

**THE INFRARED BEHAVIOUR OF THE RUNNING COUPLING  
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Approximate solutions for the gluon and ghost propagators as well as the running coupling in Landau gauge Yang–Mills theories are presented. These propagators obtained from the corresponding Dyson–Schwinger equations are in remarkable agreement with those of recent lattice calculations. The resulting running coupling possesses an infrared fixed point,  $\alpha_S(0) = 8.92/N$  for all gauge groups  $SU(N)$ . Above one GeV the running coupling rapidly approaches its perturbative form.

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**1 Motivation: Some aspects of confinement**

The investigation of Quantum Chromo Dynamics (QCD) over the last decades made obvious that a description of hadrons and their processes from the dynamics of confined quarks and gluons in one coherent approach is an outstanding complicated task. Knowledge of a completely non-perturbative running coupling would be a very important step to make towards such a QCD-based description of hadrons.

Given the experimental results in hadron physics it is evident that baryons and mesons are not elementary particles in the naive sense of the word “elementary”. The partonic substructure of the nucleon has been determined to an enormous precision leaving no doubt that the parton picture emerges from quarks and gluons, the elementary fields of QCD. This contrasts the well-known fact that these quarks and gluons have not been detected outside hadrons. This puzzle was given a name: *confinement*. The confinement hypothesis was formulated several decades ago, nevertheless, our understanding of the confinement mechanism(s) is still not satisfactory. Moreover, in contrast to other non-perturbative phenomena in QCD (*e.g.*, dynamical breaking of

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chiral symmetry,  $U_A(1)$  anomaly, and formation of relativistic bound states), it seems not even clear, at present, whether the phenomenon of confinement is at all compatible with a description of quark and gluon correlations in terms of local fields in the usual sense of quantum field theory.

One important feature of QCD is asymptotic freedom: Employing the renormalisation group (RG) in perturbative calculations clearly leads to an understanding of the experimental observation that partons interact at large (spacelike) momentum transfer only weakly. On the other hand, RG based arguments prove that perturbation theory is insufficient to account for confinement in four-dimensional field theories: Confinement requires the dynamical generation of a physical mass scale. In presence of such a mass scale, however, the RG equations imply the existence of essential singularities in physical quantities, such as the  $S$ -matrix, as functions of the coupling at  $g = 0$ . This is because the dependence of the RG invariant confinement scale on the coupling and the renormalisation scale  $\mu$  near the ultraviolet fixed point is determined by [1]

$$\Lambda = \mu \exp\left(-\int^g \frac{dg'}{\beta(g')}\right) \xrightarrow{g \rightarrow 0} \mu \exp\left(-\frac{1}{2\beta_0 g^2}\right), \quad \beta_0 > 0. \quad (1)$$

Therefore a study of the infrared behaviour of QCD amplitudes requires non-perturbative methods. In addition, as infrared singularities are anticipated, a formulation in the continuum is desirable. One promising approach to non-perturbative phenomena in QCD is provided by studies of truncated systems of its Dyson–Schwinger equations, the equations of motion for QCD Green’s functions, for recent reviews see *e.g.* [2]. As we will see in the following these studies also allow to extract a non-perturbative running coupling. One word of warning is, however, in order: A non-perturbative running coupling is not a uniquely defined object. The extension of the running coupling into the infrared domain requires in the first step its definition from a specifically chosen Green’s function. Of course, within a chosen gauge the result should be unique, and one should be able to prove this uniqueness from the Slavnov–Taylor identities of QCD. Noting that Green’s function are not gauge invariant it is not at all obvious whether the comparison of non-perturbative running couplings defined in different gauges can be meaningful at all.

