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ON THE $K^{\pm}$-MESON PRODUCTION FROM THE QUARK-GLUON PLASMA PHASE IN ULTRA-RELATIVISTIC HEAVY-ION COLLISIONS

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An abundance of the strangeness that can be induced in thermalized quark-gluon plasma (QGP) is considered as a signal of the QGP phase appearing in the intermediate state of ultra-relativistic heavy-ion collisions. As a quantitative characteristic of this signal, we take the ratio $R_{K^+K^-} = N_{K^+}/N_{K^-}$ of the multiplicities of the production of $K^{\pm}$ mesons. This ratio is evaluated for K mesons produced from thermalized QGP phase and also for K mesons produced by the quark-gluon system out of the QGP phase. For a thermalized QGP phase, the ratio $R_{K^+K^-}$ has been found a smooth function of a 3-momentum of the $K^{\pm}$ meson and in the region $160\,\mathrm{MeV} < T < 200\,\mathrm{MeV}$. We show that at $T = 175\,\mathrm{MeV}$ our prediction for the ratio $R_{K^+K^-}(q, T = 175) = 1.80^{+0.04}_{-0.18}$ agrees well with the experimental data of NA49 and NA44 collaborations on central ultra-relativistic Pb+Pb collisions at 158 GeV/A, $R^{\rm exp}_{K^+K^-} = 1.80 \pm 0.10$. For the K^+ and π^+ multiplicities we have obtained the value $R_{K^+\pi^+}(q, T = 175) = 0.134 \pm 0.014$ agreeing well with the experimental value $R^{\rm exp}_{K^+\pi^+} = 0.137 \pm 0.008$ obtained by NA35 collaboration in the nucleus-nucleus collisions at 200 GeV/A.

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1 Introduction

Nowadays there is a consensus that QCD gives a satisfactory description of strong interactions of hadrons. The important question, which is still left, concerns the properties of the QCD vacuum. One of the approaches to these properties concerns the excited vacuum states at high densities and temperatures. The quark-gluon plasma (QGP) phase of QCD [1,2] is most likely an excited

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QCD vacuum with quarks, antiquarks and gluons in the deconfined phase. There is a belief [2] that the QGP phase of the quark-gluon system can be realized in ultra-relativistic heavy-ion collisions.

As usual for the theoretical analysis of the QGP, one uses either the dynamical approach [3] or the thermal one [4]. In the dynamical approach, the QGP is treated as a non-equilibrium quark-gluon system, whose evolution obeys relativistic kinetic equations supplemented by high-energy quark-gluon interactions treated perturbatively. In the thermal approach, the QGP is approximated by the ideal quark-gluon gas thermalized after the collision of heavy-ions. In both approaches the evolution of the QGP ends by hadronization.

The experimental detection of the QGP produced in ultra-relativistic heavy-ion collisions can be carried out only by the analysis of hadrons in the final state of the reaction. However, hadrons can be produced in these collisions not only from QGP phase but also from other states of the quark-gluon system which can accompany the collisions and differ from the QGP phase. In order to avoid mistakes in the detection of the QGP phase, one needs to have distinct criteria allowing to distinguish the hadrons produced from the QGP phase from the hadrons which can appear due to other states of the quark-gluon system in ultra-relativistic heavy-ion collisions.

As has been suggested in Refs. [5,6], one can expect an abundance of strange hadrons K, Λ, Ξ produced from thermalized QGP. Such an abundance could serve as a criterion of the QGP [5,6], if the existence of the thermalized QGP in ultra-relativistic heavy-ion collisions is well justified. The arguments in favour of the formation of the thermalized QGP in ultra-relativistic heavy-ion collisions are the following: Due to ultra-relativistic energies of heavy-ions, the quark-gluon system produced in the intermediate state should contain highly relativistic and very dense quarks, antiquarks and gluons. These constituents of the quark-gluon system are almost free by virtue of *asymptotic freedom*, and an exchange of energies between them goes only via collisions. Due to very high density of the constituents the collisions between them should occur very frequently, what should lead to an equilibrium state of a quark-gluon system. Such an equilibrated state can be treated as a thermalized QGP phase of the quark-gluon system. The constituents of the thermalized QGP can be described by Fermi-Dirac and Bose-Einstein distribution functions.

The probabilities of light massless quarks $n_q(\vec{p})$ and light massless antiquarks $n_{\bar{q}}(\vec{p})$, can be defined by the Fermi-Dirac distribution functions [1,4,7]:

$$n_q(\vec{p}) = \frac{1}{\mathrm{e}^{-\nu(T) + p/T} + 1}\,, \quad n_{\bar{q}}(\vec{p}) = \frac{1}{\mathrm{e}^{\nu(T) + p/T} + 1}\,, \tag{1.1}$$

where $q = u$ or d, with momentum $\vec{p}$ and temperature T. Here, T is measured in MeV, $\nu(T) = \mu(T)/T$, $\mu(T)$ is a chemical potential of the light massless quarks $q = u$ and d depending on a temperature T [7]. A chemical potential of light antiquarks is $-\mu(T)$. In the thermalized state [1,4] a positively defined $\mu(T)$ provides an enhancement of the number of light quarks with respect to the number of light antiquarks. A chemical potential $\mu(T)$ is a phenomenological parameter of the approach which we determine below.

The probability for gluons to have a momentum $\vec{p}$ at a temperature T is given by the Bose-Einstein distribution [1,4,7]

$$n_g(\vec{p}) = \frac{1}{\mathrm{e}^{p/T} - 1}. \tag{1.2}$$

Since the strangeness of the colliding heavy-ions is zero, the densities of strange quarks and antiquarks should be equal. This implies for chemical potential $\mu_s = \mu_{\bar{s}} = 0$. In this case the probabilities of strange quarks and antiquarks can be given by

$$n_s(\vec{p}\,) = n_{\bar{s}}(\vec{p}\,) = \frac{1}{e^{\sqrt{\vec{p}^{\,2} + m_s^2}/T} + 1}, \tag{1.3}$$

where $m_s = 135\,\mathrm{MeV}$ [8] is the mass of the current strange quark/antiquark. The value of the current s-quark mass $m_s = 135\,\mathrm{MeV}$ has been successfully applied to the calculation of chiral corrections to amplitudes of low-energy interactions, form factors and mass spectra of low-lying hadrons [9] and charmed heavy-light mesons [10]. Unlike the massless antiquarks $\bar{u}$ and $\bar{d}$ for which the suppression is caused by a chemical potential $\mu(T)$, the strange quarks and antiquarks are suppressed equally by virtue of the non-zero mass m_s.

We shall describe the multiplicities N_{K^+} and N_{K^-} of the production of K^+ and K^- mesons, in a way similar to a simple coalescence model [4,6] which we call a *coalescence model of correlated quarks*. In this case the multiplicities of $K^{\pm}$ meson production can be defined in terms of quark and antiquark distribution functions

$$\begin{aligned}
N_{K^+}(\vec{q}, T) &= \; < n_u(\vec{p} - \vec{q}\,)\, n_{\bar{s}}(\vec{p}\,) > \; = \\
&= N_C V_K \int \frac{d^3p}{(2\pi)^3}\, \frac{1}{e^{-\nu(T) + |\vec{p} - \vec{q}\,|/T} + 1}\, \frac{1}{e^{\sqrt{\vec{p}^{\,2} + m_s^2}/T} + 1}, \\
N_{K^-}(\vec{q}, T) &= \; < n_{\bar{u}}(\vec{p} - \vec{q}\,)\, n_s(\vec{p}\,) > \; = \\
&= N_C V_K \int \frac{d^3p}{(2\pi)^3}\, \frac{1}{e^{\nu(T) + |\vec{p} - \vec{q}\,|/T} + 1}\, \frac{1}{e^{\sqrt{\vec{p}^{\,2} + m_s^2}/T} + 1},
\end{aligned} \tag{1.4}$$

where $\vec{q}$ is a 3-momentum of the $K^{\pm}$ mesons, $N_C = 3$ is the number of quark colour degrees of freedom, V_K is a parameter of our *coalescence model of correlated quarks* having a dimension of a spatial volume and being to some extent an intrinsic characteristic of spatial distribution of the $K^{\pm}$ mesons. We suggest to determine V_K in terms of the parameters characterizing the properties of the K-meson such as the mass $M_K = 500\,\mathrm{MeV}$ and the leptonic coupling constant $F_K = 160\,\mathrm{MeV}$ [9,10]. Due to uncertainty relations, the K-meson should be localized within a volume inversely proportional to the power of a 3-momentum. For the thermalized K-meson system, this should be a thermal 3-momentum. The thermal momentum is proportional to the K-meson mass $\sqrt{M_K}$ in the case of the Maxwell-Boltzmann gas of $K^{\pm}$ mesons. Another important intrinsic parameter of K mesons is the leptonic coupling constant $F_K = 160\,\mathrm{MeV}$ [9,10]. Thus, according to dimensional considerations, one can set $V_K = C/(F_K M_K)^{3/2}$, where C is a dimensionless parameter of our model equal for all pseudoscalar mesons. Of course, such determination of V_K is not much rigorous, but it can be useful as a working hypothesis which does not contradict to experimental data.

