HIDDEN SYMMETRY AND SEPARATION OF VARIABLES IN THE TWO-CENTRE PROBLEM WITH A CONFINEMENT-TYPE POTENTIAL

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An additional spheroidal integral of motion and a group of dynamic symmetry in a model quantum-mechanical problem of two centres $eZ_1Z_2\omega$ with Coulomb and oscillator interactions is obtained, the group properties of its solutions being studied. $P\,(3)\otimes P\,(2,1)$, $P\,(5,1)$ and $P\,(4,2)$ groups are considered as the dynamic symmetry groups of the problem, among them $P\,(3)\otimes P\,(2,1)$ group possessing the smallest number of parameters. The obtained results may appear useful at the calcuations of QQq-baryons and QQg-mesons energy spectra.

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1 Introduction

As a rule, when systems, possessing hidden (or higher dynamic) symmetry, are considered, two methods are used [1, 2]. The first of them consists in rewriting Schroedinger equation and putting it in the form where the symmetry, having been hidden before, becomes explicit. The second one implies the construction of integrals of motion which play the role of hidden symmetry group generators.

In the paper, based on the example of a physically important model of confinement-type two-centred potential we try to emphasize the deep relationship of the hidden symmetry to the possibility of separation of variables in the Schroedinger equation. The knowledge of such kind of relationships in two recent decades [3] has resulted in the intense application of the method of separation of variables to the equations of mathematical physics and led to a series of important and far from trivial results in this field of mathematics (see, for instance, [4, 5]). The method of separation of variables is much simpler compared to the two above methods. In the framework

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of this method the eigenvalues of the hidden symmetry group generators acquire the sense of separation constants, and the eigenfunctions, common for the Hamiltonian and the generators, which commutate with the Hamiltonian and with each other, are the solutions of the Schroedinger equation in the corresponding coordinates.

Below the group properties of a model quantum problem of the motion of a light particle (a gluon) in the field of two heavy particles (a quark-antiquark pair) are studied. Recently this problem has become the subject of intense studies due to its relation to a wide range of problems of hadron physics: models of baryons with two heavy quarks (QQq barions) [6] and models of heavy hybride mesons with open flavor (QQg mesons) [7]. In spite of the lack of strict theoretical substantiation, the potential models give a satisfactory description of mass spectra for heavy mesons and baryons (see e.g. [6, 7, 8] and references therein), which, according to modern views, represent bound states of quarks. While modelling the interquark interaction potential, as a rule, confinement-type potentials are used [8, 9]. One of such potentials is a so-called Cornell potential, containing a Coulomb-like term of single-gluon exchange and a term, responsible for the string interaction, providing the quark confinement. The confinement part of the potential is most often modelled by a spatial spherically symmetrical oscillator potential [6, 7]. Then in a non-relativistic approximation the motion of a light quark (gluon) in the field of two heavy quarks can be described by a stationary Schroedinger equation with a model combined potential, being the sum of the potential of two Coulomb centres and the potential of two harmonic oscillators:

$$V\left(r_1, r_2\right) = -\frac{Z_1}{r_1} - \frac{Z_2}{r_2} + \omega^2 \left(r_1^2 + r_2^2\right). \tag{1}$$

In this formula r_1 and r_2 are the distances from the particle to the fixed force centres 1 and 2, $Z_{1,2} = \frac{2}{3}\alpha_s$, α_s—the strong interaction constant, and the phenomenological parameter ω is chosen from the condition of the best agreement of the calculated mass spectra of the quark system with the experimental data. In order to avoid ambiguities, one should mention that in our consideration, concerning not only the case of purely Coulomb interaction of the light particle with each of the centres, the notion of the force centre is preserved for the $r_{1,2} = 0$ points, where the combined potential (1) has the singularities.

In the dimensionless variables the Schroedinger equation with the model potential (1) is given by

$$\widehat{H}\Psi \equiv \left[-\frac{1}{2}\triangle - \frac{Z_1}{r_1} - \frac{Z_2}{r_2} + \omega^2 \left(r_1^2 + r_2^2\right)\right] \Psi\left(\mathbf{r}; R\right) = E\left(R\right) \Psi\left(\mathbf{r}; R\right), \tag{2}$$

where r is the distance from the particle to the midpoint of the intercentre distance R, $E\left(R\right)$ and $\Psi\left(\mathbf{r}; R\right)$ are the particle energy and wave function. Hereinafter the spectral problem for the Schroedinger equation (2) with the combined potential (1) is conveniently denoted by $eZ_1Z_2\omega$. The sense of such notation follows from the fact that traditional quantum-mechanical problem of two purely Coulomb centres [10] has a standard notation eZ_1Z_2. Note that the Schroedinger equation for the eZ_1Z_2 problem can be obtained from the equation (2) by a limiting transition $\omega \to 0$.

The group properties, eigenvalue and eigenfunction spectrum for the eZ_1Z_2 problem of two Coulomb centres have been studied substantially [10, 11, 12, 13, 14, 15, 16]. Namely, the choice of a certain non-canonical basis in a group being a direct product of two groups of motions of three-dimensional spaces $P(3) \otimes P(2,1)$, or in wider groups of motions of six-dimensional spaces $P(5,1)$ and $P(4,2)$ is known to result in the necessity to solve the problem equivalent to eZ_1Z_2. The consequence of these group properties of eZ_1Z_2 solutions problem is a linear algebra of two-centred integrals, obtained in [16].

Here we show that for our case of $eZ_1Z_2\omega$ problem a generally similar situation takes place. This problem can also be considered as a problem of theory of representations of certain non-compact groups, where the function being a product of a quasiradial and a quasiangular two-centred functions by $\exp(im\alpha + i\widetilde{m}\beta)$, comprises the basis of a degenerate non-canonical representation of the group being a direct product of two three-dimentional space motion groups $P(3) \otimes P(2,1)$, or wider six-dimensional space motion groups $P(5,1)$, $P(4,2)$ etc.

The term "non-canonical representation" is used in the present paper, like in [13, 14, 15], for less studied representations where not all the operators of the complete set of the observed ones are the invariants of the subgroups of the considered group.

