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BATALIN-VILKOVISKY QUANTIZATION OF A NONCOMMUTATIVE YANG-MILLS THEORY TOY MODEL[1]

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We review the Batalin-Vilkovisky quantization procedure for Yang-Mills theory on a 2-point space.

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In this talk we give a short summary of [1], where we proposed the quantization of one of the simplest toy models for noncommutative gauge theories which is (zero dimensional) Yang–Mills theory on a 2-point space.

Noncommutative geometry constitutes one of the fascinating new concepts in current theoretical physics research with many promising applications [2–6].

We quantize the Yang–Mills theory on a 2-point space by applying the standard Batalin–Vilkovisky method [7, 8]. Somewhat surprisingly we find that despite of the model's original simplicity the gauge structure reveals infinite reducibility and the gauge fixing is afflicted with the Gribov [9] problem.

The basic idea of noncommutative geometry is to replace the notion of differential manifolds and functions by specific noncommutative algebras of functions. Following [10] we define the Yang–Mills Theory on a 2-point space in terms of the algebra $\mathbf{A} = C \oplus C$ which is represented by diagonal complex valued 2×2 matrices. The differential p-forms are constant, diagonal or offdiagonal 2×2 matrices, depending on whether p is even or odd, respectively. A nilpotent derivation $\mathbf{d}$ acting on 2×2 matrices is defined by $\mathbf{d}\, a = i \begin{pmatrix} a_{21} + a_{12} & a_{22} - a_{11} \\ a_{11} - a_{22} & a_{21} + a_{12} \end{pmatrix}$ where $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $a_{ij} \in \mathrm{C}$. The anti-Hermitean 1-forms $\mathcal{A}$ can be parametrized by

$$\mathcal{A} = \begin{pmatrix} 0 & i\phi \\ i\bar{\phi} & 0 \end{pmatrix} \tag{1}$$

and constitute the gauge fields of the model; here $\phi \in \mathrm{C}$ denotes a (constant) scalar field. The (rigid) gauge transformations of $\mathcal{A}$ are defined by

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$$\mathcal{A}^U = U^{-1}\mathcal{A}U + U^{-1}\mathbf{d}\,U \tag{2}$$

with U being a unitary element of the algebra $\mathbf{A}$. It is a constant, diagonal and unitary matrix which can be parametrized by the diagonal matrix ε

$$U = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix} = e^{i\varepsilon}, \quad \varepsilon = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \quad \alpha, \beta \in \mathrm{R}. \tag{3}$$

Due to the nonabelian form of the gauge transformations (2) the $U(1)\times U(1)$ gauge model shares many interesting features with the standard Yang–Mills theory, yet it has no physical space-time dependence and allows extremely simple calculations.

We define a scalar product for 2×2 matrices a, b by $\langle a \,|\, b\rangle = tr\, a^\dagger\, b$ where $\dagger$ denotes taking the Hermitian conjugate. The curvature $\mathcal{F}$ is defined as usual by $\mathcal{F} = \mathbf{d}\mathcal{A} + \mathcal{A}\,\mathcal{A}$ and for an action which is automatically invariant under the gauge transformations (2) one takes

$$S_{inv} = \frac{1}{2}\langle\mathcal{F}|\mathcal{F}\rangle = \big((\phi + \bar{\phi}) + \phi\,\bar{\phi}\big)^2 . \tag{4}$$

To discuss infinitesimal (zero-stage) gauge transformations we introduce a diagonal infinitesimal (zero stage) gauge parameter matrix ε_e^0 in terms of which $U \simeq \mathbf{1} + \varepsilon_e^0$. The infinitesimal (zero-stage) gauge variation of $\mathcal{A}$ derives as

$$\delta_{\varepsilon_e^0}\mathcal{A} = i\mathbf{R}^0\,\varepsilon_e^0 \quad \text{where} \quad \mathbf{R}^0 = \mathbf{D}; \tag{5}$$

here the (zero-stage) gauge generator $\mathbf{R}^0$ is defined in terms of the covariant matrix derivative $\mathbf{D}$, which acting on ε_e^0 is given by $\mathbf{D}\varepsilon_e^0 = \mathbf{d}\varepsilon_e^0 + [\mathcal{A}, \varepsilon_e^0]$.

