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OBSERVABLE NONCLASSICAL CORRELATION EFFECTS IN THE DYNAMICS OF A BIDIMENSIONALLY CONFINED ION*

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The occurrence of a peculiar correlation effect in the quantum dynamics of an ion confined in a 2D Paul microtrap is reported. We analytically find that when the ion, prepared in a SU(2) vibrational coherent state, is irradiated by two orthogonal laser beams, the time evolution of the quantum covariance between two clearly interpretable bosonic observables displays an high sensitivity to the initial state parameters. Both the nonclassical nature of this effect and a simple proposal for its detection is briefly discussed.

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1 Introduction

Over the last few years we have witnessed a very rapid development of cooling and trapping techniques both for neutral atoms and for ions [1–4]. These progresses have made it possible performing sophisticated experiments wherein several examples of couplings between a few bosonic and fermionic dynamical variables have been realised [4–9]. If, in fact, an ion confined in a miniaturised Paul trap is exposed to suitable configured laser beams, the 3D harmonic motion of the ionic centre of mass gets entangled with the time evolution of the internal degrees of freedom. Some peculiar aspects of the vibronic response stemming from such a practically environmental loss-free coupling, have been brought to light and successfully exploited for engineering several nonclassical states [6, 7, 10–20], implementing quantum logic gates [5, 21, 22], realising tomografic measurement for reconstructing the density matrix of the system [23–26], and, more in general, discovering purely quantum effects characterising the ionic quantized oscillatory motion [27–32]. Many hamiltonian models have been reported so far in the literature to describe physical properties of trapped ions. Some models explore physical situations wherein only the ionic motion along a prefixed direction of the trap is effectively influenced by the presence of the laser beams. When the induced vibronic coupling is instead extended over two (three) independent directions of the trap, then one refers to the correspondent physical scenario as to a 2D (3D) trapped ion.

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In this paper we study the motion of an ion isotropically confined in the radial plane of a Paul microtrap when it is irradiated by a properly chosen configuration of external laser beams. We show that there exist experimentally interesting conditions under which the dynamics of this system may be exactly treated. Exploiting this fact, and under suitable initial conditions, we succeed in bringing to light the existence of pairs of simple vibrational observables whose quantum correlations exhibit, at given time instants, nonclassical effects transparently reflecting the discreteness of the harmonic oscillator energy.

2 Hamiltonian model and its exact dynamics

Consider a two-level ion of mass M confined in a bidimensional isotropic harmonic potential characterised by the trap frequency ν. Indicate by $\hat{a}$ ($\hat{a}^\dagger$) and $\hat{b}$ ($\hat{b}^\dagger$) the annihilation (creation) operators of the vibrational quanta relative to the oscillatory motion along the x and y axes of the trap respectively. Accordingly, the position and momentum operators can be written as

$$\hat{X} = \sqrt{\tfrac{\hbar}{2\nu M}}\left(\hat{a}^\dagger + \hat{a}\right) \qquad \hat{Y} = \tfrac{1}{\sqrt{2\nu M}}\left(\hat{b}^\dagger + \hat{b}\right) \tag{1}$$

$$\hat{P}_x = i\sqrt{\tfrac{\hbar\nu M}{2}}\left(\hat{a}^\dagger - \hat{a}\right) \qquad \hat{P}_y = i\sqrt{\tfrac{\hbar\nu M}{2}}\left(\hat{b}^\dagger - \hat{b}\right) \tag{2}$$

It is well known that the bidimensional harmonic oscillator may be associated to a generalised Schwinger angular momentum operator $\hat{\mathbf{J}} \equiv (J_1, J_2, J_3)$ as follows

$$\hat{J}_1 = \frac{\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}}{2} \qquad \hat{J}_2 = \frac{\hat{a}^\dagger\hat{b} - \hat{b}^\dagger\hat{a}}{2i} \qquad \hat{J}_3 = \frac{\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}}{2} \tag{3}$$

Assume that the ion is driven by two laser beams applied along the two orthogonal directions $\bar{x}$ and $\bar{y}$ with an angle of $\pi/4$ relative to the x and y axis respectively, having phases $\phi_{\bar{x}} = 0$ and $\phi_{\bar{y}} = \pi$ and equal intensities and wavelengths. It is possible to demonstrate that, if the laser beams are both tuned to the second lower vibrational sideband, the physical system under scrutiny is described, in the Lamb-Dicke limit and in the interaction picture, by the following effective Hamiltonian [17]

$$\hat{H} = g\left[(\hat{a}\hat{b})\hat{\sigma}_+ + (\hat{a}^\dagger\hat{b}^\dagger)\hat{\sigma}_-\right] \tag{4}$$

where $\hat{\sigma}_z = |+\rangle\langle+| - |-\rangle\langle-|$, $\hat{\sigma}_+ = |+\rangle\langle-|$, $\hat{\sigma}_- = |-\rangle\langle+|$ are the atomic operators, $|+\rangle$ and $|-\rangle$ being the ionic excited and ground states respectively. In equation (4) g measures the strength of the interaction between the internal and external degrees of freedom and depends on physical parameters such as the laser intensity and wavelength and the amplitude of oscillation of the ionic centre of mass.