2 Spheroidal integral of motion in the problem $eZ_1Z_2\omega$

The variables in Eq. (2) can be separated by introducing an prolate spheroidal (elliptical) coordinate system $\{\xi\eta\alpha\}$ with the origin in the midpoint of R segment and foci in its endpoints [10]:

$$\left.\begin{array}{ll} \xi = (r_1 + r_2)/R, & 1 \le \xi < \infty, \\ \eta = (r_1 - r_2)/R, & -1 \le \eta \le 1, \\ \alpha = \arctan\left(\frac{x_2}{x_1}\right), & 0 \le \alpha < 2\pi \end{array}\right\} . \tag{3}$$

Here α is the angle of rotation around OX_3 axis; the origin of the Cartesian coordinate system $\{x_1, x_2, x_3\}$ is located in the midpoint of the segment R, and the axis OX_3 is directed from the centre 1 to the centre 2.

Consider the explicit form of the differential equations resulting from the procedure of the separation of variables in Eq. (2) in prolate spheroidal coordinates (3). By presenting the wave function $\Psi(\xi, \eta, \alpha; R)$ as a product $F(\xi; R)\, G(\eta; R)\, \Phi(\alpha)$ and substituting it into (2) one obtains three ordinary differential equations, linked by the separation constants λ and m:

$$\left[\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2 - 1\right)\frac{\mathrm{d}}{\mathrm{d}\xi} + a\xi + \left(p^2 - \gamma\xi^2\right)\left(\xi^2 - 1\right) - \frac{m^2}{\xi^2 - 1} + \lambda\right] F(\xi; R) = 0\,, \tag{4}$$

$$\left[\frac{\mathrm{d}}{\mathrm{d}\eta}\left(1 - \eta^2\right)\frac{\mathrm{d}}{\mathrm{d}\eta} + b\eta + \left(p^2 - \gamma\eta^2\right)\left(1 - \eta^2\right) - \frac{m^2}{1 - \eta^2} - \lambda\right] G(\eta; R) = 0\,, \tag{5}$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2} + m^2\right] \Phi(\alpha) = 0\,. \tag{6}$$

Here we use the notations

$$p = \tfrac{R}{2}\sqrt{2E'}\,, \qquad E' = E - \tfrac{\omega^2 R^2}{2}\,, \qquad \gamma = \tfrac{\omega^2 R^4}{4}\,,$$

$$a = (Z_1 + Z_2)\cdot R\,, \qquad b = (Z_2 - Z_1)\cdot R\,.$$

In order to have the complete wave function $\Psi\left(\overrightarrow{r};R\right)$ normalized, the functions $F\left(\xi;R\right)$ and $G\left(\eta;R\right)$ should obey the boundary conditions

$$|F\left(1;R\right)|<\infty\,,\qquad F\left(\infty;R\right)<\infty\,,\tag{7}$$

$$|G\left(\pm 1;R\right)|<\infty\,.\tag{8}$$

The procedure of obtaining the energy terms $E\left(R\right)$ is reduced to the following steps. At first two boundary problems are considered independently: (4), (7) for the quasiradial and (5), (8) for the quasiangular equations, $\lambda^{(\xi)}$ and $\lambda^{(\eta)}$ being considered the eigenvalues and p being left a free parameter. Each of the eigenfunctions can be conveniently characterized by two quantum numbers n, m and the eigenvalue λ, namely: n_ξ, m, $\lambda^{(\xi)}$ for $F_{n_\xi,m}\left(\xi;R\right)$ and n_κ, m, $\lambda^{(\xi)}$ for $G_{n_\eta,m}\left(\eta;R\right)$. The quantum numbers n_ξ, n_η are non-negative integers 0,1,2... and coincide with the number of nodes for $F_{n_\xi,m}\left(\xi;R\right)$, $G_{n_\eta,m}\left(\eta;R\right)$ functions on the radial ($1\le\xi<\infty$) and angular ($-1\le\eta\le 1$) intervals, respectively. The general theory of Sturm-Liuville-type one-dimensional boundary problems implies that the quantum numbers n_ξ, n_η, m, remain constant at the continuous variation of the intercentre distance R, and the eigenvalues $\lambda^{(\xi)}_{n_\xi m}\left(p,a,\gamma\right)$ or $\lambda^{(\eta)}_{n_\kappa m}\left(p,b,\gamma\right)$ are non-degenerate.

The pair of one-dimensional boundary problems for $F_{n_\xi,m}\left(\xi;R\right)$ and $G_{n_\eta,m}\left(\eta;R\right)$ is equivalent to the initial $eZ_1Z_2\omega$ problem under condition of equality of the eigenvalues $\lambda^{(\xi)}_{n_\xi m}\left(p,a,\gamma\right)$ $=\lambda^{(\eta)}_{n_\kappa m}\left(p,b,\gamma\right)$ and the account of p,a,b,γ relationship with the E,Z_1,Z_2,ω,R parameters. The eigenvalues $E_{n_\xi n_\eta m}$, $\lambda_{n_\xi n_\eta m}$ and eigenfunctions $\Psi_{n_\xi n_\eta m}\left(\mathbf{r};R\right)$ of the three-dimensional $eZ_1Z_2\omega$ problem are enumerated by a set of quantum numbers $j=\left(n_\xi n_\eta m\right)$ which are conserved at the continuous variation of Z_1, Z_2, ω, R parameters:

$$E_j\left(R\right)=E_{n_\xi n_\eta m}\left(R,Z_1,Z_2,\omega\right)\;,\tag{9}$$

$$\Psi_j\left(\mathbf{r};R\right)=N_j\left(R\right)F\left(\xi;R\right)G\left(\eta;R\right)\frac{e^{im\alpha}}{\sqrt{2\pi}}\,.\tag{10}$$

The normalization constant $N_j\left(R\right)$ is found from the condition

$$\int\limits_\Omega \mathrm{d}\Omega\Psi^*_i\Psi_j=\delta_{ij}\,,\qquad \mathrm{d}\Omega=\frac{R^3}{8}\left(\xi^2-\eta^2\right)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\alpha=\frac{R^3}{8}\mathrm{d}\tau\mathrm{d}\alpha\,,\tag{11}$$

where δ_{ij} is the Kronecker symbol, and $\Omega=\{\xi,\eta,\alpha\mid 1\le\xi<\infty,-1\le\eta\le 1,0\le\alpha<2\pi\}$. Hence, the system of functions $\{\Psi_j\left(\mathbf{r};R\right)\}$ forms a complete set of orthonormalized wave functions.

