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$SU(2)$ and $SU(1,1)$ SQUEEZING OF INTERACTING RADIATION MODES

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In this communication we discuss $SU(1,1)$ and $SU(2)$ squeezing of an interacting system of radiation modes in a quadratic medium in the framework of Lie algebra. We show that regardless of which state being initially considered, squeezing can be periodically generated.

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1 Introduction

The experiments on photon antibunching and sub-Poissonian statistics focused on the intensity or photon-number fluctuations of electromagnetic field. Recently, there was a major effort focused on the fluctuations in the quadrature amplitudes of the electromagnetic field to produce squeezed light. This light is indicated by having less noise in one field quadrature than a coherent state with an excess of noise in the conjugate quadrature such that the product of canonically conjugate variances must satisfy the uncertainty relation. Indeed, this light occupies a wide area in the studies of quantum optics theory since it has a lot of applications, e.g. in optical communication networks [1], in interferometric techniques [2], and in optical waveguide tap [3]. Moreover, generation of squeezed light has been observed in many optical processes, e.g. [4, 5]. Investigation of the squeezing properties of the radiation field is a central topic in quantum optics and noise squeezing can be measured by means of homodyne detection.

On the other hand, Lie algebras have been used to investigate the nonclassical properties of light in quantum optical systems, e.g. quantum mechanical interferometers [6], beam splitters [7] and linear directional coupler [8], since they can give powerful and systematic methods to facilitate such studies [9]. Among these nonclassical properties lies $SU(2)$ and $SU(1,1)$ squeezing [10]. The authors of [10] have shown that in the framework of a system of N two-level atoms the squeezing of angular-momentum [$SU(2)$] fluctuations is exhibited for optical transients involving the photon echo. Further, the $SU(1,1)$ fluctuations are established for general two-photon processes involving dynamical variables different from the creation and annihilation operators. Also Lie algebra techniques have been applied to problems in nonlinear optics such as a model of nonabsorbing nonlinear medium (an anharmonic oscillator) [11] or a model consisting of a degenerate parametric amplifier (nonconserving term) and an anharmonic term [12]. For the former it has been shown that squeezing is eventually revoked and the rate of revoking grows with increasing number of photon in the initial state, however, for the latter squeezing property is generally revoked by the nonabsorbing term and increased by the nonconserving term. Finally, it is convenient to point out that the Jaynes-Cummings model composed of a two-level (three-level) atom interacting with single mode (two modes) electromagnetic field has been treated also in terms of Lie algebra [13] ([14]). In all these considerations the basic point is the existence of a set of operators obeying Lie algebra.

The generation of $SU(1,1)$ CS [15, 16] and $SU(2)$ CS has been investigated for the degenerate [10] and nondegenerate parametric amplifiers [10, 17], respectively. In this communication we study $SU(1,1)$ and $SU(2)$ squeezing in terms of these states for three interacting modes in a nonlinear crystal or in any relevant device, e.g. nonlinear directional coupler.

This will be done as follows: In Section 2 we give a brief overview of the properties of $SU(1,1)$ and $SU(2)$ Lie algebras which will be used in the article. Section 3 is devoted to a discussion of the models as well as to the solution of the equation of motions. Section 4 discusses $SU(1,1)$ squeezing and $SU(2)$ squeezing. Section 5 includes conclusions and remarks.

2 Properties of $SU(1,1)$ and $SU(2)$ Lie algebras

In this section we review briefly, for future purpose, some properties of the $SU(1,1)$ and $SU(2)$ Lie algebras as well as we give the notations of $SU(1,1)$ CS [15, 16] and $SU(2)$ CS [10]. We begin by introducing the operators set $\{\hat{K}_x, \hat{K}_y, \hat{K}_z\}$ which satisfy the commutation relations

$$[\hat{K}_x, \hat{K}_y] = i\beta\hat{K}_z, \qquad [\hat{K}_y, \hat{K}_z] = i\hat{K}_x, \qquad [\hat{K}_z, \hat{K}_x] = i\hat{K}_y, \tag{1}$$

where $\beta = \pm 1$. When $\beta = -1$ this set becomes the generator of $SU(1,1)$ Lie algebra, whereas when $\beta = 1$ it becomes the generator of $SU(2)$ Lie algebra. Using the ladder operators, i.e. $\hat{K}_+, \hat{K}_-$, we can construct the operators

$$\hat{K}_x = \frac{1}{2}(\hat{K}_+ + \hat{K}_-), \qquad \hat{K}_y = \frac{1}{2i}(\hat{K}_+ - \hat{K}_-), \tag{2}$$

satisfying the commutation relation

$$[\hat{K}_-, \hat{K}_+] = 2\beta\hat{K}_z, \qquad [\hat{K}_z, \hat{K}_\pm] = \pm\hat{K}_\pm. \tag{3}$$

