# BOSON EXCITATIONS IN ELECTRON PLASMA OF METALS

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The grand canonical partition function of the electron plasma in metals is calculated by using the formalism of functional integrals. Within the framework of this formalism the Coulomb interaction between electrons is expressed naturally as an interaction mediated by bosons. The grand canonical partition function of the electron plasma is then represented as a product of two partition functions corresponding to the free electron gas and to a boson gas with a given energy spectrum and of a factor describing correlation effects between the electrons and bosons.

## 1 Introduction

Some interacting fermion systems can be described rather well by the Landau theory of normal Fermi liquids [1]. This theory applies to a fermion system with a spectrum of elementary excitations similar to that of a free Fermi gas. In more precise statement the Landau theory assumes a one to one correspondence between the states of the free Fermi gas and those of an interacting fermion system. An interacting fermion system is usually described by a Hamiltonian $H$ written as a sum of two terms

$$H = H_0 + H_I \,, \tag{1}$$

where $H_0$ describes free fermions and $H_I$ takes into account their mutual interactions. The free Hamiltonian has usually the simple form

$$H_0 = \sum_{k,\alpha} \varepsilon_{k,\alpha} a_{k,\alpha}^+ a_{k,\alpha} \,, \tag{2}$$

where $a_{k,\alpha}^+ (a_{k,\alpha})$ are the creation (annihilation) operators of fermions in the states $(k,\alpha)$ specified by the wave vector $k$ and by the spin projection $\alpha = \pm 1$ and $\varepsilon_{k,\alpha}$ denotes the energy of the fermion in the state $(k,\alpha)$. Thus, the free Hamiltonian $H_0$ defines the energy levels $\varepsilon_{k\alpha}$ as a given function of $k$ and $\alpha$.

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The Landau theory assumes that by switching on the interaction between fermions via the interaction Hamiltonian $H_I$, this interaction transforms only the values of the unperturbed energy levels $\varepsilon_{k,\alpha}$ into a new set of levels $E_{k,\alpha}$. By this assumption one expects an isomorphism between the genuine interacting fermion system and a noninteracting gas of Fermi quasiparticles with the energy levels $E_{k,\alpha}$. Therefore, the solution of the original problem represented by the interacting fermion system is reduced in the Landau theory of normal Fermi liquids to the calculation of the energy spectrum $E_{k,\alpha}$ of quasiparticles. Needles to say that the spectrum $E_{k,\alpha}$ is a complicated functional of quasiparticle distribution functions $n_{k,\alpha}$.

In general, an interacting fermion system has also collective excitations $b_c(q)$, such as plasmons in the electron plasma of metals, which are specified by the wave vector $q$. States corresponding to the collective excitations $b(q)$ have no analogy in the free Fermi gas. Therefore, the one to one correspondence assumed by the Landau theory of Fermi liquids is not strictly true. One usually gives arguments that the Landau theory is restricted to low temperatures, where an energy gap for collective excitations suppresses effects of collective modes to such an extent that they may generally be neglected in calculations of bulk properties.

However, in many interacting fermion systems their interaction Hamiltonians $H_I$ can induce transitions of fermions between states $(k', \alpha')$ and $(k, \alpha)$ associated with productions or absorptions of boson-like excitation $b(k', \alpha'; k, \alpha)$ according to the scheme

$$e(k', \alpha') \rightleftharpoons e(k, \alpha) + b(k', \alpha'; k, \alpha) \,, \tag{3}$$

where $e(k, \alpha)$ denotes a fermion (e. g. electron) in the state $(k, \alpha)$. The energy $\hbar\omega_b(k', \alpha'; k, \alpha)$ of the boson excitation $b(k', \alpha'; k, \alpha)$ in the process (3) is expected to be given by the relation

$$\hbar\omega_b(k', \alpha'; k, \alpha) \approx |\varepsilon_{k'\alpha'} - \varepsilon_{k,\alpha}| \,. \tag{4}$$

The energy spectrum (4) of the boson excitations $b(k', \alpha'; k, \alpha)$ has no energy gap. By this fact the interacting fermion system with interactions permitting transitions (3) cannot be satisfactorily described by the Landau theory of normal Fermi liquids at temperatures $T \neq 0$.

