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# QUANTUM MECHANICS ON NON-COMMUTATIVE PLANE

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One of the simplest example of non-commutative (NC) spaces is the NC plane. In this article we investigate the consequences of the non-commutativity to the quantum mechanics on a plane. We derive corrections to the standard (commutative) Hamiltonian spectrum for hydrogen-like atom and isotropic linear harmonic oscillator (LHO) and formulate the problem of the potential scattering on the NC plane. In the case of LHO we consider the non-commutativity of the momentum operators, too.

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## 1 Introduction

In recent years, the method of non-commutative geometry (NCG) was developed [1] and applied to various physical situations [2]. By the results of string theory arguments [3] the non-commutative plane has been studied extensively.

Two dimensional non-commutative quantum mechanics (NCQM) is based on a simple modification of commutation relations between the self-adjoint position operators ($\hat{\vec{x}}$) and the self-adjoint momentum operators ($\hat{\vec{p}}$) which satisfy

$$\begin{array}{rcl} [\hat{x}_a, \hat{x}_b] &=& i\theta_{ab}\,, \\ [\hat{p}_a, \hat{p}_b] &=& 0\,, \qquad\qquad a, b, \ldots \in \{1, 2\}\,, \\ [\hat{x}_a, \hat{p}_b] &=& i\hbar\delta_{ab}\,, \end{array} \tag{1}$$

where $\theta_{ab}$ is real and antisymmetric, i.e., $\theta_{ab} = \theta\epsilon_{ab}$ ($\epsilon_{ab}$ is the completely antisymmetric tensor with $\epsilon_{12} = 1$). The spatial non-commutative parameter $\theta$ is of dimension of $(length)^2$, so $\sqrt{\theta}$ may be considered as the fundamental length (Planck length?). If $\theta$ goes to zero we obtain the standard Heisenberg algebra commutation relations.

Suitable realization of commutation relations between the position operators (the first of eqs. (1)) is given by the $\star$-product (Moyal product) [4] defined as follows

$$(f \star g)(x) = \exp\left[\frac{i}{2}\theta_{ab}\partial_{x_a}\partial_{y_b}\right] f(x)g(y)|_{x=y}\,. \tag{2}$$

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The wave functions are taken as $\psi(\vec{x})$ and the operators $\hat{\vec{x}}$ and $\hat{\vec{p}}$ are realized as follows

$$\hat{x}_a\psi(\vec{x}) = x_a \star \psi(\vec{x})\,, \qquad \hat{p}_a\psi(\vec{x}) = -i\hbar\partial_a\psi(\vec{x})\,.$$

Let us remark that $|\psi(\vec{x})|^2$ can not be interpreted as the probability density to find the system in the configuration $(x_1, x_2)$. That follows from the first relation in the eq. (1).

The quantization procedure is the same as in the standard quantum mechanics. We replace the classical observables—the functions on the phase space—$A(\vec{p},\vec{x})$ by the self-adjoint operators $\hat{A} = A(\hat{\vec{p}},\hat{\vec{x}})$ which act on suitable Hilbert space. The ordering problem is like that in the standard quantum theory. The Hilbert space can be conistently taken to be the same as the Hilbert space of the corresponding commuting system, for example $L^2(\mathbb{R}^2)$: squared integrable functions on the plane with the standard Lebesgue measure. The time evolution is given by the Schrödinger equation

$$i\hbar\partial_t|\psi\rangle = \hat{H}|\psi\rangle\,,$$

where $\hat{H} = H(\hat{\vec{p}},\hat{\vec{x}})$ is the Hamiltonian. The only nontrivial part of such a formulation is to give the Hamiltonian. In what follows we shall consider two dimensional hydrogen-like atom, isotropic linear harmonic oscillator and potential scattering.

