THE NUCLEON INTERACTION AND NEUTRON MATTER FROM THE RENORMALIZATION GROUP[1]

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We show that the renormalization group decimation of modern nucleon potential models to low momenta results in a unique nucleon interaction $V_{\text{low k}}$. This interaction is free of short-ranged singularities and can be used directly in many-body calculations. The renormalization group (RG) scaling properties follow directly from the invariance of the scattering phase shifts. We discuss the RG treatment of Fermi liquids. The RG equation for the scattering amplitude in the two particle-hole channels is given at zero temperature. The flow equations are simplified by retaining only the leading term in an expansion in small momentum transfers. The RG flow is illustrated by first studying a system of spin-polarized fermions in a simple model. Finally, results for neutron matter are presented by employing the unique low momentum interaction $V_{\text{low k}}$ as initial condition of the flow. The RG approach yields the amplitude for non-forward scattering, which is of great interest for calculations of transport properties and superfluid gaps in neutron star interiors. The methods used can also be applied to condensed matter systems in the absence of long-ranged interactions.

1   Introduction

Over the past decade, there has been a great effort in deriving low-energy effective nucleon-nucleon interactions [1]. Concurrently in the condensed matter community, Shankar proposed renormalization group methods to the study of strongly interacting Fermi systems [2]. In this talk, we discuss how the renormalization group (RG) can be used for the two-nucleon system as well as the nuclear many-body problem leading to quantitatively quite encouraging results. The two problems are similar from the RG point of view. The degrees of freedom are separated by a cutoff into low and high momenta. As the cutoff is decreased, the effects of the high momentum modes on the low momentum observables are integrated out into effective couplings. For the two-body problem, this corresponds to solving the Lippmann-Schwinger equation in

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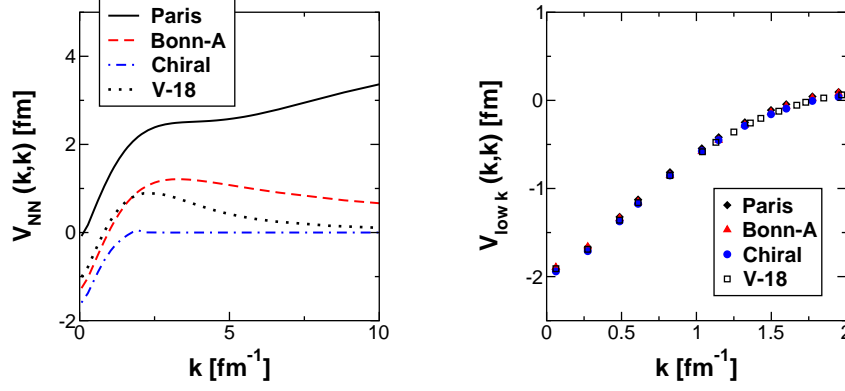[2]E-mail address: aschwenk@nuclear.physics.sunysb.edu

Fig. 1. Diagonal matrix elements of several realistic V_{NN} and $V_{low\,k}$ as a function of relative momentum in the $^1S_0$ partial wave. The $V_{low\,k}$ results are given for a cutoff $\Lambda = 2.0$ fm^{-1}. A comparison of phase shifts calculated from V_{NN} and $V_{low\,k}$ is given in [4].

the particle-particle channel, whereas for the many-body system, it constitutes a genuine two- or three-channel problem. Taking the cutoff to zero, one recovers the scattering length in the two-body case and the quasiparticle interaction of Fermi liquid theory in the matter case.

The RG approach allows us to evolve the full in-medium scattering amplitude on the Fermi surface from the vacuum two-body interaction. Compared to traditional microscopic calculations for Fermi systems, it is relatively straightforward to retain all momentum scales in the RG. In addition, our method is physically more transparent, since the effects of the high momentum modes, i.e., scattering to states far away from the Fermi surface, are completely tractable.

