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HAMILTONIAN FORMULATION AND BOUNDARY CONDITIONS IN YANG-MILLS THEORY

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The Hamiltonian formulation of pure Yang-Mills theory is analysed in the case when Gauss' law is satisfied identically by construction. It is shown that the theory has a Hamiltonian formulation in this case, provided one uses a special gauge condition, which is a natural generalisation of the Coulomb gauge condition in electrodynamics. The Hamiltonian formulation depends critically also on the boundary conditions used for the relevant field variables. Possible boundary conditions are analysed in detail. A comparison of the present formulation in the generalised Coulomb gauge with the well known Weyl gauge ($A_0 = 0$) formulation is made. It appears that the Hamiltonians in these two formulations differ from one another in a non-trivial way. It is still an open question whether these differences give rise to truly different structures upon quantisation. An extension of the formalism to include coupling to fermionic fields is briefly discussed.

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1 Introduction

This paper is concerned with analysing the Hamiltonian formulation of pure Yang-Mills theory [1] along the lines of two previous publications [2, 3], in which a new gauge condition, called the generalized Coulomb gauge condition was introduced and used to obtain a straightforward Hamiltonian formulation of Yang-Mills theory, in the case when Gauss law is satisfied identically by construction. In the previous papers certain assumptions were made concerning the spatial boundary conditions of the Yang-Mills potentials. The boundary conditions in question are very important for the elucidation of the Hamiltonian formulation. Here I will analyse the requisite boundary conditions in detail, and show that there is a set of boundary conditions at spatial infinity, which is consistent with the Hamiltonian formalism. Rigorous proofs and heavy mathematical machinery are omitted here in the interest of simplicity.

The basic quantity of Yang-Mills theory is the *action* S, which is given in terms of the gauge field $G_{\mu\nu}$ as follows,

$$S = -\frac{1}{4}\int d^4x (G_{\mu\nu}(A), G^{\mu\nu}(A)). \tag{1}$$

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where the inner product $(\ ,\ )$ is defined below. For the gauge field $G_{\mu\nu}$ I use a matrix notation (summation over repeated indices),

$$G_{\mu\nu}(A) \equiv G_{\mu\nu}^{\ a}(A)T_a = \partial_\nu A_\mu(x) - \partial_\mu A_\nu(x) - ig[A_\mu(x), A_\nu(x)], \tag{2}$$

where the quantities T_a are matrices in a convenient representation of the Lie algebra of the gauge group G,

$$[T_a, T_b] = if_{ab}^{\ c}T_c, \tag{3}$$

and the quantities A_μ are given in terms of the the gauge potential components A_μ^a,

$$A_\mu(x) = A_\mu^{\ a}T_a. \tag{4}$$

The Lie-algebra indexfree notation introduced above, i.e. the matrix notation (2) for the gauge field $G_{\mu\nu}$ as well as the notation (4) for the gauge potential A_μ, respectively, using the terminology of C. N. Yang [4], will be used systematically below.

It is assumed that the gauge group G is semisimple and compact. The inner product $(\ ,\ )$ for any two Lie algebra valued (matrix valued) quantities $A = A^aT_a$ and $B = B^aT_a$ is then expressed with the aid of the Lie algebra structure constants $f_{ab}^{\ c}$ as follows,

$$(A, B) = h_{ab}A^aB^b, \tag{5}$$

where

$$h_{ab} = -f_{ab'}^{\ \ c'}f_{bc'}^{\ \ b'}. \tag{6}$$

The quantity h_{ab} and its inverse h^{ab} are used to lower and rise Lie algebra indices, respectively. The notation used here is otherwise pretty standard or self-explanatory, with e.g. Greek letters used as indices denoting Minkowski space indices ranging from 0 to 3, and latin indices from the middle of the alphabet $(k, \ell, ...)$ denoting space indices ranging from 1 to 3. The Minkowski space metric is taken to be diagonal, with signature $(+,-,-,-)$. Unless otherwise stated, repeated indices are always summed over, be they Lie algebra-, spacetime- or space indices.