Now we proceed to establish the relationship between the symmetry properties of the $eZ_1Z_2\omega$ problem and the above separation of variables in the Schroedinger equation (2) in the prolate spheroidal coordinates (3). The very fact of such separation indicates an additional (with respect to the geometrical one) symmetry of the Hamiltonian (2), causing the existence of an additional integral of motion, whose operator commutes with $\widehat{H}$ and the operator $\widehat{L}_3$, the projection of the angular moment on the intercentral axis $\mathbf{R}$. In order to reveal it, we exclude the energy parameter p^2 and the magnetic quantum number m from the above differential equation system (4)–(6). Thus we derive the equation

$$\widehat{\lambda}\Psi_j\left(\mathbf{r};R\right)=\lambda_j\Psi_j\left(\mathbf{r};R\right),\tag{12}$$

where $\widehat{\lambda}$ denotes a differential operator

$$\widehat{\lambda} = \frac{1}{\xi^2-\eta^2}\left\{\left(\xi^2-1\right)\frac{\partial}{\partial\eta}\left(1-\eta^2\right)\frac{\partial}{\partial\eta} - \left(1-\eta^2\right)\frac{\partial}{\partial\xi}\left(\xi^2-1\right)\frac{\partial}{\partial\xi}\right\} + \\ + \left[\frac{1}{1-\eta^2} - \frac{1}{\xi^2-1}\right]\frac{\partial^2}{\partial\alpha^2} - RZ_2\frac{\xi\eta+1}{\xi+\eta} + RZ_1\frac{\xi\eta-1}{\xi-\eta} + \frac{\omega^2R^4}{4}\left(\xi^2-1\right)\left(1-\eta^2\right). \quad (13)$$

The separation constant λ_j is the eigenvalue of this operator, and the solutions of Eq. (2) are its eigenfunctions. Since in the limit $\omega \to 0$ the model $eZ_1Z_2\omega$ problem is reduced to the problem of two purely Coulomb centres eZ_1Z_2 [10], it is $a\ priori$ obvious that the operator $\widehat{\lambda}$ should be a linear combination of the operators $\widehat{L}_3$, $\widehat{P}_3^2$ and $\widehat{H}$ (here L is the orbital moment operator and $\widehat{P}_3$ is the third component of the momentum) which in the considered limit is reduced to the operator of the separation constant for the eZ_1Z_2 problem [10]. To determine the weight factors and the free constant in the mentioned linear combination we compare the expression (13) with the explicit form of the operators $\widehat{L}_3$, $\widehat{P}_3^2$ and $\widehat{H}$ in the prolate spheroidal coordinates (3). After simple but rather tedious calculations we finally obtain the algebraic expression for the separation constant operator in the $eZ_1Z_2\omega$ problem:

$$\widehat{\lambda} = -\widehat{L}^2 + x_3R\left(\frac{Z_2}{r_2} - \frac{Z_1}{r_1}\right) - \omega^2R^2\left(x_3^2 + \frac{R^2}{4}\right) + \frac{R^2}{4}\left(2\widehat{H} - \widehat{P}_3^2\right). \quad (14)$$

The fact the operator $\widehat{\lambda}$ commuting with the Hamiltonian $\widehat{H}$ and the operator $\widehat{L}_3$ of the angular moment projection onto the intercentral axis $\mathbf{R}$, can be easily verified by direct calculations of commutational relations $\left[\widehat{H}, \widehat{\lambda}\right] = \left[\widehat{\lambda}, \widehat{L}_3\right] = 0$. Thus, the operators $\widehat{H}$, $\widehat{L}_3$, $\widehat{\lambda}$ have a common complete system of eigenfunctions and can be diagonalized simultaneously. The given representation corresponds to the separation of variables in Eq. (2) in the prolate spheroidal coordinates (3): the general eigenfunction of the operators $\widehat{H}$, $\widehat{L}_3$, $\widehat{\lambda}$ is described as a product (10).

The purely geometric symmetry group of the Hamiltonian $eZ_1Z_2\omega$ is the O_2 group containing rotations around the intercentral axis $\mathbf{R}$ and reflections in the planes containing this axis. In the symmetrical case ($Z_1 = Z_2 = Z$) the $eZZ\omega$ system possesses an additional element of geometrical symmetry—the reflection in the plane, perpendicular to the $\overrightarrow{R}$ vector and cutting it at its centre.

In addition to the geometrical symmetry, the $eZ_1Z_2\omega$ problem possesses higher dynamic symmetry, related to the exact separation of variables in the Schroedinger equation (2) in the prolate spheroidal coordinates (3).

In the following subsections we show how, by means of the separation of variables method, the dynamic symmetry group of the quantum-mechanical problem $eZ_1Z_2\omega$ can be determined.

3 The representations of the group $P(3) \otimes P(2,1)$

Consider a group $P(3) \otimes P(2,1)$ being a direct product of two groups of motions of three-dimensional spaces $P(3)$ and $P(2,1)$.

We remind that the group $P(3)$ (known also as the Galilean group $E(3)$) consists of displacements (translations) and rotations (revolutions) of the Euclidean space of coordinates y_i

with a metric

$$y_i y_i = y_1^2 + y_2^2 + y_3^2, \quad i = 1, 2, 3. \tag{15}$$

Here and below the twice repeated indices imply summation.