2   RG evolution of nucleon interaction models

We start from a realistic "bare" nucleon-nucleon potential V_{NN}, which is based e.g., on multiple Yukawa interactions, where the couplings are fitted to the scattering phase shifts. By experiment, the various V_{NN} are constrained only up to a relative momentum scale of $k \sim 2.0$ fm^{-1} (corresponding to a lab energy $E_{Lab} \sim 330$ MeV) and consequently have quite different momentum components as we show in Fig. 1. We integrate out the unrestricted high momentum modes with the requirement that the effective potential reproduces the phase shift data as well as the long range wave function tails, i.e., the so-called half-on-shell (HOS) T matrix [3,4].[3] The HOS scattering amplitude is given by

$$T(k',k;k^2) = V_{NN}(k',k) + \frac{2}{\pi}\mathcal{P}\int_0^\infty \frac{V_{NN}(k',p)\,T(p,k;k^2)}{k^2 - p^2}\,p^2 dp. \tag{1}$$

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[3]The invariance of the HOS T matrix was proposed by Bogner, Kuo and Coraggio [3]. Strictly speaking, only the fully on-shell amplitude is observable. The invariance of the phase shifts $T(k,k;k^2) = -\tan\delta/k$ (without any constraints on the HOS T matrix) can be achieved by a similarity transformation on $V_{low\,k}$. The resulting low momentum interaction remains unique and the diagonal matrix elements are basically unchanged. The RG equation is then replaced by a symmetrized version of the current one, Eq. (3).
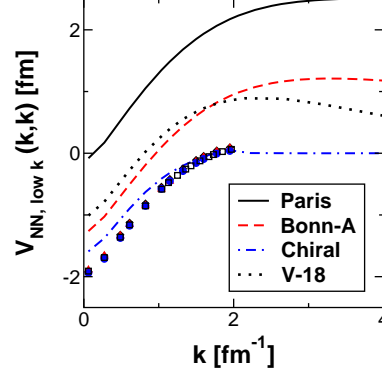
Fig. 2. The bare potential models compared to the unique $V_{\text{low k}}$. As in Fig. 1, we show the diagonal matrix elements in partial wave $^1S_0$.

After a resummation of the fast modes and subsequently removing the so-generated energy-dependence (for details see [5]), the bare potential can be replaced by the cutoff-dependent $V_{\text{low k}}$ with slow mode intermediate states only

$$T(k', k; k^2) = V_{\text{low k}}(k', k) + \frac{2}{\pi} \mathcal{P} \int_0^\Lambda \frac{V_{\text{low k}}(k', p)\, T(p, k; k^2)}{k^2 - p^2}\, p^2 dp. \tag{2}$$

Alternatively, Eq. (2) may be used as a definition of $V_{\text{low k}}$.

The scaling properties of $V_{\text{low k}}$ follow directly from the invariance $dT(k', k; k^2)/d\Lambda = 0$. This leads to the RG equation for energy-independent effective interactions [5]

$$\frac{d}{d\Lambda} V_{\text{low k}}(k', k) = \frac{2}{\pi} \frac{V_{\text{low k}}(k', \Lambda)\, T(\Lambda, k; \Lambda^2)}{1 - (k/\Lambda)^2}. \tag{3}$$

A similar RG equation for energy-dependent quasi-potentials $Q(k', k; \omega)$ was derived by Birse *et al.* [6]. In both cases, there exists a non-trivial fixed point corresponding to an infinite scattering length or a bound state at threshold [6, 7].

Starting at large cutoffs, the RG flow is used to evolve the bare potentials to low momenta. We find that, for cutoffs below the scale set by the experimental data, $\Lambda \sim 2.0$ fm^{-1}, the RG decimation leads to an unambiguous low momentum potential $V_{\text{low k}}$ [3, 4], independent of the initial bare potential, as is shown in Fig. 1. Similar results are found in all partial waves. The resulting interaction can be used as a benchmark for the interaction derived in chiral effective field theories. By comparing with the bare interaction, Fig. 2, we observe that the main effect of the RG evolution is a constant shift in momentum space (see also [8]).