It is convenient for future reference to write the action (1) in terms of a Lagrangian L. The action S is the integral of the Lagrangian L in an appropriate time interval $[x_i^0, x_f^0]$. Thus,

$$S = \int_{x_i^0}^{x_f^0} dx^0 L, \tag{7}$$

where

$$L = -\frac{1}{2}\int_V d^3\mathbf{x}\left(G_{0k}(A), G^{0k}(A)\right) - \frac{1}{4}\int_V d^3\mathbf{x}\left(G_{k\ell}(A), G^{k\ell}(A)\right). \tag{8}$$

In the expression (8) for the Lagrangian L, the quantity V is some appropriate domain in $\mathbf{R}^3$, which yet has to be specified.

As is well known, requiring the action (1) to be stationary with respect to local variations of *all* the potential components A_μ, considered as independent quantities, yields the follwing field equations,

$$\nabla_\nu(A)G^{\mu\nu}(A) = 0\ , \mu = 0, 1, 2, 3. \tag{9}$$

The "covariant gradient" $\nabla_\mu(A)$ used above in Eq. (9) is a convenient notion,

$$\nabla_\mu(A) \equiv \partial_\mu + ig\,[A_\mu, \quad], \tag{10}$$

which will be frequently used in what follows. The matrix-differential operator $\nabla_\mu(A)$ is applicable to any linear combination of the Lie-algebra matrices T_a in a given representation, $\beta^a(x)T_a$, with differentiable components β^a, and yields the covariant derivative in the adjoint representation for the array of components (β^a), independently of the particular representation used for the matrices T_a.

The non-Abelian Gauss law is obtained from the equations (9) for $\mu = 0$. Expressed in terms of the potential A the non-Abelian Gauss law reads as follows,

$$\nabla_k(A)\nabla^k(A)A^0 - \nabla_k(A)\dot{A}^k = 0, \tag{11}$$

where

$$\dot{A}_k(x) \equiv \partial_0 A_k(x). \tag{12}$$

The time derivative of any quantity will frequently in what follows be denoted by a dot on top of that quantity, as in the equation (12) above.

1.1 The Hamiltonian or Weyl gauge formalism

It is well known [6] that Yang-Mills theory can be expressed in a canonical form in the Hamiltonian gauge, or so-called Weyl gauge [7] $A_0 = 0$.

The Yang-Mills Lagrangian in the case $A_0 = 0$, which will be called L_W here, is obtained from the expression (8) by putting $A_0 = 0$ in that expression,

$$L_W = -\frac{1}{2}\int d^3\mathbf{x}(\dot{A}_k, \dot{A}^k) - \frac{1}{4}\int d^3\mathbf{x}(G_{k\ell}, G^{k\ell}). \tag{13}$$

The Lagrangian (13) describes a theory which is not quite the same as Yang-Mills theory, since Gauss' law is absent from the field equations following from the action principle with the expression (13) as Lagrangian. Gauss' law takes the following form in the case when $A_0 = 0$, as seen from Eq. (11) above,

$$\nabla_k(A)\dot{A}^k = 0. \tag{14}$$

The usual way to analyse Yang-Mills theory in the Weyl gauge $A_0 = 0$, is to proceed from the Lagrangian (13), disregarding Gauss' law to begin with. Using the variables A^a_k and $\dot{A}^a_k$ as generalised coordinates and velocities, respectively, it is then perfectly simple to derive a canonical Hamiltonian formalism for the system defined by the Lagrangian (13). The corresponding Hamiltonian H_W is,

$$H_W = -\frac{1}{2}\int d^3\mathbf{x}(\pi_k, \pi^k) + \frac{1}{4}\int d^3\mathbf{x}(G_{k\ell}, G^{k\ell}), \tag{15}$$

i.e. a simple sum of a kinetic term, depending on canonical momenta π^k_a only, and a potential, or interaction term, depending on conjugate canonical coordinates A^a_k only.

2 First attempt at a canonical formulation when Gauss' law is in force

Gauss' law, Eq. (11) above, will now be considered as an equation determining the (matrix valued) potential component A_0, for given space components $\mathbf{A}$ and $\dot{\mathbf{A}}$. This equation is a system of *linear, elliptic partial differential equations* with time x^0 acting as a parameter in an appropriate interval, with $\mathbf{x} \in V$ being the independet variables. I will discuss the solvability of Eq. (11) subsequently, but proceed now by assuming the existence of unique solution A_0, which is a *functional* of the space components $\mathbf{A}$ and their time derivatives $\partial_0 \mathbf{A} \equiv \dot{\mathbf{A}}$, i.e.

$$A_0 = A_0 \left\{ \mathbf{A}, \dot{\mathbf{A}} \right\}. \tag{16}$$

The question is then whether the Yang-Mills system, which is originally defined by the action (1), permits a Hamiltonian formulation when the potential component A_0 is a solution to Gauss' law (11), i.e. when A_0 is given in terms of $\mathbf{A}$ and $\dot{\mathbf{A}}$ by the expression (16). It is possible to get some insight into this question without specifying the actual functional form of the relation (16) in minute detail.

