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STATIONARY PARTICLE TRANSPORT IN A RANDOMLY INHOMOGENEOUS ROD

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The paper concerns the stationary solution of the one-speed transport equations of the rod model known in the theory of the neutron transport. The solution is presented in view of an arbitrary absorption cross section $\sigma(x)$ and of the scattering cross section equal to $c\sigma(x)$ where c is taken as a constant. The author treats the 'albedo problem' with a given value of the influx of particles upon the rod from the left and with no influx from the right. He recalls, at first, the exact 'albedo solution', accentuating one important statement: the transmitted fraction $\mathcal{T}$ and the reflected fraction $\mathcal{R}$ of the fluxes of the particles are functions of one single variable, namely of the 'optical length' $\Lambda = \int_0^L \mathrm{d}x\, \sigma(x)$ of the rod. (L is the geometrical length of the rod.) If $\sigma(x)$ is defined as a random function, then Λ and, consequently, $\mathcal{T}$, $\mathcal{R}$ become random variables. The statistical moments $\langle \mathcal{T}^n \rangle$ and $\langle \mathcal{R}^n \rangle$ are calculated for n = 1, 2. As $\mathcal{T}$ and $\mathcal{R}$ are nonlinear functionals of $\sigma(x)$, the random variables $\mathcal{T}$ and $\mathcal{R}$ need not be Gaussian even if $\sigma(x)$ is defined as a Gaussian random function. Conditions are discussed under which $\mathcal{T}$ and $\mathcal{R}$ may be approximated as two correlated Gaussian random variables.

1 Introduction

Recently a great deal of attention has been devoted to transport problems concerning inhomogeneous media. These problems become especially interesting if the media are treated as stochastic objects. For instance, many authors paid heed to the transport of particles in random binary mixtures (cf. e.g. [1], [2] and references quoted therein). In most investigations, the cross sections of the scattering processes inherent in these problems were assumed to fluctuate with a well defined statistics around a constant value. The transport theory in such a case encompasses applications in several fields of physics, including nuclear technology, astrophysics and climatology. We can mention, for instance, the radiative transfer through mixed regions in inertial confinement fusion pellets [3], transport of neutrons through burnable poison pellets in nuclear fuels [4], transport of

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neutral atoms in turbulent tokamak plasmas [5] or the penetration of the solar radiation through randomly distributed clouds [6].

There are, of course, various situations for which specific approximate kinetic equations may be employed. In details, these equations may differ remarkably from one another. Nevertheless, there are surely classes of problems for which a certain type of approximation appears to be preferable. In the present paper, we will solve a stochastic transport problem with two coupled linear kinetic equations. These equations, known as equations of the so-called rod model in the theory of the radiative transfer, were used productively in the literature of the last decade (cf. e.g. [7–10]). The rod-model equations describe the kinetics of an ensemble of identical non-interacting particles which are not subjected to any driving force. As a rule, these equations are related to neutral particles. The particles may impact upon a dense system of targets (such as nuclei of a given kind if the particles are neutrons) with well-defined cross sections. We assume that the density of the targets may vary randomly in space.

Section 3 of the present paper apparently involves results that are known to specialists in the field. The purpose of section 3 is only to prepare the basis for the new derivations presented in Section 4 and for the suggestions of some new ideas in Section 5.

2 Mathematical formulation of the problem

The transport problem of the present paper is defined by the equations

$$\frac{\mathrm{d}\phi^+(x)}{\mathrm{d}x} + \sigma(x)\,\phi^+(x) = \frac{c}{2}\,\sigma(x)\,\left[\phi^+(x) + \phi^-(x)\right]\,, \qquad (1^+)$$

$$-\frac{\mathrm{d}\phi^-(x)}{\mathrm{d}x} + \sigma(x)\,\phi^-(x) = \frac{c}{2}\,\sigma(x)\,\left[\phi^+(x) + \phi^-(x)\right]\,. \qquad (1^-)$$

Equations (1) are appropriate for a stationary situation since the time variable t is absent in them. This means that not only the parameter $c \geq 0$ and the function $\sigma(x) \geq 0$ are time-independent but we require, in addition, that $\partial\phi^+(x)/\partial t = \partial\phi^-(x)/\partial t = 0$, excluding thus possible relaxation processes from consideration. Equations (1), underlying the rod model, define a one-dimensional kinetic problem. We assume that $x \in (0, L)$, i.e., the axis of the rod in question coincides with the x-axis. Inside the interval $(0, L)$, we take $\sigma(x)$ as an arbitrary positive function. ($\sigma(x) \equiv 0$ outside the interval $(0, L)$.)