The discrete representation of the $SU(1,1)$ Lie group is given by

$$\begin{aligned}
&\hat{K}_z|m;k\rangle = (m+k)|m;k\rangle,\\
&\hat{K}_+|m;k\rangle = [(m+1)(m+2k)]^{\frac{1}{2}}|m+1;k\rangle,\\
&\hat{K}_-|m;k\rangle = [m(m+2k-1)]^{\frac{1}{2}}|m-1;k\rangle,
\end{aligned} \tag{4}$$

where $\hat{K}_-|0;k\rangle = 0$. On the other hand, the discrete representation of the $SU(2)$ Lie group is given by

$$\begin{aligned}
&\hat{K}_z|m;j\rangle = m|m;j\rangle,\\
&\hat{K}_+|m;j\rangle = [(j-m)(j+m+1)]^{\frac{1}{2}}|m+1;j\rangle,
\end{aligned}$$

$$\hat{K}_-|m;j\rangle = [(j+m)(j-m+1)]^{\frac{1}{2}}|m-1;j\rangle, \tag{5}$$

where $\hat{K}_-|-j;j\rangle = \hat{K}_+|-j;j\rangle = 0$.

We examine squeezing against $SU(1,1)$ CS as well as $SU(2)$ CS. In fact, there are two types of $SU(1,1)$ CS, the first one is the PCS [15] having the form

$$|\xi;k\rangle = (1-|\xi|^2)^k \sum_{m=0}^{\infty} \sqrt{\frac{\Gamma(m+2k)}{m!\Gamma(2k)}}\xi^m|m;k\rangle, \tag{6}$$

where $\xi = -\tanh(\frac{\theta}{2})\exp(-i\phi)$, with $|\xi| \in (0,1), \theta \in (-\infty,\infty), \phi \in (0,2\pi)$, Γ stands for Gamma function and k is called Bargmann index. For $k = 1/4$ and $3/4$ we get even-parity and odd-parity states, respectively. This state is a special type of squeezed vacuum state [10] which is essentially equivalent to the two-photon coherent state [18], and it possesses most of the properties of the ordinary coherent states, such as a completeness relation and a reproducing kernel. PCS can be realized in the framework of degenerate and nondegenerate parametric amplifier [17]. The second type of $SU(1,1)$ CS is the Barut-Girardello coherent state (BGCS) [16] determined by

$$|z;n\rangle = \sqrt{\frac{|z|^{2n-1}}{I_{2n-1}(2|z|)}} \sum_{m=0}^{\infty} \frac{z^m}{\sqrt{m!\Gamma(m+2n)}}|m;n\rangle, \tag{7}$$

where $I_n(..)$ is the modified Bessel function of order n. Indeed, this state is the eigenstate of $\hat{K}_-$, i.e. $\hat{K}_-|z;n\rangle = z|z;n\rangle$, and it has similar properties as the Glauber coherent state in the sense that it is not only unsqueezed state but also a minimum-uncertainty state.

$SU(2)$ CS (Bloch state) [10] is defined by

$$|\mu,j\rangle = \frac{1}{1+|\mu|^2} \sum_{m=-j}^{j} \sqrt{\frac{(2j)!}{(j-m)!(j+m)!}}\mu^{j+m}|m;j\rangle, \tag{8}$$

where $2j$ is the maximum possible number of photons and μ is a complex parameter related to the partition of photons in the $SU(2)$ CS field modes. This state is squeezed state depending on the value of μ and can be generated in a linear directional coupler in which a pure number state $|2j\rangle$ is launched into one port of the coupler and the vacuum into other [8].

The following relations will be frequently used in this work [9]

$$\langle \hat{K}_-^l(0)\hat{K}_z^m(0)\hat{K}_+^n(0)\rangle_{\rm p} = (1-|\xi|^2)^{2k}(\frac{\partial}{\partial \xi^*})^l(\frac{\partial}{\partial \xi})^n[k+|\xi|^2\frac{\partial}{\partial(|\xi|^2)}]^m\frac{1}{(1-|\xi|^2)^{2k}}, \tag{9a}$$

$$\langle \hat{K}_+^l(0)\hat{K}_z^m(0)\hat{K}_-^n(0)\rangle_{\rm b} = z^{*l}z^n\frac{1}{2|z|I_{2n-1}(2|z|)}\left(\frac{x}{2}\frac{\partial}{\partial x}\right)^m xI_{2n-1}(x)|_{x=2|z|}, \tag{9b}$$

$$\langle \hat{K}_-^l(0)\hat{K}_z^m(0)\hat{K}_+^n(0)\rangle_{\rm u2} = \frac{1}{(1+|\mu|^2)^{2j}}(\frac{\partial}{\partial \mu^*})^l(\frac{\partial}{\partial \mu})^n[|\mu|^2\frac{\partial}{\partial(|\mu|^2)}-j]^m(1+|\mu|^2)^{2j}, \tag{9c}$$

where the subscripts p, b and u2 mean that the average is performed in terms of PCS, BGCS and $SU(2)$ CS, respectively.