The Lagrangian of the Yang-Mills system, when the potential component A_0, is given by the relation (16) above, is obtained simply by inserting the solution (16) for A_0 into the Lagrangian (8) above. The resulting Lagrangian will be called L_0, and is explicitly given as follows,

$$\begin{aligned} L_0 \quad = \quad & -\frac{1}{2} \int_V d^3\mathbf{x} \left(\nabla_k(\mathbf{A}) A_0 \left\{ \mathbf{A}, \dot{\mathbf{A}} \right\} - \dot{\mathbf{A}}_k, \nabla^k(\mathbf{A}) A^0 \left\{ \mathbf{A}, \dot{\mathbf{A}} \right\} - \dot{\mathbf{A}}^k \right) \\ & -\frac{1}{4} \int_V d^3\mathbf{x} \left(G_{kl}(\mathbf{A}), G^{kl}(\mathbf{A}) \right). \end{aligned} \tag{17}$$

At this point it is appropriate to check whether the action principle involving the Lagrangian L_0 in (17) above reproduces the field equations (9). It is perfectly straightforward to verify the following result,

$$\begin{aligned} \delta \int_{x_i^0}^{x_f^0} dx^0 L_0 = & - \int_{x_i^0}^{x_f^0} dx^0 \int_V d^3\mathbf{x} \left(\delta A_k, \nabla_0(A) (\nabla^k(A) A^0 \left\{ \mathbf{A}, \dot{\mathbf{A}} \right\} - \dot{\mathbf{A}}^k) - \nabla_\ell G^{k\ell}(A) \right) \\ & - \int_{x_i^0}^{x_f^0} dx^0 \int_{\partial V} d^2\sigma_k \left(\delta A^0 \left\{ \mathbf{A}, \dot{\mathbf{A}} \right\}, \nabla^k(A) A^0 \left\{ \mathbf{A}, \dot{\mathbf{A}} \right\} - \dot{\mathbf{A}}^k \right). \end{aligned} \tag{18}$$

Now the boundary conditions for the solution A_0 to Gauss' law, i.e. the system of linear elliptic partial differential equations (11), enter into the discussion. If the surface term in Eq. (18) is non-vanishing, and not by itself a variation of some surface functional, then the Lagrangian (17) is not a valid Lagrangian in the action principle which is supposed to lead to the field equations (9) for $\mu = 1, 2, 3$, when A_0 is given by (16).

In the first place the domain V is actually considered to be all of $\mathbf{R}^3$. The integrals over V will be given the following interpretation,

$$\int_V d^3\mathbf{x} \cdot \cdot = \lim_{R \to \infty} \int_{|\mathbf{x}| < R} d^3\mathbf{x} \cdot \cdot \tag{19}$$

The vanishing of the surface term in Eq. (18) is then equivalent to the follwing,

$$\lim_{R \to \infty} \int_{|\mathbf{x}| = R} d\Omega R^2 \left(\delta A^0[\mathbf{A}, \dot{\mathbf{A}}], \nabla^{(r)}(A) A^0 \left\{ \mathbf{A}, \dot{\mathbf{A}} \right\} - \dot{\mathbf{A}}^{(r)} \right) = 0, \tag{20}$$

where the superscript (r) denotes the *radial* component of the corresponding quantity.