The multiplicity of π^+-meson production is defined in analogous way

$$\begin{aligned}
N_{\pi^+}(\vec{q}, T) &= \; < n_u(\vec{p} - \vec{q}\,)\, n_{\bar{d}}(\vec{p}\,) > \; = \\
&= N_C V_\pi \int \frac{d^3p}{(2\pi)^3}\, \frac{1}{e^{-\nu(T) + |\vec{p} - \vec{q}\,|/T} + 1}\, \frac{1}{e^{\nu(T) + p/T} + 1},
\end{aligned} \tag{1.5}$$

where $V_\pi = C/(F_\pi M_\pi)^{3/2}$, and $F_\pi = 131\,\mathrm{MeV}$ and $M_\pi = 140\,\mathrm{MeV}$ are the leptonic coupling constant and the mass of pions, respectively.

We would like to emphasize that other dimensional parameters of K and π mesons, such as charge radii r_{K^+} and r_{π^+}, can be hardly considered as intrinsic parameters of these mesons, since they are functions of F_K and F_π. For example, in the Vector Dominance Approach, the charge radii can be expressed in terms of the masses of the ρ and K^* vector mesons M_ρ and M_{K^*}, respectively. According to the Kawarabayashi-Suzuki-Fayazuddin-Riazuddin (KSFR) relations imposed by chiral symmetry [9], these masses are proportional to F_π and F_K, respectively, $M_\rho = g_\rho\, F_\pi \simeq 790\,\mathrm{MeV}$ and $M_{K^*} = g_\rho F_K \simeq 960\,\mathrm{MeV}$, where $g_\rho \simeq 6$ is the coupling constant of the $\rho\pi\pi$ interaction. Theoretical values of the masses of the ρ and K^* mesons predicted by the KSFR relations agree with experimental values $M_\rho = 770\,\mathrm{MeV}$ and $M_{K^*} = 892\,\mathrm{MeV}$ within accuracies of 3% and 8%, respectively.

In our *coalescence model of correlated quarks,* the multiplicities of K and π meson production can be represented as

$$N(\vec{q},T) = \int \frac{\mathrm{d}^3 p_q}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3 p_{\bar{q}\,'}}{(2\pi)^{3/2}}\, n_q(\vec{p}_q)\, n_{\bar{q}\,'}(\vec{p}_{\bar{q}\,'})\, \Phi(\vec{p}_q - \vec{p}_{\bar{q}\,'})\, \delta^{(3)}(\vec{q} - \vec{p}_q - \vec{p}_{\bar{q}\,'})\,, \qquad (1.6)$$

where $\vec{p}_q$ and $\vec{p}_{\bar{q}\,'}$ are the 3-momenta of the quark q and the antiquark $\bar{q}\,'$, $\Phi(\vec{p}_q - \vec{p}_{\bar{q}\,'})$ is the wave function of a relative motion of $q\bar{q}\,'$ pair and the δ-function $\delta^{(3)}(\vec{q} - \vec{p}_q - \vec{p}_{\bar{q}\,'})$ stands for the momentum conservation. The wave function $\Phi(\vec{p}_q - \vec{p}_{\bar{q}\,'})$ is responsible for bosonization of the $q\bar{q}\,'$ pair and contains all information about spontaneous breaking of chiral symmetry and quark confinement. In our approach, the wave function of a relative motion of the $q\bar{q}\,'$ pair is constant

$$\Phi(\vec{p}_q - \vec{p}_{\bar{q}\,'}) = N_C V, \qquad (1.7)$$

where $V = V_K$ or $V = V_\pi$ for the multiplicities of K and π meson production, respectively.

This approximation is admissible, since it does not contradict to confinement of quarks and antiquarks, which should accompany hadronization of quarks and antiquarks from the QGP. Indeed, the use of Fermi-Dirac distribution functions for the description of the thermalized quarks and antiquarks (1.1) provides a concentration of the values of the momentum integrals around the momenta of relative motion of the $q\bar{q}\,'$ pair of order of $p \sim T$. Effectively this confines quarks and antiquarks in the region of spatial relative distances of order $\Delta r \lesssim 1/T \sim 1\,\mathrm{fm}$.

We employ in our *coalescence model of correlated quarks* also other approximation of multiplicities of the K and π meson production from the QGP phase. According to experimental data, the phase volume of the hadrons produced in ultra-relativistic A+A collisions is not spherically symmetric with respect to the collision axis. This can imply that the thermalization of the quark-gluon system, leading to the formation of the thermalized QGP, should be different in the transversal and longitudinal direction relative to the collision axis of ultra-relativistic A+A collisions. However, due to very high complexity of the theoretical description of hadronization from the thermalized QGP phase, we suggest to approximate multiplicities of hadron production by spherically symmetrical distribution functions assuming a spherical symmetric thermalization of the quark-gluon system. As we will show below, such a simplification describes well the ratios of multiplicities of K and π meson production.

The main goal of this paper is to calculate the ratios of the multiplicities

$$R_{K^+K^-}(q,T) = \frac{N_{K^+}(\vec{q},T)}{N_{K^-}(\vec{q},T)}\,, \quad R_{K^+\pi^+}(q,T) = \frac{N_{K^+}(\vec{q},T)}{N_{\pi^+}(\vec{q},T)} \qquad (1.8)$$

with a minimum number of input parameters. Since the main input parameter for the theoretical description of the thermalized QGP is the chemical potential $\mu(T)$ of light quarks and antiquarks, we shall aim our effort to this quantity.

We suggest to treat a heavy ion as a degenerate Fermi gas. Converting formally all nucleon degrees of freedom into quark degrees of freedom we would get a degenerate Fermi gas of quarks. More generally, this is a degenerate quark-gluon system, where all gluon and antiquark degrees of freedom have died out. Heating this quark-gluon system up to temperature T and demanding the conservation of the baryon number, this corresponds to the conservation of the baryon number density in the fixed spatial volume of the ion. Thus we fix unambiguously the temperature dependence of a chemical potential of $\mu(T)$. We find that $\mu(T)$ decreases strongly for high temperatures from the value $\mu(0) = \mu_0 = 250\,\mathrm{MeV}$ obtained at zero temperature. Such a behaviour of $\mu(T)$ implies that for high temperatures the number of light antiquarks ($\bar{u}$ and $\bar{d}$) will not be suppressed noticeably with respect to the number of light quarks (u and d). Therefore, the ratio $R_{K^+K^-}(q,T)$ should tend to unity in the high-temperature limit, $R_{K^+K^-}(q,T) \to 1$ at $T \gg \mu_0$. However, the numerical estimates show that $R_{K^+K^-}(q,T) \simeq 1$ can be already reached for $T \geq \mu_0 = 250\,\mathrm{MeV}$. In turn, the ratio $R_{K^+K^-}(q,T)$ varies from 2.14 to 1.48 for intermediate temperatures $T = 160 \div 200\,\mathrm{MeV}$, as it is estimated in Sect. 6. This means that the ratio $R_{K^+K^-}(q,T)$ can be a good criterion for the signal of the thermalized QGP phase in ultra-relativistic heavy-ion collisions. In fact, one gets the ratio $R_{K^+K^-} = N_{K^+}/N_{K^-} \simeq 1$, when it is calculated for the production of $K^{\pm}$ caused by the states of the quark-gluon system different to the thermalized QGP.

In Section 2 we calculate the chemical potential $\mu(T)$. In Section 3 we discuss a possibility of the formation of the thermalized QGP in ultra-relativistic heavy-ion collisions. In Section 4 we analyse the multiplicities of the $K^{\pm}$-meson production and give the analytical formula for the ratio $R_{K^+K^-}(q,T)$ as a function of 3-momenta of the $K^{\pm}$ mesons and a temperature T. In Section 5 we analyse the multiplicities of the $\pi^{\pm}$-meson production and give the analytical formula for the ratio $R_{K^+\pi^+}(q,T)$ as a function of 3-momenta of K^+ and π^+ mesons and a temperature T. In Section 6 we make the numerical analysis of the analytical formulas obtained in Section 4 and Section 5. We show that these ratios depend smoothly on both 3-momenta of mesons and a temperature which we change from $T = 160\,\mathrm{MeV}$ to $T = 200\,\mathrm{MeV}$. We find that our theoretical formulas for the ratios of multiplicities of K and π meson production reproduce reasonably well the experimental data on the central relativistic Pb+Pb collisions at 158 GeV per nucleon by NA49 and NA44 Collaborations and the data on proton-nucleus and nucleus-nucleus collisions at 200 GeV per nucleon by NA35 Collaboration for temperature $T = 175\,\mathrm{MeV}$. In the Conclusions, we discuss the obtained results.