Equations (1) describe one-speed kinetics. Essentially, they can be interpreted as the Boltzmann kinetic equation

$$\frac{\partial\phi}{\partial t} + \dot{x}(t)\,\frac{\partial\phi}{\partial x} = \left(\frac{\partial\phi}{\partial t}\right)_{\mathrm{absorption}} + \left(\frac{\partial\phi}{\partial t}\right)_{\mathrm{generation}} \qquad (2)$$

in the stationary case when $\partial\phi/\partial t = 0$. In the rod geometry, we assume that $\dot{x}(t)$ may acquire two values only, $\dot{x}(t) = \pm\, v$. When hitting any target, the particle becomes absorbed. The rate of the absorption by targets positioned at x is $v\sigma(x)$. (When defining $\sigma(x)$, we may consider a unit perpendicular cross section of the rod.) Eventually, after each absorption event, a generation of secondaries may take place. We assume that the rate of this generation is $cv\sigma(x) > 0$ and that the x-component of the velocity of the secondaries may be, with the probability $\frac{1}{2}$ equal to v or, with the same probability, equal to $-\, v$.

Equations (1) involve two densities: $\phi^+(x) > 0$ and $\phi^-(x) > 0$ signify, respectively, the density of the particles moving from the left to the right and the particles moving from the right to the left. Accordingly, we may speak of two *states* of the particles: the state $(+)$ and the state $(-)$. Clearly, $v\phi^+(x)$ and $-v\phi^-(x)$ may be called the *fluxes*, corresponding to the particles in the state $(+)$ and in the state $(-)$. In accordance with what has been said above, $(\partial\phi^\pm/\partial t)_{\rm absorption} = -\,v\sigma(x)\phi^\pm$ and $(\partial\phi^\pm/\partial t)_{\rm generation} = \frac{1}{2}\,vc\sigma(x)[\phi^+ + \phi^-]$. Since equations (1) correspond to the stationary states, the factor v has been cancelled out. We assume, that $\mathrm{d}c/\mathrm{d}x = 0$.

Equations (1) are linear and we can easily manifest their exact solvability with any function $\sigma(x)$. Since they represent first-order equations, their solution is unique if two boundary conditions are given. We take the 'albedo-problem' conditions into account,

$$\phi^+(0) = \phi_0^+ \,, \tag{3}$$

$$\phi^-(L) = 0 \,, \tag{4}$$

assuming that $\phi_0^+ > 0$ corresponds to the known value $v\phi_0^+$ of the flux of the particles incident on the left-hand face (source face) of the rod. The values of $\phi^+(L)$ and $\phi^-(0)$ are correspondingly equal to the intensities of the forward and backward radiation of the rod. They are not given but have to be calculated.

Our first aim in the present paper is to show that the coefficient of the forward radiation

$$\mathcal{T} = \frac{\phi^+(L)}{\phi_0^+} \tag{5}$$

and the coefficient of the backward radiation

$$\mathcal{R} = \frac{\phi^-(0)}{\phi_0^+} \tag{6}$$

are simple functionals of the function $\sigma(x)$.

Our second aim is to derive some results for $\sigma(x)$ defined as a random function. Then, of course, $\phi^+(x)$ and $\phi^-(x)$, being functionals of $\sigma(x)$, must be considered as random functions and the end-point values $\phi^-(0)$ and $\phi^+(L)$ must be considered as random variables. We will assume, however, that ϕ_0^+ is a known deterministic value.

We will treat a model where $\sigma(x)$ is defined as a Gaussian random function. This model implies one formal nuisance, as it presumes the occurrence of negative values of $\sigma(x)$ for some mathematically (but not physically) possible realizations. Notwithstanding, this artefact is not important if the r.m.s. of $\sigma(x)$ is much smaller than the averaged value of $\sigma(x)$.

We will focus attention to the calculation of the average values $\langle\mathcal{T}\rangle$, $\langle\mathcal{R}\rangle$ and the variances $\langle\mathcal{T}^2\rangle - \langle\mathcal{T}\rangle^2$, $\langle\mathcal{R}^2\rangle - \langle\mathcal{R}\rangle^2$. We will also discuss the question of when $\mathcal{T}$ and $\mathcal{R}$ may be considered as Gaussian random variables, provided that $\sigma(x)$ is a Gaussian random function.

3 General solution of equations (1)

It is useful to introduce the 'optical-path' variable

$$\xi = \int_0^x \mathrm{d}x'\sigma(x') \,. \tag{7}$$

For $x = L$, we write

$$\Lambda = \int_0^L \mathrm{d}x' \sigma(x') \; . \tag{8}$$

Since $\sigma(x) > 0$, the variable ξ grows if x grows. We define the functions

$$\tilde{\phi}^+(\xi) = \phi^+(x) \; , \quad \tilde{\phi}^-(\xi) = \phi^-(x) \; . \tag{9}$$

Equations (1^+), (1^-) acquire the simple form

$$\frac{\mathrm{d}\tilde{\phi}^+(\xi)}{\mathrm{d}\xi} \; + \; \left(1 \; - \; \frac{c}{2}\right) \tilde{\phi}^+(\xi) = \frac{c}{2} \, \tilde{\phi}^-(\xi) \; , \tag{10^+}$$

$$\frac{\mathrm{d}\tilde{\phi}^-(\xi)}{\mathrm{d}\xi} \; - \; \left(1 \; - \; \frac{c}{2}\right) \tilde{\phi}^-(\xi) = - \, \frac{c}{2} \, \tilde{\phi}^+(\xi) \; . \tag{10^-}$$