The $P\,(2,1)$ group (the Poincarè group of the three-dimensional space, denoted also as $E\,(2,1)$) consists of translations and rotations of rhe pseudo-Euclidean space of coordinates y_μ with a metric

$$y_\mu y_\mu = y_4^2 + y_5^2 - y_6^2, \quad \mu = 4, 5, 6. \tag{16}$$

The infinitesimal generators of the $P\,(3)$ group

$$x_j = -i\frac{\partial}{\partial y_j}, \qquad \pounds_{jk} = -i\left(y_j\frac{\partial}{\partial y_k} - y_k\frac{\partial}{\partial y_j}\right), \qquad j, k = 1, 2, 3 \tag{17}$$

and of the $P\,(2,1)$ group

$$\begin{aligned} x_\mu &= -i\frac{\partial}{\partial y_\mu}, \mu = 4,5,6; \pounds_{46} = -i\left(y_4\frac{\partial}{\partial y_6} + y_6\frac{\partial}{\partial y_4}\right), \pounds_{56} = -i\left(y_5\frac{\partial}{\partial y_6} + y_6\frac{\partial}{\partial y_5}\right), \\ \pounds_{45} &= -i\left(y_4\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial y_4}\right) \end{aligned} \tag{18}$$

can be easily verified to satisfy the known structure relations:

$$\begin{aligned} &[x_i, x_j] = 0, [x_i, \pounds_{jk}] = i\left(\delta_{ik}x_j - \delta_{ij}x_k\right), \quad \delta_{ij} = \begin{cases} 1, & i = j = 1,2,3 \\ 0, & i \neq j \end{cases}, \\ &[\pounds_{12}, \pounds_{23}] = i\pounds_{31}, \qquad [\pounds_{31}, \pounds_{12}] = i\pounds_{23}, \qquad [\pounds_{23}, \pounds_{31}] = i\pounds_{12}, \\ &[x_\mu, x_\nu] = 0, [x_\sigma, \pounds_{\mu\nu}] = i\left(\delta_{\sigma\nu}x_\mu - \delta_{\sigma\mu}x_\nu\right), \quad \delta_{\mu\nu} = \begin{cases} 1, & \mu = \nu = 4,5 \\ -1, & \mu = \nu = 6 \\ 0, & \mu \neq \nu \end{cases}, \\ &[\pounds_{46}, \pounds_{56}] = i\pounds_{45}, \qquad [\pounds_{56}, \pounds_{45}] = i\pounds_{46}, \qquad [\pounds_{45}, \pounds_{46}] = i\pounds_{56}, \\ &[x_i, x_\mu] = 0, \qquad [\pounds_{ij}, \pounds_{\mu\nu}] = 0, \qquad [x_i, \pounds_{\mu\nu}] = 0, \qquad [x_\mu, \pounds_{ij}] = 0. \end{aligned} \tag{19}$$

Here the indices i, j, k are 1, 2, 3, and μ, ν, σ are $4, 5, 6$. Note that in order to simplify the notations the ˆ symbol over the operators is omitted since in this context no threat of ambigiuty can arise.

The differential operators (17), (18) act in the space of functions $f_j(\mathbf{y})$ which depend on the choice of the complete set of the diagonal operators in $P\,(3) \otimes P\,(2,1)$. Here j is the set of the eigenvalues of these operators. It is worth notice that in the chosen representation the functions $f_j(\mathbf{y})$ are scalar. In the general case the generators (17), (18) can possess the spin part, and $f_j(\mathbf{y})$ can be spinors, vectors, tensors, respectively.

By Fourier transformation

$$f_j\left(\mathbf{y}\right) = \int \exp\left(-i\mathbf{x}.\mathbf{y}\right) \Psi_j\left(\mathbf{x}\right) \mathrm{d}\mathbf{x} \tag{20}$$

we proceed to the x-representation and choose in the $P\,(3) \otimes P\,(2,1)$ group such set of diagonal intercommutating operators:

$$\widehat{C}_1 = x_i x_i, \qquad \widehat{C}_2 = x_i x_i \mathbf{L}^2 - x_i x_j \pounds_{ik} \pounds_{jk}, \tag{21}$$

$$\widehat{C}_3 = x_\mu x_\mu, \qquad \widehat{C}_4 = x_\mu x_\mu \mathbf{M}^2 - x_\mu x_\sigma \pounds_{\nu\mu} \pounds_{\sigma\mu}, \tag{22}$$

$$\pounds_{12}, \qquad \pounds_{45} \tag{23}$$

$$\widehat{E} = -\frac{1}{2\left(x_6^2 - x_3^2\right)} \Big[-\mathbf{M}^2 + 2\overline{a}x_6 - \mathbf{L}^2 + 2\overline{b}x_3 - 4\omega^2 \left(x_6^4 - x_3^4\right) \Big], \tag{24}$$

$$\begin{aligned} \widehat{\lambda} \;\; = \;\; & \frac{(x_3^2 - \widehat{C}_1)}{(x_6^2 - x_3^2)} \Big[-\mathbf{M}^2 + 2\overline{a}x_6 + 4\omega^2 (\widehat{C}_1^2 - x_6^4) \Big] + \\ & + \frac{(x_6^2 - \widehat{C}_1)}{(x_6^2 - x_3^2)} \Big[-\mathbf{L}^2 + 2\overline{b}x_3 - 4\omega^2 (\widehat{C}_1^2 - x_3^4) \Big]. \end{aligned} \tag{25}$$

Here

$$\mathbf{L}^2 = \pounds_{12}^2 + \pounds_{32}^2 + \pounds_{31}^2, \qquad \mathbf{M}^2 = \pounds_{46}^2 + \pounds_{56}^2 - \pounds_{45}^2,$$

ω, $\overline{a}$ and $\overline{b}$ are constants. Note that summation over the indices i, j, k is performed according to the metric (15), and over the indices μ, ν, σ—according to the metric (16).

The introduced operators (21)–(25) possess inportant properties. The operators $\widehat{C}_1$, $\widehat{C}_2$ are the Casimir operators of $P\,(3)$ group, and the operators $\widehat{C}_3$, $\widehat{C}_4$—are the Casimir operators of $P\,(2,1)$ group. One can verify by direct calculations that the operators $\widehat{C}_2$ and $\widehat{C}_4$ are equal to zero: $\widehat{C}_2 = \widehat{C}_4 = 0$. This, in turn, means that the considered representation is degenerate. Further $\pounds_{12}, \pounds_{45}$ are the invariants of uniparametric subgroups of rotations in $P\,(3)$ and $P\,(2,1)$, respectively, and $\widehat{E}$, $\widehat{\lambda}$—non-canonical diagonal operators.