2 Chemical potential of light quarks and antiquarks

Suppose that the chemical potential $\mu(T)$ is an intrinsic characteristic of the thermalized quark-gluon system, which we identify with the thermalized QGP. Then, if we assume that the thermalized QGP is an excited state of the QCD vacuum, the chemical potential $\mu(T)$ should describe the distributions of quarks and antiquarks of the thermalized QGP produced by any external state at any external conditions, not only due to ultra-relativistic heavy-ion collisions. Since any state of a thermalized system is closely related to certain external conditions, we need only to specify the external conditions of the thermalized quark-gluon system which should be the most convenient for the determination of $\mu(T)$ in order to obtain $\mu(T)$.

Let the external conditions of the thermalized quark-gluon system be caused by the state of the nuclear system, which is a heavy ion with the baryon number A. In the Fermi-gas approximation [11], a heavy ion is a degenerate gas of nucleons at $T = 0$ with the baryon density

$$n_B = \frac{A}{\frac{4\pi}{3} r_A^3} = \frac{3}{4\pi}\,\frac{1}{r_N^3} = 0.14\,\mathrm{fm}^{-3}\,, \tag{2.1}$$

coinciding with the nuclear matter density $n_{\rm N}$ [11], where $r_A = r_N\,A^{1/3}$ is the radius of a heavy ion, and $r_N = 1.2\,\mathrm{fm}$ [11,12].

Suppose that all baryon degrees of freedom are converted into quark degrees of freedom and quarks are massless. In this case we should get a degenerate Fermi gas of free quarks or more generally a degenerate free quark-gluon system, where all gluon and antiquark degrees of freedom have died out. Heating this quark-gluon system up to temperature T, we should get the thermalized system of quarks, antiquarks and gluons confined in the finite volume of a heavy ion $(4\pi/3) r_N^3\,A$.

In the low-temperature limit $T \to 0$, such conversion of baryon degrees of freedom into the quark ones required to get a system of free quarks can be understood qualitatively within a naive non-relativistic quark model, where baryons are slightly bounded three-quark states. These three valence quarks, the constituent quarks, can be treated as current quark excitations above a quark condensate produced by a cloud of current $q\bar{q}$ pairs due to spontaneous breaking of chiral symmetry.

In terms of the light quark and antiquark distribution functions (1.1), the baryon density of the thermalized quark-gluon system at a temperature T is given by [1,4]

$$\begin{aligned} n_B(T) &= \frac{1}{3}\times 2\times 2\times N_C\times [n_q(T) - n_{\bar{q}}(T)] = \\ &= \frac{4}{3}\,N_C\int\frac{\mathrm{d}^3p}{(2\pi)^3}\left[\frac{1}{\mathrm{e}^{-\nu(T) + p/T} + 1} - \frac{1}{\mathrm{e}^{\nu(T) + p/T} + 1}\right]. \end{aligned} \tag{2.2}$$

The factor $(1/3)\times 2\times 2\times N_C$ stands for the product of (*baryon charge*)$\times$ (*number of light quark flavour degrees of freedom*)$\times$(*number of spin degrees of freedom*)$\times$(*number of quark colour degrees of freedom*). Integrating over the momentum $\vec{p}$ we obtain [1]

$$n_B(T) = \frac{2}{9}\,N_C\left[\nu(T) + \frac{\nu^3(T)}{\pi^2}\right]T^3. \tag{2.3}$$

Denoting the chemical potential at zero temperature $T = 0$ as $\mu(0) = \mu_0$ we get

$$\mu_0 = \left(\frac{3\pi^2}{2}\right)^{1/3} n_{\rm B}^{1/3} = 250\,\mathrm{MeV}. \tag{2.4}$$

Here, $n_{\rm B}(0) = n_{\rm B} = 0.14\,\mathrm{fm}^{-3}$ is given by eq. (2.1), since the baryon density of nucleons should be equal to the baryon density of quarks in the fixed spatial volume due to a conservation of a baryon number. Our value of the chemical potential $\mu_0 = 250\,\mathrm{MeV}$ agrees well with the estimate $\mu_0 \sim 300\,\mathrm{MeV}$ [1].

The fluctuations of the baryon number density $n_B(T)$ caused by the fluctuations of a temperature T inside the fixed spatial volume $(4\pi/3)r_N^3\,A$ of the heavy ion should lead to the violation of the conservation of the baryon number. Since the baryon number is a good quantum number conserved for strong interactions, we impose the constraint

$$n_B(T) = n_B. \tag{2.5}$$

Using (2.3) and (2.4), the equation (2.5) can be transcribed into the form

$$\nu^3(T) + \pi^2\nu(T) - \Big(\frac{3}{N_C}\Big)\Big(\frac{\mu_0}{T}\Big)^3 = 0. \tag{2.6}$$

The cubic equation (2.6) has only one real root [13]. This defines the chemical potential $\mu(T)$ as a function of T

$$\frac{\mu(T)}{\mu_0} = \left[\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4\pi^6}{27}\left(\frac{T}{\mu_0}\right)^6}\,\right]^{1/3} - \left[-\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4\pi^6}{27}\left(\frac{T}{\mu_0}\right)^6}\,\right]^{1/3}, \tag{2.7}$$

where we have set $N_C = 3$. The chemical potential $\mu(T)$ given by eq. (2.7) guarantees the conservation of the baryon number under any fluctuations of a temperature T in the thermalized quark-gluon system confined in the fixed volume $V = (4\pi/3)\,r_N^3\,A$.

At $T \to 0$ and $T \to \infty$ the chemical potential $\mu(T)$ has the following asymptotic behaviours

$$\frac{\mu(T)}{\mu_0} = \begin{cases} 1 - \dfrac{\pi^2}{3}\dfrac{T^2}{\mu_0^2} + O(T^6)\,, & T \to 0\,, \\ \dfrac{\mu_0^2}{\pi^2}\,\dfrac{1}{T^2} + O(T^{-7})\,, & T \to \infty\,. \end{cases} \tag{2.8}$$

From (2.7) one can find that $\mu(T)$ decreases strongly when a temperature increases. Indeed, at $T = 160\,\mathrm{MeV}$ one obtains $\mu(T) \simeq \mu_0/4$, while at $T = \mu_0$ the value of the chemical potential makes up about tenth part of μ_0, i.e. $\mu(T) \simeq \mu_0/10$. This implies that at very high temperatures the function $\nu(T) = \mu(T)/T$ becomes small and the contribution of the chemical potential of light quarks and antiquarks can be treated perturbatively. This assumes that at temperatures $T \ge \mu_0 = 250\,\mathrm{MeV}$ the number of light antiquarks will not be substantially suppressed by the chemical potential relative to the number of light quarks.

Since for very high temperatures the ratio $R_{K^+K^-}(q,T)$ should be of the order of unity, one can hardly distinguish whether hadrons are produced from the thermalized QGP phase or from another states of the quark-gluon system different to the thermalized QGP for temperatures $T > \mu_0 = 250\,\mathrm{MeV}$.

Nevertheless, the ratio $R_{K^+K^-}(q,T)$ would differ noticeably from unity for intermediate temperatures $T = 160 \div 200\,\mathrm{MeV}$. Indeed, the rough estimate gives

$$R_{K^+K^-}(q,T) \sim \mathrm{e}^{2\nu(T)} = 2.14 \div 1.48 \;>\; 1. \tag{2.9}$$

That is why the experimental analysis of the ratio $R_{K^+K^-}(q,T)$ can be a good criterion for the signal of the thermalized QGP phase in ultra-relativistic heavy-ion collisions.

For rough estimate of the ratio $R_{K^+\pi^+}(q,T)$, we obtain

$$R_{K^+\pi^+}(q,T) \sim \frac{V_K}{V_\pi} \times \mathrm{e}^{\nu(T)} = 0.161 \div 0.133. \tag{2.10}$$

Our estimates (2.9) and (2.10) are in qualitative agreement with the experimental data on the central ultra-relativistic Pb+Pb at 158 GeV/A collisions by NA49 and NA44 Collaborations and with the data on proton-nucleus and nucleus-nucleus collisions at 200 GeV per nucleon by NA35 Collaboration, which are $R^{\rm exp}_{K^+K^-} = 1.80 \pm 0.10$ [14-17] and $R^{\rm exp}_{K^+\pi^+} = 0.137 \pm 0.008$ [17].