Therefore, it is instructive to follow the flow of the interaction at zero relative momentum, $V_{\text{low k}}(0, 0)$, with the cutoff. This is shown in Fig. 3. We observe that, in the $^1S_0$ channel there is a clear separation of scales. For cutoffs in the range between the mass of the lightest exchange meson, $m_\pi \approx 0.7$ fm^{-1}, and the mass scale corresponding to the short-ranged repulsion, $m_\omega \approx 4$ fm^{-1}, the interaction is basically constant. On the other hand, in the $^3S_1$ channel, where the
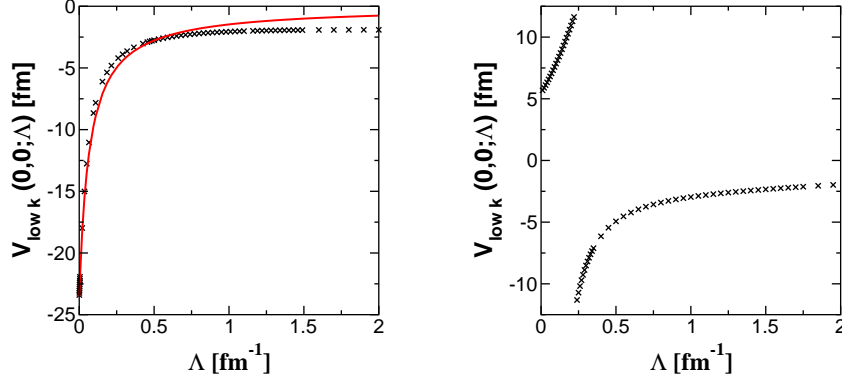
Fig. 3. The flow of $V_{\text{low k}}(0,0)$ (crosses) with the cutoff in partial waves $^1S_0$ (left) and $^3S_1$ (right figure).

tensor force contributes, a weak dependence on the cutoff reflects a less precise separation of scales due to higher order tensor interactions. As the flow is evolved to $\Lambda = 0$, $V_{\text{low k}}$ reproduces correctly the scattering lengths ($a_{^1S_0} = -23.71$ fm and $a_{^3S_1} = 5.425$ fm). The jump in the $^3S_1$ channel accounts for the compensation of the deuteron bound state. For cutoffs $\Lambda < \sqrt{m\,E_D}$, where m denotes the nucleon mass and E_D is the deuteron binding energy, the bound state is integrated out and must be included in the effective potential [4].

In Fig. 3 we also compare the exact $V_{\text{low k}}$ result with a flow prediction for small cutoffs. By retaining only the leading term in the effective range expansion, one obtains an analytic solution to the RG equation

$$V_{\text{low k}}(0,0) = \frac{1}{1/a_{^1S_0} - 2\Lambda/\pi} \quad \text{as} \quad \Lambda \to 0, \tag{4}$$

which is shown as solid line in Fig. 3. The agreement between the approximate solution and $V_{\text{low k}}$ is very good for $k < 1/|a_{^1S_0}|$.

The constant shift of $V_{\text{low k}}$ relative to the bare potentials corresponds (due to the cutoff) to a smeared delta-function and removes the short-ranged repulsion from the bare potential. Furthermore, many-body effects can be systematically included. Consequently, $V_{\text{low k}}$ is an attractive alternative to the Brueckner G matrix in microscopic many-body calculations.

3   RG for Fermi liquids in the particle-hole channels

RG methods can be applied to the nuclear many-body problem using the approach proposed by Shankar [2]. We employ the RG equation in the particle-hole channels to generate the full scattering amplitude on the Fermi surface from the two-body interaction. To this end, we impose a cutoff on the particle and hole modes around the Fermi surface. We integrate out the fast particles and fast holes with momenta $|p - k_F| > \Lambda$. The RG is initialized with the vacuum two-body interaction $V_{\text{low k}}$ at initial cutoff $\Lambda_0 = k_F$. The interaction between particles on the Fermi surface is not renormalized by particle-hole polarization effects with momentum transfers larger
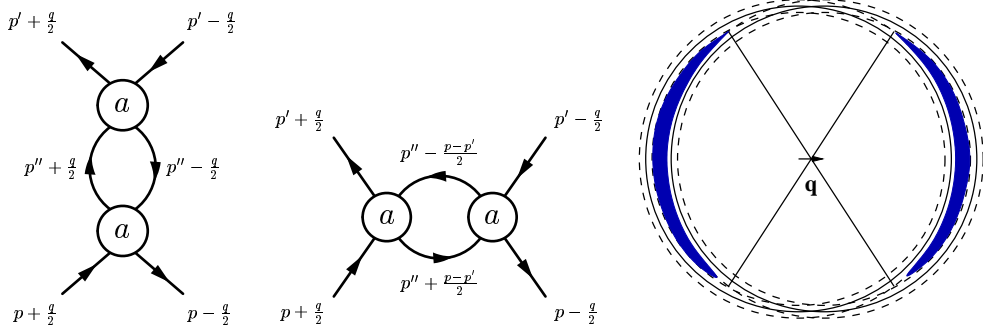
Fig. 4. The one-loop contributions to the beta functions. The Pauli principle requires to retain both the zero sound (ZS) channel with momentum transfer $\mathbf{q}$ in the particle-hole loop (left figure) and the exchange channel (ZS') with momentum transfer $\mathbf{q}' = \mathbf{p} - \mathbf{p}'$ (middle figure). The right figure shows the fast particle-hole contributions (filled shells) to the ZS flow. The dashed circles denote the cutoffs around the Fermi surface for the particle-hole pair in the loop.