The presence of boson excitations $b(k', \alpha'; k, \alpha)$ in an interacting fermion system provides us an intuitive argument that properties of the interacting fermion system can be described in some approximation by properties of a free fermion gas together with properties of a Bose gas corresponding to boson excitations $b(k', \alpha'; k, \alpha)$. To be more precise, it is expected that the general canonical partition function $\mathcal{Z}$ of the interacting fermion system involves a product of two grand canonical partition functions $\mathcal{Z}_f$ and $\mathcal{Z}_b$ corresponding to the free fermion gas and to the Bose gas of boson excitations $b(k', \alpha'; k, \alpha)$, respectively. Their forms are given by the well known formulae

$$\mathcal{Z}_f = \prod_k \prod_{\alpha=\pm 1} \left(1 + e^{-\beta(\varepsilon_{k,\alpha} - \mu)}\right) \,, \tag{5}$$

$$\mathcal{Z}_b = \prod_{k,k'} \prod_{\alpha,\alpha'=\pm 1} \left(1 - e^{-\beta\hbar\omega(k', \alpha'; k, \alpha)}\right)^{-1} \,, \tag{6}$$

where $\mu$ is the chemical potential of the fermions and $\hbar\omega(k', \alpha'; k, \alpha)$ is the energy spectrum of the boson excitations $b(k', \alpha'; k, \alpha)$ which will be presented in this paper.

Thus, the grand canonical partition function $\mathcal{Z}$ of the interacting fermion system has the form

$$\mathcal{Z} = C\mathcal{Z}_f\mathcal{Z}_b \,, \tag{7}$$

where the factor $C$ involves effects of correlations between fermions and bosons with the required accuracy corresponding to the given order of the perturbation theory.

The assumption of an one-to-one correspondence between states of the free Fermi gas and those of the interacting system definitely fails for all temperatures (even for $T = 0$) if bound states of fermions appear when the interaction is turned on. For example, a state of a superconductor is not related in a direct way to any state of the free Fermi gas for temperatures $T < T_c$, where $T_c$ is the critical temperature. In fact, for this system the basic assumption of the Landau theory of normal Fermi liquids is not valid at all. However, for such a system of interacting fermions described by BCS model of superconductivity the partition function $\mathcal{Z}$ has indeed the form (7) as it was shown in the work [3].

The purpose of this paper is to derive the partition function $\mathcal{Z}$ in the form (7) for a gas of electrons with Coulomb interactions in the presence of an uniform background of a positive charge, i. e. for the electron plasma within the simplest model of metals. Therefore, our task is to calculate the energy spectrum $\hbar\omega_b(k', \alpha'; k, \alpha)$ of the elementary Bose excitations.

## 2   Interacting electron gas

For our purpose we consider the simplest model of the electron plasma in metals. This system is an interacting electron gas in the presence of a uniform background of a positive charge in order to enforce the electric neutrality of the system. Thermodynamic properties of this system have been studied many years ago [4, 5] and are included in contents of many textbooks on quantum theory of many particle systems. As an reference textbook to this problem we refer to the excellent monograph by Fetter and Walecka [6].

The Hamiltonian $H$ for the system under considerations has the form

$$H = \sum_{k,\alpha} \frac{\hbar^2 k^2}{2m_e} a_{k,\alpha}^+ a_{k,\alpha} + \frac{2\pi e^2}{V} \sum_{\substack{k,k' \\ \alpha,\alpha'}} {\sum_q}' \frac{1}{q^2} a_{k+\frac{q}{2},\alpha}^+ a_{k'-\frac{q}{2},\alpha'}^+ a_{k'+\frac{q}{2},\alpha'} a_{k-\frac{q}{2},\alpha} \,, \tag{8}$$

where $a_{k,\alpha}(a_{k,\alpha}^+)$ is the annihilation (creation) operator of an electron in the state $(k, \alpha)$. The sum over $q$ is restricted by the condition $q \neq 0$, what is here and in what follows indicated by a prime at the summation symbol. All thermodynamic properties of the system described by the Hamiltonian (8) are derived from the grand canonical partition function

$$\mathcal{Z} = \text{Tr} \exp \left\{ -\beta(H - \mu N) \right\} \,, \tag{9}$$

where $N$ is the electron number operator, $\mu$ is the chemical potential, $\beta = 1/k_B T$, $k_B$ is the Boltzmann constant and $T$ is the temperature. For the explicit evaluation of the partition function it is very convenient to employ functional integral over Grassmann variables $a_{k,\alpha}^*(\tau)$ and $a_{k,\alpha}(\tau)$.