We have to solve these equations with respect to the boundary conditions

$$\tilde{\phi}^+(0) = \phi_0^+ \; , \tag{11}$$

$$\tilde{\phi}^-(\Lambda) = 0 \tag{12}$$

(cf. Eqs. (3) and (4)). When defining the functions

$$G(\xi) = \tilde{\phi}^+(\xi) + \tilde{\phi}^-(\xi) \; , \tag{13}$$

$$g(\xi) = \tilde{\phi}^+(\xi) - \tilde{\phi}^-(\xi) \; , \tag{14}$$

we observe that equations (10^+), (10^-) are transformed into the equations

$$\frac{\mathrm{d}G(\xi)}{\mathrm{d}\xi} = - \, g(\xi) \; , \tag{15}$$

$$\frac{\mathrm{d}g(\xi)}{\mathrm{d}\xi} = - \, \gamma^2 \, G(\xi) \; , \tag{16}$$

where we have introduced the parameter

$$\gamma = \sqrt{1 - c} \, . \tag{17}$$

Equations (15) and (16) yield us the second-order equations

$$\frac{\mathrm{d}^2 G(\xi)}{\mathrm{d}\xi^2} = \gamma^2 \, G(\xi) \; , \qquad \frac{\mathrm{d}^2 g(\xi)}{\mathrm{d}\xi^2} = \gamma^2 \, g(\xi) \; . \tag{18}$$

General solutions of these equations are

$$G(\xi) = A \exp(\gamma\xi) + B \exp(- \, \gamma\xi) \; , \qquad g(\xi) = a \exp(\gamma\xi) + b \exp(- \, \gamma\xi) \; . \tag{19}$$

After inserting these functions into equation (15) (or into equation (16)), we obtain the relations

$$a = - \, \gamma A \; , \qquad b = \gamma B \; . \tag{20}$$

Employing equations (13), (14), we write

$$\tilde{\phi}^{+}(\xi) = \frac{1}{2}\left[G(\xi) + g(\xi)\right] , \tag{21}$$

$$\tilde{\phi}^{-}(\xi) = \frac{1}{2}\left[G(\xi) - g(\xi)\right] . \tag{22}$$

Equations (13), (14) and (20), give us two equations for A and B:

$$(1-\gamma)A + (1+\gamma)B = 2\phi_0^{+} \ ,$$

$$(1+\gamma)\exp(\gamma\Lambda)\ A \ + \ (1-\gamma)\exp(-\,\gamma\Lambda)\ B = 0 \ .$$

Their solution is

$$A = -\,2\phi_0^{+}\ \frac{(1-\gamma)\exp(-\,2\gamma\Lambda)}{(1+\gamma)^2}\ \left[1 - \left(\frac{1-\gamma}{1+\gamma}\right)^2 \exp(-\,2\gamma\Lambda)\right]^{-1} \tag{23}$$

and

$$B = 2\phi_0^{+}\ \frac{1}{1+\gamma}\ \left[1 - \left(\frac{1-\gamma}{1+\gamma}\right)^2 \exp(-\,2\gamma\Lambda)\right]^{-1} . \tag{24}$$

As is seen from these expressions, it is useful to introduce the parameter

$$\alpha = \frac{1-\gamma}{1+\gamma} \,. \tag{25}$$

When using equations (19), (20) and (21), we obtain the function

$$\tilde{\phi}^{+}(\xi) = \phi_0^{+}\ \frac{\exp(-\,\gamma\xi)\ -\ \alpha^2\exp[\gamma(\xi-2\Lambda)]}{1\ -\ \alpha^2\exp(-\,2\gamma\Lambda)} \ . \tag{26$^+$}$$

Similarly, we obtain also the function

$$\tilde{\phi}^{-}(\xi) = \phi_0^{+}\ \alpha\ \frac{\exp(-\,\gamma\xi)\ -\ \exp[\gamma(\xi-\ 2\Lambda)]}{1\ -\ \alpha^2\exp(-\,2\gamma\Lambda)} \ . \tag{26$^-$}$$

Then, according to formulae (5), (6) and (9), we have got the expressions

$$\mathcal{T}(\Lambda) = (1-\ \alpha^2)\ \frac{\exp(-\,\gamma\Lambda)}{1\ -\ \alpha^2\exp(-\,2\gamma\Lambda)} \tag{27}$$

and

$$\mathcal{R}(\Lambda) = \alpha\ \frac{1\ -\ \exp(-\,2\gamma\Lambda)}{1\ -\ \alpha^2\exp(-\,2\gamma\Lambda)} \ . \tag{28}$$

If the parameter c is fixed, then the coefficients $\mathcal{T}$ and $\mathcal{R}$ are functions of the sole variable Λ, i.e., of the ‘optical length’ of the rod. (We consider Λ as a dimensionless variable. The parameters γ and α do not depend on anything else than c, cf. formulae (17), (25).) In particular, if $\Lambda\ \to\ \infty$, we obtain the values

$$\mathcal{T}(\infty) = 0 \,, \tag{29}$$

$$\mathcal{R}(\infty) = \alpha = \frac{1 - \sqrt{1-c}}{1 + \sqrt{1-c}} \,. \tag{30}$$

On the other hand, if $\Lambda \to 0$ (i.e., if the rod becomes absent at all), then, as one could expect, $\mathcal{T}(0) = 1$ and $\mathcal{R}(0) = 0$.

