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ABOUT γ^5 CHIRAL INTERACTIONS OF MASSIVE PARTICLES IN THE $(1/2,0)\oplus(0,1/2)$ REPRESENTATION

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We argue that self/anti-self charge conjugate states of the $(1/2,0)\oplus(0,1/2)$ representation possess axial charges. Furthermore, we analyze recent claims of the $\sim \boldsymbol{\sigma}\cdot[\mathbf{A}\times\mathbf{A}^*]$ interaction terms for "fermions". Finally, we briefly discuss the problem in the $(1,0)\oplus(0,1)$ representation.

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1 Introduction

The Dirac equation and the relevant theory of charged particles do *not* admit the γ^5 chiral transformation. The sign in the mass term in the Lagrangian is reversed under this type of transformations. In the mean time, the chiral transformations play significant role in our understanding of the nature of weak and strong interactions, in the problem of (un)existence of monopoles as well. Many attempts have earlier been done in order to understand the origin of the chiral (a)symmetry from the first principles, see, e. g., [1]. Recently, the authors of Ref. [2] proposed a very interesting model of the m-deformed *non-local* chiral transformations. But, they indicated at the importance of further study of chiral transformations and their relevance to the modern physics. These matters appear to be of use not only from the viewpoint of the construction of a fundamental theory for neutral particles (which is our primary purpose), but the consideration of constructs which *admit* the chiral invariance may also be useful for deeper understanding processes in QCD and other modern gauge models.

In the present article we prove that massive self/anti-self charge conjugate states in the $(1/2,0)\oplus(0,1/2)$ representation possess the axial charges (cf. also the McLennan-Case reformulation [3] of the Majorana theory [4], Refs. [5–9]). Furthermore, we present explicit examples which are relevant to the viewpoints of M. Markov [10], S. Weinberg [11] and P. A. M. Dirac [12] about the possibility of different equations for describing particles of some representation of the Lorentz group (particularly, of the $(1/2,0)\oplus(0,1/2)$ representation). In the most comprehensive and clear form it is expressed in the book, papers and lectures of S. Weinberg, e.g., [13]: "The kinematical classification of particles according to their Lorentz transformation

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properties is entirely (for finite mass) determined by their familiar representation of the rotation group. It has nothing whatever to do with the choice of one relativistic wave equation rather than another." See also [14, 15]. In our opinion, it is still required to take into account the issues related to the inversion group, namely, the theoretical possibility of unconventional representations of inversions.[2]

2 Chiral Interactions

We start from the observation that the Dirac field operator, which satisfies the Dirac equation[3]

$$[i\gamma^\mu \partial_\mu - m]\psi(x^\mu) = 0\,, \tag{1}$$

can be expanded in the following parts:

$$\psi(x^\mu) = \psi_\uparrow(x^\mu) + \psi_\downarrow(x^\mu)\ , \tag{2a}$$

$$\psi_\uparrow(x^\mu) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}\left[u_\uparrow(p^\mu)a_\uparrow(p^\mu)e^{-i\phi} + \mathcal{C}u^*_\uparrow(p^\mu)b^\dagger_\downarrow(p^\mu)e^{+i\phi}\right]\ , \tag{2b}$$

$$\psi_\downarrow(x^\mu) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}\left[u_\downarrow(p^\mu)a_\downarrow(p^\mu)e^{-i\phi} - \mathcal{C}u^*_\downarrow(p^\mu)b^\dagger_\uparrow(p^\mu)e^{+i\phi}\right]\ , \tag{2c}$$

where $\phi = (Et - \mathbf{x}\cdot\mathbf{p})/\hbar$. The charge-conjugate equation is

$$(i\gamma^\mu \partial_\mu - m)C\psi^\dagger(x^\mu) = 0\,; \tag{3}$$

and the counterparts of the "field operators" (2b,2c) are ($\vartheta_c = 0$)

$$\psi^c(x^\mu) = \psi^c_\uparrow(x^\mu) + \psi^c_\downarrow(x^\mu)\ , \tag{4a}$$

$$\psi^c_\uparrow(x^\mu) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}\left[u_\uparrow(p^\mu)b_\downarrow(p^\mu)e^{-i\phi} + \mathcal{C}u^*_\uparrow(p^\mu)a^\dagger_\uparrow(p^\mu)e^{+i\phi}\right]\ , \tag{4b}$$

$$\psi^c_\downarrow(x^\mu) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}\left[-u_\downarrow(p^\mu)b_\uparrow(p^\mu)e^{-i\phi} + \mathcal{C}u^*_\downarrow(p^\mu)a^\dagger_\downarrow(p^\mu)e^{+i\phi}\right]\ . \tag{4c}$$

[2] While this type of theories is usually called as Wigner- (or "BWW-") type, see [16, 17], the possibility of unconventional representations of inversions was first indicated by Soviet mathematical physicists [18], see also the relevant papers [19].

[3] As opposed to Ref. [3b] we use the conventional notation and metric. Namely, $g^{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$ and γ matrices are

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}, \gamma^5 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$$

in the Weyl representation. The Pauli charge-conjugation 4×4 matrix is then

$$C = e^{i\vartheta_c}\mathcal{C} = e^{i\vartheta_c}\begin{pmatrix} 0 & \Theta \\ -\Theta & 0 \end{pmatrix}, \text{ where } \Theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It has the properties

$$C = C^T\ , C^* = C^{-1}\,,$$
$$C^{-1}\gamma^\mu C = -{\gamma^\mu}^*\ , C^{-1}\gamma^5 C = -{\gamma^5}^*\ .$$