A gauge symmetry is called irreducible if the (zero stage) gauge generator $\mathbf{R}^0$ does not possess any zero mode. It is amusing to note that the Yang–Mills theory on the 2-point space reveals an infinitely reducible gauge symmetry: We observe that $\mathbf{D}\,\mathbf{d}$ is vanishing on arbitrary offdiagonal matrices. Thus there exists a zero mode ε_e^1 for the (zero-stage) gauge generator $\mathbf{R}^0$, such that

$$\mathbf{R}^0\varepsilon_e^1 = 0 \quad \text{where} \quad \varepsilon_e^1 = \mathbf{R}^1\varepsilon_o^1 \quad \text{with} \quad \mathbf{R}^1 = \mathbf{d}. \tag{6}$$

Here ε_o^1 denotes an offdiagonal, infinitesimal (first-stage) gauge parameter matrix and $\mathbf{R}^1$ the corresponding (first-stage) gauge generator. As a matter of fact an infinite tower of (higher-stage) gauge generators $\mathbf{R}^s$, $s = 1, 2, 3, \cdots$ with never ending gauge invariances for gauge invariances is arising: We define $\mathbf{R}^s = \mathbf{d}$ for $s = 1, 2, 3, \cdots$ so that for each gauge generator there exists an additional zero mode

$$\begin{array}{lll} \mathbf{R}^1\varepsilon_o^2 = 0, & \text{where} & \varepsilon_o^2 = \mathbf{R}^2\varepsilon_e^2 \\ \mathbf{R}^2\varepsilon_e^3 = 0, & \text{where} & \varepsilon_e^3 = \mathbf{R}^3\varepsilon_o^3 \\ \quad\cdots & & \quad\cdots \end{array} \tag{7}$$

due to the nilpotency $\mathbf{d}^2 = 0$.

Now we straightforwardly apply the usual field theory Batalin-Vilkovisky (BV) path integral quantization scheme [7, 8] to the 2-point model: In addition to the original gauge field $\mathcal{A} \equiv \mathcal{C}_{-1}^{-1}$ we introduce ghost fields $\mathcal{C}_s^k,\ \ \infty \geq s \geq -1,\ s \geq k \geq -1$ with k odd, as well as auxiliary ghost fields $\bar{\mathcal{C}}_s^k,\ \ \infty \geq s \geq 0,\ \ s \geq k \geq 0$ with k even. Furthermore we add Lagrange multiplier fields π_s^k, $\infty \geq s \geq 1$, $s \geq k \geq 1$ with k odd and $\bar{\pi}_s^k$, $\infty \geq s \geq 0$, $s \geq k \geq 0$ with k even; finally we introduce antifields $\mathcal{C}_s^{k*}$, $\bar{\mathcal{C}}_s^{k\,*}$. The BV-action is obtained as

$$
\begin{aligned}
S_{BV} \quad &= \quad S_{inv} + S_{aux} - \langle \mathcal{C}_{-1}^{-1*} | \mathbf{D}\, \mathcal{C}_0^{-1} \rangle - \sum_{s=1,3,5,\cdots}^{\infty} \langle \mathcal{C}_s^{-1*} | \mathbf{d}\, \mathcal{C}_{s+1}^{-1} \rangle \\
&\qquad - i \sum_{s=0,2,4,\cdots}^{\infty} \langle \mathcal{C}_s^{-1*} | \mathbf{d}\, \mathcal{C}_{s+1}^{-1} \rangle,
\end{aligned} \tag{8}
$$

where we denote by S_{aux} the auxiliary field action

$$
S_{aux} = \sum_{k=0,2,4,\cdots}^{\infty} \sum_{s=k}^{\infty} \langle \bar{\pi}_s^k | \bar{\mathcal{C}}_s^{k\,*} \rangle + \sum_{k=1,3,5,\cdots}^{\infty} \sum_{s=k}^{\infty} \langle \mathcal{C}_s^{k*} | \pi_s^k \rangle. \tag{9}
$$