Finally, we conclude this section by giving the definitions of $SU(1,1)$ and $SU(2)$ squeezing.

From Eqs. (1) we have the following uncertainty relation

$$\langle(\Delta\hat{K}_x)^2\rangle\langle(\Delta\hat{K}_y)^2\rangle \geq \frac{1}{4}|\langle\hat{K}_z\rangle|^2. \tag{10}$$

To measure $SU(1,1)$ (or $SU(2)$) squeezing, it is appropraite to introduce the function

$$S_j = \frac{\langle(\Delta\hat{K}_j)^2\rangle - \frac{1}{2}|\langle\hat{K}_z\rangle|}{\frac{1}{2}|\langle\hat{K}_z\rangle|}, \qquad j = x, y. \tag{11}$$

Maximum $SU(1,1)$ (or $SU(2)$) squeezing (100%) is obtained for $S_j = -1$.

3 Model description and exact solution

In this section we consider two types of three radiation modes interacting by somehow in a nonlinear crystal or in an optical cavity which are associating with $SU(1,1)$ and $SU(2)$ Lie algebras.

The lossless effective Hamiltonian of the first type, i.e. associated with $SU(1,1)$ Lie algebra, has the form

$$\frac{\hat{H}}{\hbar} = i\lambda_1(\hat{A}_1\hat{A}_2 - \hat{A}_1^\dagger\hat{A}_2^\dagger) + i\lambda_2(\hat{A}_1\hat{A}_3 - \hat{A}_1^\dagger\hat{A}_3^\dagger) + i\lambda_3(\hat{A}_3^\dagger\hat{A}_2 - \hat{A}_3\hat{A}_2^\dagger), \tag{12}$$

where λ_j are the coupling constants including the pump amplitude which is proportional to the second-order susceptibility of the medium $\chi^{(2)}$. This interaction mixing processes of parametric amplification and frequency conversion can be established, e.g. by means of a bulk nonlinear crystal exhibiting the second-order nonlinear properties in which three dynamical modes of frequencies $\omega_1, \omega_2, \omega_3$ are induced by three beams from lasers of these frequencies. When pumping this crystal by means of the corresponding strong coherent pump beams, as indicated in the Hamiltonian, we can approximately fulfil the phase-matching conditions for the corresponding processes, in particular if the frequencies are close each other (biaxial crystals may be helpful in such an arrangement). Also a possible use of quasi-phase matching may help in the realization, which is, however, more difficult technologically [19]. Another possibility to realize such interaction is a nonlinear symmetric directional coupler composed of two nonlinear waveguides operating by nondegenerate parametric amplification where the interaction between two waveguides can be established through the evanescent waves. More details about the quantum properties of the Hamiltonian (12) can be found in [20]. Now if we set

$$\begin{aligned}\hat{L}_x &= i(\hat{A}_1\hat{A}_2 - \hat{A}_1^\dagger\hat{A}_2^\dagger),\\ \hat{L}_y &= i(\hat{A}_1\hat{A}_3 - \hat{A}_1^\dagger\hat{A}_3^\dagger),\\ \hat{L}_z &= i(\hat{A}_3^\dagger\hat{A}_2 - \hat{A}_3\hat{A}_2^\dagger),\end{aligned} \tag{13}$$

one can easily verify that this set of operators satisfy the commutation rules (1) with $\beta = -1$, i.e. this model is associatted with $SU(1,1)$ Lie algebra.

The second type of Hamiltonian which associates with $SU(2)$ Lie algebra has the form

$$\frac{\hat{H}}{\hbar} = i\lambda_1'(\hat{A}_3^\dagger\hat{A}_2 - \hat{A}_3\hat{A}_2^\dagger) + i\lambda_2'(\hat{A}_1^\dagger\hat{A}_3 - \hat{A}_1\hat{A}_3^\dagger) + i\lambda_3'(\hat{A}_1^\dagger\hat{A}_2 - \hat{A}_1\hat{A}_2^\dagger), \tag{14}$$

where all the notations have the same meaning as before; this interaction is mixing three processes of frequency conversion. Analogously if one takes the terms involving λ_1', λ_2' and λ_3' by