3 Thermalized quark-gluon plasma

The lifetime of the thermalized QGP phase is of the order of $\tau_{\rm QGP} = (6 \div 15)\,{\rm fm/c}$. Therefore, thermalization of the quark-gluon system should occur for times $\tau_{\rm th}$ much less than $\tau_{\rm QGP}$, i.e. $\tau_{\rm QGP} \gg \tau_{\rm th}$. This can be fulfilled only in a very dense matter. Thereby, in order to be convinced that the thermalized QGP can be formed in ultra-relativistic heavy-ion collisions, we have to calculate the density of the number of the constituents of the thermalized QGP at a temperature T $n(T)$,, and to compare it with the nuclear matter density $n_{\rm N} = 0.14\,{\rm fm}^{-3}$. The existence of the thermalized QGP should correspond to $n(T) \gg n_{\rm N}$. The density $n(T)$ is determined by [1,4]

$$\begin{aligned} n(T) &= n_g(T) + n_q(T) + n_{\bar q}(T) = 2(N_C^2-1)\int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{1}{\mathrm{e}^{p/T}-1} \\ &+ 4\,N_C\int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{1}{\mathrm{e}^{-\nu(T)+p/T}+1} + 4\,N_C\int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{1}{\mathrm{e}^{\nu(T)+p/T}+1}. \end{aligned} \tag{3.1}$$

Integrating over momenta, we obtain

$$n(T) = T^3\,\frac{4N_C}{\pi^2}\left[\left(\frac{N_C^2-1}{4N_C}+\frac{1}{2}\right)\zeta(3) + \nu^2(T)\,\ln 2 + \frac{1}{6}\,\nu^3(T) - \int\limits_0^{\nu(T)} \mathrm{d}x\,\frac{(\nu(T)-x)^2}{\mathrm{e}^x+1}\right], \tag{3.2}$$

where $\zeta(3) = 1.202$ is a Riemann zeta-function [18]

$$\zeta(s) = \frac{1}{\Gamma(s)}\int\limits_0^\infty \mathrm{d}x\,\frac{x^{s-1}}{\mathrm{e}^x-1} = \frac{1}{(1-2^{1-s})}\,\frac{1}{\Gamma(s)}\int\limits_0^\infty \mathrm{d}x\,\frac{x^{s-1}}{\mathrm{e}^x+1}. \tag{3.3}$$

At $N_C = 3$, the density $n(T)$ is equal to

$$n(T) = T^3\,\frac{12}{\pi^2}\left[\frac{7}{6}\,\zeta(3) + \nu^2(T)\,\ln 2 + \frac{1}{6}\,\nu^3(T) - \int\limits_0^{\nu(T)} \mathrm{d}x\,\frac{(\nu(T)-x)^2}{\mathrm{e}^x+1}\right]. \tag{3.4}$$

We can neglect the contributions of the two last terms with accuracy better than 1% for temperatures $T \geq 160\,\mathrm{MeV}$. This yields

$$n(T) = T^3\,\frac{12}{\pi^2}\left[\frac{7}{6}\,\zeta(3) + \nu^2(T)\,\ln 2\right]. \tag{3.5}$$

Setting $T \geq 160\,\mathrm{MeV}$, we get the estimate $n(T) \geq 0.98\,\mathrm{fm}^{-3}$. This density is by a factor of 7 larger compared to the nuclear matter density $n_{\mathrm{N}} = 0.14\,\mathrm{fm}^{-3}$. Hence, we have got the inequality $n(T) \gg n_{\mathrm{N}}$ which does not rule out the formation of the thermalized QGP phase in the ultra-relativistic heavy-ion collisions.

Thus, if we treat the constituents of the thermalized QGP like rigid spheres with fixed radii, the average time of collisions of the constituents in the thermalized QGP should be much less than $D/c = (6/\pi n(T)c^3)^{1/3} \simeq 1\,\mathrm{fm/c}$, i.e. $\tau_{coll} \ll 1\,\mathrm{fm/c}$. The parameter D is a radius of a rigid sphere. In this approximation, the density $n(T)$ can be defined by $n(T) = 1/(\pi D^3/6)$, where $\pi D^3/6$ is a volume of a rigid sphere. The inequality $\tau_{coll} \ll 1\,\mathrm{fm/c}$ implies that the time of the thermalization of the quark-gluon system τ_{th}, should be $\tau_{\mathrm{th}} \leq 1\,\mathrm{fm/c}$. This value is by of order of magnitude less compared to the lifetime of the QGP, i.e. $\tau_{QGP} \geq (6 \div 15)\,\tau_{\mathrm{th}}$. Hence, these estimates confirm the possibility of the thermalization of the quark-gluon system in ultra-relativistic heavy-ion collisions. For the thermalized QGP, the energy density is determined by [1,4]:

$$\begin{aligned}\varepsilon(T) &= \varepsilon_g(T) + \varepsilon_q(T) + \varepsilon_{\bar{q}}(T) = 2(N_C^2 - 1)\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{p}{\mathrm{e}^{p/T} - 1}\\ &+ 4\,N_C\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{p}{\mathrm{e}^{-\nu(T) + p/T} + 1} + 4\,N_C\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{p}{\mathrm{e}^{\nu(T) + p/T} + 1}.\end{aligned} \tag{3.6}$$

Integrating over momenta we get [1,4]:

$$\varepsilon(T) = T^4\,N_C\left[\frac{N_C^2 - 1}{N_C}\,\frac{\pi^2}{15} + \frac{7\pi^2}{30} + \nu^2(T) + \frac{1}{2\pi^2}\,\nu^4(T)\right]. \tag{3.7}$$

At $N_C = 3$ the energy density of the thermalized QGP amounts to

$$\varepsilon(T) = T^4\left[\frac{37\pi^2}{30} + 3\,\nu^2(T) + \frac{3}{2\pi^2}\,\nu^4(T)\right]. \tag{3.8}$$

Setting $T \geq 160\,\mathrm{MeV}$, we estimate $\varepsilon(T) \geq 1.08\,\mathrm{GeV/fm^3}$. This is another confirmation of the possibility to treat the quark-gluon system produced in ultra-relativistic heavy-ion collisions as the thermalized QGP [1,4]. Since our estimates do not contradict to the existence of the thermalized QGP phase in ultra-relativistic heavy-ion collisions, we can proceed to the evaluation of the multiplicities of the $K^{\pm}$- and $\pi^{\pm}$-meson production caused by hadronization from the thermalized QGP phase.

4 Multiplicities of the $K^{\pm}$-meson production

The calculation of the multiplicities of the $K^{\pm}$-meson production caused by hadronization from the thermalized QGP we perform using eq. (1.4). These multiplicities were postulated following *(i)* the hypothesis of *coalescing* quarks and antiquarks and *(ii)* the assumption of the spherical symmetric thermalization of the quark-gluon system. Setting $N_C = 3$, we transcribe eq. (1.4) into

$$\begin{aligned} N_{K^+}(\vec{q},T) &= 3\,V_K\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{1}{\mathrm{e}^{-\nu(T)+|\vec{p}-\vec{q}\,|/T}+1}\,\frac{1}{\mathrm{e}^{\sqrt{\vec{p}^{\,2}+m_s^2}/T}+1},\\ N_{K^-}(\vec{q},T) &= 3\,V_K\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{1}{\mathrm{e}^{\nu(T)+|\vec{p}-\vec{q}\,|/T}+1}\,\frac{1}{\mathrm{e}^{\sqrt{\vec{p}^{\,2}+m_s^2}/T}+1}. \end{aligned} \tag{4.1}$$