than $2\Lambda_0$ (outside the initial cutoff shell). By evolving the scattering amplitude down to the Fermi surface, i.e., $\Lambda = 0$, we generate the full scattering amplitude for low energy excitations. In particular, we reproduce the Fermi liquid theory relations among the quasiparticle interaction and the forward scattering amplitude.[4]

The Fermi liquid RG is at least a two channel problem. This is mandated by the Pauli principle [9, 10]. The RG equations at one-loop receive contributions from the forward (ZS) and exchange (ZS') channel, which are depicted in Fig. 4. At the one-loop level, the running of the amplitude originates from the available fast particle-hole states only. The RG equation is antisymmetric by construction and reads [11] (the spin variables are suppressed for simplicity)

$$
\frac{d}{d\Lambda} a(\mathbf{q}, \mathbf{q}') = \frac{d}{d\Lambda} \left\{ g \int\limits_{\text{fast},\Lambda} \frac{d^3\mathbf{p}''}{(2\pi)^3} \frac{n_{\mathbf{p}''+\mathbf{q}/2} - n_{\mathbf{p}''-\mathbf{q}/2}}{\varepsilon_{\mathbf{p}''+\mathbf{q}/2} - \varepsilon_{\mathbf{p}''-\mathbf{q}/2}} \right\} a\left(\mathbf{q}, \frac{\mathbf{p}+\mathbf{p}'}{2} + \frac{\mathbf{q}'}{2} - \mathbf{p}''\right)
$$

$$
\times\, a\left(\mathbf{q}, \mathbf{p}'' - \frac{\mathbf{p}+\mathbf{p}'}{2} + \frac{\mathbf{q}'}{2}\right) - \left\{ \mathbf{q} \leftrightarrow \mathbf{q}' \right\}, \tag{5}
$$

where g is the spin-isospin degeneracy and the momenta are defined in Fig. 4. A resummation of the beta functions in the particle-hole channels is performed in [11]. In short, it generates $df/d\Lambda$ vertices in the beta function, where the quasiparticle interaction f receives contributions only from the ZS-irreducible flow (for details see [11]). This corresponds to the running of the vertex in one channel while integrating out fast particle-hole contributions in the second channel.

The phase space contributing to the flow, e.g., in the ZS channel, is depicted in Fig. 4. We employ an expansion in momentum transfers to solve the RG equation. It is based on the ob-

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[4]Thus, the important contributions from low-lying particle-hole excitations to the effective interaction are explicitly taken into account. The main effect of scattering in the particle-particle channel, the removal of the short-ranged repulsion, is taken care of by using $V_{\text{low k}}$ as the starting point for the RG flow. The low-lying excitations in this channel, which are responsible e.g., for superfluidity, are not included. These may be treated explicitly by employing BCS theory for the quasiparticle scattering amplitude to compute the superfluid gap.
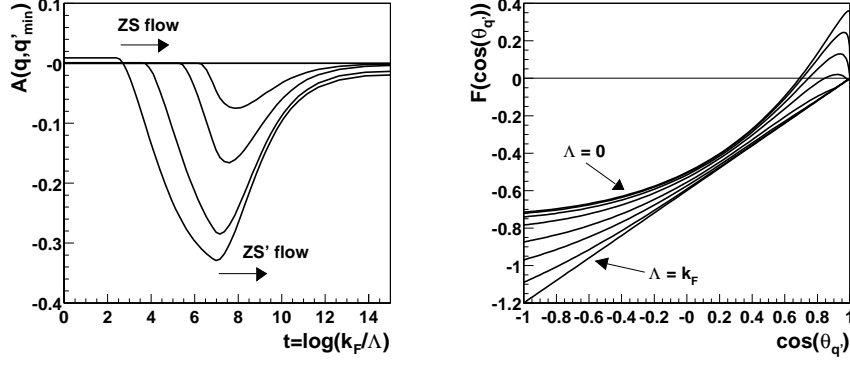
Fig. 5. Interplay of the particle-hole channels at one-loop. The left figure tracks the flow of $A(q, q'_{\min} = 0.002k_{\mathrm{F}})$ versus $t = \log(k_{\mathrm{F}}/\Lambda)$ for several values of $q \geq q'_{\min}$. The right figure shows snapshots of the running of the quasiparticle interaction $F(\cos \theta_{\mathbf{q}'})$ as function of $\cos \theta_{\mathbf{q}'}$ at different cutoffs.