$\hat{L}'_x, \hat{L}'_y$ and $\hat{L}'_z$, respectively, it is easy to prove that these operators satisfy the commutation rules (1) with $\beta = 1$. For completeness, it would be convenient to mention that pair creation and annihilation operators $\hat{A}_j\hat{A}_k$ and $\hat{A}_j^\dagger\hat{A}_k^\dagger$ $(j \neq k)$ of the two-mode field form elements of the $SU(1,1)$ Lie group; on the other hand operators $\hat{A}_j^\dagger\hat{A}_k$ and $\hat{A}_j\hat{A}_k^\dagger$ form elements of the $SU(2)$ Lie group, e.g. in the lossless beam splitter [7]. So one can note that (12) includes three terms, two of them represent $SU(1,1)$ Hamiltonian (parametric amplifiers, $\hat{L}_x, \hat{L}_y$) and the third one forms $SU(2)$ Hamiltonian ($\hat{L}_z$). Hamiltonian in (14) is formally a sum of three $SU(2)$ Hamiltonians. We assume that the used optical crystal is pumped simultaneously in two different regimes by corresponding laser beams. Now in this paper we treat the systems (12) and (14) by unified model that exploits their underlying Lie algebra similarity. The unified model is

$$\frac{\hat{H}}{\hbar} = \alpha_1\hat{K}_x + \alpha_2\hat{K}_y + \alpha_3\hat{K}_z, \tag{15}$$

where $\alpha_j, j = 1,2,3$ are parameters specializing which model is considered. In other words, the Lie algebras results for either the models (12) or (14) can be recovered from our general formula (15) by taking $\beta = +1$ or -1 for $\hat{K}_j = \hat{L}_j$ or $\hat{K}_j = \hat{L}'_j, j = x, y, z$, respectively, and specializing the constants α_j to the particular values that they have in the corresponding models (λ_j or $\lambda'_j, j = 1,2,3$). Now the energy of the system is proportional to Lie algebra generators. It would be of interest to mention that a similar model of (15) has been considered in [21] for semiclassical Dicke model and the ideal parametric amplifier, however the treatments there have been given in the framework of pseudospin vector and/or pseudotensor and consequently simple geometrical arguments have been performed to explain the phenomena. Indeed, the model (15) is quite general for any operator system can fulfill the $SU(1,1)$ or $SU(2)$ Lie algebra rules. To discuss the dynamical behaviour of the model we may solve the Heisenberg equations of motion for the Hamiltonian (15) which are

$$\begin{aligned}
\frac{d\hat{K}_x}{dt} &= -\alpha_3\hat{K}_y + \beta\alpha_2\hat{K}_z,\\
\frac{d\hat{K}_y}{dt} &= \alpha_3\hat{K}_x - \beta\alpha_1\hat{K}_z,\\
\frac{d\hat{K}_z}{dt} &= -\alpha_2\hat{K}_x + \alpha_1\hat{K}_y.
\end{aligned} \tag{16}$$

The matrix representation of the solutions of these equations is

$$\begin{bmatrix} \hat{K}_x(t) \\ \hat{K}_y(t) \\ \hat{K}_z(t) \end{bmatrix} = \begin{bmatrix} R_1(t,\beta) & J^{(-)}(t,\beta) & \beta S^{(+)}(t) \\ J^{(+)}(t,\beta) & R_2(t,\beta) & \beta V^{(-)}(t) \\ S^{(-)}(t) & V^{(+)}(t) & R_3(t,\beta=1) \end{bmatrix} \begin{bmatrix} \hat{K}_x(0) \\ \hat{K}_y(0) \\ \hat{K}_z(0) \end{bmatrix}, \tag{17}$$

where

$$\begin{aligned}
R_j(t,\beta) &= \cos(gt) + 2\frac{\beta\alpha_j^2}{g^2}\sin^2(\frac{gt}{2}), \quad j = 1,2,3,\\
J^{(\pm)}(t,\beta) &= 2\frac{\beta\alpha_1\alpha_2}{g^2}\sin^2(\frac{gt}{2}) \pm \frac{\alpha_3}{g}\sin(gt),\\
S^{(\pm)}(t) &= 2\frac{\alpha_1\alpha_3}{g^2}\sin^2(\frac{gt}{2}) \pm \frac{\alpha_2}{g}\sin(gt),
\end{aligned}$$

$$V^{(\pm)}(t)=2\frac{\alpha_2\alpha_3}{g^2}\sin^2(\frac{gt}{2})\pm\frac{\alpha_1}{g}\sin(gt), \tag{18}$$

and $g=(\alpha_3^2+\beta\alpha_1^2+\beta\alpha_2^2)^{\frac{1}{2}}$. It is easy to check that the commutation relations (1) are still valid for solutions (17). Moreover, this solution is periodic with period $4\pi/g$, i.e. $\hat{K}_j(t+\frac{4n\pi}{g})=\hat{K}_j(t), n=0,1,2,\ldots$ provided that g is real. It is reasonable mentioning that one can alternatively work in the Schrödinger picture where the operators remain unchanged but the state vector of the model becomes time-dependent, i.e. $|\psi(t)\rangle=\exp(-it\hat{H})|\psi(0)\rangle$ where $|\psi(0)\rangle$ is the initial state of the system. Then using the disentanglement theorem of $SU(1,1)$ or $SU(2)$ Lie algebra [9] the problem can be treated in an algebric way.