By using the standard rules [7] we express $\mathcal{Z}$ by the following functional integral

$$\mathcal{Z} = \int \mathcal{D}(a^*, a) \exp\left\{ -\int_0^\beta d\tau \left[ \sum_{k,\alpha} a^*_{k,\alpha}(\tau) \left( \frac{\partial}{\partial\tau} + \xi_{\vec{k}} \right) a_{k,\alpha}(\tau) + \right.\right.$$

$$\left.\left. +\frac{2\pi e^2}{V} \sum_{\substack{k,k' \\ \alpha,\alpha'}} \sideset{}{'}\sum_q \frac{1}{q^2} a^*_{k+\frac{q}{2},\alpha}(\tau) a^*_{k'-\frac{q}{2},\alpha'}(\tau) a_{k'+\frac{q}{2},\alpha'}(\tau) a_{k-\frac{q}{2},\alpha}(\tau) \right] \right\} , \qquad (10)$$

where

$$\xi_k = \frac{\hbar^2 k^2}{2m} - \mu \qquad (11)$$

and the Grassmann variables satisfy the antiperiodic conditions

$$a^*_{k,\alpha}(\tau + \beta) = -a^*_{k,\alpha}(\tau), \ a_{k,\alpha}(\tau + \beta) = -a_{k,\alpha}(\tau) . \qquad (12)$$

The functional integral formalism employed in statistical physics permits one to rewrite the Coulomb interaction between electrons as the interaction between electrons mediated by an electric field represented by its corresponding electric scalar potential $\varphi(\mathbf{x}, \tau)$. Within this formalism we express the partition function (10) as the following functional integral

$$\mathcal{Z} = \mathcal{Z}_0^{-1} \int \mathcal{D}(a^*, a, \varphi) \exp\left\{ -\int_0^\beta d\tau \left[ \frac{V}{8\pi} \sideset{}{'}\sum_q q^2 \varphi_q(\tau)\varphi_{-q}(\tau) + \right.\right.$$

$$\left.\left. +\sum_{k,\alpha} a^*_{k,\alpha}(\tau) \left( \frac{\partial}{\partial\tau} + \xi_{\vec{k}} \right) a_{k,\alpha}(\tau) + ie \sum_{k\neq k',\alpha} \varphi_{k-k'}(\tau) a^*_{k'\alpha}(\tau) a_{k\alpha}(\tau) \right] \right\} . \qquad (13)$$

Here $\varphi_q(\tau)$ are commutative (Grassmann even) integration variables satisfying the periodic conditions

$$\varphi_q(\tau + \beta) = \varphi_q(\tau) \qquad (14)$$

and the constant $\mathcal{Z}_0$ is defined by the functional integral

$$\mathcal{Z}_0 = \int \mathcal{D}(\varphi) \exp\left\{ -\int_0^\beta d\tau \frac{V}{8\pi} \sideset{}{'}\sum_q q^2 \varphi_q(\tau)\varphi_{-q}(\tau) \right\} . \qquad (15)$$

The integration variables $\varphi_q(\tau)$ are in fact the Fourier transforms of the scalar potential $\varphi(\mathbf{x}, \tau)$. The partition function $\mathcal{Z}$ expressed by the formula (13) corresponds indeed to a system of electrons interacting with the electric field represented by the scalar potential.

One may convince himself that the partition functions $\mathcal{Z}$ defined by (10) and (13) are identical quantities. By the exact and explicit integration over $\varphi_q(\tau)$ variables in the formula (13) one recovers indeed exactly the functional integral (10). However, we can equally well integrate first

exactly and explicitly over the anticommuting variables $a^*_{k,\alpha}(\tau)$ and $a_{k,\alpha}(\tau)$ in the functional integral (13). We obtain the result

$$
\mathcal{Z} = \mathcal{Z}_0^{-1} \int \mathcal{D}(\varphi) \exp\left\{-\int_0^\beta d\tau \frac{V}{8\pi} \sum_q{}' q^2 \varphi_q(\tau)\varphi_{-q}(\tau)\right\} \det \mathcal{F}(\varphi) , \tag{16}
$$

where $\mathcal{F}(\varphi)$ is the functional matrix, the matrix elements of which are as follows

$$
\langle k', \alpha', \tau'|\mathcal{F}(\varphi)|\, k, \alpha, \tau\rangle = \left[\left(\frac{\partial}{\partial\tau} + \xi_k\right)\delta_{k',k} + ie\varphi_{k'-k}(\tau)\right]\delta_{\alpha',\alpha}\delta(\tau' - \tau) . \tag{17}
$$