Below, we show that the multiplicities eq. (4.1) are functions of $\lambda = \mathrm{e}^{\nu(T)}$, $\lambda_s = \mathrm{e}^{m_s/T}$ and $\lambda_K = \mathrm{e}^{\,q/T}$,

$$\begin{aligned} N_{K^+}(\vec{q},T) &= N_{K^+}(\lambda,\lambda_K,\lambda_s),\\ N_{K^-}(\vec{q},T) &= N_{K^+}(\lambda^{-1},\lambda_K,\lambda_s), \end{aligned} \tag{4.2}$$

where q is a 3-momentum of $K^{\pm}$ mesons. Integrating over directions of a momentum $\vec{p}$ we obtain

$$\begin{aligned} N_{K^+}(\lambda,\lambda_K,\lambda_s) &= \frac{3m_s^2TV_K}{4\pi^2\ln\lambda_K}\int\limits_0^{\varphi(q)}\mathrm{d}\varphi\frac{\mathrm{sh}\varphi\,\mathrm{ch}\varphi}{1+\lambda_s^{\mathrm{ch}\varphi}}\Big[F\Big(\lambda,\lambda_K^{-1},\lambda_s^{\mathrm{sh}\varphi}\Big)-F\Big(\lambda,\lambda_K^{-1},\lambda_s^{-\mathrm{sh}\varphi}\Big)\\ &+(\ln\lambda_K-\ln\lambda_s\mathrm{sh}\varphi)\,\ln\Big(1+\lambda\lambda_K^{-1}\lambda_s^{\mathrm{sh}\varphi}\Big)-(\ln\lambda_K+\ln\lambda_s\mathrm{sh}\varphi)\ln\Big(1+\lambda\lambda_K^{-1}\lambda_s^{-\mathrm{sh}\varphi}\Big)\Big]\\ &+\frac{3m_s^2TV_K}{4\pi^2\ln\lambda_K}\int\limits_{\varphi(q)}^{\infty}\mathrm{d}\varphi\,\frac{\mathrm{sh}\varphi\,\mathrm{ch}\varphi}{1+\lambda_s^{\mathrm{ch}\varphi}}\Big[F\Big(\lambda,\lambda_K,\lambda_s^{-\mathrm{sh}\varphi}\Big)-F\Big(\lambda,\lambda_K^{-1},\lambda_s^{-\mathrm{sh}\varphi}\Big)-(\ln\lambda_K-\ln\lambda_s\\ &\times\mathrm{sh}\varphi)\ln\Big(1+\lambda\lambda_K\lambda_s^{-\mathrm{sh}\varphi}\Big)-(\ln\lambda_K+\ln\lambda_s\mathrm{sh}\varphi)\ln\Big(1+\lambda\lambda_K^{-1}\lambda_s^{-\mathrm{sh}\varphi}\Big)\Big], \end{aligned} \tag{4.3}$$

where we have used the abbreviation $p = m_s\,\mathrm{sh}\varphi$ and denoted

$$\varphi(q) = \ln\left(\frac{q}{m_s}+\sqrt{1+\frac{q^2}{m_s^2}}\right). \tag{4.4}$$

For the integration over directions $\vec{p}$ we have used

$$\int\frac{\mathrm{d}x\,x}{\lambda^{-1}\,\mathrm{e}^{x}+1} = -x\,\ln\Big(1+\lambda\,\mathrm{e}^{-x}\Big)-\int\limits_0^{\lambda}\mathrm{d}z\,\frac{\ln\Big(1+z\,\mathrm{e}^{-x}\Big)}{z}. \tag{4.5}$$

The functions $F(\lambda, \lambda_K^{\pm 1}, x)$ are defined by

$$F(\lambda, \lambda_K^{\pm 1}, x) = \int\limits_0^{\lambda} \mathrm{d}z\, \frac{\ln(1 + \lambda_K^{\pm 1}\, x\, z)}{z}. \tag{4.6}$$

Using eq. (4.2) we obtain the ratio of the multiplicities of the $K^{\pm}$-meson production

$$\begin{aligned}
R_{K^+K^-}(q,T) &= \frac{N_{K^+}(\lambda, \lambda_K, \lambda_s)}{N_{K^+}(\lambda^{-1}, \lambda_K, \lambda_s)} = \\
&= \Bigg\{ \int\limits_0^{\varphi(q)} \mathrm{d}\varphi\, \frac{\mathrm{sh}\varphi\, \mathrm{ch}\varphi}{1 + \lambda_s^{\mathrm{ch}\varphi}} \Big[F\Big(\lambda, \lambda_K^{-1}, \lambda_s^{\mathrm{sh}\varphi}\Big) - F\Big(\lambda, \lambda_K^{-1}, \lambda_s^{-\mathrm{sh}\varphi}\Big) \\
&+ (\ln\lambda_K - \ln\lambda_s\, \mathrm{sh}\varphi)\, \ln\Big(1 + \lambda\, \lambda_K^{-1}\, \lambda_s^{\mathrm{sh}\varphi}\Big) \\
&- (\ln\lambda_K + \ln\lambda_s\, \mathrm{sh}\varphi)\, \ln\Big(1 + \lambda\, \lambda_K^{-1}\, \lambda_s^{-\mathrm{sh}\varphi}\Big)\Big] \\
&+ \int\limits_{\varphi(q)}^{\infty} \mathrm{d}\varphi\, \frac{\mathrm{sh}\varphi\, \mathrm{ch}\varphi}{1 + \lambda_s^{\mathrm{ch}\varphi}} \Big[F\Big(\lambda, \lambda_K, \lambda_s^{-\mathrm{sh}\varphi}\Big) - F\Big(\lambda, \lambda_K^{-1}, \lambda_s^{-\mathrm{sh}\varphi}\Big) \\
&- (\ln\lambda_K - \ln\lambda_s\, \mathrm{sh}\varphi)\, \ln\Big(1 + \lambda\, \lambda_K\, \lambda_s^{-\mathrm{sh}\varphi}\Big) \\
&- (\ln\lambda_K + \ln\lambda_s\, \mathrm{sh}\varphi)\, \ln\Big(1 + \lambda\, \lambda_K^{-1}\, \lambda_s^{-\mathrm{sh}\varphi}\Big)\Big]\Bigg\} \\
&\times \Bigg\{ \int\limits_0^{\varphi(q)} \mathrm{d}\varphi\, \frac{\mathrm{sh}\varphi\, \mathrm{ch}\varphi}{1 + \lambda_s^{\mathrm{ch}\varphi}} \Big[F\Big(\lambda, \lambda_K^{-1}, \lambda_s^{\mathrm{sh}\varphi}\Big) - F\Big(\lambda, \lambda_K^{-1}, \lambda_s^{-\mathrm{sh}\varphi}\Big) \\
&+ (\ln\lambda_K - \ln\lambda_s\, \mathrm{sh}\varphi)\, \ln\Big(1 + \lambda^{-1}\, \lambda_K^{-1}\, \lambda_s^{\mathrm{sh}\varphi}\Big) \\
&- (\ln\lambda_K + \ln\lambda_s\, \mathrm{sh}\varphi)\, \ln\Big(1 + \lambda^{-1}\, \lambda_K^{-1}\, \lambda_s^{-\mathrm{sh}\varphi}\Big)\Big] \\
&+ \int\limits_{\varphi(q)}^{\infty} \mathrm{d}\varphi\, \frac{\mathrm{sh}\varphi\, \mathrm{ch}\varphi}{1 + \lambda_s^{\mathrm{ch}\varphi}} \Big[F\Big(\lambda, \lambda_K, \lambda_s^{-\mathrm{sh}\varphi}\Big) - F\Big(\lambda, \lambda_K^{-1}, \lambda_s^{-\mathrm{sh}\varphi}\Big) \\
&- (\ln\lambda_K - \ln\lambda_s\, \mathrm{sh}\varphi)\, \ln\Big(1 + \lambda^{-1}\, \lambda_K\, \lambda_s^{-\mathrm{sh}\varphi}\Big) \\
&- (\ln\lambda_K + \ln\lambda_s\, \mathrm{sh}\varphi)\, \ln\Big(1 + \lambda^{-1}\, \lambda_K^{-1}\, \lambda_s^{-\mathrm{sh}\varphi}\Big)\Big]\Bigg\}^{-1}.
\end{aligned} \tag{4.7}$$

The ratio $R_{K^+K^-}(q,T)$ depends on 3-momenta of the $K^{\pm}$ mesons q and a temperature T in terms of λ, λ_K and λ_s.

For high momenta ($q \to \infty$), the multiplicities of the $K^{\pm}$-meson production can be substantially simplified. We would like to emphasize that infinite momenta $q \to \infty$ do not correspond to the thermodynamic description. In fact, we deal only with momenta $q \gg T$. However, since the momentum integrals describing the multiplicities of K and π meson productions are concentrated around the relative momenta of quarks and antiquarks, which are of order $p \sim T$, the limit $q \to \infty$ should be understood as a mathematical idealization of the inequality $q \gg T$.

