and $j_{\phi}(m)$. The other integrals $j_{\sigma}(m)$ and $J_u(m)$ are of a similar nature but involve besides Φ_{m,d^*} further scaling functions of the free theory like the one pertaining to the convolution $(\nabla_{\parallel}G * \nabla_{\parallel}G)(\boldsymbol{x})$ [= line with an insertion of $(\nabla_{\parallel}\phi)^2$].

The RG equations implied by the reparametrizations (11) and the μ -invariance of the bare theory can be exploited in a familiar manner to derive the scaling properties of correlation and vertex functions. For details the reader is referred to the original papers [5, 6], where the ϵ expansions of the exponents are also utilized to obtain estimates for the numerical values of the critical exponents in the physical interesting cases with d = 3. The resulting values agree, in particular, remarkably well with the results of a very recent Monte Carlo investigation [34] of the ANNNI model, but also with some (though not all) experimental results and other theoretical estimates.

So far we assumed perfect isotropy of the derivative terms of the Hamiltonian (3) in x_{\parallel} -space. This is unrealistic in the case of lattice systems. Unless there is only one parallel direction (m = 1) or none, we should allow for a more general q_{\parallel}^4 term, making the replacement

$$(\triangle_{\parallel}\phi)^2 \to w_a T^a_{ijkl} (\partial_i \partial_j \phi) \partial_k \partial_l \phi = (\triangle_{\parallel}\phi)^2 + w \sum_{i=1}^m (\partial_i^2 \phi)^2 + \dots$$
(13)

in \mathcal{H} , where T_{ijkl}^a are the respective fourth-rank tensors compatible with the symmetry of the system. In the case of cubic symmetry, the only other term on the right-hand side besides the symmetric (first) contribution is the second one. To order ϵ^2 , the dimensionless variable w is found to be relevant at the isotropic w = 0 fixed point (the coefficient of the ϵ^2 term of its RG eigenexponent is positive) [50]. Unfortunately, the RG analysis for general $w \neq 0$ that is required to decide whether a new fixed point exists to which the flow leads is rather complicated and still remains to be completed.

6 Concluding remarks

Almost 25 years after the discovery of Lifshitz points and the introduction of the continuum models (3), systematic field-theoretic RG analyses beyond one-loop order of the universality classes these models represent have finally become feasible. This could open the way to accurate quantitative field-theory investigations and detailed comparisons with experimental results. Hopefully, this will also trigger further experimental work on the critical behavior at Lifshitz points. On the theoretical side, the progress reported here in applying the field-theoretic RG to the study of continuum models like (3) could lead to further insights regarding the question as to whether and under what conditions anisotropic scale invariance implies additional symmetries.

Acknowledgement: My own work on Lifshitz points has benefited from the very pleasant and fruitful collaboration with Mykola Shpot. I am also indebted to Royce K. P. Zia for joining us in our work [50] on the effects of anisotropic q_{\parallel}^4 terms.

Finally, I would like to acknowledge the support by the Deutsche Forschungsgemeinschaft via Sonderforschungsbereich 237.