## 2 Hydrogen-like atom

Two dimensional hydrogen-like atom in NCQM is defined by the following Hamiltonian (the Einsten's summation convention in latin indeces is used).

$$\hat{H} = \frac{1}{2}\hat{p}_a\hat{p}_a + U_0\ln\left(\frac{\sqrt{\hat{x}_a\hat{x}_a}}{r_0}\right) \equiv \frac{1}{2}\hat{p}_a\hat{p}_a + U(\hat{\vec{x}})\,, \tag{3}$$

where $U_0$ and $r_0$ are the positive constants. We define the new operators $\tilde{x}_a$, $\tilde{p}_a$

$$\begin{array}{rcl} \tilde{x}_a & = & \hat{x}_a + \frac{1}{2\hbar}\theta_{ab}\hat{p}_b\,, \\ \tilde{p}_a & = & \hat{p}_a\,, \end{array} \tag{4}$$

which satisfy the usual canonical commutation relations

$$\begin{array}{rcl} [\tilde{x}_a,\tilde{x}_b] & = & 0\,, \\ [\tilde{p}_a,\tilde{p}_b] & = & 0\,, \\ [\tilde{x}_a,\tilde{p}_b] & = & i\hbar\delta_{ab}\,. \end{array} \tag{5}$$

The representation of the above commutation relations in the $L^2(\mathbb{R}^2)$ is well known:

$$(\tilde{x}_a f)(\vec{x}) = x_a f(\vec{x})\,, \qquad (\tilde{p}_a f)(\vec{x}) = -i\hbar\partial_a f(\vec{x})\,. \tag{6}$$

If we replace hat operators by tilde operators in the Hamiltonian (3) (to simplify calculations we shall use the system of units in which the mass of electron and Planck constant are equal to one) and expand the potential to the Taylor series, we obtain the first $\theta$-order time independent Schrödinger equation in polar coordinates $(r,\phi)$

$$\Delta\psi = 2\left[U_0\ln\left(\frac{r}{r_0}\right) - \epsilon + \theta\frac{iU_0}{2r^2}\partial_\phi\right]\psi\,, \tag{7}$$

where $\epsilon$ is the Hamiltonian's eigenvalue. We note that $-i\partial_\phi$ is the $z$ component ($L_z$) of the angular momentum operator. Standard separation of variables $\psi(r,\phi) = R(r)\exp(im\phi)$ leads to the radial Schrödinger equation

$$\frac{1}{r}(rR')' + \left[2(\epsilon - U(r)) - \frac{m^2 - mU_0\theta}{r^2}\right] R = 0\,, \tag{8}$$

where $m$ (orbital quantum number) is an arbitrary integer.
In the case of $\theta = 0$ we obtain radial equation for 2D hydrogen-like atom in commutative quantum mechanics. It is an important fact that the structure of equation (8) does not change. So we can state that if

$$\epsilon^C = \epsilon^C(n, m^2) \tag{9}$$

is the spectrum of Hamiltonian in the commutative case then in NC quantum mechanics the spectrum is given by

$$\epsilon(n, m^2) = \epsilon^C(n, m^2 - mU_0\theta)\,, \tag{10}$$

where $n$ is the principial quantum number. We have derived the approximative formulae for the commutative spectrum (see Appendix). The results are
(i) For the states with $|m| \gg n$, $n = 1, 2, \ldots$

$$\epsilon(n, m^2) = \frac{U_0}{2}\left\{1 + \ln\left[\frac{m^2 - mU_0\theta - 1/4}{r_0^2 U_0}\right] + \sqrt{2}\frac{n - 1/2}{\sqrt{m^2 - mU_0\theta - 1/4}}\right\}. \tag{11}$$

(ii) For high excited ($n \gg 1$) states with zero orbital momentum ($m = 0$)

$$\epsilon(n, m^2 = 0) = \frac{U_0}{2}\left\{\ln\left[\frac{2\pi}{U_0 r_0^2}\right] + 2\ln(n - 1/2)\right\}. \tag{12}$$

## Linear Stark effect

In the commutative theory the potential energy of an electron in an external electrostatic field oriented along the $x_1$ axis is given by $\delta\hat{H} = eEx_1$, where $e$ is the electric charge of the electron and $E$ is the intensity of the electric field. In the non-commutative theory it holds

$$\delta\hat{H} = eE\hat{x}_1 = eE(x_1 + i\frac{\theta}{2}\partial_{x_2})\,. \tag{13}$$