Thus, the contribution of the integrals over the region $\varphi(q) \le \varphi < \infty$ can be neglected relative to the contribution of the integrals over the region $0 \le \varphi \le \varphi(q)$ for $q \to \infty$ or $\lambda_K \to \infty$. Keeping only the leading terms in the λ_K expansion we arrive to

$$N_{K^\pm}(\lambda, \lambda_K, \lambda_s) = \frac{3T^3 V_K}{4\pi^2}\,\lambda^{\pm 1}\,\lambda_K^{-1} = \frac{3T^2 V_K}{4\pi^2}\,\mathrm{e}^{\pm\nu(T)}\,\mathrm{e}^{-q/T}, \tag{4.8}$$

where the factor $\mathrm{e}^{-q/T}$ testifies the transition of the thermalized QGP into the thermalized ultra-relativistic gas of $K^\pm$ mesons.

Taking the ratio $R_{K^+K^-}(q,T)$ in the limit $q \to \infty$, we obtain

$$\lim_{q\to\infty} R_{K^+K^-}(q,T) = R_{K^+K^-}(\infty,T) = \lambda^2. \tag{4.9}$$

We notice that the ratio of the multiplicities $R_{K^+K^-}(q,T)$ does not depend on the momenta of the $K^\pm$-mesons. It is defined only by the chemical potential of the light quarks. Further we will show that this result agrees well with the experimental data on central ultra-relativistic Pb+Pb collisions at 158 GeV per nucleon for temperatures $160\,\mathrm{MeV} \le T \le 200\,\mathrm{MeV}$.

The ratio $R_{K^+K^-}(\infty,T)$ given by eq. (4.9) differs from the result obtained by Koch, Müller and Rafelski (see eq. (6.29) of Ref. [4b]) by the factor $\lambda_s^2 = \exp(2\mu_s/T)$, the squared frugality of strange quarks, where μ_s is a chemical potential of strange quarks. In the case of chemical equilibrium which we follow in our approach, $\mu_s = 0$ and $\lambda_s = \lambda_{\bar{s}}^{-1} = 1$.

In the limit $q \to 0$, the multiplicities of the $K^\pm$-meson production behave like

$$N_{K^\pm}(\lambda, 1, \lambda_s) \;=\; \frac{3m_s^3 V_K}{2\pi^2}\int\limits_0^\infty \mathrm{d}\varphi\,\frac{\mathrm{sh}^2\varphi\,\mathrm{ch}\varphi}{1+\lambda_s^{\mathrm{ch}\varphi}}\,\frac{1}{1+\lambda^{\mp 1}\,\lambda_s^{\mathrm{sh}\varphi}}. \tag{4.10}$$

It is easy to see that the main contribution to $N_{K^\pm}(\lambda, 1, \lambda_s)$ comes from the region $\varphi(q) \le \varphi < \infty$. Thus, in the limit $q \to 0$, the ratio of the multiplicities amounts to

$$R_{K^+K^-}(0,T) = \left[\int\limits_0^\infty \mathrm{d}\varphi\,\frac{\mathrm{sh}^2\varphi\,\mathrm{ch}\varphi}{1+\lambda_s^{\mathrm{ch}\varphi}}\,\frac{1}{1+\lambda^{-1}\,\lambda_s^{\mathrm{sh}\varphi}}\right]\left[\int\limits_0^\infty \mathrm{d}\varphi\,\frac{\mathrm{sh}^2\varphi\,\mathrm{ch}\varphi}{1+\lambda_s^{\mathrm{ch}\varphi}}\,\frac{1}{1+\lambda\,\lambda_s^{\mathrm{sh}\varphi}}\right]^{-1} \tag{4.11}$$

The numerical analysis (see Section 6) shows that $R_{K^+K^-}(0,T)$ calculated for $160\,\mathrm{MeV} \le T \le 200\,\mathrm{MeV}$ differs insignificantly from $R_{K^+K^-}(q,T)$ for $q \gg T$ approximated by (4.9).

5 Multiplicity of the $\pi^\pm$-meson production

In our approach the multiplicities $N_{\pi^+}(\vec{q},T)$ and $N_{\pi^-}(\vec{q},T)$ of the production of the π^+ and π^- mesons are equal and defined by eq. (1.5). Integrating over directions of momentum $\vec{p}$, we get the expression depending only on λ and $\lambda_\pi = \exp(q/T)$

$$N_{\pi^+}(\lambda, \lambda_\pi) = \frac{3V_\pi T^3}{4\pi^2 \ln\lambda_\pi} \int\limits_0^{\lambda_\pi} \frac{\mathrm{d}x\, x}{\lambda\,\mathrm{e}^x + 1} \Big[F\Big(\lambda, \lambda_\pi^{-1}, \mathrm{e}^x\Big) - F\Big(\lambda, \lambda_\pi^{-1}, \mathrm{e}^{-x}\Big)$$
$$+ (\ln\lambda_\pi - x)\,\ln\Big(1 + \lambda\,\lambda_\pi^{-1}\,\mathrm{e}^x\Big) - (\ln\lambda_\pi + x)\,\ln\Big(1 + \lambda\,\lambda_\pi^{-1}\,\mathrm{e}^{-x}\Big)\Big]$$
$$+ \frac{3VT^3}{4\pi^2 \ln\lambda_\pi} \int\limits_{\lambda_\pi}^{\infty} \frac{\mathrm{d}x\, x}{\lambda\,\mathrm{e}^x + 1} \Big[F\Big(\lambda, \lambda_\pi, \mathrm{e}^{-x}\Big) - F\Big(\lambda, \lambda_\pi^{-1}, \mathrm{e}^{-x}\Big)$$
$$- (\ln\lambda_\pi + x)\,\ln\Big(1 + \lambda\,\lambda_\pi^{-1}\,\mathrm{e}^{-x}\Big) - (\ln\lambda_\pi - x)\,\ln\Big(1 + \lambda\,\lambda_\pi^{-1}\,\mathrm{e}^{-x}\Big)\Big]. \tag{5.1}$$

The ratio $R_{K^+\pi^+}(q,T)$ of the multiplicities of the K^+ and π^+ meson production is given by

$$R_{K^+\pi^+}(q,T) = \frac{N_{K^+}(\lambda, \lambda_K, \lambda_s)}{N_{\pi^+}(\lambda, \lambda_\pi)} =$$
$$= \frac{m_s^2}{T^2}\frac{V_K}{V_\pi}\Bigg\{ \int\limits_0^{\varphi(q)} \mathrm{d}\varphi\, \frac{\mathrm{sh}\varphi\,\mathrm{ch}\varphi}{1 + \lambda_s^{\mathrm{ch}\varphi}} \Big[F\Big(\lambda, \lambda_K^{-1}, \lambda_s^{\mathrm{sh}\varphi}\Big) - F\Big(\lambda, \lambda_K^{-1}, \lambda_s^{-\mathrm{sh}\varphi}\Big)$$
$$+ (\ln\lambda_K - \ln\lambda_s\,\mathrm{sh}\varphi)\,\ln\Big(1 + \lambda\,\lambda_K^{-1}\,\lambda_s^{\mathrm{sh}\varphi}\Big) - (\ln\lambda_K + \ln\lambda_s\,\mathrm{sh}\varphi)\,\ln\Big(1 + \lambda\,\lambda_K^{-1}\,\lambda_s^{-\mathrm{sh}\varphi}\Big)\Big]$$
$$+ \int\limits_{\varphi(q)}^{\infty} \mathrm{d}\varphi\, \frac{\mathrm{sh}\varphi\,\mathrm{ch}\varphi}{1 + \lambda_s^{\mathrm{ch}\varphi}} \Big[F\Big(\lambda, \lambda_K, \lambda_s^{-\mathrm{sh}\varphi}\Big) - F\Big(\lambda, \lambda_K^{-1}, \lambda_s^{-\mathrm{sh}\varphi}\Big)$$
$$- (\ln\lambda_K - \ln\lambda_s\,\mathrm{sh}\varphi)\,\ln\Big(1 + \lambda\,\lambda_K\,\lambda_s^{-\mathrm{sh}\varphi}\Big) - (\ln\lambda_K + \ln\lambda_s\,\mathrm{sh}\varphi)\,\ln\Big(1 + \lambda\,\lambda_K^{-1}\,\lambda_s^{-\mathrm{sh}\varphi}\Big)\Big]\Bigg\}$$
$$\times \Bigg\{ \int\limits_0^{\ln\lambda_\pi} \frac{\mathrm{d}x\, x}{\lambda\,\mathrm{e}^x + 1} \Big[F\Big(\lambda, \lambda_\pi^{-1}, \mathrm{e}^x\Big) - F\Big(\lambda, \lambda_\pi^{-1}, \mathrm{e}^{-x}\Big)$$
$$+ (\ln\lambda_\pi - x)\,\ln\Big(1 + \lambda\,\lambda_\pi^{-1}\,\mathrm{e}^x\Big) - (\ln\lambda_\pi + x)\,\ln\Big(1 + \lambda\,\lambda_\pi^{-1}\,\mathrm{e}^{-x}\Big)\Big]$$
$$+ \int\limits_{\ln\lambda_\pi}^{\infty} \frac{\mathrm{d}x\, x}{\lambda\,\mathrm{e}^x + 1} \Big[F\Big(\lambda, \lambda_\pi\,\mathrm{e}^{-x}\Big) - F\Big(\lambda, \lambda_\pi^{-1}, \mathrm{e}^{-x}\Big)$$
$$- (\ln\lambda_\pi - x)\,\ln\Big(1 + \lambda\,\lambda_\pi\,\mathrm{e}^{-x}\Big) - (\ln\lambda_\pi + x)\,\ln\Big(1 + \lambda\,\lambda_\pi^{-1}\,\mathrm{e}^{-x}\Big)\Big]\Bigg\}^{-1}. \tag{5.2}$$