Both $\psi_\uparrow$, $\psi^c_\uparrow$ and $\psi_\downarrow$, $\psi^c_\downarrow$ can be used to form self/anti-self charge conjugate *field operators* in the coordinate representation after regarding corresponding superpositions. For instance,

$$\Psi^S = \frac{\psi_\uparrow + \psi^c_\uparrow}{2} = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}\left[u_\uparrow(p^\mu)\frac{a_\uparrow + b_\downarrow}{2}e^{-i\phi} + \mathcal{C}u^*_\uparrow(p^\mu)\frac{a^\dagger_\uparrow + b^\dagger_\downarrow}{2}e^{+i\phi}\right], \quad (5a)$$

$$\Psi^A = \frac{\psi_\downarrow - \psi^c_\downarrow}{2} = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}\left[u_\downarrow(p^\mu)\frac{a_\downarrow + b_\uparrow}{2}e^{-i\phi} - \mathcal{C}u^*_\downarrow(p^\mu)\frac{a^\dagger_\downarrow + b^\dagger_\uparrow}{2}e^{+i\phi}\right], \quad (5b)$$

$$\widetilde{\Psi}^S = \frac{\psi_\downarrow + \psi^c_\downarrow}{2} = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}\left[u_\downarrow(p^\mu)\frac{a_\downarrow - b_\uparrow}{2}e^{-i\phi} + \mathcal{C}u^*_\downarrow(p^\mu)\frac{a^\dagger_\downarrow - b^\dagger_\uparrow}{2}e^{+i\phi}\right], \quad (5c)$$

$$\widetilde{\Psi}^A = \frac{\psi_\uparrow - \psi^c_\uparrow}{2} = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}\left[u_\uparrow(p^\mu)\frac{a_\uparrow - b_\downarrow}{2}e^{-i\phi} - \mathcal{C}u^*_\uparrow(p^\mu)\frac{a^\dagger_\uparrow - b^\dagger_\downarrow}{2}e^{+i\phi}\right]. \quad (5d)$$

As opposed to K. M. Case [3b] we introduce the interaction with the 4-vector potential in the beginning and substitute $\partial_\mu \to \nabla_\mu = \partial_\mu - ieA_\mu$ in the equation (1). For the sake of generality we assume that the 4-vector potential is a *complex* field $A_\mu = C_\mu + iB_\mu$, what is the extension of this concept comparing with the usual quantum-field consideration.[4] Following the logics of Refs. [3, 5] (the separation of different *chirality* sub-spaces) we should consider additional equations for $\gamma^5\psi_{\uparrow\downarrow}$ and $\gamma^5\psi^c_{\uparrow\downarrow}$, i.e., the following set

$$[i\gamma^\mu(\partial_\mu - ieC_\mu + eB_\mu) - m]\,(\psi_\uparrow + \psi_\downarrow) = 0\,, \quad (6a)$$

$$[i\gamma^\mu(\partial_\mu + ieC_\mu + eB_\mu) - m]\,(\psi^c_\uparrow + \psi^c_\downarrow) = 0\,, \quad (6b)$$

$$[i\gamma^\mu(\partial_\mu - ieC_\mu + eB_\mu) + m]\,\gamma^5(\psi_\uparrow + \psi_\downarrow) = 0\,, \quad (6c)$$

$$[i\gamma^\mu(\partial_\mu + ieC_\mu + eB_\mu) + m]\,\gamma^5(\psi^c_\uparrow + \psi^c_\downarrow) = 0\,. \quad (6d)$$

Due to symmetries of the Dirac equations one can proceed in various ways. For instance, let us introduce the following linear combinations

$$\psi_1 = \psi^c_\uparrow - \gamma^5\psi_\downarrow\,, \quad \psi_2 = \psi_\downarrow + \gamma^5\psi^c_\uparrow\,, \quad (7a)$$

$$\psi_3 = \psi^c_\downarrow + \gamma^5\psi_\uparrow\,, \quad \psi_4 = \psi_\uparrow - \gamma^5\psi^c_\downarrow\,, \quad (7b)$$

which can be used to represent solutions of Eqs. (6a-6d). Then we proceed with simple algebraic transformations of the set (6a-6d) to obtain ($\widetilde{\nabla}_\mu \equiv \partial_\mu + eB_\mu$)

$$i\gamma^\mu\widetilde{\nabla}_\mu(\psi_1 - \gamma^5\psi_4) - e\gamma^\mu C_\mu(\gamma^5\psi_2 + \psi_3) - m(\gamma^5\psi_2 + \psi_3) = 0\,, \quad (8a)$$

$$i\gamma^\mu\widetilde{\nabla}_\mu(\gamma^5\psi_2 + \psi_3) - e\gamma^\mu C_\mu(\psi_1 - \gamma^5\psi_4) - m(\psi_1 - \gamma^5\psi_4) = 0\,. \quad (8b)$$

Other two equations are obtained after multiplying (8a,8b) by the γ^5 matrix. For the first sight one can conclude that we obtain different physical excitations (due to mathematically different dynamical equations with different interactions) depending on constraints which we impose on functions $\psi_{1,2,3,4}$. Let us impose $\widetilde{\Psi}^S = 0$ and $\widetilde{\Psi}^A = 0$, see Eqs. (5c,5d). They are considered here to be equivalent to either the constraints on the creation/annihilation operators[5] $a_\uparrow(p^\mu) =$

[4] In the modern textbooks on the classical/quantum field theory the 4-vector potential in the coordinate representation is usually the *real* function(al). We still note that different choices of a) relations between the left- and right- parts of the momentum-space bispinors; b) relations between creation and annihilation operators in the field operator; and c) metrics would induce ones to change this conclusion for interactions of various field configurations which ones consider.

[5] Of course, this is true only if one works with IR representations of the Wigner-Jordan (anti)commutation rules.

$b_\downarrow(p^\mu)$ and $a_\downarrow(p^\mu) = b_\uparrow(p^\mu)$ or the constraints $\psi_\uparrow = \psi_\uparrow^c \equiv \psi^s$ and $\psi_\downarrow = -\psi_\downarrow^c \equiv \psi^a$.[6] The functions $\psi_{1,2,3,4}$ become to be interrelated by the conditions

$$\psi_1 = \psi^s - \gamma^5\psi^a\,, \quad \psi_2 = \psi^a + \gamma^5\psi^s\,, \psi_3 \equiv \gamma^5\psi_1\,, \quad \psi_4 \equiv \gamma^5\psi_2\,. \tag{9}$$

It is the simple procedure to show that ψ_1 presents itself self-charge conjugate field and ψ_2, the anti-self charge conjugate field.[7] As the result one obtains

$$i\gamma^\mu D_\mu^\ast\psi_1 - m\gamma^5\psi_2 \quad = \quad 0\,, \tag{10a}$$
$$i\gamma^\mu D_\mu\psi_2 + m\gamma^5\psi_1 \quad = \quad 0\,, \tag{10b}$$

where the lengthening derivative is now defined

$$D_\mu = \partial_\mu - ie\gamma^5 C_\mu + eB_\mu\,.$$

Equations for the Dirac conjugated counterparts of $\psi_{1,2}$ read

$$i\partial_\mu\overline{\psi}_1\gamma^\mu - m\overline{\psi}_2\gamma^5 \quad = \quad 0\,, \tag{11a}$$
$$i\partial_\mu\overline{\psi}_2\gamma^\mu + m\overline{\psi}_1\gamma^5 \quad = \quad 0\,. \tag{11b}$$