$\delta\hat{H}$ is considered as the small perturbation to the Hamiltonian (3). Linear Stark effect is the change in the hydrogen-like atom energy levels due to the perturbation shift of (13) computed within the first order of the perturbation theory. It contains two contributions, the commutative one: $\delta\epsilon^C_{nm} = \langle nm|eEx|nm\rangle$ and the noncommutative one: $\delta\epsilon^{NC}_{nm} = \langle nm|eEi\frac{\theta}{2}\partial_{x_2}|nm\rangle$. They both are equal to zero. It follows from the fact that the eigenstates $|nm\rangle$ of the unperturbated Hamiltonian (3) are not degenerated—this fact allows us to write the energetical changes as we have done above. Indeed, the states $|nm\rangle$ are of the form $f_{nm}(r)e^{im\phi}$, so we have

$$\delta\epsilon^C_{nm} \sim \int_{-\infty}^{+\infty} \mathrm{d}x_1 \int_{-\infty}^{+\infty} \mathrm{d}x_2 f^*_{nm} e^{-im\phi} x_1 f_{nm} e^{im\phi} = 0\,,$$

and

$$\delta\epsilon_{nm}^{NC} \sim \int_{-\infty}^{+\infty} \mathrm{d}x_1 \int_{-\infty}^{+\infty} \mathrm{d}x_2 f_{nm}^* e^{-im\phi} \left( \frac{x_2}{r} f_{nm}' e^{im\phi} + f_{nm} ime^{im\phi} \frac{x_1}{r^2} \right) = 0\,.$$

## 3 Linear harmonic oscillator

Now, let us consider a slight generalization of the commutation relations (1) between the canonically conjugated positions and momenta

$$\begin{aligned} [\hat{x}_a, \hat{x}_b] &= i\theta_{ab}\,, \\ [\hat{p}_a, \hat{p}_b] &= i\kappa_{ab}\,, \\ [\hat{x}_a, \hat{p}_b] &= i\hbar \left(1 + \tfrac{\theta\kappa}{4\hbar^2}\right) \delta_{ab}\,. \end{aligned} \tag{14}$$

The role of $\kappa_{ab} = \kappa\epsilon_{ab}$ is similar that of $\theta_{ab}$ with the difference between the two quantities that $\kappa$ is of dimension of $(momentum)^2$ and $\sqrt{\kappa}$ plays the role of fundamental momentum. We stress that the eq. (14) with $\theta_{ab} = 0$ is valid for a particle moving in a magnetic field in the standard quantum mechanics, but in this case the momenta are not conjugated to coordinates as it is the case here.

We would like to find some linear transformation $(\tilde{\vec{x}}, \tilde{\vec{p}}) \to (\hat{\vec{x}}, \hat{\vec{p}})$ so that tilde operators satisfy canonical commutation relations (5). In other words, we would like to transform the canonical form defined by (5) to the form defined by (14). It is possible only if $4\hbar^2 \neq \kappa\theta$, because in this case the form defined by (14) is singular. The transformation $(T)$ in question is of the form

$$\begin{aligned} \hat{x}_a &= \tilde{x}_a - \tfrac{\theta}{2\hbar}\epsilon_{ab}\tilde{p}_b\,, \\ \hat{p}_a &= \tilde{p}_a + \tfrac{\kappa}{2\hbar}\epsilon_{ab}\tilde{x}_b\,. \end{aligned} \tag{15}$$

Any other transformation $T'$ of the tilde operators to the hat operators can be written as $T' = T \circ U$, where $U$ is from the group of transformations which preserve the form defined by (5), i.e. $U \in Sp(2,\mathbb{R})$ (for more informations about the symplectic groups and their use in physics see for example [8]). The operator realization of tilde operators in $L^2(\mathbb{R}^2)$ is the standard one given by (6).

Let us consider the isotropic linear harmonic oscillator Hamiltonian in the NC quantum mechanics

$$\hat{H} = \frac{1}{2M}\hat{p}_a\hat{p}_a + \frac{M}{2}\omega^2\hat{x}_a\hat{x}_a\,. \tag{16}$$

We express the hat operators in the LHO Hamiltonian in terms of the tilde operators using the formulae (15). We get

$$\begin{aligned} \hat{H} &= \tilde{p}_a\tilde{p}_a \left( \frac{1}{2M} + \frac{1}{2}M\omega^2\frac{\theta^2}{4\hbar^2} \right) + \tilde{x}_a\tilde{x}_a \left( \frac{1}{2}M\omega^2 + \frac{1}{2M}\frac{\kappa^2}{4\hbar^2} \right) \\ &\quad + \tilde{p}_a\epsilon_{ab}\tilde{x}_b \left( \frac{1}{2M}\frac{\kappa}{\hbar} + \frac{1}{2}M\omega^2\frac{\theta}{\hbar} \right)\,. \end{aligned} \tag{17}$$