Thus, *if* the surface term (20) vanishes for all admissible variations of the independent generalised coordinates $\mathbf{A}$ and velocities $\dot{\mathbf{A}}$, respectively, then the variational principle

$$\delta \int_{x_i^0}^{x_f^0} dx^0 L_0 = 0, \tag{21}$$

leads to the following equations of motion,

$$\nabla_0(A)(\nabla^k(A)A^0 \left\{\mathbf{A}, \dot{\mathbf{A}}\right\} - \dot{\mathbf{A}}^k) - \nabla_\ell G^{k\ell}(A) = 0, \tag{22}$$

as is evident from the relation (18). Needless to say, the equations (22) are nothing but the field equations (9) for $\mu = 1, 2, 3$, with A_0 given by the formal solution (16) to Gauss' law (11). The vanishing of the surface term (20) depends on the assumed asymptotic behaviour of the *independent* variables $\mathbf{A}$ and $\dot{\mathbf{A}}$, as well as on the boundary conditions (at spatial infinity) of the *dependent* variable A_0. I will return to this question below, and continue for the time being by assuming that the relation (20) is valid. Then the Lagrangian L_0 given in (17) ought to be a suitable starting point for the construction of a Hamiltonian and the corresponding canonical variables by means of a Legendre transform in the usual way.

The formal definition of the canonical momentum P_k^a conjugate to the coordinate A_a^k is,

$$P_k^a(x^0, \mathbf{x}) \equiv \frac{\delta L_0}{\delta \dot{A}_a^k(x^0, \mathbf{x})} = \left(\nabla_k(A)A_0 \left\{\mathbf{A}, \dot{\mathbf{A}}\right\}\right)^a (x^0, \mathbf{x}) - \dot{A}_k^a(x^0, \mathbf{x}), \tag{23}$$

where the condition (20) has been used in the calculation of the functional derivative of L_0 in (23) above. In view of the fact that A_0 in Eq. (23) satisfies Gauss' law (11), one finds immediately from (23) that

$$\nabla^k(A)P_k(x^0, \mathbf{x}) \equiv 0. \tag{24}$$

Now one is supposed to be able to solve Eq. (23) for the generalised velocity $\dot{A}_k^a$ in terms of $\mathbf{A}$ and $\mathbf{P}$, respectively. But this is impossible, since, Eq. (23) can not be solved for the quantity Γ defined below,

$$\Gamma(x^0, \mathbf{x}) \equiv \nabla_k(A)\dot{A}^k(x^0, \mathbf{x}), \tag{25}$$

i.e. if one tries to derive an equation for the quantity Γ defined above from Eq. (23), one gets a completely vacuous identity for this quantity, as a result of Eq. (24).

It would seem then, that the canonical formalism breaks down for the case at hand. However this is not necessarily the case. The difficulty described above can be avoided if one can manage to make A_0 *independent* of the generalized velocity variables $\dot{\mathbf{A}}$. This can be accomplished in the present situation ($A_0 \neq 0$) by imposing the following *gauge condition*,

$$\nabla_k(A)\dot{A}^k(x^0, \mathbf{x}) = 0. \tag{26}$$

The condition (26) is the *generalized Coulomb gauge condition* referred to previously. The fact that this is actually a *proper gauge condition*, free from Gribov ambiguities [5], under appropriate boundary conditions, was discussed in Ref. [2].

However, if one uses the gauge condition (26) then the generalised velocities $\dot{A}^k$ are no longer independent quantities, and then one cannot use the formula (23) as it stands for the construction of the canonical momentum variables. An alternative procedure which leads to a proper canonical formalism will be given below.

3 Gauss' law and asymptotic conditions

I now assume that the generalized Coulomb gauge condition (26 is in force. Then Gauss' law, Eq. (11), takes the following form,

$$\nabla_k(A)\nabla^k(A)A^0 = 0. \tag{27}$$

If one demands that the solution to Eq. (27) vanishes roughlty speaking faster than $\mid \mathbf{x} \mid^{-\frac{1}{2}}$ for $\mid \mathbf{x} \mid \rightarrow \infty$, then the solution vanishes identically. This can be seen as follows. Take the inner product of Eq. (27) with A_0 and integrate over $\mathbf{x}$. Using the ordinary divergence theorem one readily obtains the following result,

$$\lim_{R\rightarrow\infty}\int_{|\mathbf{x}|<R} d^3\mathbf{x}(\nabla_k(A)A_0, \nabla^k(A)A_0) = \lim_{R\rightarrow\infty}\int_{|\mathbf{x}|=R} d\Omega R^2(A_0, \nabla^{(r)}(A)A_0). \tag{28}$$