One can propose the Lagrangian for free fields $\psi_{1,2}$ and their Dirac conjugates (cf. with the concept of the extra Dirac equations in Ref. [9d] and with the spin-1 case, Ref. [20]):[8]

$$\begin{aligned} L^{free} \quad &= \quad \frac{i}{2}\left[\overline{\psi}_1\gamma^\mu\partial_\mu\psi_1 - \partial_\mu\overline{\psi}_1\gamma^\mu\psi_1 + \overline{\psi}_2\gamma^\mu\partial_\mu\psi_2 - \partial_\mu\overline{\psi}_2\gamma^\mu\psi_2\right] + \\ &- \quad m\left[\overline{\psi}_1\gamma^5\psi_2 - \overline{\psi}_2\gamma^5\psi_1\right]\;; \end{aligned} \tag{12}$$

and the terms of the interaction:

$$L^{int} = -e(\overline{\psi}_1\gamma^\mu\gamma^5\psi_1 - \overline{\psi}_2\gamma^\mu\gamma^5\psi_2)C_\mu + ie(\overline{\psi}_1\gamma^\mu\psi_1 + \overline{\psi}_2\gamma^\mu\psi_2)B_\mu \tag{13}$$

The conclusion that self/anti-self charge conjugate can possess the axial charge (of opposite values) is *in accordance* with the conclusions of Refs. [5, 7] and with the old ideas of R. E. Marshak [21]. It is remarkable feature of this model that we did *not* assume that self/anti-self charge conjugate fields are massless.

One can come to this conclusion on using another way of speculations. Equations (1,3) with interaction can be presented in two-component form (ψ = column$(\phi \quad \chi)$ and $\sigma^\mu = (\mathbf{1}_{2\times 2}, -\sigma^i)$):

$$i\sigma^\mu\nabla_\mu\chi - m\phi = 0\,, \tag{14a}$$
$$i\widetilde{\sigma}^\mu\nabla_\mu\phi - m\chi = 0\,, \tag{14b}$$
$$i\sigma^\mu\nabla_\mu^\ast(-\Theta\phi^\dagger) - m(\Theta\chi^\dagger) = 0\,, \tag{14c}$$
$$i\widetilde{\sigma}^\mu\nabla_\mu^\ast(\Theta\chi^\dagger) - m(-\Theta\phi^\dagger) = 0\,, \tag{14d}$$

[6] As opposed to the above, one can wish to put the constraints $\Psi^S = \Psi^A = 0$ (or even more general ones), which are considered to be equivalent to $a_\uparrow = -b_\downarrow$ and $a_\downarrow = -b_\uparrow$. Thus, one can reformulate the formulas in the rest of the paper. In my opinion, the physical content, which is relevant to the aims of the *present article*, will not be changed. So, the constraints are used *here* only for the purposes of simplicity and clarity.

[7] The operator of the charge conjugation and the chirality γ^5 operator (chosen as above) are the *anti-commuting* operators.

[8] At this point we still leave the room for other kinds of the Lagrangians describing self/anti-self charge conjugate states, see below and cf. [7].

with the hermitian conjugation acting on the q- numbers (it acts on the c- numbers as the complex conjugation). Introducing other bispinors

$$\mathcal{R}^{S,A} = \begin{pmatrix} \phi \\ \mp\Theta\phi^\dagger \end{pmatrix} , \quad \mathcal{L}^{S,A} = \begin{pmatrix} \pm\Theta\chi^\dagger \\ \chi \end{pmatrix} , \tag{15}$$

and combining the second and the third equation, and then, the first and the fourth equations, one can arrive at the equations for new bispinors:

$$i\gamma^\mu D_\mu \mathcal{R}^{S,A} - m\,\mathcal{L}^{S,A} = 0\,, \tag{16a}$$
$$i\gamma^\mu D_\mu^* \mathcal{L}^{S,A} - m\,\mathcal{R}^{S,A} = 0\,. \tag{16b}$$

After taking into account relations between $\mathcal{R}^S$ and $\mathcal{R}^A$ (and between $\mathcal{L}^S$ and $\mathcal{L}^A$) we can obtain two sets:

$$i\gamma^\mu D_\mu^* \mathcal{L}^S - m\gamma^5 \mathcal{R}^A = 0\,, \quad i\gamma^\mu D_\mu \mathcal{R}^A + m\gamma^5 \mathcal{L}^S = 0\,; \tag{17}$$

and/or

$$i\gamma^\mu D_\mu \mathcal{R}^S + m\gamma^5 \mathcal{L}^A = 0\,, \quad i\gamma^\mu D_\mu^* \mathcal{L}^A - m\gamma^5 \mathcal{R}^S = 0\,. \tag{18}$$

They are precisely the equations which we obtained before (cf. (10a,10b) and the equations multiplied by the γ^5 matrix). If we now impose the Majorana *anzatz* [9] $\phi = e^{i\alpha}\Theta\chi^\dagger$ on all four equations we obtain that the γ^5 interaction terms seem to disappear. Due to our previous research [5–7], which was based on other postulates (see also below), we are sure in the necessity of modifications of the Dirac theory for neutral particles and in the presence of the γ^5 interactions. In fact, Majorana *anzatz* (e. g., with $\alpha = 0$) is connected with the interrelations between field operators $\psi_{1,2}$ above. So, we may loose some information. How to solve the problem rigorously? See below.