Gauge fixing conditions similar to the usual Feynman gauge are implemented by introducing the gauge fixing fermion $\Psi = \Psi_\delta + \Psi_\pi$

$$
\begin{aligned}
\Psi_\delta \quad &= \quad \sum_{s=0,2,4,\cdots}^{\infty} \ \sum_{k=0,2,4,\cdots\ k \leq s} \Big(-\langle \bar{\mathcal{C}}_s^k \,|\, \boldsymbol{\delta} \mathcal{C}_{s-1}^{k-1} \,\rangle + \langle \boldsymbol{\delta} \bar{\mathcal{C}}_{s+1}^k \,|\, \mathcal{C}_{s+2}^{k+1} \,\rangle \\
&\qquad + i \langle \bar{\mathcal{C}}_{s+1}^k \,|\, \boldsymbol{\delta} \mathcal{C}_s^{k-1} \,\rangle + i \langle \boldsymbol{\delta} \bar{\mathcal{C}}_s^k \,|\, \mathcal{C}_{s+1}^{k+1} \,\rangle \Big), \\
\Psi_\pi \quad &= \quad \frac{1}{2} \sum_{s=0,2,4,\cdots}^{\infty} \ \sum_{k=0,2,4,\cdots\ k < s} \big(\langle \bar{\mathcal{C}}_s^k \,|\, \pi_s^{k+1} \,\rangle + \langle \bar{\pi}_s^k \,|\, \mathcal{C}_s^{k+1} \,\rangle \\
&\qquad + i \langle \bar{\mathcal{C}}_{s+1}^k \,|\, \pi_{s+1}^{k+1} \,\rangle + i \langle \bar{\pi}_{s+1}^k \,|\, \mathcal{C}_{s+1}^{k+1} \,\rangle \big) + \frac{1}{2} \sum_{k=0,2,4,\cdots}^{\infty} \langle \bar{\mathcal{C}}_k^k \,|\, \bar{\pi}_k^k \,\rangle.
\end{aligned} \tag{10}
$$

By $\boldsymbol{\delta}$ we denote a nilpotent coderivative operator $\boldsymbol{\delta}\, a = i \begin{pmatrix} a_{12} - a_{21} & -a_{11} - a_{22} \\ -a_{11} - a_{22} & -a_{12} + a_{21} \end{pmatrix}$ where $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $a_{ij} \in \mathrm{C}$. We eliminate the antifields by using the gauge fixing fermion Ψ via

$$
\langle \mathcal{C}_s^{k*} | = \frac{\partial \Psi}{\partial | \mathcal{C}_s^k \rangle}, \quad | \bar{\mathcal{C}}_s^{k\,*} \rangle = \frac{\partial \Psi}{\partial \langle \bar{\mathcal{C}}_s^k |}, \tag{11}
$$

so that the gauge fixed action S_Ψ reads

$$\begin{aligned} S_\Psi \;=\; & S_{inv} - i\langle \bar{\mathcal{C}}_0^0 \,|\, \boldsymbol{\delta}\mathbf{D}\, \mathcal{C}_0^{-1}\rangle \\ & - i \sum_{s=1,3,5,\cdots}^{\infty} \langle \bar{\mathcal{C}}_{s+1}^0 \,|\, \boldsymbol{\delta}\mathbf{d}\, \mathcal{C}_{s+1}^{-1}\rangle + \sum_{s=0,2,4,\cdots}^{\infty} \langle \bar{\mathcal{C}}_{s+1}^0 \,|\, \boldsymbol{\delta}\mathbf{d}\, \mathcal{C}_{s+1}^{-1}\rangle \\ & + \sum_{k=0,2,4,\cdots}^{\infty} \; \sum_{s=k+1,\, odd}^{\infty} \big(i\langle \bar{\pi}_s^k \,|\, \pi_s^{k+1}\rangle + \langle \bar{\pi}_s^k \,|\, (i\boldsymbol{\delta}\mathcal{C}_{s-1}^{k-1} \;+\; \mathbf{d}\mathcal{C}_{s+1}^{k+1})\rangle \\ & \qquad\qquad + \langle (i\boldsymbol{\delta}\bar{\mathcal{C}}_{s-1}^{k} \;-\mathbf{d}\bar{\mathcal{C}}_{s+1}^{k+2}) \,|\, \pi_s^{k+1}\rangle \big) \\ & + \sum_{k=0,2,4,\cdots}^{\infty} \; \sum_{s=k+2,\, even}^{\infty} \big(\langle \bar{\pi}_s^k \,|\, \pi_s^{k+1}\rangle + \langle \bar{\pi}_s^k \,|\, (-\boldsymbol{\delta}\mathcal{C}_{s-1}^{k-1} \;+\; i\mathbf{d}\mathcal{C}_{s+1}^{k+1})\rangle \\ & \qquad\qquad + \langle (\boldsymbol{\delta}\bar{\mathcal{C}}_{s-1}^{k} \;+i\mathbf{d}\bar{\mathcal{C}}_{s+1}^{k+2}) \,|\, \pi_s^{k+1}\rangle \big) \\ & + \sum_{k=0,2,4,\cdots}^{\infty} \langle \bar{\pi}_k^k \,|\, (-\boldsymbol{\delta}\mathcal{C}_{k-1}^{k-1} \;+ i\mathbf{d}\mathcal{C}_{k+1}^{k+1} \;+ \frac{1}{2}\bar{\pi}_k^k)\rangle. \end{aligned} \tag{12}$$