References

- [1] R. M. Hornreich: J. Magn. Magn. Mater. 15-18 (1980) 387
- [2] W. Selke: in: *Phase Transitions and Critical Phenomena*, eds. C. Domb, J. L. Lebowitz, vol. 15, pp. 1–72. Academic, London (1992)
- [3] C. Mergulhão, Jr., C. E. I. Carneiro: Phys. Rev. B 58 (1998) 6047
- [4] C. Mergulhão, Jr., C. E. I. Carneiro: Phys. Rev. B 59 (1999) 13 954
- [5] H. W. Diehl, M. Shpot: Phys. Rev. B 62 (2000) 12 338; cond-mat/0006007
- [6] M. Shpot, H. W. Diehl: Nucl. Phys. B 612 (2001) 340; cond-mat/0106105
- [7] H. W. Diehl, M. Shpot: J. Phys. A 34 (2001) 9101; cond-mat/0106105
- [8] H. W. Diehl, M. Shpot: J. Phys. A 35 (2002, to be published); cond-mat/0204267
- [9] R. M. Hornreich, M. Luban, S. Shtrikman: Phys. Rev. Lett. 35 (1975) 1678
- [10] E. Abraham, I. E. Dzyaloshinskii: Solid State Commun. 23 (1977) 883
- [11] C. Hartzstein, V. Zevin, M. Weger: J. Phys. (France) 41 (1980) 677
- [12] Y. Shapira, C. Becerra, N. Oliveira, Jr., T. Chang: Phys. Rev. B 24 (1981) 2780
- [13] A. Zieba, M. Slota, M. Kucharczyk: Phys. Rev. B 61 (2000) 3435
- [14] M. E. Fisher, W. Selke: Phys. Rev. Lett. 44 (1980) 1502; Philos. Trans. R. Soc. London, Ser. A 302 (1981) 1
- [15] R. B. Griffiths, J. C. Wheeler: Phys. Rev. A 2 (1970) 1047
- [16] P. M. Chaikin, T. C. Lubensky: *Principles of condensed matter theory*: Cambridge University Press, Cambridge (GB) (1995)
- [17] For a more complete set of references, see Refs. [1] and [2].
- [18] Y. M. Vysochanskii, V. Y. Slivka: Usp. Fiz. Nauk 162 (1992) 139: [Sov. Phys. Usp. 35, 123-134]
- [19] J. H. Chen, T. C. Lubensky: Phys. Rev. A 14 (1976) 1202
- [20] A. Aharony, D. Mukamel: J. Phys. C 13 (1980) L255
- [21] J. Lajzerowicz, J. J. Niez: J. Phys. (Paris) Lett. 40 (1979) L165
- [22] F. S. Nogueira: Phys. Rev. B 62 (2000) 14 559
- [23] F. S. Bates, W. Maurer, T. P. Lodge, M. F. Schulz, M. W. Matsen: Phys. Rev. Lett. 75 (1995) 4429
- [24] P. C. Hohenberg, B. I. Halperin: Rev. Mod. Phys. 49 (1977) 435
- [25] See R. Folk's contribution to these proceedings.
- [26] B. Schmittmann, R. K. P. Zia: Statistical Mechanics of Driven Diffusive Systems, vol. 17 of Phase Transitions and Critical Phenomena: Academic, London (1995)

- [27] J. Krug: Adv. Phys. 46 (1997) 139
- [28] M. Henkel: Phys. Rev. Lett. 78 (1997) 1940
- [29] M. Henkel: *Conformal invariance and critical phenomena*: Texts and monographs in physics. Springer, Berlin (1999)
- [30] A. M. Polyakov: Pisma ZhETP 12 (1970) 538; [JETP Lett. 12, 381-383 (1970)]
- [31] A. A. Belavin, A. M. Polyakov, A. B. Zamalodchikov: Nucl. Phys. B 241 (1984) 333
- [32] J. L. Cardy: In: *Phase Transitions and Critical Phenomena*, eds. C. Domb, J. L. Lebowitz, vol. 11, pp. 55–126. Academic, London (1987)
- [33] L. Frachebourg, M. Henkel: Physica A 195 (1993) 577
- [34] M. Pleimling, M. Henkel: Phys. Rev. Lett. 87 (2001) 125702
- [35] M. Henkel, M. Pleimling, C. Godrèche, J.-M. Luck: Phys. Rev. Lett. 87 (2001) 265701
- [36] W. Selke: Z. Phys. 29 (1978) 133
- [37] W. Selke: J. Phys. C 13 (1980) L261
- [38] K. Kaski, W. Selke: Phys. Rev. B 31 (1985) 3128
- [39] C. S. O. Yokoi, M. D. Coutinho-Filho, S. R. Salinas: Phys. Rev. B 29 (1964) 6341
- [40] G. S. Grest, J. Sak: Phys. Rev. B 17 (1978) 3607
- [41] T. Garel, P. Pfeuty: J. Phys. C 9 (1976) L246
- [42] N. D. Mermin, H. Wagner: Phys. Rev. Lett. 17 (1966) 1133
- [43] T. C. Lubensky: Phys. Rev. Lett. 29 (1972) 3885
- [44] D. Mukamel, M. Luban: Phys. Rev. B 18 (1978) 3631
- [45] J. Cardy: Scaling and Renormalization in Statistical Physics: Cambridge Lecture Notes in Physics. Cambridge University Press, Cambridge, Great Britain (1996)
- [46] D. Mukamel: J. Phys. A 10 (1977) L249
- [47] R. M. Hornreich, A. D. Bruce: J. Phys. A 11 (1978) 595
- [48] J. Sak, G. S. Grest: *Phys. Rev.* B 17 (1978) 3602
- [49] L. C. de Albuquerque, M. M. Leite: J. Phys. A 34 (2001) L327; 35 (2002) 1807
- [50] H. W. Diehl, M. Shpot, R. K. P. Zia: to be published