## 2 A possible definition of the non-perturbative running coupling in Landau gauge

As stated above, the definition of the non-perturbative running coupling rests on a specifically chosen Green’s function. To this end we note that the ghost-gluon vertex in Landau gauge acquires no independent renormalisation, in specialist language  $\tilde{Z}_1 = 1$ . This relates the charge renormalisation constant  $Z_g$  to the ones for the gluon and ghost wave functions,  $1 = \tilde{Z}_1 = Z_g \sqrt{\tilde{Z}_3} \tilde{Z}_3$ : The gluon leg provides a factor  $\sqrt{\tilde{Z}_3}$ , the two ghost legs  $\tilde{Z}_3$ . As we will demonstrate in the following this allows for a definition of the running coupling resting solely on the properties of the gluon and ghost propagators.

In linear covariant gauges the gluon propagator is of the form<sup>5</sup>

$$D_{\mu\nu}^{\text{Gluon}} = \frac{Z(k^2)}{k^2} \left( \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + \xi \frac{k_\mu k_\nu}{k^4}. \quad (2)$$

The Lorentz condition  $\partial_\mu A_\mu^a = 0$  is strictly implemented only in the limit  $\xi \rightarrow 0$  which defines the Landau gauge. Then the gluon propagator is strictly transverse with respect to the gluon

<sup>5</sup>As usual in these studies a Wick rotation to Euclidean space has been employed.

momentum  $k$ . The gluon dressing, also in the non-perturbative domain, is solely described by the function  $Z(k^2)$ . The Faddeev–Popov ghosts are introduced in the process of quantization: The functional integral over these ghosts is a representation of the Jacobian factor induced in the generating functional when enforcing the gauge condition. The scalar ghost fields belong to the trivial representation of the connected part of the Lorentz group. As local fields with space-like anti-commutativity, they violate the spin-statistics theorem and are thus necessarily unphysical. The general form of their propagator reads

$$D_{\text{Ghost}} = - \frac{G(k^2)}{k^2}. \quad (3)$$

A comparison to the tree level form of these propagators reveals that  $Z(k^2) \rightarrow \text{const.}$  and  $G(k^2) \rightarrow \text{const.}$ , up to the perturbative logarithms, for asymptotically large momenta  $k^2$ .

In the next step we discuss the employed non-perturbative subtraction scheme and its relation to the definition of the running coupling. As already stated, the starting point is the following identity for the renormalisation constants

$$\tilde{Z}_1 = Z_g Z_3^{1/2} \tilde{Z}_3 = 1, \quad (4)$$

which holds in Landau gauge. It follows that the product  $g^2 Z(k^2) G^2(k^2)$  is RG invariant. In absence of any dimensionful parameter this (dimensionless) product is therefore a function of the running coupling  $\bar{g}$ ,

$$g^2 Z(k^2) G^2(k^2) = f(\bar{g}^2(t_k, g)), \quad t_k = \frac{1}{2} \ln k^2 / \mu^2. \quad (5)$$

Here, the running coupling  $\bar{g}(t, g)$  is the solution of  $d/dt \bar{g}(t, g) = \beta(\bar{g})$  with  $\bar{g}(0, g) = g$  and the Callan–Symanzik  $\beta$ -function  $\beta(g) = -\beta_0 g^3 + \mathcal{O}(g^5)$ . The perturbative momentum subtraction scheme is asymptotically defined by  $f(x) \rightarrow x$  for  $x \rightarrow 0$ . This is realized by independently setting

$$Z(\mu^2) = 1 \quad \text{and} \quad G(\mu^2) = 1 \quad (6)$$

for some asymptotically large subtraction point  $k^2 = \mu^2$ . If the quantity  $g^2 Z(k^2) G^2(k^2)$  is to have a physical meaning, *e.g.*, in terms of a potential between static colour sources, it should be independent under changes  $(g, \mu) \rightarrow (g', \mu')$  according to the RG for arbitrary scales  $\mu'$ . Therefore,

$$g^2 Z(\mu'^2) G^2(\mu'^2) \stackrel{!}{=} g'^2 = \bar{g}^2(\ln(\mu'/\mu), g), \quad (7)$$