Formulae (27) and (28) are valid for any deterministic function $\sigma(x)$, but they are also valid if $\sigma(x)$ is defined as a random function.

4 Statistics of the coefficients $\mathcal{T}$ and $\mathcal{R}$ if $\boldsymbol{\sigma(x)}$ is a Gaussian random function

If the cross section varies randomly along the x-axis, then Λ is a random variable. In what follows, we will assume that $\sigma(x)$ is a statistically homogeneous random function. This means that the average values of products $\sigma(x_1)\sigma(x_2)\ldots\sigma(x_m)$ ($m = 1, 2, \ldots$) do not depend on x if we replace all values of x_i by $x_i + x$ ($i = 1, 2, \ldots, m$).

In this section, we will exemplify $\sigma(x)$ by a Gaussian random function. From the probabilistic point of view, any Gaussian random function is defined completely if its mean and its autocorrelation function are defined. If σ_1 and σ_2 are two (possibly correlated) Gaussian random variables, then $\sigma_1 + \sigma_2$ is also a Gaussian random variable. Generalizing this statement, we may say that if $\sigma(x)$ is a Gaussian random function, then $\int_0^L \mathrm{d}x\sigma(x)$ is a Gaussian random variable. We define the autocorrelation function $\eta_\sigma^2 W_\sigma(x'' - x')$ of the random function $\sigma(x)$:

$$\eta_\sigma^2 W_\sigma(x'' - x') = \langle[\sigma(x') - \langle\sigma\rangle][\sigma(x'') - \langle\sigma\rangle]\rangle \,. \tag{31}$$

We assume that the function $W_\sigma(x)$ is positive for real values of x. Moreover, we require that

$$W_\sigma(0) = 1 \,, \tag{32}$$

$$W_\sigma(x) \;\to\; 0 \quad \text{if} \quad |x| \;\to\; \infty \,. \tag{33}$$

We prefer to demonstrate our calculations with the function

$$W_\sigma(x) = \exp\left(- \frac{|x|}{\lambda} \right) , \tag{34}$$

although any other reasonable autocorrelation function $W_\sigma(x)$ could be taken into consideration. Actually, realistic autocorrelation functions may resemble the function $\sim \exp(-|x|/\lambda)$, the function $\sim \exp(-x^2/\lambda^2)$ or another decreasing (for $x > 0$) function. The 'width' of the autocorrelation function – in our case λ – is the 'correlation length' of the cross section. On the other hand, $\eta_\sigma > 0$ means the r.m.s. deviation of the random values $\sigma(x)$. (In a sense, η_σ defines the 'vertical' randomness, whilst $\lambda > 0$ defines the 'horizontal' randomness of $\sigma(x)$.)

The mean value of Λ is

$$\langle\Lambda\rangle = \int_0^L \mathrm{d}x\langle\sigma(x)\rangle = \langle\sigma\rangle L \,. \tag{35}$$

The variance of Λ is

$$[\eta_\Lambda(L)]^2 = \langle[\Lambda - \langle\Lambda\rangle]^2\rangle = \int_0^L \int_0^L \mathrm{d}x'\mathrm{d}x''\langle[\sigma(x') - \langle\sigma\rangle]\langle[\sigma(x'') - \langle\sigma\rangle]\rangle$$

$$= \eta_\sigma^2 \int_0^L \int_0^L \mathrm{d}x' \mathrm{d}x'' W_\sigma(x'' - x') \, . \tag{36}$$

Hence, the probability density of the Gaussian variable Λ (for the given value of L) is

$$P_{\mathrm{optlength}}(\Lambda) = \frac{1}{\sqrt{2\pi}\, \eta_\Lambda(L)} \, \exp\left(- \, \frac{(\Lambda - \langle\sigma\rangle L)^2}{2[\eta_\Lambda(L)]^2} \right) . \tag{37}$$

In the special case when $W_\sigma(x)$ is given by formula (34), we can calculate $\eta_\Lambda(L)$ explicitly:

$$[\eta_\Lambda(L)]^2 =$$

$$\eta_\sigma^2 \int_0^L \int_0^L \mathrm{d}x' \mathrm{d}x'' \exp\left(- \, \frac{|x'' - x'|}{\lambda} \right) = 2\eta_\sigma^2 \int_0^L \mathrm{d}x \int_0^x \mathrm{d}x' \exp\left(- \, \frac{x - x'}{\lambda} \right)$$

$$= 2\eta_\sigma^2 \lambda \left\{ L \, - \, \lambda \left[1 \, - \, \exp\left(- \, \frac{L}{\lambda} \right) \right] \right\} . \tag{38}$$