The Dirac $\delta$-function $\delta(\tau' - \tau)$ which enters in the relation (17) above is defined in terms of its spectral representation

$$
\delta(\tau' - \tau) = \frac{1}{\beta} \sum_{\nu=-\infty}^{+\infty} \exp[i\omega_\nu(\tau' - \tau)] , \ \omega_\nu = \frac{\pi}{\beta}(2\nu + 1) , \tag{18}
$$

where the summation index $\nu$ runs through all integer values and $\omega_\nu$ are the so-called Matsubara's frequencies for fermions. Next we decompose the functional matrix $\mathcal{F}(\varphi)$ into two parts as

$$
\mathcal{F}(\varphi) = \mathcal{F}_0 + e\mathcal{F}_1(\varphi) , \tag{19}
$$

where the matrices $\mathcal{F}_0$ and $\mathcal{F}_1(\varphi)$ have the following matrix elements

$$
\langle k', \alpha', \tau'|\mathcal{F}_0|\, k, \alpha, \tau\rangle = \left(\frac{\partial}{\partial\tau} + \xi_k\right)\delta_{k',k}\delta_{\alpha',\alpha}\delta(\tau' - \tau) , \tag{20}
$$

$$
\langle k', \alpha', \tau'|\mathcal{F}_1(\varphi)|\, k, \alpha, \tau\rangle = i\varphi_{k'-k}(\tau)\delta_{\alpha',\alpha}\delta(\tau' - \tau) . \tag{21}
$$

In this notation we express $\det \mathcal{F}(\varphi)$ entering (16) as the following product of two determinants

$$
\det \mathcal{F}(\varphi) = \det \mathcal{F}_0 \cdot \det[\mathbf{1} + e\mathcal{F}_0^{-1}\mathcal{F}_1(\varphi)] . \tag{22}
$$

The first factor, $\det \mathcal{F}_0$, is independent on the integration variables $\varphi_q(\tau)$ and it is exactly equal to the partition function $\mathcal{Z}_f$ of the free electron gas given by the relation (5), i. e.,

$$
\mathcal{Z}_f = \prod_k \prod_{\alpha=\pm 1} \left[1 + e^{-\beta(\varepsilon_{k,\alpha}-\mu)}\right] . \tag{23}
$$

By employing the well-known formula we express the second factor in the product (22) by the relations

$$
\begin{aligned}
\det[\mathbf{1} + e\mathcal{F}_0^{-1}\mathcal{F}_1(\varphi)] &= \exp\left\{\text{Tr} \ln[\mathbf{1} + e\mathcal{F}_0^{-1}\mathcal{F}_1(\varphi)]\right\} \\
&= \exp\left\{-\text{Tr} \sum_{n=1}^\infty (-1)^n \frac{1}{n}[e\mathcal{F}_0^{-1}\mathcal{F}_1(\varphi)]^n\right\} .
\end{aligned} \tag{24}
$$

The matrix $\mathcal{F}_0^{-1}$ entering relation (22) and (24) has the matrix elements defined by

$$
\begin{aligned}
\langle k', \alpha', \tau'|\mathcal{F}_0^{-1}|\, k, \alpha, \tau\rangle &= -\delta_{k',k}\delta_{\alpha',\alpha}\frac{1}{\beta}\sum_\nu \frac{1}{i\omega_\nu - \xi_k} \exp[i\omega_\nu(\tau' - \tau)] \\
&\equiv \delta_{k',k}\delta_{\alpha',\alpha}G_k(\tau' - \tau) .
\end{aligned} \tag{25}
$$