and,  $f(x) \equiv x, \forall x$ . This can thus be adopted as a physically sensible definition of a non-perturbative running coupling in the Landau gauge. In the scheme summarized in the next section, it is not possible to realize  $f(x) \equiv x$  by simply extending the perturbative subtraction scheme (6) to arbitrary values of the scale  $\mu$ , as this would imply a relation between the functions  $Z$  and  $G$  which is inconsistent with the leading infrared behaviour of the solutions as will become evident from the discussion presented in the next section. For two independent functions the condition (6) is in general too restrictive to be used for arbitrary subtraction points. Rather,

in extending the perturbative subtraction scheme, one is allowed to introduce functions of the coupling such that

$$Z(\mu^2) = f_A(g) \quad \text{and} \quad G(\mu^2) = f_G(g) \quad \text{with} \quad f_G^2 f_A = 1, \quad (8)$$

and the limits  $f_{A,G} \rightarrow 1$ ,  $g \rightarrow 0$ . Using this it is straightforward to see that for  $k^2 \neq \mu^2$  one has  $(t_k = (\ln k^2/\mu^2)/2)$ ,

$$\begin{aligned} Z(k^2) &= \exp \left\{ -2 \int_g^{\bar{g}(t_k, g)} dl \frac{\gamma_A(l)}{\beta(l)} \right\} f_A(\bar{g}(t_k, g)), \\ G(k^2) &= \exp \left\{ -2 \int_g^{\bar{g}(t_k, g)} dl \frac{\gamma_G(l)}{\beta(l)} \right\} f_G(\bar{g}(t_k, g)). \end{aligned} \quad (9)$$

Here  $\gamma_A(g)$  and  $\gamma_G(g)$  are the anomalous dimensions of gluons and ghosts, respectively, and  $\beta(g)$  is the Callan–Symanzik  $\beta$ -function. Eq. (4) corresponds to the following identity for these scaling functions in Landau gauge:

$$2\gamma_G(g) + \gamma_A(g) = -\frac{1}{g} \beta(g). \quad (10)$$

One thus verifies that the product  $g^2 Z G^2$  indeed gives the running coupling (*i.e.*, Eq. (5) with  $f(x) \equiv x$ ). Perturbatively, at one-loop level Eq. (10) is realized separately, *i.e.*,  $\gamma_G(g) = -\delta \beta(g)/g$  and  $\gamma_A(g) = -(1 - 2\delta) \beta(g)/g$  with  $\delta = 9/44$  for  $N_f = 0$  and arbitrary  $N_c$ . Non-perturbatively one can still separate these contributions from the anomalous dimensions by introducing an unknown function  $\epsilon(g)$ ,

$$\gamma_G(g) =: -(\delta + \epsilon(g)) \frac{\beta(g)}{g} \Rightarrow \gamma_A(g) = -(1 - 2\delta - 2\epsilon(g)) \frac{\beta(g)}{g}. \quad (11)$$

This allows to rewrite Eqs. (9) as follows:

$$\begin{aligned} Z(k^2) &= \left( \frac{\bar{g}^2(t_k, g)}{g^2} \right)^{1-2\delta} \exp \left\{ -4 \int_g^{\bar{g}(t_k, g)} dl \frac{\epsilon(l)}{l} \right\} f_A(\bar{g}(t_k, g)), \\ G(k^2) &= \left( \frac{\bar{g}^2(t_k, g)}{g^2} \right)^\delta \exp \left\{ 2 \int_g^{\bar{g}(t_k, g)} dl \frac{\epsilon(l)}{l} \right\} f_G(\bar{g}(t_k, g)). \end{aligned} \quad (12)$$

This is also possible in the presence of quarks. In this case one has  $\delta = \gamma_0^G/\beta_0 = 9N_c/(44N_c - 8N_f)$  for  $N_f$  flavours in Landau gauge. The above representation of the renormalisation functions expresses clearly that regardless of possible contributions from the unknown function  $\epsilon(g)$ , the resulting exponentials cancel in the product  $G^2 Z$ . For a parameterisation of the renormalisation functions, these exponentials can of course be absorbed by a redefinition of the functions  $f_{A,G}$ . The only effect of such a redefinition is that the originally scale independent functions  $f_{A,G}(\bar{g}(t_k, g))$  will acquire a scale dependence by this, if  $\epsilon \neq 0$ .