It is easy to verify that the total number of excitations

$$\hat{N} = \hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b} + \hat{\sigma}_z + 1 \tag{5}$$

and the difference of vibrational quanta, relative to the x and y harmonic motion, $\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b} = 2\hat{J}_z$ are constants of motion. Let's denote with $|n_a,\, n_b\rangle = |n_a\rangle|n_b\rangle$ the simultaneous eigenstates of $\hat{a}^\dagger\hat{a}$ and $\hat{b}^\dagger\hat{b}$ such that:

$$\hat{a}^\dagger\hat{a}|n_a n_b\rangle = n_a|n_a n_b\rangle \qquad \hat{b}^\dagger\hat{b}|n_a n_b\rangle = n_b|n_a n_b\rangle \tag{6}$$

Suppose to prepare the ion at $t = 0$ in the state $|\Psi_N(0)\rangle = |\tau = 1, j_0 = \frac{N}{2}\rangle|-\rangle$, where

$$|\tau = 1, j_0 = \frac{N}{2}\rangle \equiv \frac{1}{2^{N/2}} \sum_{k=0}^{N} \begin{pmatrix} N \\ k \end{pmatrix}^{1/2} |N-k, k\rangle \equiv \equiv \sum_{k=0}^{N} P_k |N-k, k\rangle \tag{7}$$

is a SU(2) coherent state [34, 35]. It is not difficult to see that

$$|\tau = 1, j_0 = \frac{N}{2}\rangle = e^{+i\frac{\pi}{2}\hat{J}_2}|0, N\rangle \tag{8}$$

and that, from Eqs. (1), (2) and (3) the operator $\hat{J}_y$ is proportional to the z-component of the ionic adimensional angular momentum operator $\hat{L}_z \equiv \frac{1}{\hbar}\left(\hat{X}\hat{P}_y - \hat{Y}\hat{P}_x\right) \equiv \left(\hat{a}^\dagger\hat{b} - \hat{b}^\dagger\hat{a}\right) = 2\hat{J}_2$. Thus, the generation of the initial state $|\tau = 1, j_0 = \frac{N}{2}\rangle$ amounts at realising a Fock state of the ion motion along the direction $\bar{x}$.

It is worth noting that Fock states of the unidimensional ionic c.m. motion in a Paul trap have been experimentally realised by Meekhof et al. [6, 33].

The states $|N-k, k\rangle$ appearing in Eq. (7) are eigenstates of the operator $\left(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}\right)$ all pertaining to the common eigenvalue $N = 2j_0$ representing the initial total number of vibrational quanta.

If, at $t = 0$, we turn on the laser fields realising the Hamiltonian model (4), then at any subsequent instant of time t, the state of the system, in the Schrödinger picture and apart from an overall phase factor, can be written down as follows

$$|\Psi_N(t)\rangle = |\varphi_-(t)\rangle|-\rangle - i|\varphi_+(t)\rangle|+\rangle \tag{9}$$

with

$$|\varphi_-(t)\rangle = \sum_{k=0}^{N} P_k \cos(f_k t)|N-k, k\rangle \tag{10}$$

$$|\varphi_+(t)\rangle = \sum_{k=1}^{N-1} P_k \sin(f_k t)|N-k-1, k-1\rangle \tag{11}$$

where

$$f_k = 2g\sqrt{(N-k)k} \tag{12}$$

are the Rabi frequencies. Eq. (9) clearly shows that the laser-ion interaction, and in particular the momentum exchange between the classical laser beams and the ion, entangles the ionic internal and external degrees of freedom.

3 Time evolution of momenta e positions covariances

We now concentrate our attention on the component $\hat{J}_1$ of the Schwinger angular momentum operator. Since the system is initially prepared in an eigenstate of the operator $\hat{N}$ defined by equation (5), taking into consideration that the total number of excitations is a constant of motion, then the state $|\Psi_N(t)\rangle$ as given by equation (9) is an eigenstate of $\hat{N}$ pertaining to the initial eigenvalue $2j_0$. This implies, by virtue of equations (1) and (2), that the expectation values of the operators $\hat{X}$, $\hat{Y}$, $\hat{P}_x$ and $\hat{P}_y$ vanishes at any t. Accordingly it is not difficult to convince oneself that, using the notation $\langle A(t)\rangle \equiv \langle\Psi_N(t)|A|\Psi_N(t)\rangle$, the relations