Assuming now

$$A_0(x^0,\mathbf{x}) \mid_{|\mathbf{x}|=R} \sim R^{-\gamma}\ ,\ \ \nabla^{(r)}(A)A_0(x^0,\mathbf{x}) \mid_{|\mathbf{x}|=R} \sim R^{-\gamma-1}\ , \gamma > \frac{1}{2}, \tag{29}$$

one finds that the limiting value of the right hand side of Eq. (28) is zero. Since the inner product $(\ ,\ )$ is positive definite, one then concludes from Eq. (28) that A_0 is *covariantly constant*. However, since A_0 vanishes at infinity by assumption, the covariant constant is zero, i.e.

$$A_0(x^0,\mathbf{x}) \equiv 0. \tag{30}$$

Needless to say, the asymptotic condition (29) above can be somewhat refined; all that is needed to obtain the result (30) is that the right hand side of Eq. (28) vanishes, which guarantees that A_0 is covariantly constant, and then that A_0 has the limiting value 0 at infinity (or at some finite point), so that the covariant constant in question actually vanishes.

However, the class of functions with (roughly speaking) the asymptotic behaviour (29) does not exhaust the class of possible solutions to Eq. (27). It is also possible to consider functions A_0 which approach a non-vanishing *constant* matrix at space infinity,

$$\lim_{|\mathbf{x}|\rightarrow\infty} A_0(x^0,\mathbf{x}) = \Lambda \equiv \Lambda^a T_a, \tag{31}$$

where the real quantities Λ^a are absolute constants. In addition to Eq. (31) one can impose an asymptotic condition on the radial derivative of A_0,

$$\lim_{|\mathbf{x}|\rightarrow\infty} \mid \mathbf{x} \mid \frac{\partial}{\partial \mid \mathbf{x} \mid} A_0(x^0,\mathbf{x}) = 0. \tag{32}$$

Requiring the validity of the asymptotic conditions (31) and (32) and using standard arguments of potential theory [8], one now derives an integral representation for the function A_0 involving the (ordinary) Laplacian of that function,

$$A_0(x^0,\mathbf{x}) = \Lambda - \frac{1}{4\pi}\int_{\mathbf{R}^3} d^3\mathbf{y}\frac{1}{\mid \mathbf{x}-\mathbf{y} \mid}\nabla^2_{\mathbf{y}}A_0(x^0,\mathbf{y}), \tag{33}$$

Using the the present form of Gauss' law, Eq. (27), one then converts (33) into an integral equation for the determination of A_0. For this purpose it is convenient to introduce some new notation,

$$U^{ka}{}_b(x^0,\mathbf{y}) := 2gf^a{}_{bc}A^{kc}(x^0,\mathbf{y}), \tag{34}$$

and

$$V^a{}_b(x^0,\mathbf{y}) := 2gf^a{}_{bc}\frac{\partial}{\partial y^k}A^{kc}(x^0,\mathbf{y}) + g^2 f^a{}_{c'd}f^{c'}{}_{ba'}A_k{}^d(x^0,\mathbf{y})A^{ka'}(x^0,\mathbf{y}). \tag{35}$$

One then obtains the following integral equations,

$$A_0^a(x^0,\mathbf{x}) = \Lambda^a - \frac{1}{4\pi}\int_{\mathbf{R}^3}\frac{d^3\mathbf{y}}{\mid \mathbf{x}-\mathbf{y}\mid}\left\{U^{ka}{}_b(x^0,\mathbf{y})\frac{\partial A_0^b(x^0,\mathbf{y})}{\partial y^k} + V^a{}_b(x^0,\mathbf{y})A_0^b(x^0,\mathbf{y})\right\} \tag{36}$$

The integral equations (36) constitute the starting point for the proof of existence of solutions to the present form of Gauss' law, Eq. (27). For this one needs naturally also to specify conditions on the potential components A_k^a, which, together with their space derivatives determine the (unique) solution to Eq. (36). All this is a part of the "heavy mechinery" mentioned in the Introduction, which I will omit in this paper. However it is appropriate to note the following asymptotic conditions, which are needed for the existence of solutions A_0^a to Eq. (36),

$$A_k^a(x^0,\mathbf{x}) \;\sim\; \frac{1}{\mid \mathbf{x}\mid^{1+\epsilon}}\;,\;\; \frac{\partial}{\partial x^\ell}A_k^a(x^0,\mathbf{x}) \;\sim\; \frac{1}{\mid \mathbf{x}\mid^{2+\epsilon}}\;,\; \epsilon>0. \tag{37}$$