By substituting the expression for $\mathbf{L}^2 - 2\overline{b}x_3 - 4\omega^2 x_3^4$ (or $\mathbf{M}^2 - 2\overline{a}x_6 + 4\omega^2 x_6^4$) from (24) into (25) $\widehat{\lambda}$ can be given by

$$\widehat{\lambda} = -\mathbf{L}^2 + 2\left(\widehat{C}_1 - x_3^2\right)\left[\widehat{E} - 2\omega^2\left(\widehat{C}_1 + x_3^2\right)\right] + 2\overline{b}x_3, \tag{26}$$

or also

$$\widehat{\lambda} = \mathbf{M}^2 + 2\left(\widehat{C}_1 - x_6^2\right)\left[\widehat{E} - 2\omega^2\left(\widehat{C}_1 + x_6^2\right)\right] - 2\overline{a}x_6. \tag{27}$$

Our further goal is to construct the basis of eigenvectors $\Psi_j\left(\overrightarrow{x}\right)$ in which the complete set of operators (21)–(25) is diagonal in $P\,(3) \otimes P\,(2,1)$ group. For this purpose we introduce a new coordinate system in the x-space:

$$\begin{aligned} x_1 &= \tfrac{R}{2}\sqrt{1-\eta^2}\cos\alpha, \quad x_2 = \tfrac{R}{2}\sqrt{1-\eta^2}\sin\alpha, \quad x_3 = \tfrac{R}{2}\eta, \\ x_4 &= \tfrac{R}{2}\sqrt{\xi^2-1}\cos\beta, \quad x_5 = \tfrac{R}{2}\sqrt{\xi^2-1}\sin\beta, \quad x_6 = \tfrac{R}{2}\xi, \end{aligned} \tag{28}$$

where

$$0 \le R < \infty, \quad 1 \le \xi < \infty, \quad -1 \le \eta < +1, \quad 0 \le \alpha, \beta \le 2\pi. \tag{29}$$

Having omitted the intermediate calculations, we write the final expressions for the operators (17), (18) in the new variables (28):

$$\begin{aligned}
\pounds_{23} &= -i\left(\sqrt{1-\eta^2}\sin\alpha\tfrac{\partial}{\partial\eta} - \eta\tfrac{\cos\alpha}{\sqrt{1-\eta^2}}\tfrac{\partial}{\partial\alpha}\right),\\
\pounds_{31} &= -i\left(-\sqrt{1-\eta^2}\cos\alpha\tfrac{\partial}{\partial\eta} - \eta\tfrac{\sin\alpha}{\sqrt{1-\eta^2}}\tfrac{\partial}{\partial\alpha}\right), \quad \pounds_{12} = -i\tfrac{\partial}{\partial\alpha},\\
\pounds_{46} &= -i\left(\sqrt{\xi^2-1}\sin\beta\tfrac{\partial}{\partial\xi} + \xi\tfrac{\cos\beta}{\sqrt{\xi^2-1}}\tfrac{\partial}{\partial\beta}\right),\\
\pounds_{56} &= -i\left(\sqrt{\xi^2-1}\cos\beta\tfrac{\partial}{\partial\xi} - \xi\tfrac{\sin\beta}{\sqrt{\xi^2-1}}\tfrac{\partial}{\partial\beta}\right), \qquad \pounds_{45} = -i\tfrac{\partial}{\partial\beta}.
\end{aligned} \tag{30}$$

It is seen from these formulae that the operators $\pounds_{12}$ and $\pounds_{45}$, belonging to the complete set of intercommutating operators (21)–(25), in the coordinate system (28) depend only on the variables α and β. Hence, we obtain the following relations:

$$\begin{aligned}
-i\tfrac{\partial}{\partial\alpha}\Psi_j(\xi,\eta,R,\alpha,\beta) &= m_j\Psi_j(\xi,\eta,R,\alpha,\beta),\\
-i\tfrac{\partial}{\partial\beta}\Psi_j(\xi,\eta,R,\alpha,\beta) &= \widetilde{m}_j\Psi_j(\xi,\eta,R,\alpha,\beta),
\end{aligned} \tag{31}$$

where m_j, $\widetilde{m}_j$ are the eigenvalues of the $\pounds_{12}$, $\pounds_{45}$, operators, respectively. The common solution of Eqs. (31) can now be given in the multiplicative form

$$\Psi_j(\xi,\eta,R,\alpha,\beta) = \varphi(\xi,\eta,R)\,e^{im_j\alpha + i\widetilde{m}_j\beta}. \tag{32}$$

The rest of the operators from the complete set (21)–(25) in the coordinate system (28) with the account of (31) are given by

$$\widehat{C}_1 = \frac{R^2}{4}, \quad \widehat{C}_2 = 0, \quad \widehat{C}_3 = -\frac{R^2}{4}, \quad \widehat{C}_4 = 0, \tag{33}$$

$$\begin{aligned}
\widehat{E} = &-\frac{2}{R^2(\xi^2-\eta^2)}\left[\frac{\partial}{\partial\xi}\left(\xi^2-1\right)\frac{\partial}{\partial\xi} + a\xi - \frac{\omega^2R^4}{4}\xi^4 - \frac{\widetilde{m}_j^2}{\xi^2-1}\right] - \\
&-\frac{2}{R^2(\xi^2-\eta^2)}\left[\frac{\partial}{\partial\eta}\left(1-\eta^2\right)\frac{\partial}{\partial\eta} + b\eta + \frac{\omega^2R^4}{4}\eta^4 - \frac{m_j^2}{1-\eta^2}\right],
\end{aligned} \tag{34}$$

$$\begin{aligned}
\widehat{\lambda} = &-\frac{\left(1-\eta^2\right)}{\left(\xi^2-\eta^2\right)}\left[\frac{\partial}{\partial\xi}\left(\xi^2-1\right)\frac{\partial}{\partial\xi} + a\xi + \frac{\omega^2R^4}{4}\left(1-\xi^4\right) - \frac{\widetilde{m}_j^2}{\xi^2-1}\right] + \\
&+\frac{\left(\xi^2-1\right)}{\left(\xi^2-\eta^2\right)}\left[\frac{\partial}{\partial\eta}\left(1-\eta^2\right)\frac{\partial}{\partial\eta} + b\eta - \frac{\omega^2R^4}{4}\left(1-\eta^4\right) - \frac{m_j^2}{1-\eta^2}\right],
\end{aligned} \tag{35}$$

note that $a = \overline{a}R$, $b = \overline{b}R$.