Further arguments in aid of our reasoning are given by several constructs which appeared recently [5–7, 9]. The Ahluwalia reformulation of the McLennan-Case construct was presented in 1994 [6]. The following type-II spinors have been defined in the momentum space:

$$\lambda^{S,A}(p^\mu) = \begin{pmatrix} \zeta_\lambda \Theta_{[j]} \phi_L^*(p^\mu) \\ \phi_L(p^\mu) \end{pmatrix} , \quad \rho^{S,A}(p^\mu) = \begin{pmatrix} \phi_R(p^\mu) \\ (\zeta_\rho \Theta_{[j]})^* \phi_R^*(p^\mu) \end{pmatrix} . \tag{19}$$

In our choice of the operator of the charge conjugation ($\vartheta_c = 0$) the phase factors $\zeta_{\lambda,\rho}$ are defined as ± 1, for λ^S (ρ^S), and ∓ 1, for λ^A (ρ^A), respectively. One can find relations between the type-II

[9] α is an arbitrary phase factor. It is easy to note that in the case of $\alpha = 0$ the Majorana *anzatz* results in $\mathcal{R}^S = +\mathcal{L}^S$ and $\mathcal{R}^A = -\mathcal{L}^A$. But, in the case $\alpha = \pi$ one obtains $\mathcal{R}^S = -\mathcal{L}^S$ and $\mathcal{R}^A = +\mathcal{L}^A$.

spinors and the Dirac spinors. They are listed here[10]

$$\lambda^S_\uparrow (p^\mu) = +\rho^S_\downarrow (p^\mu) = +\frac{1-\gamma^5}{2} u_\uparrow (p^\mu) + \frac{1+\gamma^5}{2} u_\downarrow (p^\mu)\,, \tag{20a}$$

$$\lambda^S_\downarrow (p^\mu) = -\rho^S_\uparrow (p^\mu) = -\frac{1+\gamma^5}{2} u_\uparrow (p^\mu) + \frac{1-\gamma^5}{2} u_\downarrow (p^\mu)\,, \tag{20b}$$

$$\lambda^A_\uparrow (p^\mu) = -\rho^A_\downarrow (p^\mu) = +\frac{1-\gamma^5}{2} u_\uparrow (p^\mu) - \frac{1+\gamma^5}{2} u_\downarrow (p^\mu)\,, \tag{20c}$$

$$\lambda^A_\downarrow (p^\mu) = +\rho^A_\uparrow (p^\mu) = +\frac{1+\gamma^5}{2} u_\uparrow (p^\mu) + \frac{1-\gamma^5}{2} u_\downarrow (p^\mu)\,. \tag{20d}$$

Positive-energy solutions are assumed in Ref. [6] to be presented by, e. g., self charge conjugate λ^S spinors, negative-energy solutions, by anti-self charge conjugate λ^A spinors:[11, 12]

$$\nu(x^\mu) = \lambda^S(x^\mu) + \lambda^A(x^\mu) \equiv \tag{21}$$

$$\equiv \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2p_0}\sum_\eta \left[\lambda^S_\eta(p^\mu) c_\eta(p^\mu)\exp(-ip\cdot x) + \lambda^A_\eta(p^\mu) d^\dagger_\eta(p^\mu)\exp(+ip\cdot x)\right] .$$

Of course, one can construct the field operator composed of $\rho^{S,A}$ bispinors, e. g.,

$$\widetilde{\nu}(x^\mu) = \rho^A(x^\mu) + \rho^S(x^\mu) \equiv \tag{22}$$

$$\equiv \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2p_0}\sum_\eta \left[\rho^A_\eta(p^\mu) e_\eta(p^\mu)\exp(-ip\cdot x) + \rho^S_\eta(p^\mu) f^\dagger_\eta(p^\mu)\exp(+ip\cdot x)\right] .$$

One of surprising features of this construct [6, 7] is the fact that dynamical equations take eight-component form from the beginning. As shown there the equations for self/anti-self charge conjugate states read:[13]

$$i\gamma^\mu \partial_\mu \lambda^S(x^\mu) + m\rho^A(x^\mu) = 0 \quad , \tag{23a}$$

$$i\gamma^\mu \partial_\mu \rho^A(x^\mu) + m\lambda^S(x^\mu) = 0 \quad ; \tag{23b}$$

and

$$i\gamma^\mu \partial_\mu \lambda^A(x^\mu) - m\rho^S(x^\mu) = 0 \quad , \tag{24a}$$

$$i\gamma^\mu \partial_\mu \rho^S(x^\mu) - m\lambda^A(x^\mu) = 0 \quad . \tag{24b}$$

They can be written in the 8-component form as follows (see formulas (21) in [7] for Γ matrices):

$$\left[i\Gamma^\mu \partial_\mu + m\right]\Psi_{(+)}(x^\mu) = 0 \quad , \tag{25a}$$

$$\left[i\Gamma^\mu \partial_\mu - m\right]\Psi_{(-)}(x^\mu) = 0 \quad , \tag{25b}$$

[10] *Note added* (13/IX/2000): A form of connection between λ_η, ρ_η spinors and the Dirac spinors u_σ, v_σ has indeed been communicated to me by D. V. Ahluwalia in 1997. However, it is known since 1995, see Ref. [9a,b,formulas (22a-22d) and (67-70) respectively] and cf. with the formula (7) in *Hadronic J.* **20** (1997) 435-448.

[11] Let me remind that the sign of the phase in the field operator is considered to be invariant if we restrict ourselves by the orthochroneous proper Poincaré group. This fact has been used at the stage of writing the dynamical equations (23a,23b,24a,24b), see below.

[12] Field operators in this construct may be not self/anti-self charge conjugate operator. So, the notation (S, A) used in the formulas in the coordinate space indicates *only* the presence of self/anti-self charge conjugate *states* and does not refer to the properties of the field operator.

[13] $\vartheta_c = 0$ again. The sign in the mass term depends on this phase factor.

where

$$\Psi_{(+)}(x^\mu) = \begin{pmatrix} \rho^A(x^\mu) \\ \lambda^S(x^\mu) \end{pmatrix} \quad , \quad \Psi_{(-)}(x^\mu) = \begin{pmatrix} \rho^S(x^\mu) \\ \lambda^A(x^\mu) \end{pmatrix} \quad . \tag{26}$$