We can now eliminate the Lagrange multiplier fields π_s^k and $\bar{\pi}_s^k$ and arrive at

$$\begin{aligned} S_\Psi \longrightarrow \quad & S_{inv} + \frac{1}{2}\langle \mathcal{A} \,|\, \mathbf{d}\boldsymbol{\delta}\, \mathcal{A}\rangle - i\langle \bar{\mathcal{C}}_0^0 \,|\, (\boldsymbol{\delta}\mathbf{D} \;+ \mathbf{d}\boldsymbol{\delta})\, \mathcal{C}_0^{-1}\rangle \\ & - i \sum_{s=1,3,5,\cdots}^{\infty} \langle \bar{\mathcal{C}}_{s+1}^0 \,|\, (\boldsymbol{\delta}\mathbf{d} + \mathbf{d}\boldsymbol{\delta})\, \mathcal{C}_{s+1}^{-1}\rangle \\ & + \sum_{s=0,2,4,\cdots}^{\infty} \langle \bar{\mathcal{C}}_{s+1}^0 \,|\, (\boldsymbol{\delta}\mathbf{d} + \mathbf{d}\boldsymbol{\delta})\, \mathcal{C}_{s+1}^{-1}\rangle \\ & - i \sum_{k=0,2,4,\cdots}^{\infty} \; \sum_{s=k+1,\, odd}^{\infty} \langle \bar{\mathcal{C}}_{s+1}^{k+2} \,|\, (\boldsymbol{\delta}\mathbf{d} \;+ \mathbf{d}\boldsymbol{\delta})\, \mathcal{C}_{s+1}^{k+1}\rangle \\ & + \sum_{k=0,2,4,\cdots}^{\infty} \; \sum_{s=k+2,\, even}^{\infty} \langle \bar{\mathcal{C}}_{s+1}^{k+2} \,|\, (\boldsymbol{\delta}\mathbf{d} \;+ \mathbf{d}\boldsymbol{\delta})\, \mathcal{C}_{s+1}^{k+1}\rangle \\ & + \frac{1}{2} \sum_{k=0,2,4,\cdots}^{\infty} \langle \mathcal{C}_{k+1}^{k+1} \,|\, (\boldsymbol{\delta}\mathbf{d} \;+ \mathbf{d}\boldsymbol{\delta})\, \mathcal{C}_{k+1}^{k+1}\rangle. \end{aligned} \tag{13}$$

All the higher-stage ghost contributions can be integrated away as $\boldsymbol{\delta}\mathbf{d} \;+ \mathbf{d}\boldsymbol{\delta} \;=\; 4 \cdot \mathbf{1}$ and we simply obtain

$$S_\Psi \longrightarrow S_{inv} + \frac{1}{2}\langle \mathcal{A} \,|\, \mathbf{d}\boldsymbol{\delta}\, \mathcal{A}\rangle - i\langle \bar{\mathcal{C}}_0^0 \,|\, (\boldsymbol{\delta}\mathbf{D} \;+ \mathbf{d}\boldsymbol{\delta})\, \mathcal{C}_0^{-1}\rangle. \tag{14}$$

We summarize that the zero dimensional Yang–Mills theory model on a 2-point space reveals infinite reducibility; after applying the standard BV-quantization procedure the action finally contains invertible quadratic parts for the gauge field, as well as for the ghost fields. A closer inspection [1] shows that the model suffers from a Gribov problem [9].

We expect that our present investigations will lead to a study of the renormalization effects at higher orders; it may also be possible to compare the perturbative calculations with explicit analytic integrations (for related attempts see [11]).

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