Though in order to solve the question concerning the eigenfunctions of the complete set of operators (21)–(25) one can use their explicit form (33)–(35), we give the expressions for the operators (26), (27) in the new coordinates as well, since they will also be used for another purpose:

$$\widehat{\lambda}=\left[\frac{\partial}{\partial\eta}\left(1-\eta^2\right)\frac{\partial}{\partial\eta}+b\eta+\left(1-\eta^2\right)\frac{R^2\widehat{E}}{2}-\frac{\omega^2R^4}{4}\left(1-\eta^4\right)-\frac{m_j^2}{1-\eta^2}\right], \tag{36}$$

$$\widehat{\lambda}=-\left[\frac{\partial}{\partial\xi}\left(\xi^2-1\right)\frac{\partial}{\partial\xi}+a\xi+\left(\xi^2-1\right)\frac{R^2\widehat{E}}{2}+\frac{\omega^2R^4}{4}\left(1-\xi^4\right)-\frac{\widetilde{m}_j^2}{\xi^2-1}\right]. \tag{37}$$

Now we show how, using the separation of variables method, one can find $\Psi_j\left(\xi,\eta,R,\alpha,\beta\right)$ functions which are common eigenfunctions of the operators (31), (33)–(35) and (36), (37). The application of this method is based on the properties of $\widehat{\lambda}$ operator, expressed by Eq. (36), (37). We choose the basis of eigenvectors Ψ_j where the operator $\widehat{\lambda}$ is diagonal:

$$\widehat{\lambda}\Psi_j=\lambda_j\Psi_j \tag{38}$$

and represent Ψ_j in the form of a product

$$\Psi_j\equiv\Psi_j\left(\xi,\eta,R,\alpha,\beta\right)=N_j\left(R\right)F_j\left(\xi;R\right)G_j\left(\eta;R\right)\frac{e^{im_j\alpha+i\widetilde{m}_j\beta}}{\sqrt{2\pi}}, \tag{39}$$

where λ_j are the eigenvalues of the operator $\widehat{\lambda}$, and $N_j\left(R\right)$ is a normalizing factor. After the separation of variables in (38) a pair of ordinary differential equations for the unknown functions $F_j\left(\xi;R\right), G_j\left(\eta;R\right)$, is obtained:

$$\left[\frac{\partial}{\partial\xi}\left(\xi^2-1\right)\frac{\partial}{\partial\xi}+a\xi+\frac{R^2E_j}{2}\left(\xi^2-1\right)+\frac{\omega^2R^4}{4}\left(1-\xi^4\right)+\lambda_j-\frac{\widetilde{m}_j^2}{\xi^2-1}\right]F_j\left(\xi;R\right)=0, \tag{40}$$

$$\left[\frac{\partial}{\partial\eta}\left(1-\eta^2\right)\frac{\partial}{\partial\eta}+b\eta+\frac{R^2E_j}{2}\left(1-\eta^2\right)-\frac{\omega^2R^4}{4}\left(1-\eta^4\right)-\lambda_j-\frac{m_j^2}{1-\eta^2}\right]G_j\left(\eta;R\right)=0. \tag{41}$$

Here E_j are the eigenvalues of the operator $\widehat{E}$. Since the operator $\widehat{\lambda}$ commutes with all the operators (21)–(24), the eigenfunctions (39) of the operator $\widehat{\lambda}$ are also the eigenfunctions of the operators (21)–(24) in the coordinate system (28).

The invariance of the Hamiltonian of the $eZ_1Z_2\omega$ problem with respect to the $P\left(3\right)\otimes P\left(2,1\right)$ group is now obvious. Indeed, at $m_j=\widetilde{m}_j=m$ the system of equations (40), (41) coincides with the system (4)–(6). Hence, at given $a,b,\omega,R,m_j,\widetilde{m}_j$ the determination of the eigenvalues $E_j=E_j\left(R\right)$, $\lambda_j=\lambda_j\left(R\right)$ and limited in the corresponding ranges (29) eigenfunctions $F_j\left(\xi;R\right)$, $G_j\left(\eta;R\right)$ of the complete set of the commutating operators (31), (33)–(35) is reduced at $m_j=\widetilde{m}_j=m$ to the solution of the problem, completely equivalent to the quantum-mechanical problem $eZ_1Z_2\omega$. In this case the common eigenfunctions (39) of the complete set (31) and (33)–(35), which comprise the basis of the degenerate non-canonical representation of

$P(3) \otimes P(2,1)$ group, coincide within the normalizing factor with the two-centred functions (10) multiplied by $\exp(im\beta)$. The expressions (34), (35) for the operators $\widehat{E}$ and $\widehat{\lambda}$ coincide at $m_j = \widetilde{m}_j = m$ with the expressions for the operators of energy $\widehat{H}$ and separation constant $\widehat{\lambda}$ (see (13)) in the $eZ_1Z_2\omega$ problem in the prolate spheroidal coordinate system (3). The variable R, being used to express the Casimir operators of $P(3) \otimes P(2,1)$ group, in the $eZ_1Z_2\omega$ problem is equal to the intercentral distance.

The operators (31), (33)–(35) are Hermitian in the scalar product

$$\langle \Psi_i \mid \Psi_j \rangle = \int\limits_{\overline{\Omega}} \Psi_i^* \Psi_j \mathrm{d}\overline{\Omega}, \tag{42}$$

where $\overline{\Omega}$ corresponds to the range (29) and the volume element $\mathrm{d}\overline{\Omega} = \xi \mathrm{d}\xi \cdot \mathrm{d}\eta \cdot \mathrm{d}\alpha \cdot \mathrm{d}\beta$. Thus, the representation, corresponding to the set (33)–(35) and (36), (37), is unitary.