By taking into account the relations (16)–(25) we can express the partition function $\mathcal{Z}$ for the interacting electron gas as the following functional integral over commuting variables $\varphi_q(\tau)$

$$\mathcal{Z} = \mathcal{Z}_0^{-1} \mathcal{Z}_f \int \mathcal{D}(\varphi) \exp\left\{-S_0(\varphi) - S_{int}(\varphi)\right\} , \tag{26}$$

where the effective actions $S_0(\varphi)$ and $S_{int}(\varphi)$ are defined by

$$S_0(\varphi) = \int_0^\beta \mathrm{d}\tau \frac{V}{8\pi} {\sum_q}' \left\{ q^2 \varphi_q(\tau)\varphi_{-q}(\tau) - \right.$$

$$\left. - e^2 \sum_k \int_0^\beta \mathrm{d}\tau' G_{k+\frac{q}{2}}(\tau - \tau') G_{k-\frac{q}{2}}(\tau' - \tau)\varphi_q(\tau')\varphi_{-q}(\tau) \right\} \tag{27}$$

$$S_{int}(\varphi) = \mathrm{Tr} \sum_{n=3}^\infty (-1)^n \frac{e^n}{n} [\mathcal{F}_0^{-1}\mathcal{F}_1(\varphi)]^n . \tag{28}$$

Until now all our calculations were exact. From this point we start to employ a perturbation theory in which the action $S_0(\varphi)$ is treated exactly and the action $S_{int}(\varphi)$ as a perturbation. By using the expressions (27) and (28) we can carry out the functional integration over the commuting variables $\varphi_q(\tau)$ in the functional integral (26) with the following result

$$\mathcal{Z} = \exp\left\{ \sum_{n=1}^\infty \frac{1}{n!} \langle S_{int}^n(\varphi)\rangle_{0,connected} \right\} \mathcal{Z}_f \prod_q [\det(\mathbf{1} - M_q)]^{-1/2} . \tag{29}$$

Here $M_q$ is a functional matrix the matrix elements of which are given by

$$\langle \tau'|M_q|\tau\rangle = \frac{2v_q}{V} \sum_k G_{k+\frac{q}{2}}(\tau - \tau') G_{k-\frac{q}{2}}(\tau' - \tau) , \tag{30}$$

$$v_q = \frac{4\pi e^2}{q^2} \tag{31}$$

and $\langle S_{int}^n(\varphi)\rangle_{0,connected}$ denotes the connected part of the statistical average $\langle S^n(\varphi)\rangle_0$ averaged with the unperturbed action $S_0(\varphi)$. Perhaps we should mention that in the process of the derivation of the formula (29) the factor $\mathcal{Z}_0^{-1}$ entering (26) has been canceled and the cluster expansion theorem

$$\langle \mathrm{e}^{-S_{int}(\varphi)}\rangle_0 = \exp\left\{ \sum_{n=1}^\infty \frac{(-1)^n}{n!} \langle S_{int}^n(\varphi)\rangle_{0,connected} \right\} \tag{32}$$

has been employed.

The $\det(\mathbf{1} - M_q)$ can be explicitly calculated with the result

$$\det(\mathbf{1} - M_q) = \exp\left\{\mathrm{Tr}\,\ln[\mathbf{1} - M_q]\right\} = \exp\left(-\frac{2\beta v_q}{V} \sum_k f_{k+\frac{q}{2}} f_{k-\frac{q}{2}} + \right.$$

$$\left. + \sum_q \sum_{\substack{\nu=-\infty \\ \nu\neq 0}}^{+\infty} \{\ln[1 + F_q(i2\pi\nu)] - F_q(i2\pi\nu)\} + \ln[1 + F_q(0)] - F_q(0) \right) , \tag{33}$$
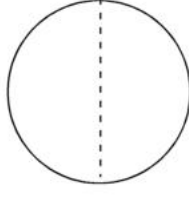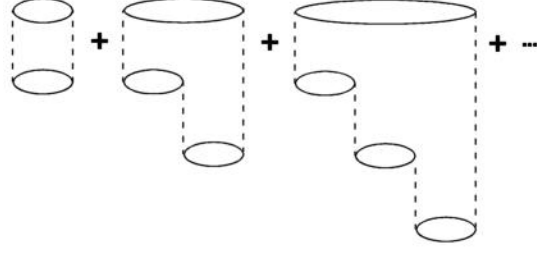
Fig. 1. The exchange diagram