Based on the results of the present section together with those of the 2nd section we discuss the $SU(1,1)$ and $SU(2)$ squeezing in the following section.

4 $SU(1,1)$ and $SU(2)$ squeezing

First, we consider the $SU(1,1)$ squeezing and investigate fluctuations in terms of PCS and BGCS. For this purpose, the relations (6), (7), (9a-b) and (17) should be used. After some calculations the quadrature variances $\langle(\Delta\hat{K}_x(t))^2\rangle_{\rm p}$ and $\langle(\Delta\hat{K}_y(t))^2\rangle_{\rm p}$ as well as $\langle\hat{K}_z(t)\rangle_{\rm p}$ for PCS can be written in the following forms

$$\begin{aligned}\langle(\Delta\hat{K}_x(t))^2\rangle_{\rm p}=2k\Big\{&|f(t,-1)|^2+\frac{[S^{(+)}(t)-\xi^*f(t,-1)-\xi f^*(t,-1)]^2}{(1-|\xi|^2)^2}\\&+\frac{S^{(+)}(t)[\xi^*f(t,-1)+\xi f^*(t,-1)-S^{(+)}(t)]}{(1-|\xi|^2)}\Big\},\end{aligned} \tag{19a}$$

$$\begin{aligned}\langle(\Delta\hat{K}_y(t))^2\rangle_{\rm p}=2k\Big\{&|g(t,-1)|^2+\frac{[V^{(-)}(t)-\xi^*g(t,-1)-\xi g^*(t,-1)]^2}{(1-|\xi|^2)^2}\\&+\frac{V^{(-)}(t)[\xi^*g(t,-1)+\xi g^*(t,-1)-V^{(-)}(t)]}{(1-|\xi|^2)}\Big\},\end{aligned} \tag{19b}$$

and

$$\langle\hat{K}_z(t)\rangle_{\rm p}=\frac{k}{(1-|\xi|^2)}\left\{(1+|\xi|^2)R_3(t,1)+2[\xi^*h(t)+\xi h^*(t)]\right\}, \tag{19c}$$

where we have used the following abbreviations

$$\begin{aligned}&f(t,\beta)=\frac{1}{2}[R_1(t,\beta)-iJ^{(-)}(t,\beta)],\qquad g(t,\beta)=\frac{1}{2}[J^{(+)}(t,\beta)-iR_2(t,\beta)],\\&h(t)=\frac{1}{2}[S^{(-)}(t)-iV^{(+)}(t)].\end{aligned} \tag{20}$$

Of course, $\beta=-1$ in the present case.

It is easy to check that relations (19) reduce to those of [9, 17] at $t=0$. From (11) and (19) it is evident that the fluctuations are independent of the value of k. In Figs. 1a-c we have plotted the squeezing factors $S_j(t)$ given by (11) after substituting from (19) against time t for shown values of the parameters. Further, in these figures first quadrature is always represented by the solid curve, whereas second quadrature is represented by the dashed curve. Now apart from the case $\phi=\frac{\pi}{4}$ which will be discussed shortly, one can observe that at $t=0$ there is squeezing in the $\hat{K}_x$ quadrature as expected since PCS are a type of squeezed states depending on the