One can reveal the possibility of the γ^5 phase transformations [7]. The Lagrangian [7, Eq.(24)], which (like in the Dirac construct) becomes to be equal to zero on the solutions of the dynamical equations[14]

$$\begin{aligned} \mathcal{L} &= \frac{i}{2}\Big[\overline{\lambda^S}\gamma^\mu\partial_\mu\lambda^S - (\partial_\mu\overline{\lambda^S})\gamma^\mu\lambda^S + \overline{\rho^A}\gamma^\mu\partial_\mu\rho^A - (\partial_\mu\overline{\rho^A})\gamma^\mu\rho^A + \\ &+ \overline{\lambda^A}\gamma^\mu\partial_\mu\lambda^A - (\partial_\mu\overline{\lambda^A})\gamma^\mu\lambda^A + \overline{\rho^S}\gamma^\mu\partial_\mu\rho^S - (\partial_\mu\overline{\rho^S})\gamma^\mu\rho^S\Big] + \\ &+ m\Big[\overline{\lambda^S}\rho^A + \overline{\rho^A}\lambda^S - \overline{\lambda^A}\rho^S - \overline{\rho^S}\lambda^A\Big] \end{aligned} \tag{27}$$

is invariant with respect to the phase transformations:

$$\lambda'(x^\mu) \rightarrow (\cos\alpha - i\gamma^5\sin\alpha)\lambda(x^\mu) \quad , \tag{28a}$$
$$\overline{\lambda}'(x^\mu) \rightarrow \overline{\lambda}(x^\mu)(\cos\alpha - i\gamma^5\sin\alpha) \quad , \tag{28b}$$
$$\rho'(x^\mu) \rightarrow (\cos\alpha + i\gamma^5\sin\alpha)\rho(x^\mu) \quad , \tag{28c}$$
$$\overline{\rho}'(x^\mu) \rightarrow \overline{\rho}(x^\mu)(\cos\alpha + i\gamma^5\sin\alpha) \quad . \tag{28d}$$

Obviously, the 4-spinors $\lambda^{S,A}(p^\mu)$ and $\rho^{S,A}(p^\mu)$ remain in the space of self/anti-self charge conjugate states.[15] In terms of the field functions $\Psi_{(\pm)}(x^\mu)$ the transformation formulas recast as follows ($Ł^5 = \mathrm{diag}(\gamma^5 \quad -\gamma^5)$ and $\overline{\Psi}_{(\pm)} \equiv \Psi^\dagger_{(\pm)}\Gamma^0$)

$$\Psi'_{(\pm)}(x^\mu) \rightarrow \left(\cos\alpha + iŁ^5\sin\alpha\right)\Psi_{(\pm)}(x^\mu) \quad , \tag{29a}$$
$$\overline{\Psi}'_{(\pm)}(x^\mu) \rightarrow \overline{\Psi}_{(\pm)}(x^\mu)\left(\cos\alpha - iŁ^5\sin\alpha\right) \quad . \tag{29b}$$

Let us proceed further with the local gradient transformations (gauge transformations) in the Majorana construct. When we are interested in them one must introduce the compensating field of the 4-vector potential [7]

$$\partial_\mu \rightarrow \nabla_\mu = \partial_\mu + ieŁ^5 A_\mu \quad , \tag{30a}$$
$$A'_\mu(x) \rightarrow A_\mu(x) - \frac{1}{e}\,\partial_\mu\alpha \quad . \tag{30b}$$

Therefore, equations describing interactions of the λ^S and ρ^A with 4-vector potential are the following

$$i\gamma^\mu\partial_\mu\lambda^S(x^\mu) + e\gamma^\mu\gamma^5 A_\mu\lambda^S(x^\mu) + m\rho^A(x^\mu) = 0 \quad , \tag{31a}$$
$$i\gamma^\mu\partial_\mu\rho^A(x^\mu) - e\gamma^\mu\gamma^5 A_\mu\rho^A(x^\mu) + m\lambda^S(x^\mu) = 0 \quad . \tag{31b}$$

[14] The overline implies the Dirac conjugation.

[15] Usual phase transformations like that which were applied to the Dirac field will destroy self/anti-self charge conjugacy. The origin lies in the fact that the charge conjugation operator is *not* a linear operator and it includes the operation of complex conjugation.

The second-order equations follow immediately form the set (31a,31b)[16]

$$\left\{ \left(i\widehat{\partial} - e\widehat{A}\gamma^5\right) \left(i\widehat{\partial} + e\widehat{A}\gamma^5\right) - m^2 \right\} \lambda^S (x^\mu) = 0 \quad , \tag{32a}$$

$$\left\{ \left(i\widehat{\partial} + e\widehat{A}\gamma^5\right) \left(i\widehat{\partial} - e\widehat{A}\gamma^5\right) - m^2 \right\} \rho^A (x^\mu) = 0 \quad ; \tag{32b}$$

with the notation being used: $\widehat{A} \equiv \gamma^\mu A_\mu = \gamma^0 A^0 - (\boldsymbol{\gamma} \cdot \mathbf{A})$. After algebraic transformations in the spirit of [22, 23] one obtains

$$\left\{ \Pi_\mu^+ \Pi^{\mu\,+} - m^2 + \frac{e}{2} \gamma^5 \Sigma^{\mu\nu} F_{\mu\nu} \right\} \lambda^{S,A} (x^\mu) = 0 \quad , \tag{33a}$$

$$\left\{ \Pi_\mu^- \Pi^{\mu\,-} - m^2 - \frac{e}{2} \gamma^5 \Sigma^{\mu\nu} F_{\mu\nu} \right\} \rho^{A,S} (x^\mu) = 0 \quad , \tag{33b}$$

where the "covariant derivative" operators acting in the $(1/2, 0) \oplus (0, 1/2)$ representation are now defined

$$\Pi_\mu^\pm = \frac{1}{i} \partial_\mu \mp e\gamma^5 A_\mu \quad , \tag{34}$$

and

$$\Sigma^{\mu\nu} = \frac{i}{2} \left[\gamma^\mu\, , \, \gamma^\nu\right]_- \quad . \tag{35}$$

Thus, we see that the second-order equations for the particles described by the field operator $\nu(x^\mu)$ (Eq. (46) in [6] and Eq. (21) of this paper), which interact with the 4-vector potential, have the same form for positive- and negative-energy parts. The same is true in the case of the use of the field operator composed from ρ^A and ρ^S. One can see the difference with the Dirac case; namely, the presence of γ^5 matrix in the "Pauli term" and in the lengthening derivatives. Next, we are able to decouple the set (33a,33b) for the up- and down- components of the bispinors in the coordinate representation. For instance, the up- and the down- parts of the $\nu^{^{DL}}(x) =$ column$(\xi \quad \eta)$ interact with the vector potential in the following manner:

$$\begin{cases} \left[\pi_\mu^+ \pi^{\mu\,+} - m^2 + \frac{e}{2} \sigma^{\mu\nu} F_{\mu\nu}\right] \xi(x^\mu) = 0 \quad , \\ \left[\pi_\mu^- \pi^{\mu\,-} - m^2 - \frac{e}{2} \widetilde{\sigma}^{\mu\nu} F_{\mu\nu}\right] \eta(x^\mu) = 0 \quad , \end{cases} \tag{36}$$

where already one has $\pi_\mu^\pm = i\partial_\mu \pm eA_\mu$, $\sigma^{0i} = -\widetilde{\sigma}^{0i} = i\sigma^i$, $\sigma^{ij} = \widetilde{\sigma}^{ij} = \epsilon_{ijk}\sigma^k$. Of course, introducing the operator composed of the ρ states one can write corresponding equations for its up- and down- components and, hence, restore the Feynman-Gell-Mann equation [24, Eq.(3)] and its charge conjugate (if one considers that A^μ and $F^{\mu\nu}$ are the real fields).[17] In fact, this way

[16] The case of λ^A and ρ^S is similar.