The last term in the Hamiltonian is proportional to the $z$ component of the angular momentum ($L_z = \epsilon_{ab}\tilde{x}_a\tilde{p}_p$). The spectrum $\epsilon_{n_1n_2}$ of this type of Hamiltonian is well known, so we shall write down the result

$$\begin{aligned}\epsilon_{n_1n_2} &= \sqrt{\hbar^2\omega^2\left(1+\frac{\kappa\theta}{4\hbar^2}\right)^2+\frac{1}{4}\left(\frac{\kappa}{M}-\theta M\omega^2\right)^2}(n_1+n_2+1) \\ &\quad -\frac{1}{2}\left(\frac{\kappa}{M}+\theta M\omega^2\right)^2(n_1-n_2)\,,\end{aligned} \tag{18}$$

where $n_1, n_2 = 0, 1, 2\ldots$
If $\kappa$ and $\theta$ aprroach zero we recover the standard spectrum for the two-dimensional LHO.

## 4 Potential scattering on NC-plane

The theory of the best-known quantum systems—linear harmonic oscillator and hydrogen-like atom is formulated in the two previous sections of this work. It is natural that the next step will be the formulation of the potential scattering on non-commutative plane. Our goal is to give the NC correction to the commutative cross-section.

In the first step we remind the computation of the differential cross-section in 2D commutative quantum mechanics. Let $V(\vec{r}),\ \vec{r}\in\mathbb{R}^2$ be the potential energy vanishing at infinity. We consider the particle of the mass $M$ incident (coming from the infinity) with the wave vector $\vec{k}$, say $\vec{k}=(k,0)$. We expect the out-state (at $r\to\infty$) will be of the form

$$\psi(\vec{r}) \sim e^{ikx} + f(\phi)\frac{e^{ikr}}{\sqrt{r}}\,, \tag{19}$$

where $r = |\vec{r}|$ and a function $f$ depends on a variable $\phi \in [0, 2\pi)$—the polar angle. The differential cross-section $\mathrm{d}\sigma/\mathrm{d}\phi$ is expressed in terms of $f$ as follows

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\phi} = |f(\phi)|^2\,. \tag{20}$$

We have to solve the following integral equation

$$\psi(\vec{r}) = e^{ikx} + \int \mathrm{d}^2\vec{r}'\,G_k(\vec{r},\vec{r}')U(\vec{r}')\psi(\vec{r}')\,, \tag{21}$$

where $U = \frac{2M}{\hbar^2}V$ and $G_k$ is the Green's function of the operator $\Delta^{(2)}+k^2$ obeying the boundary condition (19)[3]. The solution to the eq. (21) in the Born approximation is given by

$$\psi(\vec{r}) = e^{ikx} + \frac{e^{ikr}}{\sqrt{r}}\frac{e^{-i\pi/4}}{i\sqrt{8\pi k}}\int \mathrm{d}^2\vec{r}'\,e^{-i\vec{k}'.\vec{r}'}U(\vec{r}')e^{i\vec{k}.\vec{r}'}\,, \tag{22}$$

where $\vec{k}' = k\frac{\vec{r}}{r}$ and $\vec{k}=(k,0)$. We identify the function $f$ from the above equation and the eq. (19), and we get

$$f(\phi) = \frac{e^{-i\pi/4}}{i\sqrt{8\pi k}}\int \mathrm{d}^2\vec{r}'\,e^{-i\vec{k}'\vec{r}'}U(\vec{r}')e^{i\vec{k}\vec{r}'} = \frac{e^{-i\pi/4}}{i\sqrt{8\pi k}}\int \mathrm{d}^2\vec{r}'\,e^{i\vec{q}.\vec{r}'}U(\vec{r}')\,, \tag{23}$$

[3] $G$ is given by the formula $G(\vec{r},\vec{r}') = G(|\vec{r}-\vec{r}'|) = \frac{1}{4i}H_0^{(1)}(k|\vec{r}-\vec{r}'|)$, where $H_0^{(1)}$ is the Hankel's function.

where $\vec{q} = \vec{k} - \vec{k}'$. The right-hand side of the eq. (23) depends on the scattering angle $\phi$ via $q$, because $q = 2k\sin(\phi/2)$.