I will now denote the solution of the system of integral equations (36) by $A_0\{\mathbf{A}\}$, and the corresponding Lagrangian by L_{00} (compare with Eq. (17)) ,

$$\begin{aligned} L_{00} \;\;=\;\; & -\frac{1}{2}\int_V d^3\mathbf{x}\left(\nabla_k(\mathbf{A})A_0\{\mathbf{A}\} - \dot{\mathbf{A}}_k, \nabla^k(\mathbf{A})A^0\{\mathbf{A}\} - \dot{\mathbf{A}}^k\right) \\ & -\frac{1}{4}\int_V d^3\mathbf{x}\left(G_{kl}(\mathbf{A}), G^{kl}(\mathbf{A})\right). \end{aligned} \tag{38}$$

The Lagrangian (38) will now in the next section be used to derive the Hamiltonian for the Yang-Mills system under the condition that the generalized Coulomb gauge (26) is in force. I implement this condition as a *constraint*, by means of a (matrix valued) Lagrange multiplier field $C(x)$, which is used to modify the Lagrangian (38) as follows,

$$L_{00} \to L' = L_{00} + \int_V d^3\mathbf{x}(C(x), \nabla_k(A)\dot{A}^k). \tag{39}$$

4 Canonical coordinates, momenta and Hamiltonian

I now make a direct transition to a Hamiltonian formulation using the modified Lagrangian (39) above, in a manner described in the general case by Berezin [9]. The starting point is the familiar definition of canonical momentum variables π_k^a,

$$\pi_k^a(x^0,\mathbf{x}) \equiv \frac{\delta L'}{\delta \dot{A}_a^k(x^0,\mathbf{x})} = \left(\nabla_k(A)A_0\{\mathbf{A}\}\right)^a - \dot{A}_k^a - \left(\nabla_k(A)C\right)^a, \tag{40}$$

The equations (40) above, *together* with the constraint equations (26) are now supposed to be solved for the quantities $\dot{A}_k^a$ and C^a in terms of the canonical coordinates A_a^k and momenta π_k^a, respectively. Using Eqns. (26) and (27), one finds immediately from Eq. (40) that

$$-\nabla_k(A)\nabla^k(A)C = \nabla^k(A)\pi_k, \tag{41}$$

which, together with appropriate boundary conditions, defines the quantity C as an x-dependent functional of $\mathbf{A}$ and π, respectively,

$$C = C\left\{\mathbf{A}, \pi\right\}(x^0, \mathbf{x}). \tag{42}$$

It is certainly desirable that the solution $C\left\{\mathbf{A}, \pi\right\}$ to Eq. (41) be *unique*. The uniqueness is guaranteed if one uses the following boundary condition,

$$\lim_{R\to\infty} \int_{|\mathbf{x}|=R} d\Omega R^2 (C(x), \nabla^{(r)}(A)C(x)) = 0. \tag{43}$$

The proof of uniqueness of the solution to Eq. (41) under the condition (43) has essentially already been given above in connection with Eq. (27). Namely, if there are two distinct solutions to Eq. (41), then their difference satisfies the corresponding homogeneous equation, which is precisely of the form (27). However, under the condition (43), the homogeneous equation in question has only the trivial zero solution, as demonstrated in the discussion following Eq. (27). Hence the solution $C\left\{\mathbf{A}, \pi\right\}$ is unique.

One now straightforwardly expresses the generalised velocity in terms of coordinate- and momentum variables,

$$\dot{A}_k^a = \left(\nabla_k(A_0\left\{\mathbf{A}\right\} - C)\right)^a - \pi_k^a. \tag{44}$$

The construction of the Hamiltonian H then proceeds in the usual way. The relation defining the Hamiltonian H is the following,

$$H = \int_V d^3\mathbf{x}(\pi_k, \dot{A}^k) - L_{00}, \tag{45}$$

where the quantity $\dot{A}^k$ ocurring in the expressions in (45) should be given in terms of canonical variables by the expression (44). It should be observed, that it is indeed the Lagrangian L_{00} which enters in the definition of the Hamiltonian H above, since at this stage the constraint (26) is an identity.