$$\langle\hat{J}_1\rangle \propto \langle\hat{X}\hat{Y}\rangle - \langle\hat{X}\rangle\langle\hat{Y}\rangle \equiv C(\hat{X},\hat{Y},t) \tag{13}$$

$$\langle\hat{J}_1\rangle \propto \langle\hat{P}_x\hat{P}_y\rangle - \langle\hat{P}_x\rangle\langle\hat{P}_y\rangle \equiv C(\hat{P}_x,\hat{P}_y,t) \tag{14}$$

hold at any t. In words, the mean value of $\hat{J}_1$, at any time instant t, is proportional both to the covariance $C(\hat{X},\hat{Y},t)$ between the position operators $\hat{X}$, $\hat{Y}$ and to the covariance $C(\hat{P}_x,\hat{P}_y,t)$ between the momentum operators $\hat{P}_x$, $\hat{P}_y$. It is then interesting to consider in more detail the time evolution of $\langle\hat{J}_1(t)\rangle$ since it gives direct information on momenta and positions covariances.

From Eqs. (3), (9), (10) and (11) one gets

$$\begin{aligned}\langle\hat{J}_1(t)\rangle &= \frac{N}{2}\sum_{k=0}^{N-1}\left|P_k^{(N-1)}\right|^2\cos[(f_k-f_{k+1})t]+\\ &+ \frac{N}{2}\sum_{k=0}^{N-1}\left|P_k^{(N-1)}\right|^2\left[\sqrt{\frac{k}{k+1}\frac{N-k-1}{N-k}}-1\right]\sin(f_k t)\sin(f_{k+1}t)\end{aligned} \tag{15}$$

where

$$\left|P_k^{(N-1)}\right|^2 = \frac{(N-1)!}{(N-k-1)!k!}\frac{1}{2^{N-1}} \tag{16}$$

It is possible to show, by lengthy calculations, that the modulus of the second term appearing in the right hand side of Eq. (15) is always ≤ 1. In light of this consideration Eq. (15) can be cast in the form

$$\langle\hat{J}_1(t)\rangle = \frac{N}{2}\sum_{k=0}^{N-1}\left|P_k^{(N-1)}\right|^2\cos[(f_k-f_{k+1})t]+O(1) \tag{17}$$

Figure 1 displays the time evolution of $\langle\hat{J}_1(t)\rangle$ in correspondence to two different exemplary values of the initial total number of vibrational quanta, $N=20$ and $N=21$.

These plots, obtained from numerical computation of the exact expression (17), suggest the existence of a rich dynamical behaviour and clearly display the occurrence of constructive and destructive interference, at specific time instants, among the oscillatory terms present in the right hand side of Eq. (17) .

In order to get an analytical confirmation of such a behaviour, we look for a closed expression of $\langle\hat{J}_1(t)\rangle$. To this end we recall, first of all, that $\left|P_k^{(N-1)}\right|^2$ is a binomial distribution peaked

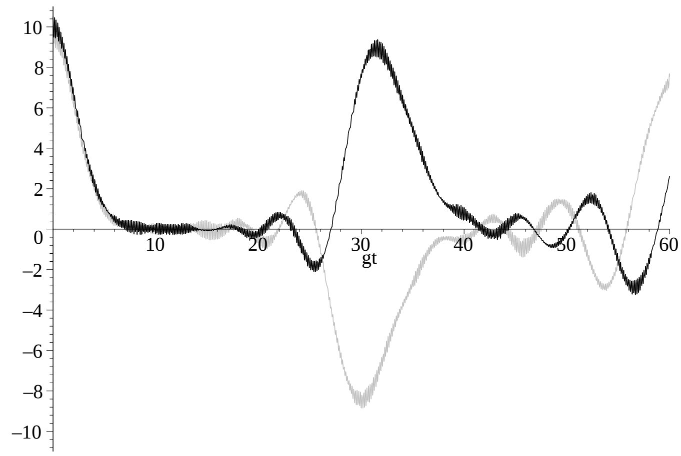

Fig. 1. Time evolution of $\langle \hat{J}_1(t) \rangle$ in correspondence $N = 20$ (gray line) and $N = 21$ (black line)

on its mean value $\langle k \rangle = \frac{N-1}{2}$ with a variance of $\frac{\sqrt{N-1}}{2}$. We then may assume that, in the limit $N \gg 1$, only the terms satisfying the inequality $\frac{N-1}{2} - \frac{\sqrt{N-1}}{2} \leq k \leq \frac{N-1}{2} + \frac{\sqrt{N-1}}{2}$ effectively contribute to the sum appearing in Eq. (17). Moreover, if $N \gg 1$, it is possible to linearize, with respect to k, the frequency difference $f_k - f_{k+1}$, obtaining the following expression

$$f_k - f_{k+1} = g\left(\frac{2k - N + 1}{N}\right) + gO\left(\frac{1}{N^2}\right) \tag{18}$$