Fig. 2. Contribution of the ring diagrams to the correlation energy

where

$$f_k = (e^{\beta \xi_k} + 1)^{-1} \tag{34}$$

is the Fermi function, $\nu$ are integers and

$$F_q(z) = \frac{2\beta v_q}{V} \sum_k \frac{f_{k+\frac{q}{2}} - f_{k-\frac{q}{2}}}{z - \beta(\xi_{k+\frac{q}{2}} - \xi_{k-\frac{q}{2}})} = \frac{2\beta^2 v_q}{V} \sum_k \frac{(f_{k+\frac{q}{2}} - f_{k-\frac{q}{2}})(\xi_{k+\frac{q}{2}} - \xi_{k-\frac{q}{2}})}{z^2 - \beta^2(\xi_{k+\frac{q}{2}} - \xi_{k-\frac{q}{2}})^2}$$

$$= -\frac{2v_q\beta^2}{V} \sum_k \frac{|f_{k+\frac{q}{2}} - f_{k-\frac{q}{2}}|}{z^2 - \beta^2(\xi_{k+\frac{q}{2}} - \xi_{k-\frac{q}{2}})^2} |\xi_{k+\frac{q}{2}} - \xi_{k-\frac{q}{2}}| \tag{35}$$

is an analytic function in the complex $z$-plane, except for pairs of poles at $\mathrm{Re}\, z = x_p = \pm z_p(k, q)$, where

$$z_p(k, q) = \beta |\xi_{k+\frac{q}{2}} - \xi_{k-\frac{q}{2}}| . \tag{36}$$

From the expressions (33) and (35) one identifies that the product $[\prod_q \det(\mathbf{1} - M_q)]^{-1/2}$ in the partition function (29) is exactly the same as the contribution coming from the exchange diagram and from the infinite sum of the so-called ring diagrams [6] depicted on Fig. 1 and Fig. 2, respectively.

In our approach these contributions are shown as if they had an origin in a noninteracting boson system described by the action (27) corresponding to a system of independent harmonic oscillators. These terms contributing to the grand canonical potential $\Omega = -k_B T \ln \mathcal{Z}$ have the notations $\Omega_1$ and $\Omega_r$ as in [6]. The term

$$\Omega_1 = -\frac{1}{V} \sum_{q,k} v_q f_{k+\frac{q}{2}} f_{k-\frac{q}{2}} \tag{37}$$

comes from the exchange diagram and the term

$$\Omega_r = \frac{1}{2} k_B T \sum_q \sum_{\nu=-\infty}^{+\infty} \{\ln[1 + F_q(2\pi i\nu)] - F_q(2\pi i\nu)\} \tag{38}$$
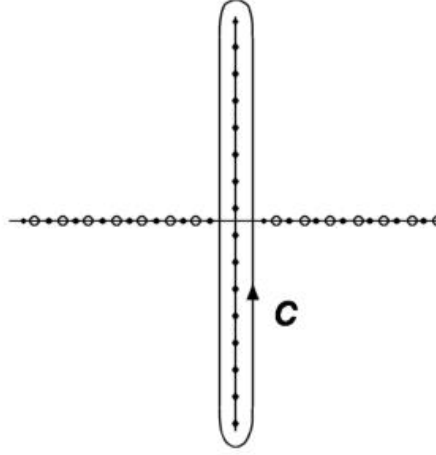
Fig. 3. The integration path $C$ in the contour integral (40). The full dots denote the poles of the integrand and the open dots denote nodes of the function $1 + F_q(z)$
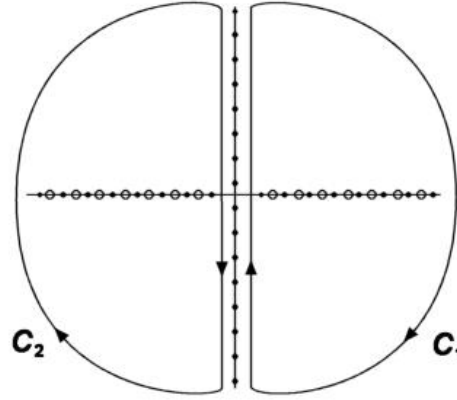
Fig. 4. The two integration paths $C_1$ and $C_2$ are equivalent to the path on Fig. 3

corresponds to the infinite sum of the mentioned ring diagrams.