[17] One can connect the Feynman-Gell-Mann field with $\nu(x^\mu)$ and $\widetilde{\nu}(x^\mu)$ defined in (21,22). For instance,

$$\Psi^{FGM} = \frac{1+\gamma^5}{2} \widetilde{\nu} \pm \frac{1-\gamma^5}{2} \nu \, , \tag{37a}$$

$$(\Psi^{FGM})^c = \frac{1+\gamma^5}{2} \nu \pm \frac{1-\gamma^5}{2} \widetilde{\nu} \, . \tag{37b}$$

But the signs are not fixed in the framework of this consideration (due to the fact that the Feynman-Gell-Mann equations are of the second order and the left-hand side operator (see Eq. (3) in [24]) commutes with the γ^5 matrix.

will lead us to the consideration which is identical to the recent papers [25]. It was based on the linearization procedure for 2-spinors, which is similar to that used by Feshbach and Villars [26] in order to derive the Hamiltonian form of the Klein-Gordon equation. Some insights in the interaction issues with the 4-vector potential in the eight-component equation have been made there: for instance, while explicit form of the wave functions slightly differ from the Dirac case, the hydrogen atom spectrum is the same to that in the usual Dirac theory [23, p.66,74-75]. Next, like in the paper [27] the equations of [25] presume a non-CP-violating[18] electric dipole moment of the corresponding states.

Next, by using the relations (20a-20d) one can deduce how is the ν operator connected with the Dirac field operator and its charge conjugate. In the particular case when $(a_\downarrow + b_\uparrow)/2 = (a_\uparrow - b_\downarrow)/2 \equiv c_\uparrow = d_\downarrow$ and $(a_\uparrow + b_\downarrow)/2 = (a_\downarrow - b_\uparrow)/2 \equiv c_\downarrow = d_\uparrow$ one has

$$\nu(x^\mu) = +\frac{1}{2}(\psi_\downarrow(x^\mu) - \psi^c_\uparrow(x^\mu)) - \frac{\gamma^5}{2}(\psi_\uparrow(x^\mu) + \psi^c_\downarrow(x^\mu))\,. \tag{38}$$

The operator composed of ρ spinors is then expressed[19]

$$\widetilde{\nu}(x^\mu) = +\frac{1}{2}(\psi_\downarrow(x^\mu) + \psi^c_\uparrow(x^\mu)) + \frac{\gamma^5}{2}(\psi_\uparrow(x^\mu) - \psi^c_\downarrow(x^\mu))\,. \tag{39}$$

Other fields which we use in order to obtain dynamical equations are

$$\nu^c(x^\mu) = -\frac{1}{2}(\psi_\uparrow(x^\mu) - \psi^c_\downarrow(x^\mu)) + \frac{\gamma^5}{2}(\psi_\downarrow(x^\mu) + \psi^c_\uparrow(x^\mu))\,, \tag{40a}$$

$$\gamma^5\nu(x^\mu) = -\frac{1}{2}(\psi_\uparrow(x^\mu) + \psi^c_\downarrow(x^\mu)) + \frac{\gamma^5}{2}(\psi_\downarrow(x^\mu) - \psi^c_\uparrow(x^\mu))\,, \tag{40b}$$

$$\gamma^5\nu^c(x^\mu) = +\frac{1}{2}(\psi_\downarrow(x^\mu) + \psi^c_\uparrow(x^\mu)) - \frac{\gamma^5}{2}(\psi_\uparrow(x^\mu) - \psi^c_\downarrow(x^\mu))\,, \tag{40c}$$

$$\widetilde{\nu}^c(x^\mu) = +\frac{1}{2}(\psi_\uparrow(x^\mu) + \psi^c_\downarrow(x^\mu)) + \frac{\gamma^5}{2}(\psi_\downarrow(x^\mu) - \psi^c_\uparrow(x^\mu))\,, \tag{40d}$$

$$\gamma^5\widetilde{\nu}(x^\mu) = +\frac{1}{2}(\psi_\uparrow(x^\mu) - \psi^c_\downarrow(x^\mu)) + \frac{\gamma^5}{2}(\psi_\downarrow(x^\mu) + \psi^c_\uparrow(x^\mu))\,, \tag{40e}$$

$$\gamma^5\widetilde{\nu}^c(x^\mu) = +\frac{1}{2}(\psi_\downarrow(x^\mu) - \psi^c_\uparrow(x^\mu)) + \frac{\gamma^5}{2}(\psi_\uparrow(x^\mu) + \psi^c_\downarrow(x^\mu))\,. \tag{40f}$$

After rather tiresome calculational procedure one obtains the dynamical equations in this approach

$$i\gamma^\mu\widetilde{\nabla}_\mu(\nu - \nu^c) - e\gamma^\mu\gamma^5 C_\mu(\nu - \nu^c) - m\gamma^5(\widetilde{\nu} + \widetilde{\nu}^c) = 0\,, \tag{41a}$$

$$i\gamma^\mu\widetilde{\nabla}_\mu(\widetilde{\nu} + \widetilde{\nu}^c) + e\gamma^\mu\gamma^5 C_\mu(\widetilde{\nu} + \widetilde{\nu}^c) + m\gamma^5(\nu - \nu^c) = 0\,, \tag{41b}$$

$$i\gamma^\mu\gamma^5\widetilde{\nabla}_\mu(\nu - \nu^c) - e\gamma^\mu C_\mu(\nu - \nu^c) + m(\widetilde{\nu} + \widetilde{\nu}^c) = 0\,, \tag{41c}$$

$$i\gamma^\mu\gamma^5\widetilde{\nabla}_\mu(\widetilde{\nu} + \widetilde{\nu}^c) + e\gamma^\mu C_\mu(\widetilde{\nu} + \widetilde{\nu}^c) - m(\nu - \nu^c) = 0\,, \tag{41d}$$