By straightforward calculation one finally obtains the Hamiltonian expressed in terms of canonical variables from the definition (45),

$$\begin{aligned} H \quad = \quad & -\frac{1}{2}\int_V d^3\mathbf{x}\left(\pi_k, \pi^k\right) + \frac{1}{4}\int_V d^3\mathbf{x}\left(G_{kl}(A), G^{kl}(A)\right) \\ & + \int_V d^3\mathbf{x}\left(\pi_k, \nabla^k(A)A^0\left\{\mathbf{A}\right\}\right) + \frac{1}{2}\int_V d^3\mathbf{x}\left(\nabla_k(A)C, \nabla^k(A)C\right), \end{aligned} \tag{46}$$

where the quantity C is the appropriate solution to the system of linear elliptic partial differential equations (41), as discussed previously.

5 The Hamiltonian equations of motion

The Hamiltonian equations of motion are obtained by functional differentiation of the Hamiltonian (46) with respect to the canonical momenta and coordinates, respectively.

The equations of motion for the coordinates are as follows,

$$\dot{A}^a_k(x^0,\mathbf{x}) \equiv \frac{\delta H}{\delta \pi^k_a(x^0,\mathbf{x})} = (\nabla_k(A)(A_0\{\mathbf{A}\} - C))^a - \pi^a_k, \tag{47}$$

which agree precisely with the expressions (44) as they should. In the calculation leading to Eq. (47) one encounters the following surface term,

$$\lim_{R\to\infty}\int_{|\mathbf{x}|<R} d^3\mathbf{x}\partial_k(C,\nabla^k(A)\delta_\pi C) = \lim_{R\to\infty}\int d\Omega R^2(C,\nabla^{(r)}(A)\delta_\pi C), \tag{48}$$

which must vanish in order that the functional derivative in Eq. (47) be well defined. The surface term (48) in question vanishes as it should, in view of the general boundary conditions (43) imposed on the quantity $C\{\mathbf{A},\pi\}$.

The calculation of the functional derivative of the Hamiltonian (46) with respect to the generalized coordinate variables A^a_k is fairly lengthy, but nevertheless straightforward. In the course of the calculation one finds that a surface term analoguous to Eq. (20) has to vanish in order that the functional derivative in question be well defined, i.e. that

$$\lim_{R\to\infty}\int_{|\mathbf{x}|=R} d\Omega R^2\left(\delta A^0[\mathbf{A}],\nabla^{(r)}(A)A^0\{\mathbf{A}\} - \dot{\mathbf{A}}^{(r)}\right) = 0. \tag{49}$$

The condition (49) is quite non-trivial. It should be remembered, that the variation of the quantity A_0 can not be declared to vanish outside some finite region, since A_0 is a dependent variable, determined by Gauss' law, i.e. in the present case by Eq. (36). Under essentially the conditions (37) with $\epsilon > 0$, I have been able to prove, that the iterative solution of the Eqns. (36), which are equivalent to the system of partial differential equations (27) with the boundary conditions (31) and (32), is such that, for large $|\ \mathbf{x}\ |$,

$$\delta A_0(\mathbf{A})(x^0,\mathbf{x}) = O\left(\frac{1}{|\ \mathbf{x}\ |}\right), \tag{50}$$

for any *local* variations $\delta\mathbf{A}$. If one assumes that $\epsilon > 1$ in the discussion above, it is essentially trivial to show conclusively that Eq. (50) is valid. However, I will proceed by assuming that Eq. (50) is valid also if one merely takes $\epsilon > 0$, which is a plausible assumption as discussed above.

If the conditions (37) and (50) are valid, or more precisely if the condition (49) is in force, then one obtains,

$$\begin{aligned}\dot{\pi}^k_a(x^0,\mathbf{x}) \equiv -\frac{\delta H}{\delta A^a_k(x^0,\mathbf{x})} \quad &= \quad -ig[A_0\{\mathbf{A}\} - C,\nabla^k(A)C + \pi^k]_a \\ &\quad + \left(\nabla_\ell G^{k\ell}(A)\right)_a - ig[C,\nabla^k(A)A_0\{\mathbf{A}\}]_a.\end{aligned} \tag{51}$$

The pairs of equations, (47) and (51) are supposed to be equivalent to the original field equations (9). I will analyse this equivalence below. For this purpose it is convenient to note that the equations (47) and (51) admit a *constant of motion*,

$$\partial_0(\nabla_k(A)\pi^k) = 0, \tag{52}$$

which is crucial for the establishing of the equivalence between Eqns. (9) and the Hamiltonian equations of motion (47) and (51). Recalling the equation (41) defining the quantity C together with the boundary conditions (43), one finds using Eq. (52) that

$$\nabla_k(A)\nabla^k(A)C = K_1, \tag{53}$$

where K_1 denotes a constant.