In the case of a radial symetric function $V$ we have

$$f(\phi) = \frac{e^{-i\pi/4}}{i\sqrt{8\pi k}} \int_0^\infty \mathrm{d}r' r' U(r') \int_0^{2\pi} \mathrm{d}\alpha e^{iqr'\cos(\alpha)} = \frac{2\pi e^{-i\pi/4}}{i\sqrt{8\pi k}} \int_0^\infty \mathrm{d}r' r' U(r') J_0(qr')\,, \tag{24}$$

where $J_0$ is the Bessel's function of the first kind[4].

The noncommutativity of the plane according to the eqs. (1) can be implemented to the scattering problem in the same way as it has been done in the introduction and the second section. We start from the Schrödinger equation $H\psi = E\psi$, where $H = \hat{p}_a\hat{p}_a/(2M) + V(\hat{\vec{r}}) = \tilde{p}_a\tilde{p}_a/(2M) + V(\hat{\vec{r}})$. Then we replace the hat operators in $V$ by the tilde operators according to the transformations (4) and we perform the expansion of the function $V$ into the Taylor series in powers of $\theta$. In the special case of the radial symetric potential $V$ we have

$$V(\sqrt{\hat{x}_a\hat{x}_a}) = V(r) - \frac{\theta}{2\hbar}\frac{1}{r}\frac{\mathrm{d}V(r)}{\mathrm{d}r}\epsilon_{ab}\tilde{x}_a\tilde{p}_b + O(\theta^2)\,. \tag{25}$$

So, we can state, that the first NC correction to the $f$ function is given by the following formula

$$f^{NC}(\phi) - f(\phi) = \frac{\theta}{2}\frac{e^{-i\pi/4}}{\sqrt{8\pi k}} \int \mathrm{d}^2\vec{r}' e^{i\vec{q}.\vec{r}'}\frac{1}{r'}\frac{\mathrm{d}U(r')}{\mathrm{d}r'}\epsilon_{ab}x'_a k_b\,. \tag{26}$$

After some rearrargements we get the final formula

$$f^{NC}(\phi) - f(\phi) = -\frac{\theta}{2}\frac{i\pi e^{-i\pi/4}}{\sqrt{8\pi}}\frac{1}{\sqrt{k}}\cot(\frac{\phi}{2}) \int_0^\infty \mathrm{d}r' r' \frac{\mathrm{d}U(r')}{\mathrm{d}r'}\frac{\mathrm{d}J_0(qr')}{\mathrm{d}r'}\,. \tag{27}$$

Note that for the potential proportional to $\ln(r)$ the scattering states do not exist, so there is no analogue of the Rutherford formula within the 2D quantum mechanics. But one can investigate the scattering of the charged particle (electron) on the neutral atom and obtain the NC correrction.

## 5 Conclusion

In this paper we have presented the results on the 2D systems within non-commutative quantum mechanics for the hydrogen-like atom, the isotropic LHO and the potential scattering problem. We have obtained the corrections depending on the parameter of the space noncommutativity to the classical (commutative) spectra of related Hamiltonians and cross-section. We note that 2D hydrogen-like atom has the analogy in 3D-motion of a charged particle around the homogenous charged straight line. Some others interesting results in NC quantum theory can be found in [6,9].

[4] We shall use this Bessel's function in what follows, so we remind that for any complex $x$ the value of $J_0(x)$ is

$$J_0(x) = \sum_{k=0}^\infty \frac{(-1)^k}{(k!)^2}\left(\frac{x}{2}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\alpha e^{ix\cos(\alpha)}.$$

The main goal of the NC quantum mechanics is to find a measurable effect. Unfortunately, we cannot hope that our corrections to the energy levels of the systems in question could be experimentally verified because of its dimensionality (LHO) and experimental complications with the realization of the 3D analogy of the hydrogen-like atom. In [6] the modification of the energy levels and Lamb shift for 3D hydrogen-like atom due to the presence of the non-commutative plane in $\mathbb{R}^3$ were presented as "measurable". But, this type of space non-commutativity is not applicable if we insist on the isotropy of the space. In spite of this it would be desirable to use 3D space non-commutativity which preserves rotational symmetry of the hydrogen-like atom problem.

The starting point of our treatment to the NC quantum mechanics are the nontrivial commutation relations between the position operators. The violation of the space parity is explicitly shown in the formulae (11), (18) and (27). It is well-known that the parity is violated if one considers the external magnetic field as well as that the CP is preserved in this case. In our two noncommutative cases (commutative/noncommutative momenta) the space parity as well as the CP are violated. The reason of the CP violation is that $\theta$ (and $\kappa$ too) are not related to the electric charge as the magnetic field. In addition, we have found an analogue to the relativistic quantum mechanics in the (perturbative) formula (11): one obtains nonzero imaginary part of the energy for very large values of $U_0\theta$ and properly small $m$. This effect is well-known from the theories of Klein-Gordon and Dirac hydrogen-like atom.