We focus attention to the case when the correlation length of the fluctuations of $\sigma(x)$ is much shorter than the length of the rod, i.e., $\lambda \ll L$. Therefore, if we accept the approximation of $W_\sigma(x)$ by the exponential function (expression (34)), we obtain the simple result

$$[\eta_\Lambda(L)]^2 \;\approx\; 2\eta_\sigma^2 \lambda (L - \lambda) \;\approx\; 2\eta_\sigma^2 \lambda L \tag{39}$$

and then

$$P_{\mathrm{optlength}}(\Lambda) \;\approx\; \frac{1}{\sqrt{4\pi \eta_\sigma^2 \lambda L}} \, \exp\left(- \, \frac{(\Lambda - \langle\sigma\rangle L)^2}{4\eta_\sigma^2 \lambda L} \right) . \tag{40}$$

Now, let us discuss the conditions under which the Gaussian randomness of the cross section $\sigma(x)$ and of the ‘optical lenght’ Λ may be considered as a sound concept. Firstly, we have to assume that

$$\eta_\sigma \;\ll\; \langle\sigma\rangle \, . \tag{41}$$

Then, in calculating averaged values of functionals of $\sigma(x)$, we may tolerate the rare occurrence of negative values of σ in the Gaussian statistics which are forbidden physically. (Owing to the negligible weight of the non-physical negative values of σ, their influence upon the averaged values of the functionals of $\sigma(x)$ is negligible.) Secondly, we have equally to assume that

$$\eta_\Lambda \;\ll\; \langle\Lambda\rangle \, , \tag{42}$$

since Λ is also always positive. It is easy to see that condition (42) is satisfied very well if $\lambda \ll L$. Namely, condition (42) requires that $2\eta_\sigma^2\lambda \;\ll\; \langle\sigma\rangle^2 \, L$ and, indeed, $(\eta_\sigma/\langle\sigma\rangle)^2(\lambda/L) \;\ll\; 1$. Condition (42) says that when using the substitution

$$\Lambda = \langle\Lambda\rangle + \chi \, , \tag{43}$$

we may take χ as a small variable. Then we may develop $\mathcal{T}(\Lambda)$, $\mathcal{R}(\Lambda)$ into the Taylor series,

$$\mathcal{T}(\Lambda) = T_0 + T_1\, \chi + T_2\, \chi^2 + \ldots \, , \qquad \mathcal{R}(\Lambda) + R_0 + R_1\, \chi + R_2\, \chi^2 + \ldots$$

and, since the probability function $P(\langle\Lambda\rangle+\chi)$ is even in the variable χ, we may write

$$\langle\mathcal{T}\rangle = T_0 + T_2\langle\,\chi^2\rangle + \mathcal{O}(\langle\chi^4\rangle)\,, \qquad \langle\mathcal{R}\rangle = R_0 + R_2\langle\,\chi^2\rangle + \mathcal{O}(\langle\chi^4\rangle)\,,$$

$$\langle\mathcal{T}^2\rangle = T_0^2 + (T_1^2 + 2T_0T_2)\langle\chi^2\rangle + \mathcal{O}(\langle\chi^4\rangle)\,, \qquad \langle\mathcal{R}^2\rangle = R_0^2 + (R_1^2 + 2R_0R_2)\langle\chi^2\rangle + \mathcal{O}(\langle\chi^4\rangle)\,.$$

These equations give us the variances

$$[\eta_{\mathcal{T}}(L)]^2 = \langle\mathcal{T}^2\rangle \,-\, \langle\mathcal{T}\rangle^2 = T_1^2\,\langle\chi^2\rangle \,+\, \mathcal{O}(\langle\chi^4\rangle)\,, \tag{44}$$

$$[\eta_{\mathcal{R}}(L)]^2 = \langle\mathcal{R}^2\rangle \,-\, \langle\mathcal{R}\rangle^2 = R_1^2\,\langle\chi^2\rangle \,+\, \mathcal{O}(\langle\chi^4\rangle)\,. \tag{45}$$

Explicitly,

$$T_0 \;\equiv\; T_0(L) = \mathcal{T}(\langle\Lambda\rangle) = (1-\,\alpha^2)\,\frac{\exp(-\,\gamma\langle\sigma\rangle L)}{1\,-\,\alpha^2\exp(-\,2\gamma\langle\sigma\rangle L)}\,, \tag{46}$$

$$R_0 \;\equiv\; R_0(L) = \mathcal{R}(\langle\Lambda\rangle)) = \alpha\,\frac{1\,-\,\exp(-\,2\gamma\langle\sigma\rangle L)}{1\,-\,\alpha^2\exp(-\,2\gamma\langle\sigma\rangle L)}\,, \tag{47}$$