[18] This is possible due to the Wigner “doubling” of the components of the wave function.
[19] Of course, certain relations between creation/annihilation operators of various field operators are again assumed.

or

$$(1+\gamma^5)\left[i\gamma^\mu \widetilde{\nabla}_\mu(\nu - \nu^c + \widetilde{\nu} + \widetilde{\nu}^c) + e\gamma^\mu C_\mu(\nu - \nu^c - \widetilde{\nu} - \widetilde{\nu}^c) + m(\nu - \nu^c - \widetilde{\nu} - \widetilde{\nu}^c)\right] = 0\,, \tag{42a}$$

$$(1+\gamma^5)\left[i\gamma^\mu \widetilde{\nabla}_\mu(\nu - \nu^c - \widetilde{\nu} - \widetilde{\nu}^c) + e\gamma^\mu C_\mu(\nu - \nu^c + \widetilde{\nu} + \widetilde{\nu}^c) - m(\nu - \nu^c + \widetilde{\nu} + \widetilde{\nu}^c)\right] = 0\,, \tag{42b}$$

$$(1-\gamma^5)\left[i\gamma^\mu \widetilde{\nabla}_\mu(\nu - \nu^c + \widetilde{\nu} + \widetilde{\nu}^c) - e\gamma^\mu C_\mu(\nu - \nu^c - \widetilde{\nu} - \widetilde{\nu}^c) - m(\nu - \nu^c - \widetilde{\nu} - \widetilde{\nu}^c)\right] = 0\,, \tag{42c}$$

$$(1-\gamma^5)\left[i\gamma^\mu \widetilde{\nabla}_\mu(\nu - \nu^c - \widetilde{\nu} - \widetilde{\nu}^c) - e\gamma^\mu C_\mu(\nu - \nu^c + \widetilde{\nu} + \widetilde{\nu}^c) + m(\nu - \nu^c + \widetilde{\nu} + \widetilde{\nu}^c)\right] = 0\,. \tag{42d}$$

Thus, one can see the operators

$$\nu(x^\mu) - \nu^c(x^\mu) \quad \text{and} \quad (\widetilde{\nu}(x^\mu) + \widetilde{\nu}^c(x^\mu))$$

also satisfy the equations of the type (10a,10b). From the formulas (38,39) one can figure out, why do Eqs. (17,18) and (23a-24b) have different forms and how are $\mathcal{R}^{S,A}$, $\mathcal{L}^{S,A}$, the self/anti-self charge conjugate operators, and $\lambda^{S,A}$, $\rho^{S,A}$, the operators answering for the self/anti-self charge conjugate states, connected?

3 The term $\boldsymbol{\sigma}\cdot[\mathbf{A}\times\mathbf{A}^*]$: To Be or Not To Be?

The possibility of terms as $\sim \boldsymbol{\sigma}\cdot[\mathbf{A}\times\mathbf{A}^*]$, Ref. [28, 29], appears to be related to the matters of chiral interactions. As we are now convinced, the Dirac field operator can be always presented as a superposition of the self- and anti-self charge conjugate field operators. The anti-self charge conjugate part can give the self charge conjugate part after multiplying by the γ^5 matrix and *vice versa*. In the equations (9) we are able to put further constraints to extract the self-conjugate states. For instance,

$$\gamma^5\psi_2 = \psi_1 \tag{43}$$

or, the anti-self charge conjugate states:

$$\gamma^5\psi_1 = -\psi_2\,. \tag{44}$$

Hence, one has[20]

$$[i\gamma^\mu D^*_\mu - m]\psi_1^s = 0\,, \tag{46}$$

[20] The anti-self charge conjugate field function ψ_2 can also be used. The equation has then the form:

$$[i\gamma^\mu D^*_\mu + m]\psi_2^a = 0\,. \tag{45}$$

or[21]

$$[i\gamma^\mu D_\mu - m]\psi_2^a = 0\,, \tag{48}$$

Both equations lead to the terms of interaction such as $\sim \boldsymbol{\sigma}\cdot[\mathbf{A}\times\mathbf{A}^*]$ provided that the 4-vector potential is considered as a complex function(al). In fact, from (46) we have:

$$i\sigma^\mu\nabla_\mu\chi_1 - m\phi_1 = 0 \tag{49a}$$
$$i\widetilde{\sigma}^\mu\nabla^*_\mu\phi_1 - m\chi_1 = 0\,. \tag{49b}$$

And, from (48) we have

$$i\sigma^\mu\nabla^*_\mu\chi_2 - m\phi_2 = 0 \tag{50a}$$
$$i\widetilde{\sigma}^\mu\nabla_\mu\phi_2 - m\chi_2 = 0\,. \tag{50b}$$

The meanings of σ^μ and $\widetilde{\sigma}^\mu$ are obvious from the definition of γ matrices. From the above set we extract the terms as $\pm e^2\sigma^i_{Pauli}\sigma^j_{Pauli}A_iA^*_j$, which lead to the discussed terms [28, 29].

Furthermore, one can come to the same conclusions not applying to the constraints on the creation/annihilation operators (which we have chosen previously for clarity and simplicity). It is possible to work with self/anti-self charge conjugate fields $\mathcal{R}^{S,A}$ and $\mathcal{L}^{S,A}$ and *two* Majorana *anzatzen*, see equations (16a,16b).

Thus, in the considered cases it is the γ^5 transformation which distinguishes various field configurations (helicity, self/anti-self charge conjugate properties etc) in the coordinate representation.

It would be interesting to compare the above arguments for derivation of the Esposito-Recami-Evans term with those which have been used in [29]. We would also like to note that in the submitted Esposito-Recami paper the terms of the type $\sim \boldsymbol{\sigma}\cdot[\mathbf{A}\times\mathbf{A}^*]$ can be reduced to $(\boldsymbol{\sigma}\cdot\nabla)\mathcal{V}$, where $\mathcal{V}$ is the scalar potential.