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## Appendix

In this section we briefly describe how to find the formulae (11) and (12). Let us start with the equation (8), whose structure does not change when we put $\theta = 0$ (the commutative case).

Let us first analyze the case of $m \neq 0$. If we introduce into eq. (8) a new function $\chi(r) = \sqrt{r}R(r)$, we get

$$\chi'' + [2\epsilon - 2U_{eff}]\,\chi = 0\,, \qquad \chi \in L^2(R^+)\,, \tag{28}$$

where the effective potential $U_{eff}$ is given by

$$U_{eff} = U + \frac{m^2 - 1/4}{2r^2}\,.$$

We expand $U_{eff}$ into the powers of $(r - r_k)$, where $r_k = [(m^2 - 1/4)/U_0]^{1/2}$ is the point at which $U_{eff}$ is minimal. Harmonic approximation gives

$$U^h_{eff}(r) = U_{eff}(r_k) + \frac{1}{2}\Omega^2(r - r_k)^2\,, \qquad \Omega^2 = 2\frac{U_0^2}{m^2 - 1/4}\,. \tag{29}$$

It can be easily shown that the harmonic approximation is valid only if the following inequality is fulfilled

$$\left|\frac{U_{eff} - U^h_{eff}}{U^h_{eff}}\right| \approx \frac{5}{3}\frac{(2n+1)^{1/2}}{(2m^2 - 1/2)^{1/4}} \ll 1\,, \tag{30}$$

where $n = 1, 2, \ldots$ is the principal quantum number. Fur the LHO with the effective potential (27) one obtains eq. (11) with $\theta = 2$.

In the second step we analyze the high excited states with zero angular momentum $m$. Substitution

$$v = \ln\left(\frac{r}{r_0}\right) - \frac{\epsilon}{U_0}\,, \qquad v \in R\,,$$

in (8) leads to

$$\hat{R}''(v) - \left[\Xi v e^{2v}\right]\hat{R}(v) = 0\,, \qquad \Xi(\epsilon) = 2U_0 r_0^2 \exp\left(\frac{2\epsilon}{U_0}\right)\,. \tag{31}$$

Now, we shall investigate the asymtotic solutions to the above boundary problem. In the case of $v \to -\infty$ the only acceptable solution is

$$\hat{R}(v)_- = const\,. \tag{32}$$

In the other extreme case of $v \to +\infty$ we have the WKB [5] solution

$$\hat{R}(v)_+ \sim V_0^{-1/4} \exp\left(-V_4^{1/2}\right)\,, \qquad V_0 = \Xi v e^{2v}\,. \tag{33}$$

It is important that the WKB limit is applicable for $v > v_m$, where $v_m$ is determined by the standard WKB condition

$$\frac{2}{2V_0}\,\left|\,\frac{\mathrm{d}(V_0)^{1/2}}{\mathrm{d}v}\,\right| \ll 1\,. \tag{34}$$

For high excited states with $\Xi \gg 1$ $v_m$ can be approximated by

$$|v_m| \sim \ln(\Xi) \qquad \text{and} \qquad v_m < 0\,. \tag{35}$$

The full WKB function $\hat{R}^{WKB}$ is then of the form

$$\hat{R}(v)^{WKB} = \begin{cases} C_0 p^{-1/2} \sin\left(\int_v^0 p'\mathrm{d}v' + \pi/4\right) & ; \quad v_m < v < 0\,, \\ \frac{1}{2} C_0 q^{-1/2} \exp\left(-\int_0^v q'\mathrm{d}v'\right) & ; \quad v > 0\,, \end{cases} \tag{36}$$

where $p = (-V_0)^{1/2}$ and $q = (V_0)^{1/2}$. The continuity condition of the wave function in $v_m$ leads to the following analogy of the Bohr-Sommerfeld quantization rule

$$\sqrt{\Xi}\int_{-\infty}^{0} |v|^{1/2} e^{v}\mathrm{d}v = \pi(n - 1/2)\,, \tag{37}$$

from which we, using (29), obtain instantly the spectrum formula (12).

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