One of the possible consequences of the above group interpretation of the solutions of the $eZ_1Z_2\omega$ problem consists in the calculation of the matrix elements of generators (30) in the non-canonical basis (39) being reduced to the calculation of the two-centred integrals over the variable ξ and the similar integrals over the variable η. This circumstance is the base for the deduction (without the use of the explicit form of the solutions of the system of Eqs. (4)–(6) of a specific linear algebra of two-centred integrals. It consists of a sum of two independent subalgebras: one—for the radial integrals, containing polynomials over ξ, $\sqrt{\xi^2-1}$ and $\frac{\partial}{\partial \xi}$, and the other—for the angular integrals, containing polynomials over η, $\sqrt{1-\eta^2}$ and $\frac{\partial}{\partial \eta}$. But in specific quantum-mechanical calculations of the energies and wave functions of various states of three-quark systems the calculations of two-centred integrals, containing the derivatives over R, are required. The standard way, resulting in the construction of the algebra of such kind of integrals, consists in the extension of the semiordinary group $P(3) \otimes P(2,1)$ to the ordinary one $P(5,1)$, being realized by the motions of the six-dimensional coordinate space y_ν with the metric

$$y_\nu y_\nu = y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 - y_6^2. \tag{43}$$

Having complemented the set of generators (17), (18) by nine more generators

$$\begin{aligned} &\pounds_{j4} = -i\left(y_j \frac{\partial}{\partial y_4} - y_4 \frac{\partial}{\partial y_j}\right), \qquad j = 1,2,3, \\ &\pounds_{j5} = -i\left(y_j \frac{\partial}{\partial y_5} - y_5 \frac{\partial}{\partial y_j}\right), \quad \pounds_{j6} = -i\left(y_j \frac{\partial}{\partial y_6} + y_6 \frac{\partial}{\partial y_j}\right), \end{aligned} \tag{44}$$

we proceed to the x-representation and choose the set of diagonal commutating operators, corresponding to the set (21)–(25). Additional diagonal operators, arising in the group $P(5,1)$ due to the degeneracy of the chosen representation do not result in any new relations. In the coordinate system (28) we obtain the same equations (40), (41) which are reduced to the problem (4)–(6) and whose solutions in the case of the group $P(5,1)$ will be realized on the cone

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - x_6^2 = 0. \tag{45}$$

In this case the generators (44) in the coordinate representation in the coordinate system (28) are given by

$$\begin{aligned}
\pounds_{14} &= -i\sqrt{(\xi^2-1)(1-\eta^2)}\Bigg[\cos\alpha\cos\beta\left(\xi\frac{\partial}{\partial\xi}+\eta\frac{\partial}{\partial\eta}\right)-\frac{\cos\alpha\sin\beta}{\xi^2-1}\frac{\partial}{\partial\beta}+\\
&\quad+\frac{\cos\beta\sin\alpha}{1-\eta^2}\frac{\partial}{\partial\alpha}-R\cos\alpha\cos\beta\frac{\partial}{\partial R}\Bigg],\\
\pounds_{24} &= -i\sqrt{(\xi^2-1)(1-\eta^2)}\Bigg[\sin\alpha\cos\beta\left(\xi\frac{\partial}{\partial\xi}+\eta\frac{\partial}{\partial\eta}\right)-\frac{\sin\alpha\sin\beta}{\xi^2-1}\frac{\partial}{\partial\beta}-\\
&\quad-\frac{\cos\beta\cos\alpha}{1-\eta^2}\frac{\partial}{\partial\alpha}-R\sin\alpha\cos\beta\frac{\partial}{\partial R}\Bigg],\\
\pounds_{34} &= i\sqrt{\xi^2-1}\Bigg[\cos\beta\left(\xi\eta\frac{\partial}{\partial\xi}-\left(1-\eta^2\right)\frac{\partial}{\partial\eta}\right)-\frac{\eta\sin\beta}{\xi^2-1}\frac{\partial}{\partial\beta}-R\eta\cos\beta\frac{\partial}{\partial R}+\\
&\quad+\frac{\sin\alpha\sin\beta}{1-\eta^2}\frac{\partial}{\partial\alpha}-R\cos\alpha\sin\beta\frac{\partial}{\partial R}\Bigg],\\
\pounds_{25} &= -i\sqrt{(\xi^2-1)(1-\eta^2)}\Bigg[\sin\alpha\sin\beta\left(\xi\frac{\partial}{\partial\xi}+\eta\frac{\partial}{\partial\eta}\right)+\frac{\sin\alpha\cos\beta}{\xi^2-1}\frac{\partial}{\partial\beta}-\\
&\quad-\frac{\cos\alpha\sin\beta}{1-\eta^2}\frac{\partial}{\partial\alpha}-R\sin\alpha\sin\beta\frac{\partial}{\partial R}\Bigg],\\
\pounds_{35} &= -i\sqrt{\xi^2-1}\Bigg[\sin\beta\left(\xi\eta\frac{\partial}{\partial\xi}-\left(1-\eta^2\right)\frac{\partial}{\partial\eta}\right)+\frac{\eta\cos\beta}{\xi^2-1}\frac{\partial}{\partial\beta}-R\eta\sin\beta\frac{\partial}{\partial R}\Bigg],\\
\pounds_{16} &= -i\sqrt{1-\eta^2}\Bigg[\cos\alpha\left(-\left(\xi^2-1\right)\frac{\partial}{\partial\xi}-\xi\eta\frac{\partial}{\partial\eta}\right)-\frac{\xi\sin\alpha}{1-\eta^2}\frac{\partial}{\partial\alpha}+R\xi\cos\alpha\frac{\partial}{\partial R}\Bigg],\\
\pounds_{26} &= -i\sqrt{1-\eta^2}\Bigg[\sin\alpha\left(-\left(\xi^2-1\right)\frac{\partial}{\partial\xi}-\xi\eta\frac{\partial}{\partial\eta}\right)+\frac{\xi\cos\alpha}{1-\eta^2}\frac{\partial}{\partial\alpha}+R\xi\sin\alpha\frac{\partial}{\partial R}\Bigg],\\
\pounds_{36} &= -i\Bigg[-\eta\left(\xi^2-1\right)\frac{\partial}{\partial\xi}+\xi\left(1-\eta^2\right)\frac{\partial}{\partial\eta}+\xi\eta R\frac{\partial}{\partial R}\Bigg].
\end{aligned} \tag{46}$$