For the truncation scheme presented in the next section it is possible, however, to obtain explicitly scale independent equations thus showing that the solutions for the renormalisation functions  $G$  and  $Z$  obey one-loop scaling at all scales [3]. In particular, this implies that the products  $g^{2\delta} G$  and  $g^{2(1-2\delta)} Z$  are separately RG invariants (as they are at one-loop level). As for the renormalisation scale dependence, the non-perturbative nature of the result is therefore buried entirely in the result for the running coupling.

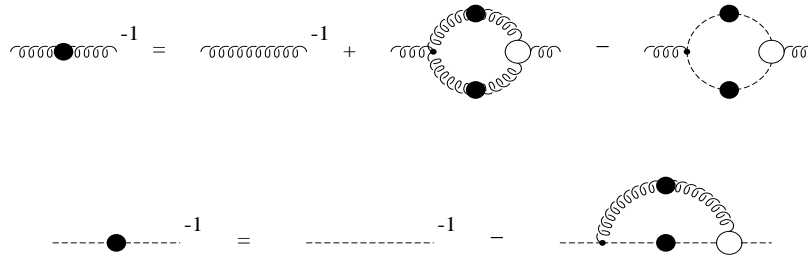


Fig. 1. Diagrammatic representation of the truncated gluon and ghost Dyson–Schwinger equations studied in this letter. Terms with four–gluon vertices have been dismissed. Open circles denote vertex functions (which are approximated by bare ones) and fully drawn circles denote dressed propagators.

### 3 Infrared exponents for gluons and ghosts

Having at hand a non-perturbative definition of the running coupling which employs only the properties of the gluon and ghost propagators, an equally non-perturbative method is then required to determine these propagators. Their infrared singularity structures are particularly well accessible by non-perturbative continuum methods. To this end the Landau gauge Dyson–Schwinger equations (DSEs) have been solved analytically in the infrared [3–8]. Necessarily, however, DSE studies will always be subject to truncations in order to obtain the closed system of equations to be studied. Thus, the quality of such a study crucially depends on the justification of the truncations. Some uncertainty about these truncations remains, however, even in the most ambitious study. On the other hand, corresponding lattice calculations, see *e.g.* [9–11], include all non-perturbative physics but are limited for small momenta by the finite lattice volume. Despite their respective shortcomings both these approaches agree encouragingly well in the general observations: there is clear evidence for an infrared finite or even vanishing gluon propagator and a diverging ghost propagator. This is in accordance with the Kugo–Ojima confinement criterion, which in Landau gauge includes the statement that the ghost propagator should be more singular than a simple pole [12].

For the purpose of this talk we will concentrate on the truncation scheme for DSEs presented in refs. [7, 8]. The infrared behaviour of the ghost and gluon system was studied analytically for a wide class of non-perturbatively dressed ghost-gluon vertex functions in ref. [6]. Based on few and general assumptions about the generic structure of this vertex, it was thereby concluded that its dressing should not affect the qualitative findings. We therefore restrict to bare three-point vertices for simplicity. All contributions from explicit four-gluon vertices are neglected in addition. A diagrammatical representation of the resulting system of equations is presented in Fig. 1. This truncation provides to one-loop order the correct anomalous dimensions of the ghost and gluon dressing functions,  $Z(k^2)$  and  $G(k^2)$ , in the ultraviolet region of momentum. On the other hand, it reproduces the infrared exponents found in [5, 6] which are close to the ones extracted from lattice calculations [9–11]. The numerical solution to the truncated DSEs for the gluon and the ghost propagators<sup>6</sup> proved to be compatible with only one out of the two solutions reported in the infrared analysis of ref. [5]. This thus demonstrates that not every analytical

<sup>6</sup>A detailed description of corresponding numerical techniques can be found in refs. [4, 7, 13].