servation that the dependence on Landau angle in the ZS' flow (the "induced interaction" [9]) is small. For $q, q' \ll k_{\mathrm{F}}$, the beta functions may be obtained analytically. This leads to a flow equation, which in a schematic notation, is of the form

$$\frac{d}{d\Lambda} a(\mathbf{q}, \mathbf{q}') = \Theta(q - 2\Lambda)\, \beta_{\mathrm{ZS}}[a, \mathbf{q}, \Lambda] - \Theta(q' - 2\Lambda)\, \beta_{\mathrm{ZS'}}[a, \mathbf{q}', \Lambda], \qquad (6)$$

where $\beta_{\mathrm{ZS}^{(\prime)}}[a, \mathbf{q}^{(\prime)}, \Lambda]$ is derived in detail in [13]. We note that the flow starts for $\Lambda \leq q^{(\prime)}/2$ (see also Fig. 4). This can be used to switch off either the ZS or ZS' contributions. E.g., by setting $q = 0$, we retain only the exchange contributions. These correspond to the induced interaction of Babu and Brown [9].[5] We use this observation to define the quasiparticle interaction and the forward scattering amplitude of Fermi liquid theory (FLT) in the RG approach,

$$f_{\mathrm{FLT}}(\mathbf{q}') = \lim_{\Lambda \to 0} a(q = 0, \mathbf{q}', \Lambda) \quad \text{and} \quad a_{\mathrm{FLT}}(\mathbf{q}') = \lim_{q \to 0} a(\mathbf{q}, \mathbf{q}', \Lambda = 0). \qquad (7)$$

This ambiguity in the limiting procedure corresponds to the long wavelength-low energy singularity in the particle-hole channel, which was used by Landau to eliminate the ZS contribution.

4   Results for a toy model and neutron matter

We demonstrate the interplay of the particle-hole channels at the one-loop level for a toy system of spin-polarized fermions. The RG is solved at zero temperature and for three dimensions in the small-momentum transfer approximation ($q, q' \ll k_{\mathrm{F}}$). The complete phase space in two dimensions was studied in [12]. We choose the initial two-body interaction to be (in units where

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[5]We note that the RG approach yields, in a Pauli principle conserving approximation, the effective interaction and scattering amplitude for finite scattering angles. In fact, to the best of our knowledge, it constitutes the only microscopic approach where both momentum transfers are treated on an equal footing.
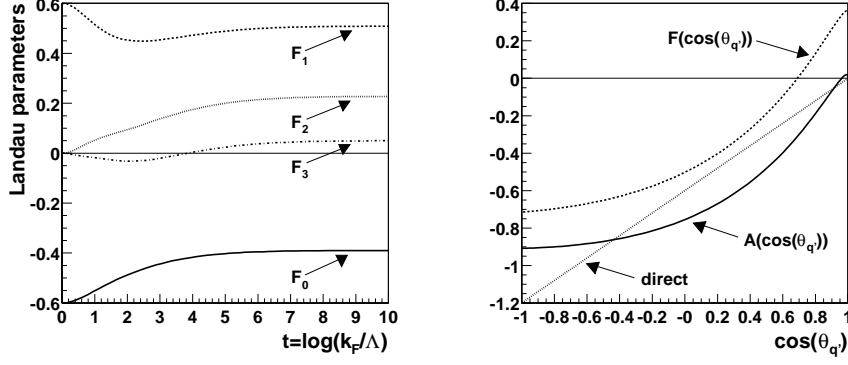
Fig. 6. The left figure shows the flow of the first four Landau parameters. The right figure gives the solution for the quasiparticle interaction and the forward scattering amplitude and the initial "direct" term, Eq. (8).

the density of states $m\,k_F/2\pi^2 = 1$)

$$a(\mathbf{q},\mathbf{q}',\Lambda=k_F) = \frac{1}{2}\big(\cos\theta_{\mathbf{q}'}-\cos\theta_{\mathbf{q}}\big), \quad \text{where} \quad q^{(\prime)} = k_F\sqrt{2-2\cos\theta_{\mathbf{q}^{(\prime)}}} \qquad (8)$$

and the angle $\theta_{\mathbf{q}'}$ reduces to the Landau angle for $q=0$. This toy interaction has only $F_0 = -6/10$ and $F_1 = 6/10$ non-vanishing Fermi liquid parameters.