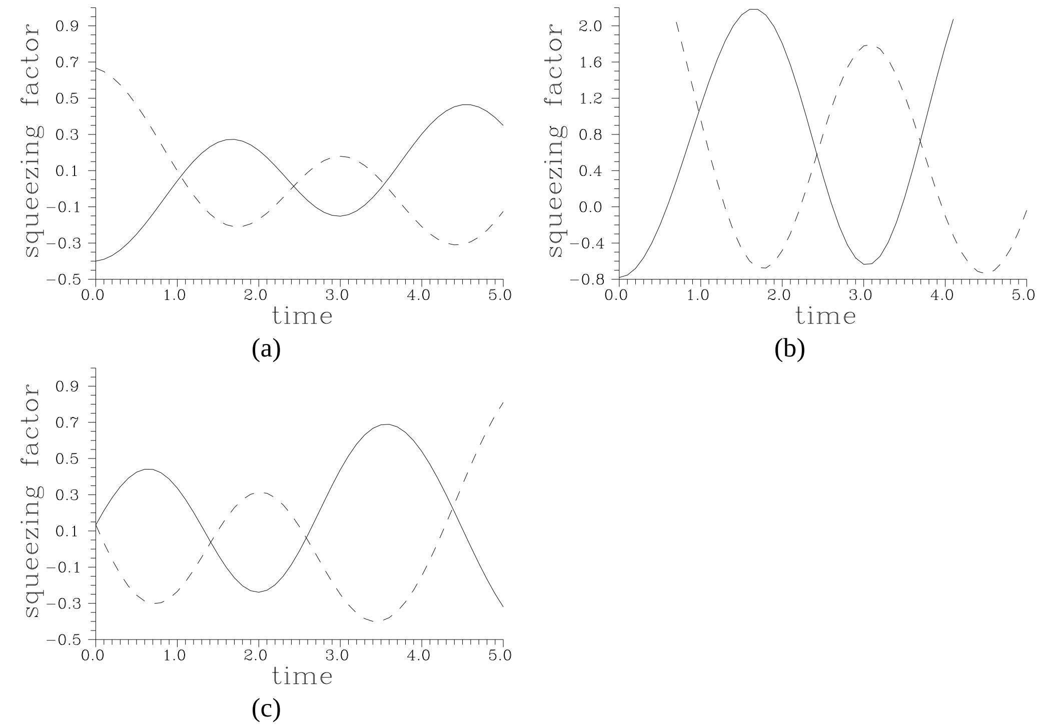

Fig. 1. Squeezing factor $S_j(t)$ of PCS against time t for $(\lambda_1, \lambda_2, \lambda_3) = (0.1, 0.25, 1)$ and for (a) $(\phi, |\xi|) = (\frac{\pi}{2}, 0.5)$; (b) $(\phi, |\xi|) = (\frac{\pi}{2}, 0.8)$; (c) $(\phi, |\xi|) = (\frac{\pi}{4}, 0.5)$. In these figures first quadrature is always represented by the solid curve, whereas second quadrature is represented by the dashed curve.

value of ϕ. When the time increases exchange of energy between modes starts to play a role, and consequently squeezing transfers to the second quadrature, and in the first one it disappears. This behaviour is periodically repeated as the interaction time increases. Further, it is clear that larger the parameter $|\xi|$, greater the squeezing which can be obtained. It is important mentioning that squeezing can be realized even if the initial states are not squeezed. This fact is demonstrated for the case $\phi = \frac{\pi}{4}$ where PCS are not squeezed (this is clear from fig.1c at $t = 0$), however, at later times periodical squeezing is generated which can be switched between the two quadratures. As we have shown before such behaviour can periodically appear with period $4\pi/g$. Indeed such behaviours require that $\lambda_3^2 > \lambda_1^2 + \lambda_2^2$, otherwise the initial squeezing of PCS will vanish when the interaction time increases since the solutions (17) in this case include hyperbolic functions which are monotonically increasing.

We proceed by focusing the attention on the behaviour of BGCS [16] specified by (7). The required quantities to discuss $SU(1,1)$ squeezing related to this state are

$$\langle(\Delta\hat{K}_x(t))^2\rangle_{\rm b} = 2|f(t,-1)|^2\left[n + \frac{|z|I_{2n}(2|z|)}{I_{2n-1}(2|z|)}\right] - S^{(+)}(t)[z^* f(t,-1) + z f^*(t,-1)]$$

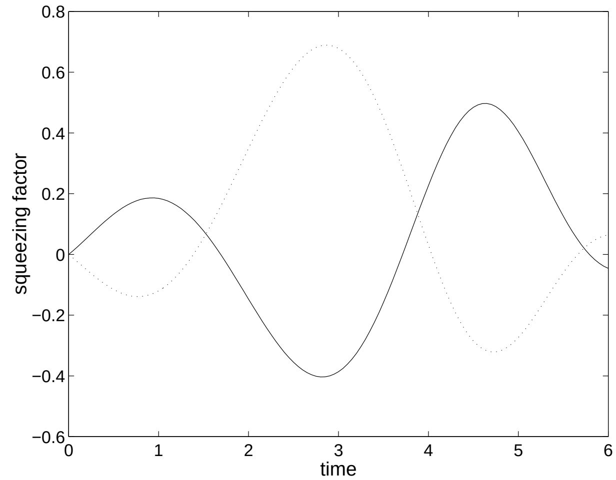

Fig. 2. Squeezing factor $S_j(t)$ of BGCS against time t for $n = 2, z = 10\exp(i\pi)$ and $(\lambda_1, \lambda_2, \lambda_3) = (0.1, 0.25, 1)$: first quadrature (solid curve), second quadrature (dot curve).