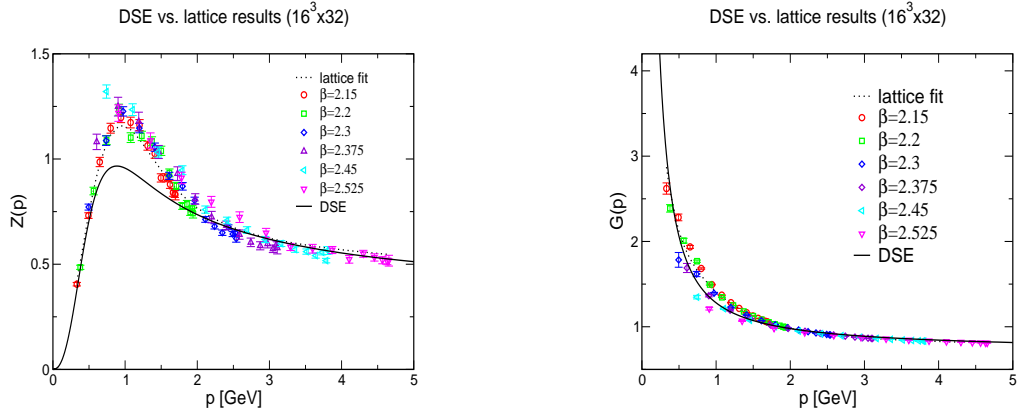


Fig. 2. Solutions of the Dyson-Schwinger equations compared to recent lattice results for two colours [11].

solution for asymptotically small momenta necessarily connects to a numerical solution for finite momenta.

The corresponding infrared behaviour of the propagators is given by  $D_{\text{Gluon}}(k^2) \sim (k^2)^{2\kappa-1}$  and  $D_{\text{Ghost}}(k^2) \sim (k^2)^{-\kappa-1}$  with  $\kappa = (93 - \sqrt{1201})/98 \approx 0.595$ : One obtains a weakly infrared vanishing gluon propagator and a strongly infrared enhanced ghost propagator. In Fig. 2 a comparison of the numerical solution of DSEs with the results of recent lattice calculations [11] for the case of two colours is shown. As the solutions on the lattice include all non-perturbative effects, the results shown in Fig. 2 suggest that the omission of the two-loop diagrams in the truncated DSEs mostly effects the region around the bending point at 1 GeV. Given the limitations of both methods the qualitative and partly even quantitative agreement is remarkable. The combined evidence of the two methods points strongly towards an infrared vanishing or finite gluon propagator and an infrared singular ghost propagator in Landau gauge.

#### 4 Infrared fixed point and $\alpha_S(\mu^2)$

The non-perturbative definition of the running coupling given in section 2 can be summarized as follows:

$$\alpha_S(k^2) = \alpha_S(\mu^2) Z(k^2; \mu^2) G^2(k^2; \mu^2). \quad (13)$$

Here we have made explicit the dependence of the propagator functions on the renormalisation point. An important point to notice in the results described in the last section is the unique relation between the gluon and ghost infrared behaviour. This is no accident: Consistency of the DSEs require that the product  $Z(k^2)G^2(k^2)$  goes to a constant in the infrared.

Correspondingly we find an infrared fixed point of the running coupling:

$$\alpha_S(0) = \frac{2\pi}{3N_c} \frac{\Gamma(3-2\kappa)\Gamma(3+\kappa)\Gamma(1+\kappa)}{\Gamma^2(2-\kappa)\Gamma(2\kappa)}, \quad \kappa = \frac{93 - \sqrt{1201}}{98}. \quad (14)$$