and

$$\begin{aligned} T_1 \;\equiv\; T_1(L) &= \left.\frac{\partial\mathcal{T}(\Lambda)}{\partial\Lambda}\right|_{\Lambda=\langle\Lambda\rangle} \\ &= -\,\gamma\,(1-\,\alpha^2)\,\frac{\exp(-\,\gamma\langle\sigma\rangle L)\,[1\,+\,\alpha^2\exp(-\,2\gamma\langle\sigma\rangle L)]}{[1\,-\,\alpha^2\exp(-\,2\gamma\langle\sigma\rangle L)]^2}\,, \end{aligned} \tag{48}$$

$$\begin{aligned} R_1 \;\equiv\; R_1(L) = \left.\frac{\partial\mathcal{R}(\Lambda)}{\partial\Lambda}\right|_{\Lambda=\langle\Lambda\rangle} &= -\,\frac{1-\,\alpha^2}{\alpha}\,\frac{\partial}{\partial\Lambda}\,\left.\frac{1}{1-\,\alpha^2\exp(-\,2\gamma\Lambda)}\right|_{\Lambda=\langle\Lambda\rangle} \\ &= 2\gamma\alpha\,(1-\,\alpha^2)\,\frac{\exp(-\,2\gamma\langle\sigma\rangle L)}{[1\,-\,\alpha^2\exp(-\,2\gamma\langle\sigma\rangle L)]^2}\,. \end{aligned} \tag{49}$$

In the course of calculating expression (49), we have transformed formula (28) into the form

$$\mathcal{R}(\Lambda) = \frac{1}{\alpha}\left(1\,-\,\frac{1-\,\alpha^2}{1\,-\,\alpha^2\exp(-\,2\gamma\Lambda)}\right)\,. \tag{28$'$}$$

If we neglect higher-order terms $\mathcal{O}(\chi)$ in the Taylor series for $\mathcal{T}(\Lambda)$ and $\mathcal{R}(\Lambda)$, writing approximately

$$\mathcal{T}(\Lambda) \;\approx\; T_0(L) + T_1(L)\,\chi\,, \qquad \mathcal{R}(\Lambda) = R_0(L) + R_1(L)\,\chi\,, \tag{50}$$

we define, in fact, $\mathcal{T}(\Lambda)$ and $\mathcal{R}(\Lambda)$ as two *Gaussian* random variables. When employing formula (37), or formula (40), we can easily calculate the variance $\langle\chi^2\rangle$:

$$\begin{aligned} &\langle\chi^2\rangle = \\ &\quad\int_{-\infty}^{\infty}\mathrm{d}\chi\,\chi^2\,P_{\mathrm{optlength}}(\langle\Lambda\rangle+\chi) = [\eta_\Lambda(L)]^2 \;\approx\; 2\eta_\sigma^2\lambda(L-\lambda) \;\approx\; 2\eta_\sigma^2\lambda L\,. \end{aligned} \tag{51}$$

(Cf. expression (39).)

If the length L of the rod is finite, the random variables $\mathcal{T}(\Lambda)$ and $\mathcal{R}(\Lambda)$ are correlated. Indeed, we may define the correlation matrix

$$\mathbf{C} = \begin{pmatrix} \langle(\mathcal{T} - \langle\mathcal{T}\rangle)^2\rangle & \langle(\mathcal{T} - \langle\mathcal{T}\rangle)(\mathcal{R} - \langle\mathcal{R}\rangle)\rangle \\ \langle(\mathcal{T} - \langle\mathcal{T}\rangle)(\mathcal{R} - \langle\mathcal{R}\rangle)\rangle & \langle(\mathcal{R} - \langle\mathcal{R}\rangle)^2\rangle \end{pmatrix} \tag{52}$$

with the components

$$C_{\mathcal{T}\mathcal{T}} = \langle(\mathcal{T} - \langle\mathcal{T}\rangle)^2\rangle \approx [T_1(L)]^2 \langle\chi^2\rangle, \tag{52.1}$$

$$C_{\mathcal{R}\mathcal{R}} = \langle(\mathcal{R} - \langle\mathcal{R}\rangle)^2\rangle \approx [R_1(L)]^2 \langle\chi^2\rangle \tag{52.2}$$

and

$$C_{\mathcal{T}\mathcal{R}} = C_{\mathcal{R}\mathcal{T}} = \langle(\mathcal{T} - \langle\mathcal{T}\rangle)(\mathcal{R} - \langle\mathcal{R}\rangle)\rangle \approx T_1(L)R_1(L) \langle\chi^2\rangle . \tag{52.3}$$

If L is finite, the value of $C_{\mathcal{T}\mathcal{R}}$ is not equal to zero. Only if $L \to \infty$, then $C_{\mathcal{T}\mathcal{R}} \to 0$, i.e., the random variable $\mathcal{T}(L)$ becomes non-correlated with the random variable $\mathcal{R}(L)$.