$$+|z|S^{(+)2}(t)\left[|z|\left(1-\frac{I_{2n}^2(2|z|)}{I_{2n-1}^2(2|z|)}\right)+(1-2n)\frac{I_{2n}(2|z|)}{I_{2n-1}(2|z|)}\right], \qquad (21a)$$

$$\langle(\Delta\hat{K}_y(t))^2\rangle_{\rm b} = 2|g(t,-1)|^2\left[n+\frac{|z|I_{2n}(2|z|)}{I_{2n-1}(2|z|)}\right] - V^{(-)}(t)[z^*g(t,-1)+zg^*(t,-1)]$$
$$+|z|V^{(-)2}(t)\left[|z|\left(1-\frac{I_{2n}^2(2|z|)}{I_{2n-1}^2(2|z|)}\right)+(1-2n)\frac{I_{2n}(2|z|)}{I_{2n-1}(2|z|)}\right], \qquad (21b)$$

and

$$\langle\hat{K}_z(t)\rangle_{\rm b} = R_3(t,1)\left[n+\frac{|z|I_{2n}(2|z|)}{I_{2n-1}(2|z|)}\right] + z^*h(t) + zh^*(t), \qquad (21c)$$

where $f(t,-1), g(t,-1)$ and $h(t)$ are given in (20). As we mentioned earlier BGCS is similar to the Glauber coherent state, i.e. it is a minimum-uncertainty state. However, it has been shown that the superposition of such states (even- and odd-BGCS) can produce squeezing as a result of the quantum mechanical interference between the components of the state in phase space [22]. Also in the model under discussion this state can evolve to produce squeezing (see Fig. 2 for shown values of the parameters). It is clear that squeezing is generated and interchanged between the two components provided that $\lambda_3^2 > \lambda_1^2 + \lambda_2^2$.

Second, we study the $SU(2)$ squeezing in terms of $SU(2)$ CS (8) as we did before. After straightforward calculations the quadrature variances $\langle(\Delta\hat{K}_x(t))^2\rangle_{\rm u2}$ and $\langle(\Delta\hat{K}_y(t))^2\rangle_{\rm u2}$ as well as $\langle\hat{K}_z(t)\rangle_{\rm u2}$ are

$$\langle(\Delta\hat{K}_x(t))^2\rangle_{\rm u2} = 2j\left\{\frac{[S^{(+)}(t)-\mu^*f(t,1)-\mu f^*(t,1)][\mu^*f(t,1)+\mu f^*(t,1)]}{(1+|\mu|^2)}\right.$$
$$\left.+|f(t,1)|^2+\frac{|\mu|^2[S^{(+)}(t)-\mu^*f(t,1)-\mu f^*(t,1)]^2}{(1+|\mu|^2)^2}\right\}, \qquad (22a)$$

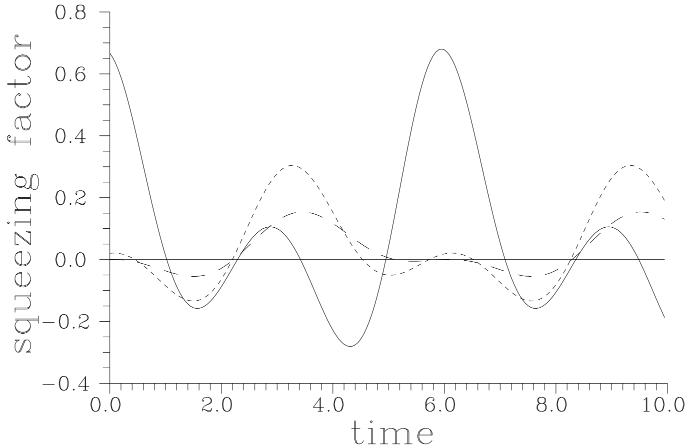

Fig. 3. Squeezing factor $S_1(t)$ (first quadrature) of $SU(2)$ squeezing against time t for $(\lambda_1, \lambda_2, \lambda_3) = (0.1, 0.25, 1)$, $\phi = \frac{\pi}{2}$ and $|\mu| = 0.5$ (solid curve), 10 (long-dashed curve), 100 (short-dashed curve). Straightline is the squeezing bound.