For high momenta ($q \to \infty$), which should be understood as a mathematical idealization of the inequality $q \gg T$, the multiplicities of the $\pi^{\pm}$-meson production behave like the multiplicities of the $K^{\pm}$-meson production

$$N_{\pi^{\pm}}(\lambda, \lambda_\pi) = \frac{3T^3 V_\pi}{4\pi^2}\,\lambda_\pi^{-1} = \frac{3T^3 V_\pi}{4\pi^2}\,\mathrm{e}^{-q/T}, \tag{5.3}$$

where the factor $\exp(-q/T)$ testifies the production of $\pi^{\pm}$ mesons from the thermalized QGP in the state of the thermalized ultra-relativistic $\pi^{\pm}$-meson gas at a temperature T.

Taking into account eq. (4.8) we obtain the ratio $R_{K^+\pi^+}(q,T)$ in the limit $q \to \infty$

$$\lim_{q\to\infty} R_{K^+\pi^+}(q,T) = R_{K^+\pi^+}(\infty,T) = \frac{V_K}{V_\pi}\, \mathrm{e}^{\nu(T)}. \tag{5.4}$$

As we will show below this relation agrees well with the experimental data on nucleus-nucleus ultra-relativistic collisions at 200 GeV per nucleon (NA35 Collaboration).

In the ratio the parameter C is canceled, as it has been intended, and the ratio $R_{K^+\pi^+}(q,T)$ is determined only by well established parameters of K^+ and π^+ mesons such as masses, $M_K = 500\,\mathrm{MeV}$ and $M_\pi = 140\,\mathrm{MeV}$, and leptonic coupling constants, $F_K = 160\,\mathrm{MeV}$ and $F_\pi = 131\,\mathrm{MeV}$.

6 Numerical analysis of the ratios $R_{K^+K^-}(q,T)$ and $R_{K^+\pi^+}(q,T)$

The numerical analysis of the ratio $R_{K^+K^-}(q,T)$ (4.7) testifies that $R_{K^+K^-}(q,T)$ varies slightly around the value $R_{K^+K^-}(\infty,T) = \lambda^2$, when 3-momenta of the $K^{\pm}$ mesons take values from interval $0 \le q < 10^3\,\mathrm{GeV}$. This satisfies the relation (4.9) at $q \ge 10^3\,\mathrm{GeV}$.

The ratio (4.7) depends also smoothly on temperature T for $160\,\mathrm{MeV} \le T < 200\,\mathrm{MeV}$. The numerical results read

$$\begin{aligned} R_{K^+K^-}(q,T=160) &= 2.14^{+0.13}_{-0.30},\\ R_{K^+K^-}(q,T=175) &= 1.80^{+0.04}_{-0.18},\\ R_{K^+K^-}(q,T=190) &= 1.58^{+0.02}_{-0.13}, \end{aligned} \tag{6.1}$$

where the upper and the lower values correspond to the maximum and the minimum of the ratio $R_{K^+K^-}(q,T)$, respectively.

Comparing the theoretical values (6.1) with the experimental data on central ultra-relativistic Pb+Pb collisions at 158 GeV per nucleon [14–17]

$$R^{\exp}_{K^+K^-} = 1.80 \pm 0.10\,, \tag{6.2}$$

one can see that our *coalescence model of correlated quarks*, supplemented by the assumption of the spherical symmetric thermalization of the quark-gluon system, describes well the experimental data on the production of $K^{\pm}$-mesons at the temperature $T = 175\,\mathrm{MeV}$.

For $K^{\pm}$ mesons, which can be produced in ultra-relativistic heavy-ion collisions due to mechanisms having no relation to the thermalized QGP, we obtain that the ratio $R_{K^+K^-}$ is independent on the 3-momentum of the $K^{\pm}$-mesons and is equal to $R_{K^+K^-} = 1.10 \pm 0.01$. Our estimate (6.1) testifies that the intermediate state in ultra-relativistic heavy-ion collisions should evolve via the thermalized QGP phase with a reasonable probability.

For 3-momenta $0 \le q < \infty$ the ratio $R_{K^+\pi^+}(q,T)$ of the multiplicities of the K^+ and π^+ meson production varies very smoothly and it is given by

$$\begin{aligned} R_{K^+\pi^+}(q,T=160) &= 0.144 \pm 0.017,\\ R_{K^+\pi^+}(q,T=175) &= 0.134 \pm 0.014,\\ R_{K^+\pi^+}(q,T=190) &= 0.128 \pm 0.011, \end{aligned} \tag{6.3}$$

where $\pm\Delta$ corresponds to the minimum and the maximum values of the ratio. Comparing the theoretical ratios (6.3) with the experimental data given by NA35 Collaboration on proton-nucleus and nucleus-nucleus collisions at 200 GeV/A [17]

$$R^{\rm exp}_{K^+\pi^+} = 0.137 \pm 0.008, \tag{6.4}$$

one can see that the best agreement is obtained again for $T = 175\,{\rm MeV}$.

7 Conclusions

Resuming the obtained results we would like to accentuate that the calculation of the chemical potential $\mu(T)$ as a function of T has allowed to diminish the number of input parameters for the description of the thermalized QGP phase. The value of the chemical potential at zero temperature $\mu(0) = \mu_0 = 250\,{\rm MeV}$ agrees well with the estimate $\mu_0 \sim 300\,{\rm MeV}$ [1].

For the theoretical analysis of multiplicities of meson production from the thermalized QGP phase we have used the approach based on the hypothesis of *coalescing quarks and antiquarks*. We have called this approach as the *coalescence model of correlated quarks*. For the simplification of the theoretical formulas for multiplicities we have assumed that *(i)* the wave function of the relative motion of a quark q and antiquark $\bar{q}\,'$ *coalescing* into the meson with a quark structure $q\bar{q}\,'$ is constant and *(ii)* the thermalization of the quark-gluon system produced in the ultra-relativistic heavy-ion collisions is spherically symmetric.

We have shown that the multiplicities of meson production defined in the *coalescence model of correlated quarks* acquire the shape of the Maxwell-Boltzmann distribution functions in the ultra-relativistic limit. This testifies the availability of the ideal multi-component meson gas at a temperature T in the hadronic phase of the thermalized quark-gluon system.

We have found that the ratio $R_{K^+K^-}(q,T)$ is a smooth function of 3-momenta q of the $K^{\pm}$ mesons. It varies slightly around the value $R_{K^+K^-}(\infty,T) = \exp(2\mu(T)/T)$. At $T = 175\,{\rm MeV}$, we have calculated $R_{K^+K^-}(q,T=175) = 1.80^{+0.04}_{-0.18}$. This result agrees well with the experimental data on central ultra-relativistic Pb+Pb collisions at 158 GeV/A (NA49 and NA44 Collaborations) and ultra-relativistic nucleus-nucleus collisions at 200 GeV/A (NA35 Collaboration) [14–17], $R^{\rm exp}_{K^+K^-} = 1.80 \pm 0.10$. Moreover, the thermodynamic parameters of our fit of experimental data $T = 175\,{\rm MeV}$ and $\mu(T=175) = 51\,{\rm MeV}$ are in qualitative agreement with the experimental ones [15] $T \sim 170\,{\rm MeV}$ and $\mu \sim 85\,{\rm MeV}$.

We would like to accentuate that infinite momenta ($q \to \infty$), which do not correspond to the thermodynamic description, should be understood only as a mathematical idealization of the regime $q \gg T$. This is justified by the concentration of the momentum integrals, describing the multiplicities of meson production, around the momenta $p \sim T$ of a relative motion of *coalescing* quarks and antiquarks.