The sum

$$Y_q = \sum_{\substack{\nu = -\infty \\ \nu \neq 0}}^{+\infty} \{\ln[1 + F_q(i2\pi\nu)] - F_q(i2\pi\nu)\} \tag{39}$$

over all integer $\nu$ with the exception $\nu = 0$, which enters the partition function (29) through the relation (33) can be evaluated by employing the theory of analytic functions, namely, by the contour integral

$$Y_q = \frac{1}{2\pi i} \oint_C dz \left(\frac{1}{e^z - 1} - \frac{1}{z}\right) \{\ln[1 + F_q(z)] - F_q(z)\} \ . \tag{40}$$

Here the closed curve $C$ encircles the imaginary axis $\mathrm{Im} z = y$ in the complex $z$-plane as it is depicted on Fig. 3.

It is now easy to verify that the contour $C$ may be deformed as shown on Fig. 4 into two closed contours $C_1$ and $C_2$, because the contributions along the arcs vanish.

Thus the contour $C$ in the contour integral (40) can be formally regarded as an union of two contours $C_1$ and $C_2$. The contours $C_1$ and $C_2$ encircle singularities of the integrand (40) which are situated on the real axis in the complex $z$-plane. These singularities are the poles $\pm z_p(k, q)$ of the function $F_q(z)$ given by the relation (36) and nodes (i.e., zero points) of the function $1 + F_q(z)$. The expression (35) implies that between each of the two neighbour poles of $F_q(z)$

on the real axis there exist nodes at points $x_n = \pm z_n(k,q)$ given approximately by the following relation

$$z_n(k,q) = z_p(k,q)+$$

$$+ \frac{v_q\beta}{V}|f_{k+\frac{q}{2}} - f_{k-\frac{q}{2}}| \left\{ 1 - \frac{v_q\beta}{V} \frac{|f_{k+\frac{q}{2}} - f_{k-\frac{q}{2}}|}{z_p(k,q)} \right\} + O(e^6) \,. \tag{41}$$

We can use the expressions (36) and (41) and the argument principle from the theory of the complex function analysis to evaluate the integral (40), with the following result

$$Y_q = 2 \sum_k \left\{ \ln \left[ \frac{z_p(k,q)}{z_n(k,q)} \frac{\sinh \frac{1}{2} z_n(k,q)}{\sinh \frac{1}{2} z_p(k,q)} \right] - \right. \tag{42}$$

$$\left. - \left[ \frac{1}{2} \coth \frac{1}{2} z_p(k,q) - \frac{1}{z_p(k,q)} \right] \frac{v_q\beta}{V} |f_{k+\frac{q}{2}} - f_{k-\frac{q}{2}}| \right\} \,. \tag{43}$$

By using the relation (41) we derive the formula

$$\ln \prod_k \frac{z_p(k,q)}{z_n(k,q)} = -\ln \left[ 1 + \frac{v_q}{V} \sum_k \frac{|f_{k+\frac{q}{2}} - f_{k-\frac{q}{2}}|}{|\xi_{k+\frac{q}{2}} - \xi_{k-\frac{q}{2}}|} \right] = -\ln[1 + \frac{1}{2} F_q(0)] \,. \tag{44}$$

With this result we can write

$$\prod_q \det(\mathbf{1} - M_q)^{-\frac{1}{2}} = e^{-\beta\Omega_1} \prod_q \left\{ \frac{1 + \frac{1}{2} F_q(0)}{[1 + F_q(0)]^{\frac{1}{2}}} \right\} \prod_{k,q} \left\{ \frac{\sinh \frac{1}{2} z_p(k,q)}{\sinh \frac{1}{2} z_n(k,q)} \right\} \times$$

$$\times \exp \left\{ \frac{1}{2} \sum_{k,q} \frac{v_q\beta}{V} |f_{k+\frac{q}{2}} - f_{k-\frac{q}{2}}| \coth \frac{1}{2} z_p(k,q) \right\} \,. \tag{45}$$

From the last expression we immediately see the relation

$$\prod_{k,q} \left[ \sinh \frac{1}{2} z_n(k,q) \right]^{-1} = \mathcal{Z}_b \prod_{k,q} 2 e^{-\frac{1}{2} z_n(k,q)} \,, \tag{46}$$

where $\mathcal{Z}_b$ is the partition function of a noninteracting bose gas

$$Z_b = \prod_{k,q} (1 - e^{-\beta\hbar\omega(k,q)})^{-1} \tag{47}$$

with the energy spectrum of boson excitations given by

$$\hbar\omega(k,q) = |\xi_{k+\frac{q}{2}} - \xi_{k-\frac{q}{2}}|+$$

$$+ \frac{v_q}{V} |f_{k+\frac{q}{2}} - f_{k-\frac{q}{2}}| \left\{ 1 + \frac{v_q}{V} \frac{f_{k+\frac{q}{2}} - f_{k-\frac{q}{2}}}{\xi_{k+\frac{q}{2}} - \xi_{k-\frac{q}{2}}} \right\} + O(e^6) \,. \tag{48}$$

Needless to say that the expression (46) represents the partition function of independent harmonic oscillators with the frequencies $\omega(k, q)$.