So, if the equations of motion for the canonical variables A and π, respectively, are in force, then the quantity C is determined by the equation (53), where the constant K_1 is so far unknown.

Eliminating the canonical momentum variables π between the equations (47) and (51) one obtains the following equations,

$$\nabla_\nu(A)G^{k\nu}(A) + \nabla^k(A)\dot{C} + 2ig\left[\dot{A}^k, C\right] = 0\,,\ k = 1, 2, 3. \tag{54}$$

At this point one should recall that Gauss' law is in force by construction,

$$\nabla_k(A)G^{0k}(A) = 0. \tag{55}$$

Were it not for the terms involving the quantity C in Eqns. (54) one could now conclude that the Hamiltonian equations of motion (47) and (51) are equivalent to the original field equations (9). In order to obtain the required equivalence one must demand that

$$\nabla^k(A)\dot{C} + 2ig\left[\dot{A}^k, C\right] = 0, \tag{56}$$

simultaneously with the Hamiltonian equations of motion. In this situation the quantity C is determined by the partial differential equations (53). However, it does not appear to be possible to prove that the unique solution to (53) also satisfies (56) if the constant K_1 has an arbitrary value, different from zero. However if

$$K_1 = 0, \tag{57}$$

then substituting this value in Eq. (53) one obtains the final equation determining the quantity C, when all the other equations of motion are in force,

$$\nabla_k(A)\nabla^k(A)C = 0. \tag{58}$$

But Eq. (58) has only the trivial solution $C = 0$ in the class of functions satisfying the boundary conditions (43) as discussed previously. Hence if the constant of integration K_1 in Eq. (53) equals zero, then Eq. (56) is trivially true, whence the Hamiltonian equations (47) and (51) are completely equivalent to the original field equations (9). Thus by demanding that the condition (57) be valid, one obtains *complete equivalence* between the original field equations (9) and the Hamiltonian equations of motion (47) and (51) which have been obtained using the generalized Coulomb gauge condition (26).

It is now appropriate to return to the Hamiltonian (46). This Hamiltonian differs in a non-trivial way from the Hamiltonian in the Weyl gauge, i.e. from the expression (15), mainly due to the C -dependent terms in the former expression, but also due to the fact that a non-trivial A_0-dependence is possible in the case of the Hamiltonian (46). The fact that the functional $C\{\mathbf{A}, \pi\}$ actually becomes zero when all the equations of motion are in force, does not mean that one can

set C equal to zero in the Hamiltonian (46) and use the resulting expression to generate the appropriate equations of motion by means of functional differentiation. The *functional* dependence of the quantity $C\{\mathbf{A}, \pi\}$ in the Hamiltonian (46) is essential in order to generate the proper equations of motion.

Finally I comment briefly on the possibilities to generalize the considerations in this paper to a situation in which one couples the Yang-Mills field to e.g. a fermionic field. The field equations (9) then get replaced by the following,

$$\nabla_\nu(A)G^{\mu\nu}(A) = J^\mu \equiv g\bar{\psi}\gamma^\mu T_a \psi T^a, \tag{59}$$

to which one has to add the field equations for the fermionic fields. Also in this case is it possible to use the generalized Coulomb gauge condition (26) so that the Gauss' law takes the form

$$\nabla_k(A)\nabla^k(A)A_0 = J_0. \tag{60}$$

The construction of the canonical variables and Hamiltonian in this case, along similar lines to those presented here for the pure Yang-Mills case, does not meet with any difficulties of principle. Likewise, coupling the Yang-Mills field to a scalar field also gives rise to a system which has a Hamiltonian formulation. Details of these constructs as well as some mathematical detail only briefly touched upon in this paper will be given in future publications [10].

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