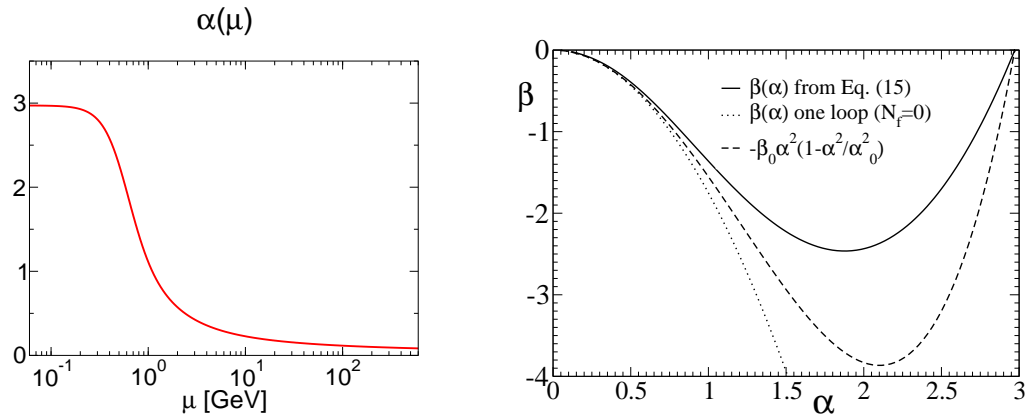


Fig. 3. The strong running coupling (left panel) and the corresponding  $\beta$  function (right panel, fully drawn line). The latter is compared to the one-loop  $\beta$  function (dotted line) and a polynomial in  $\alpha$  (dashed line).

For the gauge group SU(3) the corresponding numerical value is  $\alpha_S(0) \approx 2.972$ . Of course, this result depends on the employed truncation scheme. In ref. [6], assuming the infrared dominance of ghosts it has been shown that the tree-level vertex result  $\alpha_S(0) \approx 2.972$ , among the general class of dressed ghost-gluon vertices considered in the infrared, provides the maximal value for  $\alpha_S(0)$ . If the exponent  $\kappa$  is chosen in an interval between 0.5 and 0.7 (as strongly suggested by lattice results) one obtains  $\alpha_S(0) > 2.5$  [6].

In the case of three colours values for physical scales can be obtained by requiring the experimental value  $\alpha_S(M_Z^2 = (91.2\text{GeV})^2) = 0.118$ . Together with the numerical solutions for the gluon and the ghost propagators we can summarize our knowledge of the running strong coupling in the following fit:

$$\alpha_S(x) = \frac{\alpha_S(0)}{\ln(e + a_1 x^{a_2} + b_1 x^{b_2})}, \quad (15)$$

$$\alpha_S(0) = 2.972, \quad a_1 = 5.292\text{GeV}^{-2a_2}, \quad a_2 = 2.324, \quad b_1 = 0.034\text{GeV}^{-2b_2}, \quad b_2 = 3.169.$$

In Fig. 3 the running coupling and the corresponding  $\beta$  function are shown.

Finally, we note that indications for an infrared finite coupling have recently also been obtained within the background field method from the Exact Renormalisation Group Equations [14].

## 5 Outlook: $\alpha_S(\mu^2)$ and physical observables

To summarize: We have provided a non-perturbative definition for the running strong coupling in the Landau gauge. The underlying picture is related to confinement of transverse gluons, see *e.g.* [15] and references therein. Loosely speaking, one can summarize this by stating that gluons are confined by the Faddeev–Popov ghosts. The most interesting result for the running coupling is the existence of an infrared fixed point. The occurrence of this fixed point can hereby be traced back to very general properties of the ghost Dyson–Schwinger equation [6]. Taking only one-loop terms in the gluon Dyson–Schwinger equation into account one obtains that  $\alpha_S(\mu^2) \propto$

$1/N_c$  with  $N_c$  being the number of colours. For three colours the corresponding critical value for  $\alpha_S(0)$  slightly depends on the approximations used. Together with the results of lattice calculations we confidently conclude that  $\alpha_S(0) \approx 3$  or slightly lower.

Of course, a phenomenological verification of the presented picture would be highly welcome. From so-called analytic perturbation theory, see *e.g.* [16] and references therein, it is known that an infrared finite coupling allows for an almost straightforward calculation of amplitudes relevant in inclusive  $\tau$  decay, in  $e^+e^-$ -annihilation into hadrons, in inelastic lepton-hadron scattering etc. The interesting point hereby is that related observables might give bounds on allowed values for  $\alpha_S(0)$ .

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