Thus, we have shown that the partition function $\mathcal{Z}$ of interacting electron gas has indeed the form (7) involving the product of the partition functions $\mathcal{Z}_f$ and $\mathcal{Z}_b$ corresponding to the free electron gas and to the noninteracting bose gas as it was intuitively expected in the introduction. All remaining factors entering the expression (45) and those given by (32) can be assembled into the factor $C$ in the expression (7).

In order to derive a compact formula for the product (45), we again use (41) to derive the expression

$$
\prod_k \left\{ \frac{\sinh \frac{1}{2} z_p(k, q)}{\sinh \frac{1}{2} z_n(k, q)} \right\} = \prod_k \left\{ 1 + \frac{1}{2} \coth \frac{1}{2} z_p(k, q) \left[ \frac{\beta v_q}{V} |f_{k+\frac{q}{2}} - f_{k-\frac{q}{2}}| + O(\frac{e^6}{V^2}) \right] \right\}^{-1}
$$

$$
= \left\{ 1 + \frac{\beta}{2} \frac{v_q}{V} \sum_k \coth \frac{1}{2} z_p(k, q) |f_{k+\frac{q}{2}} - f_{k-\frac{q}{2}}| \right\}^{-1} . \tag{49}
$$

The identity

$$
\coth \frac{1}{2} z_p(k, q) |f_{k+\frac{q}{2}} - f_{k-\frac{q}{2}}| = (f_{k+\frac{q}{2}} - f_{k-\frac{q}{2}}) \coth \frac{1}{2} \beta (\xi_{k+\frac{q}{2}} - \xi_{k-\frac{q}{2}})
$$

$$
= (1 - f_{k+\frac{q}{2}}) f_{k-\frac{q}{2}} + f_{k+\frac{q}{2}} (1 - f_{k-\frac{q}{2}}) \tag{50}
$$

is used to express $\prod\limits_q \det(\mathbf{1} - M_q)^{-1/2}$ in the following form

$$
\prod_q \det(\mathbf{1} - M_q)^{-1/2} = \mathrm{e}^{-\beta \Omega_1} \times \exp \sum_q \left\{ \ln \left[ \frac{1 + \frac{1}{2} F_q(0)}{[1 + F_q(0)]^{\frac{1}{2}}} \right] - \ln[1 + X_q] + X_q \right\} , \tag{51}
$$

where $X_q$ is defined by

$$
X_q = \frac{\beta v_q}{V} \sum_k f_{k+\frac{q}{2}} (1 - f_{k-\frac{q}{2}}) . \tag{52}
$$

We have made the explicit and exact summation over $\nu$ in the sum (39). This is a completely new result in the theory of interacting electron gas. Therefore, we have obtained a new representation of the contribution to the grand-canonical potential $\Omega_r$ coming from the ring diagrams

$$
\Omega_r = \frac{1}{\beta} \sum_q \left\{ \ln[1 + X_q] - X_q + \frac{1}{2} \ln \left[ \frac{1 + F_q(0)}{[1 + \frac{1}{2} F_q(0)]^2} \right] \right\} . \tag{53}
$$

## 3  Conclusion

In this work we have investigated the simplest model of the electron plasma in metals. We have focused on the structure of its grand canonical partition function by using the formalism of functional integrals. We have explicitly shown that the partition function contains a product of two partition functions corresponding to the free electron gas and to a noninteracting boson gas

as well as a factor describing correlation effects between the electrons and bosons. In addition to this result we have obtained a new representation for the infinite sum of the ring diagram contributions by carrying out the sum over Matsubara's frequencies exactly and explicitly.

The formalism of functional integrals offers very elegant solutions of problems from the field of quantum statistical physics. It is especially advantageous to use it to calculate the partition function of many-particle systems. Besides this work, this has been done in [3], where it is shown that the partition function of BCS model has a similar structure. We expect the future of the presented technique in application of it for more realistic models.

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