The dual probability density for the pair of the Gaussian random variables $\mathcal{T}$, $\mathcal{R}$ reads:

$$P_{\text{radiation}}(\mathcal{T}, \mathcal{R}) = \frac{1}{2\pi\,(C_{\mathcal{T}\mathcal{T}}C_{\mathcal{R}\mathcal{R}} - C_{\mathcal{T}\mathcal{R}}^2)^{1/2}} \times$$

$$\exp\left(- \frac{C_{\mathcal{R}\mathcal{R}}(\mathcal{T} - \langle\mathcal{T}\rangle)^2 \; - \; 2C_{\mathcal{T}\mathcal{R}}((\mathcal{T} - \langle\mathcal{T}\rangle)((\mathcal{R} - \langle\mathcal{R}\rangle) \; + \; C_{\mathcal{T}\mathcal{T}}(\mathcal{R} - \langle\mathcal{R}\rangle)^2}{2\,(C_{\mathcal{T}\mathcal{T}}C_{\mathcal{R}\mathcal{R}} - C_{\mathcal{T}\mathcal{R}}^2)} \right) . \tag{53}$$

Clearly, the probability densities of the variables $\mathcal{T}$ and $\mathcal{R}$, if they are taken separately, read:

$$P_{\text{transmission}}(\mathcal{T}) =$$

$$\int_{-\infty}^{\infty} \mathrm{d}\mathcal{R}\; P_{\text{radiation}}(\mathcal{T}, \mathcal{R}) = \frac{1}{(2\pi\; C_{\mathcal{T}\mathcal{T}})^{1/2}} \; \exp\left(- \frac{(\mathcal{T} - \langle\mathcal{T}\rangle)^2}{2C_{\mathcal{T}\mathcal{T}}} \right), \tag{54}$$

$$P_{\text{reflection}}(\mathcal{R}) =$$

$$\int_{-\infty}^{\infty} \mathrm{d}\mathcal{T}\; P_{\text{radiation}}(\mathcal{T}, \mathcal{R}) = \frac{1}{(2\pi\; C_{\mathcal{R}\mathcal{R}})^{1/2}} \; \exp\left(- \frac{(\mathcal{R} - \langle\mathcal{R}\rangle)^2}{2C_{\mathcal{R}\mathcal{R}}} \right). \tag{55}$$

(We have formally allowed negative values of $\mathcal{T}$ and $\mathcal{R}$ in the integrals of formulae (54) and (55) and we have done it intentionally, being aware of the negligible weight of such values in the Gaussian distribution represented by the function $P_{\text{radiation}}(\mathcal{T}, \mathcal{R})$.)

5 Concluding remarks

In the present paper, we have derived some explicit results ensuing from the linear stationary one-speed transport equations of the 'rod model' with a stochastically defined cross section $\sigma(x)$. Although the rod model evidently oversimplifies the theory of the radiative transfer in some points, it was treated by many authors (cf. quotations in the Introduction). The one-dimensionality of the rod model is, of course, disputable from the point of view of physics but, on the other hand, the drawbacks of this model are often felt to be counterbalanced by its mathematical simplicity, for it enables to derive analytical results. If the rod model is accepted, then, in the one-speed formulation of the kinetic equations for a system of non-interacting identical particles (such as the neutrons), we simply define two densities, $\phi^+(x)$ and $\phi^-(x)$, the first for the particles moving with a given velocity from the left to the right and the second for the particles moving with the same velocity from the right to the left.

In the present paper, we have defined the cross section $\sigma(x)$ as a statistically homogeneous *Gaussian* random function. In contrast to Prinja and Gonzales-Aller [8], who had also treated a Gaussian randomness of $\sigma(x)$, we have not not put emphasis on a derivation of an equation for the averaged densities $\langle\phi^+(x)\rangle$ and $\langle\phi^-(x)\rangle$. Namely, this would rather be superfluous in our problem since, as we have shown for a rod of length L, the averaged values $\langle\mathcal{T}\rangle$ and $\langle\mathcal{R}\rangle$ (as well as $\langle\mathcal{T}^2\rangle$ and $\langle\mathcal{R}^2\rangle$) of the fractions of the particle flux emitted, respectively, by the far face of the rod forwards and by the source face of the rod backwards, can be calculated *directly*, without any prerequisity of formulating an equation for the averaged functions $\langle\phi^+(x)\rangle$ and $\langle\phi^-(x)\rangle$. This, in fact, means that we have circumvented any 'closure problem'.

In principle, the manner of how we have derived such quantities as the statistical moments $\langle\mathcal{T}^n\rangle$ and $\langle\mathcal{R}^n\rangle$ (of the transmitted and reflected fraction of the particles, $\mathcal{T}$ and $\mathcal{R}$, respectively) can be applied even if the random function $\sigma(x)$ is non-Gaussian. For instance, we could define the randomness of $\sigma(x)$ in a Poissonian way, assuming, in the case of a heterogeneous rod with two media alternating randomly along its axis, that the values of $\sigma(x)$ may fluctuate between two values, $\sigma_1 > 0$ and $\sigma_2 > 0$. (Mathematically, we then consider intervals whose lengths are random, defined with Poissonian probabilities.)