$$\langle(\Delta\hat{K}_y(t))^2\rangle_{\mathrm{u2}} = 2j\left\{\frac{[V^{(+)}(t) - \mu^* g(t,1) - \mu g^*(t,1)][\mu^* g(t,1) + \mu g^*(t,1)]}{(1+|\mu|^2)}\right.$$
$$\left. + |g(t,1)|^2 + \frac{|\mu|^2[V^{(+)}(t) - \mu^* g(t,1) - \mu g^*(t,1)]^2}{(1+|\mu|^2)^2}\right\}, \qquad (22b)$$

and

$$\langle\hat{K}_z(t)\rangle_{\mathrm{u2}} = \frac{2j}{(1+|\mu|^2)^2}\left\{R_3(t,1)(|\mu|^2 - 1) + 2[\mu^* h(t) + \mu h^*(t)]\right\}, \qquad (22c)$$

where $f(t,1), g(t,1)$ and $h(t)$ are given in (20). In Fig. 3 we have plotted squeezing factor of the first quadrature against time t for the shown values of the parameters. From this figure one can observe that there is no initial squeezing and this is in contrast with $SU(1,1)$ squeezing case (compare solid curves in Figs. 1 and Fig. 3). As a result of the interaction of the field with the material media squeezing can occur periodically with maximum value smaller than for $SU(1,1)$ squeezing. Also one may observe that the degree of squeezing decreases as the values of $|\mu|$ increase (i.e. decreasing the initial mean photon number) and this is in contrast with $SU(1,1)$ squeezing where the opposite situation is established for a given $|\xi|$ (as it is well known that the initial mean photon number increases as $|\xi|$ increases). In other words, when the initial mean photon number increases the degrees of squeezing of both $SU(1,1)$ and $SU(2)$ squeezing increase, too.

5 Conclusions and remarks

In this work we have studied $SU(1,1)$ and $SU(2)$ squeezing of interacting systems of radiation modes in a quadratic medium in the framework of Lie algebra. Particular examples have been given for three mode case, however, the model is quite general and may be applied to any Hamiltonian consisting of a set of operators obeying these kinds of Lie algebra. In other words, from Hamiltonian (15) one can recognize that the boson operators are not explicitly involved and the models become indistinguishable. This means that if we have a model (say) which includes several modes, but its Hamiltonian can be represented as a linear combination of $SU(1,1)$ or $SU(2)$ squeezing Lie algebra generators, the behaviour of the degree of squeezing of this model can be the same as discussed here. For the considered models we have shown that squeezing is reached for both PCS, BGCS and $SU(2)$ CS, and can be periodically recovered provided that g is real. We conclude this article by referring to [23] where two kinds of two-mode squeezing (sum and difference squeezing) have been discussed. Sum squeezing is described by operators which form a representation of the $SU(1,1)$ Lie algebra, whereas operators of difference squeezing form $SU(2)$ Lie algebra. Both of these kinds can be turned into normal squeezing and consequently can be detected. Unfortunately, this situation cannot be established here, where the Hamiltonian itself is represented in terms of the quadrature operators and any modification in the quadratures should be reflected in the structure of the Hamiltonian. More illustratively, the used quadratures in this article are represented bilinearly in bosonic operators and consequently they can be converted into normal squeezing. That is restricting ourselves on $SU(1,1)$ squeezing and considering modes 1 and 2 are strong, they can be replaced by $|\Gamma_j|\exp(i\phi_j), j = 1,2$ where $|\Gamma_j|$ and ϕ_j are their amplitudes and phases, further taking $\phi_2 = \phi_1 + \pi/2$. In this case the quadratures (13) reduce to those of normal squeezing as

$$\hat{L}_x = -|\Gamma_2|[\hat{A}_3\exp(-i\phi_1) + \hat{A}_3^\dagger\exp(i\phi_1)], \quad \hat{L}_y = i|\Gamma_1|[\hat{A}_3\exp(-i\phi_1) - \hat{A}_3^\dagger\exp(i\phi_1)],$$
$$\hat{L}_z = -2|\Gamma_1||\Gamma_2|. \tag{23}$$

However, the price is payed that the Hamiltonian becomes a linear combination of creation and annihilation operators which cannot provide squeezing as well as the rules of the Lie algebra are not established. In conclusion, we have showed that special types of three modes interacting bilinearly in a nonlinear crystal can provide squeezing. This can be achieved in sum (difference)-frequency generation where the interaction arises from the second-order polarizability of the nonlinear medium. Squeezing in the quadratures $\hat{K}_j, j = x,y$ of the input field can be observed by studying the standard quadrature of the output field [24], e.g. through heterodyne detector. Moreover, such realization seems to be more feasible using the $SU(2)$ and $SU(1,1)$ interferometers [6]. In the case of $SU(1,1)$ interferometer the beam splitters of a conventional interferometer have been replaced by the four-wave mixers and consequently it has a simpler construction than the $SU(2)$ interferometer. Indeed, this fact together with periodic solution of equations of motion with the period $4\pi/g$ can be used for obtaining squeezing on a rather long time scale.

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