Inserting Eq. (18) into Eq. (17) and exploiting the identity

$$\sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) \cos(xm + a) = 2^n \cos^n\left(\frac{x}{2}\right) \cos\left(x\frac{n}{2} + a\right) \tag{19}$$

finally yields

$$\langle \hat{J}_1(t) \rangle \simeq \frac{N}{2} \cos^{N-1}\left(\frac{gt}{N}\right) \tag{20}$$

As a consequence of the approximated approach followed to derive Eq. (20) from Eq. (17) with the help of Eq. (18), expression (20) has a limited temporal validity, satisfying the condition

$$gt \ll N^2 \tag{21}$$

From Eq. (20) it is easy to deduce that at the time instant $t_N = \frac{\pi N}{g}$, compatible with (21), we get

$$\langle \hat{J}_1(t) \rangle \simeq (-1)^{N-1} \frac{N}{2} \tag{22}$$

This result clearly evidences that our system possesses an inherent peculiar sensibility to the parity of the initial total number N of vibrational quanta. In fact, Eq. (22) says that there exists a

N-dependent instant of time t_N at which $\langle \hat{J}_1(t) \rangle$ assumes values crucially related to the evenness or oddness of N. In more detail, $\langle \hat{J}_1(t_N) \rangle$ is equal to $-N/2$ when N is even and $+N/2$ when N is odd. Remembering the link between $\langle \hat{J}_1(t) \rangle$ and the quantum covariances $C(\hat{X}, \hat{Y}, t)$ and $C(\hat{P}_x, \hat{P}_y, t)$ and taking into consideration that the maximum (minimum) mean value of $\hat{J}_1$ is $N/2$ ($-N/2$), this result can be readily interpreted in the following way. The dynamics of the system under scrutiny is characterised by the existence of a N-dependent instant of time at which the ionic oscillatory motions along the x and y axis are maximally or minimally correlated in position and momentum in dependence on the oddness or evenness of N respectively.

4 Discussion and conclusive remarks

In this paper we have analysed the dynamics of the quantity $\langle \hat{J}_1(t_N) \rangle$ which, as put into evidence in Section 3, gives physically transparent information on the spatial and momentum correlations between the two oscillatory motions along the axis of the trap. We have presented a new quantum effect in its dynamics showing that there exists an instant of time t_N at which this quantity undergo variations proportional to N further to a change of one quantum only in the initial total number $N \gg 1$ of vibrational quanta. Such a peculiar dependence on N may be traced back to the specific set of Rabi frequencies characterising the dynamics of our system and is therefore a phenomenon having a nonclassical origin being directly related to the discreteness of the quantum energy of the harmonic oscillator. We now intend to address briefly the question of the measurability of $\langle \hat{J}_1(t_N) \rangle$. Introducing the annihilation operators along the two bisectors $\bar{x}$ and $\bar{y}$ of the trap axis x and y

$$\hat{a}_{\bar{x}} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{b}) \qquad \hat{b}_{\bar{y}} = \frac{1}{\sqrt{2}}(-\hat{a} + \hat{b}) \tag{23}$$

it is immediate to deduce that the operator $\hat{J}_1$, as given by Eq. (3), can be written down as follows

$$\hat{J}_1 = \frac{\hat{n}_{\bar{x}} - \hat{n}_{\bar{y}}}{2} \tag{24}$$

where $\hat{n}_{\bar{x}} = \hat{a}^\dagger_{\bar{x}} \hat{a}_{\bar{x}}$ and $\hat{n}_{\bar{y}} = \hat{b}^\dagger_{\bar{y}} \hat{b}_{\bar{y}}$ are the number operators relative to the motion along the $\bar{x}$ and $\bar{y}$ directions. According to Eq. (24), measuring $\langle \hat{J}_1(t) \rangle$ is equivalent at measuring the mean vibrational quanta along the directions $\bar{x}$ and $\bar{y}$. it is worth noting that the measurement of $\langle \hat{n} \rangle$ along a given direction is currently performed in the experiments and in particular it is deduced from the population analysis of motional states [6, 33]. Moreover several other methods for reconstructing the vibrational distribution have been recently proposed [36–38], including QND techniques [38, 39].

These considerations legitimate us to say that we have brought to light for the first time a new measurable nonclassical correlation effect in the dynamics of an ion isotropically confined in a 2D microtrap.

Before concluding we wish to point out that, by virtue of the rotational invariance of the system with respect to the z-axis of the trap, our results, here presented prefixing a pair of orthogonal directions in the radial plane of the trap, may be generalised getting rid of the assumptions concerning the directions of both the laser beams and the initial oscillatory motion.

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