Finally, consider the basis in the group $P(4,2)$—the group of motions of the six-dimensional coordinate space y_μ with a metric

$$y_\mu y_\mu = y_1^2+y_2^2+y_3^2-y_4^2-y_5^2+y_6^2. \tag{47}$$

We introduce the infinitesimal generators of this group

$$\begin{aligned}
x_j &= -i\tfrac{\partial}{\partial y_j}, & L_{jk} &= -i\left(y_j\tfrac{\partial}{\partial y_k}-y_k\tfrac{\partial}{\partial y_j}\right), \quad j,k=1,2,3,6,\\
L_{\mu k} &= -i\left(y_\mu\tfrac{\partial}{\partial y_k}+y_k\tfrac{\partial}{\partial y_\mu}\right), & L_{45} &= -i\left(y_4\tfrac{\partial}{\partial y_5}-y_5\tfrac{\partial}{\partial y_4}\right), \quad \mu=4,5
\end{aligned} \tag{48}$$

and proceed in the x-representation to a new coordinate system

$$\begin{aligned}
x_1 &= \tfrac{R}{\sqrt2}\sqrt{1-\eta^2}\cos\alpha\cos\gamma, & x_2 &= \tfrac{R}{\sqrt2}\sqrt{1-\eta^2}\sin\alpha\cos\gamma, & x_3 &= \tfrac{R}{\sqrt2}\eta\cos\gamma,\\
x_4 &= \tfrac{R}{\sqrt2}\sqrt{\xi^2-1}\cos\beta\sin\gamma, & x_5 &= \tfrac{R}{\sqrt2}\sqrt{\xi^2-1}\sin\beta\sin\gamma, & x_6 &= \tfrac{R}{\sqrt2}\xi\sin\gamma,
\end{aligned} \tag{49}$$

where $\alpha,\ \beta$ run from 0 to π, and γ—from 0 to $\frac{\pi}{2}$. By calculating the expressions for the generators (48) in the x-representation in the new coordinates (49), we finally obtain that L_{jk} $(j,k=1,2,3)$, $L_{56,}\ L_{46},\ L_{45}$ have the same form as $\pounds_{jk}$ $(j,k=1,2,3)$, $\pounds_{56,}\ \pounds_{46},\ \pounds_{45}$ in the $P\,(3)\otimes P\,(2,1)$ group in the coordinate system (28). Now following the above scheme of constructing the complete set of the commutating operators of Eq. (21)–(25)-type, we obtain at $\cos\gamma=\sin\gamma=\frac{1}{\sqrt{2}}$, $\frac{\partial}{\partial\gamma}=0$ a problem, completely equivalent to the $eZ_1Z_2\omega$ problem.

A similar consideration of wider groups, e.g. conformal groups of six-dimensional spaces (43), (47) or a group being a direct product of two conformal groups of spaces (15) and (16), results at the choice of the corresponding set of the commutating operators (of Eq. (21)–(25)-type) to a problem, equivalent to the $eZ_1Z_2\omega$ problem. The calculation of the matrix elements of the generators of these groups is reduced to the calculation of two-centred integrals, some of which contain the first and the second derivatives over R.

Without a detailed consideration of all aspects of the chosen representations of the mentioned groups note only that all these representations are non-canonical representations in group theory. For instance, for a group of three-dimensional rotations the non-canonical representations were first considered in [17] in the context of the quantum theory of asymmetric rotator.

Note also that the considered groups $P\,(3)\otimes P\,(2,1)$, $P\,(5,1)$, $P\,(4,2)$ are the groups of motions (translations and rotations) of the corresponding spaces, not the groups of rotations like in the case of the hydrogen-like atom groups.

4 Conclusions and final remarks

Summarizing the results of the work, we focus on its most important points. By means of the separation of variables method an additional spheroidal integral of motion $\widehat{\lambda}$ is constructed, whose eigenvalues are the separation constant in the model quantum-mechanical $eZ_1Z_2\omega$ problem. This has enabled the dynamic symmetry groups of this problem to be determined and the group properties of its solutions to be studied. $P\,(3)\otimes P\,(2,1)$, $P\,(5,1)$, and $P\,(4,2)$ groups are considered as such dynamic groups, among them $P\,(3)\otimes P\,(2,1)$ possessing the smallest number of parameters.

While searching for the eigenfunctions of the complete set of intercommutating operators in the $P\,(3)\otimes P\,(2,1)$ group in the case of degenerate unitary representations of this group a problem is shown to occur, quite equivalent to the quantum-mechanical $eZ_1Z_2\omega$ problem. In this case the energy operator $\widehat{E}$ is not the Casimir operator of this group and, accordingly, does not commutate with all generators of this group. The operator $\widehat{E}$ and the operator $\widehat{\lambda}$, corresponding to the “additional” operator of the separation constant λ_j, commutating with $\widehat{E}$, are included into the set of the diagonal operators, determining the non-canonical basis in the considered group. The space of the chosen representation of this dynamic group covers the whole spectrum of the energy values for the two-centre $eZ_1Z_2\omega$ problem.

The developed group treatment of the model $eZ_1Z_2\omega$ problem is related to the group treatment of the tradiational quantum-mechanical problem of two Coulomb centres eZ_1Z_2 [10, 11, 12, 13, 14, 15, 16]. But its consequence is a more rich linear algebra of two-centred integrals, which contains the corresponding linear algebra of the eZ_1Z_2 problem as a partial case (i. e. at $\omega=0$). A separate publication will be devoted to the construction of such algebra while here we only note that the presence of the both mentioned algebras enables and essentially simplifies the quantum-mechanical calculations of matrix elements and effective potentials in the three-body problem with Coulomb and oscillatory interactions [10]. In particular, the obtained results may appear useful at the calculations of the energy spectra of QQq-baryons and QQg-mesons. Note also that the model $eZ_1Z_2\omega$ problem can at certain conditions be treated as a step to the solution of a relativized Schroedinger equation [18] with a two-centred confinement-type potential (1).

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