4 Generalizations to Higher Spin Representations and Conclusions

As we have learnt the γ^5 interactions is intimately related to the question of defining the self/anti-self charge conjugate states. But, as we discussed in [9c] (see also [6]) it is impossible to introduce self/anti-self charge conjugate momentum-space objects in the $(1,0)\oplus(0,1)$ representation. One can see difficulties of introducing the analogues of $\psi_{1,2,3,4}$ in the $(1,0)\oplus(0,1)$ representation, for instance, from these formulas:

$$\psi_\uparrow(x^\mu) = \int\frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}\left[u_\uparrow(p^\mu)a_\uparrow(p^\mu)e^{-i\phi} + \mathcal{C}u^*_\uparrow(p^\mu)b^\dagger_\downarrow(p^\mu)e^{+i\phi}\right]\,, \tag{51a}$$
$$\psi_\downarrow(x^\mu) = \int\frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}\left[u_\downarrow(p^\mu)a_\downarrow(p^\mu)e^{-i\phi} + \mathcal{C}u^*_\downarrow(p^\mu)b^\dagger_\uparrow(p^\mu)e^{+i\phi}\right]\,, \tag{51b}$$
$$\psi_\rightarrow(x^\mu) = \int\frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}\left[u_\rightarrow(p^\mu)a_\rightarrow(p^\mu)e^{-i\phi} - \mathcal{C}u^*_\rightarrow(p^\mu)b^\dagger_\rightarrow(p^\mu)e^{+i\phi}\right]\,, \tag{51c}$$

[21] The self charge conjugate field function ψ_1 also can be used. The equation has the form:

$$[i\gamma^\mu D_\mu + m]\psi_1^s = 0\,. \tag{47}$$

As readily seen in the cases of alternative choices we have opposite "charges" in the terms of the type $\sim \boldsymbol{\sigma}\cdot[\mathbf{A}\times\mathbf{A}^*]$ and in the mass terms.

and

$$\psi^c_\uparrow(x^\mu) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}\left[-u_\uparrow(p^\mu)b_\downarrow(p^\mu)e^{-i\phi} + \mathcal{C}u^*_\uparrow(p^\mu)a^\dagger_\uparrow(p^\mu)e^{+i\phi}\right] \,, \tag{52a}$$

$$\psi_\downarrow(x^\mu) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}\left[-u_\downarrow(p^\mu)b_\uparrow(p^\mu)e^{-i\phi} + \mathcal{C}u^*_\downarrow(p^\mu)a^\dagger_\downarrow(p^\mu)e^{+i\phi}\right] \,, \tag{52b}$$

$$\psi_\rightarrow(x^\mu) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}\left[u_\rightarrow(p^\mu)b_\rightarrow(p^\mu)e^{-i\phi} + \mathcal{C}u^*_\rightarrow(p^\mu)a^\dagger_\rightarrow(p^\mu)e^{+i\phi}\right] \,, \tag{52c}$$

The equation $S^c_{[1]}\psi(x^\mu) = e^{i\alpha}\psi(x^\mu)$ with

$$S^c = \mathcal{C}\mathcal{K} = e^{i\vartheta^c_{[1]}}\begin{pmatrix} 0 & \Theta_{[1]} \\ -\Theta_{[1]} & 0\end{pmatrix}\mathcal{K}\,, \quad \Theta_{[1]} = \begin{pmatrix} 0 & 0 & 1\\ 0 & -1 & 0 \\ 1 & 0 & 0\end{pmatrix} \tag{53}$$

defined in Ref. [6, 17], has no solutions in the field of complex numbers. We use the following analogues of the formulas of the footnote 3 for the Barut-Muzinich-Williams matrices

$$\mathcal{C}^T = -\mathcal{C}\,, \quad \mathcal{C}^* = \mathcal{C} = -\mathcal{C}^{-1} \tag{54a}$$

$$\mathcal{C}\gamma^{\mu\nu^*}\mathcal{C}^{-1} = -\gamma^{\mu\nu}\,, \quad \mathcal{C}\gamma^{5^*}\mathcal{C}^{-1} = -\gamma^5\,. \tag{54b}$$

Furthermore, in the Majorana representation of $\gamma^{\mu\nu}$ matrices the operator of the charge conjugation ($\vartheta_c = 0$) is equal to

$$S^c_{[1]} = \gamma^5_{MR}\mathcal{K} = i\gamma^5_{WR}\gamma^0_{WR}\mathcal{K} = \begin{pmatrix} 0 & i \\ -i & 0\end{pmatrix}\mathcal{K}\,. \tag{55}$$

Thus, if one implies that $\wp_{u,v} = i(\partial/\partial t)/E = \pm 1$ the Sankaranarayanan-Good equation of Ref. [17] transforms as follows:

$$[\gamma^{\mu\nu}\partial_\mu\partial_\nu + \wp_{u,v}m^2]\psi(x^\mu) = 0 \Rightarrow [\gamma^{\mu\nu}\partial_\mu\partial_\nu + \wp_{u,v}m^2]\psi^c(x^\mu) = 0\,. \tag{56}$$

Please notice that the operator $\wp_{u,v}$ defined above has the following property with respect to the ordinary complex conjugation $\wp^*_{u,v} = -\wp_{u,v}$, cf. [30].

Finally, in the Majorana representation the analogues of (6a-6d) have the form:

$$\left[\gamma^{\mu\nu}\nabla_\mu\nabla_\nu + \wp_{u,v}m^2\right]\psi(x^\mu) = 0\,, \tag{57a}$$

$$\left[\gamma^{\mu\nu}\nabla^*_\mu\nabla^*_\nu + \wp_{u,v}m^2\right]\gamma^5\psi^*(x^\mu) = 0\,, \tag{57b}$$

$$\left[\gamma^{\mu\nu}\nabla_\mu\nabla_\nu - \wp_{u,v}m^2\right]\gamma^5\psi(x^\mu) = 0\,, \tag{57c}$$

$$\left[\gamma^{\mu\nu}\nabla^*_\mu\nabla^*_\nu - \wp_{u,v}m^2\right]\psi^*(x^\mu) = 0\,. \tag{57d}$$

It is seen after calculations which are similar to above that the combinations $\psi + \psi^*$ and $\gamma^5(\psi - \psi^*)$ may have interaction of the “axial” form.

Concluding, we state that the γ^5 interaction is indispensable element of $(1/2, 0) \oplus (0, 1/2)$ representation space (and, presumably, of all the representations of the type $(j, 0) \oplus (0, j)$). We discussed important physical consequences of the presence of this interaction for the particles of this representation and found relations to other models. The proper account of such terms may lead to deeper understanding of the nature of particle interactions in the modern gauge theories, of the structure of the Fock space and reasons for introduction of the latter as well.

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