In Fig. 5, we show the running of the scattering amplitude and the quasiparticle interaction with $t = \log(k_F/\Lambda)$. The left graph of Fig. 5 shows the interference of the ZS and the ZS' channel. The phase space is open for $\Lambda \leq q, q'/2$. Accordingly we observe the onset of the ZS channel first in the running of $A(q,q'_{\min}) = m^\star/m\,a(q,q'_{\min})$. The ZS' channel sets in later and counteracts the ZS channel, as the two channels are related by exchange. Next we switch off the ZS channel. In other words, we study the running of the quasiparticle interaction $F(\mathbf{q}') = m^\star/m\,f(\mathbf{q}')$. This is shown in the right graph of Fig. 5. Due to the factor $\Theta(q'-2\Lambda)$ in the beta function, the flow at small angles (small q') sets in only for small values of the cutoff Λ.

In Fig. 6, we show the flow of the $l \leq 3$ Fermi liquid parameters. The Fermi liquid parameters F_l are obtained by projecting the quasiparticle interaction on Legendre polynomials $F(\cos\theta_{\mathbf{q}'}) = \sum_l F_l P_l(\cos\theta_{\mathbf{q}'})$. The solution of the RG flow naturally generates higher Fermi liquid parameters, corresponding to higher powers of momentum transfer. Thus, it is very convenient to solve the RG equations without projecting on a set of coupling constants. The quasiparticle interaction and the forward scattering amplitude obtained in the RG are shown in Fig. 6.

The adaption of the method to a realistic system, like neutron matter is straightforward; we include spin and employ $V_{\text{low k}}$ as initial condition. The details of the calculation are given in [13]. Here, we show only the $l = 0$ Fermi liquid parameters in comparison with the calculation of Wambach *et al.* [14]. We note the importance of the large spin Fermi liquid parameter $G_0 \approx 0.7-0.8$. It induces spin-density fluctuations which increase the incompressibility, related to $1 + F_0$, considerably, as can be seen in Fig. 7. Strikingly, our results for the Fermi liquid parameters in this simple RG calculation agree very well with Wambach *et al.* [14].

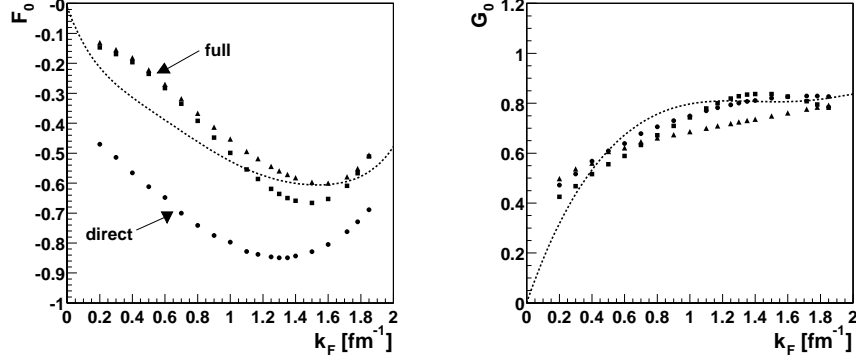In addition to Fermi liquid theory, the RG approach makes the full scattering amplitude for

Fig. 7. The density dependence of the $l = 0$ Fermi liquid parameters of neutron matter F_0 (left figure) and G_0 (right figure), $\mathcal{F}(\cos\theta_{\mathbf{q}'}) = \sum_l (F_l + G_l\,\sigma\cdot\sigma')\,P_l(\cos\theta_{\mathbf{q}'})$. Each graph contains the initial condition of the flow (circles) and the full RG solution [13] (squares for a density independent $z_{k_{\mathrm{F}}}^2 = 0.9$ and triangles with an adaptive $z_{k_{\mathrm{F}}}$ factor) in comparison with the study of Wambach *et al.* [14] (dashed line). For details regarding the $z_{k_{\mathrm{F}}}$ factor, see [13]. Note that for the toy model we have set $z_{k_{\mathrm{F}}} = 1$.

finite q accessible. The q dependence has been used to calculate the superfluid $^1\mathrm{S}_0$ pairing gap including polarization effects for neutron matter [13]. We conclude by remarking that the RG method is a very promising tool for studying a wide range of nuclear many-body problems.

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