For the first time, the ratio $R_{K^+K^-}(q,T)$ has been calculated as a function of a chemical potential of light quarks and a temperature by Koch, Müller and Rafelski [4b]. Comparing our result, given by eqs. (4.7) and (6.1), with that obtained by Koch, Müller and Rafelski (see eq. (6.29) and Fig. 6.7 of Ref. [4b]) we argue that in the thermalized QGP phase *(i)* strange quarks and antiquarks are in equilibrium state that provides a vanishing value of their chemical potential, *(ii)* the ratio of multiplicities is a smooth function of 3-momenta of $K^{\pm}$ mesons, *(iii)* the most important values of a temperature should exceed $T = 160\,\mathrm{MeV}$ and *(iv)* a chemical potential of light quarks is a well-defined function of T decreasing as T^{-2}.

The ratio $R_{K^+\pi^+}(q,T)$ of the multiplicities of the K^+ and π^+-meson production has been found a smooth function of the 3-momenta q, the mean value of which depends on $V_K/V_\pi = (F_\pi M_\pi/F_K M_K)^{3/2} = 0.109$. This gives the ratio $R_{K^+\pi^+}(q,T)$ agreeing well with experimental data obtained by NA35 Collaboration [17], $R^{\rm exp}_{K^+\pi^+} = 0.137 \pm 0.008$. The best agreement we get for $T = 175\,\mathrm{MeV}$ and $q \gg T$, $R_{K^+\pi^+}(q,T = 175) = 0.134 \pm 0.014$.

The first calculation of the ratio $R_{K^+\pi^+}(q,T)$ in the thermalized QGP has been performed by Glendenning and Rafelski [19]. Unlike our approach, by skipping the intermediate quark-gluon stage of the evolution of the thermalized QGP, Glendenning and Rafelski have postulated that the mesonic phase of the thermalized QGP is the ideal Bose gas of K and π mesons. This has led to the numerical value of the ratio $R(K^+\pi^+) \approx 0.3$ (see Fig. 2 of Ref. [19]) computed for $160\,\mathrm{MeV} \le T \le 180\,\mathrm{MeV}$, which does not agree with contemporary experimental data. We explain such a discrepancy by a loss of the quark-antiquark origin of the π^+-meson production, which is retained in our *coalescence model of correlated quarks*.

Supposing that the production of $K^\pm$ mesons in ultra-relativistic heavy-ion collisions can be caused by mechanisms which have no relation to the thermalized QGP, we have calculated the ratio $R_{K^+K^-} = 1.10 \pm 0.01$. Therefore, the numerical estimate $R_{K^+K^-}(q,T) \simeq 1.80$, obtained due to hadronization from the thermalized QGP phase, testifies that the intermediate state for ultra-relativistic heavy-ion collisions should proceed via the QGP phase with reasonable probability.

In the conclusion we would like to accentuate that we have considered the multiplicities of meson production from the thermalized QGP phase for the QGP at rest. This does not contradict to pioneering papers on the thermalized QGP [1–6,19] (see also [21,22]). In the case of the thermalized QGP moving with a constant 4-velocity $u^\mu = (\gamma, \gamma\vec{u}\,)$, where $\gamma = 1/\sqrt{1-u^2}$ is the Lorentz factor, the multiplicities of meson production should be defined by using quark and antiquark distribution functions in the Jüttner form [23]

$$\begin{aligned} N_{K^+}(\vec{q},T) &= 3V_K \int \frac{{\rm d}^3p}{(2\pi)^3}\,\frac{1}{{\rm e}^{-\nu(T) + \gamma\,(|\vec{p}-\vec{q}\,| - \vec{u}\cdot(\vec{p}-\vec{q}\,))/T} + 1}\\ &\quad\times\,\frac{1}{{\rm e}^{\gamma\,(\sqrt{\vec{p}^{\,2}+m^2_s} - \vec{u}\cdot\vec{p})/T} + 1},\\ N_{K^-}(\vec{q},T) &= 3V_K \int \frac{{\rm d}^3p}{(2\pi)^3}\,\frac{1}{{\rm e}^{\nu(T) + \gamma\,(|\vec{p}-\vec{q}\,| - \vec{u}\cdot(\vec{p}-\vec{q}\,))/T} + 1}\\ &\quad\times\,\frac{1}{{\rm e}^{\gamma\,(\sqrt{\vec{p}^{\,2}+m^2_s} - \vec{u}\cdot\vec{p})/T} + 1}. \end{aligned} \tag{7.1}$$

For ultra-relativistic QGP the Lorentz factor $\gamma \gg 1$ but and the main contribution to the integrals comes from the momenta $|\vec{p}\,| \sim T$, therefore, the multiplicities of the K-meson production can be reduced to the more simple form

$$\begin{aligned} N_{K^+}(\vec{q},T) &= 3V_K\,{\rm e}^{+\nu(T)} \int \frac{{\rm d}^3p}{(2\pi)^3}\,{\rm e}^{-\gamma\,(|\vec{p}-\vec{q}\,| + \sqrt{\vec{p}^{\,2}+m^2_s} - \vec{u}\cdot(2\vec{p}-\vec{q}\,))/T},\\ N_{K^-}(\vec{q},T) &= 3V_K\,{\rm e}^{-\nu(T)} \int \frac{{\rm d}^3p}{(2\pi)^3}\,{\rm e}^{-\gamma\,(|\vec{p}-\vec{q}\,| + \sqrt{\vec{p}^{\,2}+m^2_s} - \vec{u}\cdot(2\vec{p}-\vec{q}\,))/T}. \end{aligned} \tag{7.2}$$

Since the momentum integrals coincide, the ratio of multiplicities is equal to

$$R_{K^+K^-}(\vec{q},T) = \frac{N_{K^+}(\vec{q},T)}{N_{K^-}(\vec{q},T)} = \mathrm{e}^{2\nu(T)} = 1.80. \tag{7.3}$$

This confirms our result obtained in Section 6.

The multiplicity of the π^+-meson production should be

$$\begin{aligned} &N_{\pi^+}(\vec{q},T) = \\ &= 3V_\pi \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, \frac{1}{\mathrm{e}^{-\nu(T) + \gamma\,(|\vec{p}-\vec{q}\,| - \vec{u}\cdot(\vec{p}-\vec{q}\,))/T} + 1}\, \frac{1}{\mathrm{e}^{\gamma\,(|\vec{p}\,| - \vec{u}\cdot\vec{p})/T} + 1}. \end{aligned} \tag{7.4}$$

For $\gamma \gg 1$ the r.h.s. of (7.4) reduces to

$$N_{\pi^+}(\vec{q},T) = 3V_\pi \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, \mathrm{e}^{-\gamma\,(|\vec{p}-\vec{q}\,| + |\vec{p}\,| - \vec{u}\cdot(2\vec{p}-\vec{q}\,))/T}. \tag{7.5}$$

Making a shift of variables $\vec{p}-\vec{q} \to \vec{p}$ and taking into account that $|\vec{q}\,| \gg T$, we obtain

$$\begin{aligned} N_{K^+}(\vec{q},T) &= 3V_K\, \mathrm{e}^{+\nu(T)}\, \mathrm{e}^{-u\cdot q/T} \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, \mathrm{e}^{-2\gamma\,(|\vec{p}\,| - \vec{u}\cdot\vec{p})/T}, \\ N_{\pi^+}(\vec{q},T) &= 3V_\pi\, \mathrm{e}^{-u\cdot q/T} \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, \mathrm{e}^{-2\gamma\,(|\vec{p}\,| - \vec{u}\cdot\vec{p})/T}, \end{aligned} \tag{7.6}$$

where we have neglected the terms of order $O(m_s/|\vec{q}\,|)$ and $O(|\vec{p}\,|/|\vec{q}\,|)$. This is correct, since $|\vec{p}\,| \sim m_s \sim T$ that testifies that these ratios are of the same order $m_s/|\vec{q}\,| \sim |\vec{p}\,|/|\vec{q}\,| \sim T/|\vec{q}\,| \ll 1$.

Taking the ratio $R_{K^+\pi^+}(\vec{q},T)$ of multiplicities defined by (7.6) we get

$$R_{K^+\pi^+}(\vec{q},T) = \frac{N_{K^+}(\vec{q},T)}{N_{\pi^+}(\vec{q},T)} = \frac{V_K}{V_\pi}\, \mathrm{e}^{+\nu(T)} = 0.147. \tag{7.7}$$

This result agrees with the number obtained in Section 6.

Thus, we can conclude that our theoretical predictions for the ratios of K and π meson multiplicities, obtained for the thermalized QGP at rest, do not contradict to the case of the QGP moving with ultra-relativistic hydrodynamical velocity.

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