Undoubtedly, probabilistic methods in the theory of the radiative transfer are very up to date [11]. In more advanced theories of the radiative transfer where the scattering cosines μ may span continuously the whole interval $[-1, 1]$ (not necessarily being equal only to the margin values $\mu = \pm 1$ as in the rod model) and where $\sigma(\mathbf{r})$ may be a random function in space, the direct way that we have exercised in calculating the means $\langle\mathcal{T}^n\rangle$, $\langle\mathcal{R}^n\rangle$ is generally applicable under one crucial premise: we have to be able to express $\mathcal{T}$ and $\mathcal{R}$, before their averaging, as functionals of the cross section $\sigma(\mathbf{r})$. At present, we dare not say for which other equations of the radiative transfer, excepting the equations of the rod model, an analytical derivation of such functionals is viable. For some newly revisited one-speed transport equations consult, e.g., the recent paper by Williams [12].

In the rod-model theory, we can, of course, readily use many various empirical probability-density functions $P^{\mathrm{empiric}}_{\mathrm{optlength}}(\Lambda)$ of the random optical length Λ of the rod. We can, for instance, choose $P^{\mathrm{empiric}}_{\mathrm{optlength}}(\Lambda)$ in the form of the log-normal or of the Cauchy (Lorentz) probability density, or of other known probability densities (cf. e.g. [13]), with the possibility to define these probability densities in an asymmetric way respecting the positiveness of the random values of Λ from the very outset. Then conditions like inequalities (41) and (42) need not be stipulated at all.

A fundamental question arises, however, if a priori one does not define the probability density for Λ but rather the random function $\sigma(x)$. A general – possibly non-Gaussian – random function $\sigma(x)$ is defined completely by its *characteristic functional* [14, 15]

$$\mathcal{Z}(\{\sigma(x)\};\{u(x)\}) = \left\langle \exp[\mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}x\, u(x)\sigma(x)] \right\rangle . \tag{56}$$

Here $u(x)$ is an arbitrary real function which may be subjected to variations. The angular brackets, as above, mean the averaging with respect to the randomness of $\sigma(x)$. If functional (56) is *prescribed*, then *all* moment functions of $\sigma(x)$ can be calculated by the functional differentiation:

$$\langle \sigma(x_1)\sigma(x_2)\ldots\sigma(x_m)\rangle = \left(\frac{1}{\mathrm{i}}\right)^m \left.\frac{\delta^m \mathcal{Z}(\{\sigma(x)\};\{u(x)\})}{\delta u(x_1)\delta u(x_2)\ldots\delta u(x_m)}\right|_{u(x)\,\equiv\,0} \tag{57}$$

($m = 1, 2, \ldots$). Now, recalling the definition of Λ by formula (8), we can write the mth statistical moment of Λ as the functional

$$\langle \Lambda^m\rangle = \int_0^L \mathrm{d}x_1 \int_0^L \mathrm{d}x_2 \ldots \int_0^L \mathrm{d}x_m\, \langle\sigma(x_1)\sigma(x_2)\ldots\sigma(x_m)\rangle$$

$$= \left(\frac{1}{\mathrm{i}}\right)^m \int_0^L \mathrm{d}x_1 \int_0^L \mathrm{d}x_2 \ldots \int_0^L \mathrm{d}x_m \left.\frac{\delta^m \mathcal{Z}(\{\sigma(x)\};\{u(x)\})}{\delta u(x_1)\delta u(x_2)\ldots\delta u(x_m)}\right|_{u(x)\,\equiv\,0} . \tag{58}$$

When the moments $\langle\Lambda^m\rangle$ are at hand, we can readily construct cumulants of the random variable Λ and, as the probability density $P_{\mathrm{optlength}}(\Lambda)$ is a simple series of the cumulants [14], we may state that the functional $\mathcal{Z}(\{\sigma(x)\};\{u(x)\})$ defines the function $P_{\mathrm{optlength}}(\Lambda)$ uniquely.

In the case when $\sigma(x)$ is a Gaussian random function, everything becomes simplified since only two cumulants (namely for $m = 1$ and for $m = 2$) do not vanish. That is why our analysis in section 4 was relatively simple.

In the non-Gaussian case, we have offered actually two ways which may be followed in investigating the rod model. One way may be characterized as students' way or engineers' way: it is tempting to guess whether various probability densities $P^{\mathrm{empiric}}_{\mathrm{optlength}}(\Lambda)$ are adequate enough or not. The second way, which we more recommend to follow, is based on the ideas leading to formula (57). This is a senior probabilist's way: the probability density $P_{\mathrm{optlength}}(\Lambda)$ is not to be guessed but has to be calculated from properties of the random function $\sigma(x)$ provided that these properties have been uniquely defined in advance.

Finally, let us point out that we have only dealt with the stationary solution of the transport equations of the rod model. If these equations are reformulated to non-stationary ones, new problems arise. We hope to treat them in the future.

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