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PULSED NONLINEAR OSCILLATOR – CLASSICAL AND QUANTUM DYNAMICS*

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We discuss a system comprising an anharmonic oscillator permanently excited by a series of ultra-short coherent pulses. Assuming that the system was initially in the vacuum state we investigate and compare its classical and quantum dynamics. Moreover, we compare the picture based on the single classical trajectory with the "averaged" one.

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1 The quantum model

Models involving a nonlinear oscillator has been a subject numerous papers ([1]– [8] and the references quoted therein). We discuss the model identical to that discussed in [6] and [7]. It contains an anharmonic oscillator (the oscillator can be interpreted as the third-order Kerr medium), located inside a one-mode, high-Q cavity. The cavity is irradiated by a seriesof ultra-short classical pulses. The evolution of our system is governed by the following Hamiltonian (in the interaction picture):

$$H = H_{NL} + H_K \quad , \tag{1}$$

where

$$H_{NL} = \frac{\chi}{2}\left(a^{\dagger}\right)^2 a^2 \tag{2}$$

and

$$H_K = \epsilon\left(a + a^{\dagger}\right)\sum_{k=1}^{\infty}\delta(t - kT) \quad . \tag{3}$$

The quantity χ denotes nonlinearity of the oscillator and can be interpreted as the third-order nonlinear susceptibility of the Kerr medium (anharmonic oscillator), ϵ measures the strength of

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the external field – cavity field interaction, whereas T is the time between two subsequent pulses. We use units of $\hbar = 1$. In fact, the Hamiltonian (1) is the same as that for the system discussed in the paper [6]. Moreover, we assume that the time T is long enough to exceed $1/\omega$, where ω is the external field frequency. Thanks to this assumption the series of ultra-short pulses can be approximated by a series of *Dirac delta* "functions" [Eq.(3)] and the rapid oscillations are washed out in the interaction picture. We note, that between two subsequent pulses the evolution of the system is governed by the Hamiltonian H_{NL}. Thus, we define the "free" nonlinear evolution operator U_{NL} as follows:

$$U_{NL} = e^{-i\chi T\hat{n}(\hat{n}-1)} \quad , \tag{4}$$

where $\hat{n} = a^\dagger a$ is the photon number operator. Moreover, we introduce the "kick" evolution operator U_K responsible for changes of the system under the influence of the laser pulses.

$$U_K = e^{-i\epsilon(a^\dagger + a)} \quad . \tag{5}$$

Thus, the evolution of the system from the moment of time just after the k-*th* pulse up to the time after the $(k+1)$-*th* pulse is governed by the unitary evolution operator

$$U = U_{NL} U_K \quad . \tag{6}$$

Acting n times with the operator U on the wave-function $|\Psi\rangle$ defined for $t = 0$, we obtain the wave-function describing the system at the moment of time just after the n-*th* laser pulse:

$$|\Psi_n\rangle = U^n \, |\Psi(t=0)\rangle \quad . \tag{7}$$

Obviously, these calculations can be easily performed numerically in the same way as in [6].

2 Classical dynamics

Since we are interested in comparing the classical and quantum mechanical models, we have to introduce the appropriate formulas describing classical evolution of the system. We start our considerations from the equation of motion for the annihilation operator a:

$$\frac{da}{dt} = -i\chi a^\dagger a a \quad . \tag{8}$$

Of course, we assume at this stage that damping is absent. Therefore, the number of photons is an integral of motion during the time between two laser pulses, and the operator $a^\dagger a$ can be treated in Eq. (8) as constant. Hence, equation (8) is easily solvable and the solution for the operator a is of the form:

$$a(\tau) = e^{-i\chi a^\dagger a \tau} a \quad . \tag{9}$$

It is obvious that τ is restricted to moments of time belonging to the interval between two laser "kicks". To include the influence of the laser pulse on the operator a we have to transform Eq.(9) using the evolution operator U_K. Since U_K is the well known shift operator, equation (9) becomes

$$a_{k+1} = e^{-i\chi T(a_k^\dagger + i\epsilon)(a_k - i\epsilon)T}(a_k - i\epsilon) \quad . \tag{10}$$

We now replace the creation and annihilation operators $a^\dagger$ and a by the complex numbers α^* and α respectively:

$$\alpha_{k+1} = (\alpha_k - i\epsilon)e^{-i(\chi|\alpha_k - i\epsilon|^2)T} \quad . \tag{11}$$

This equation allows us to find the classical map of the system and, consequently, the energy $|\alpha|^2$ for the system discussed here. This method varies from that used in [4], where the cumulant method has been applied.

Since we will be comparing the classical and quantum models and will assume that the quantum system is initially in the vacuum state, we shall need to find a way to simulate the quantum vacuum state in classical picture. We will follow the procedure proposed by Milburn and Holmes [8]. Instead of studying single classical trajectories we will investigate an ensemble of trajectories, all starting from a circle with the center at $\alpha = 0$ and the radius equal to 0.5. These trajectories are chosen randomly (the starting points of the trajectories are generated in Monte-Carlo fashion) from within the above mentioned circle. Next, we will calculate an "average" trajectory that will be treated as the counterpart of the quantum trajectory.

3 Results and discussion

In this part we briefly discuss the quantum and classical dynamics of our system corresponding to various parameters describnig the system. Thus, Fig. 1 corresponds to weak external excitations ($\epsilon = \pi/25$) and the time $T = \pi$. In Fig. 1 we compare the dynamics of the classical mean energy $n = |\alpha|^2$ (Fig. 1a,b) and of average photon number $\langle a^\dagger a \rangle$ (Fig. 1c) for the system. Moreover, Fig. 1d depicts classical, stroboscopic map of the evolution of the complex amplitude α introduced in (11). We see that the single classical trajectory (trajectory starting from a single point - Fig. 1a) is regular and oscillates between 0 and ~ 0.3. However, as we have mentioned, the "averaged" trajectory contains information from all the trajectories starting from the circle centered at the point Re$(\alpha) =$ Im$(\alpha) = 0$. Therefore this trajectory (Fig. 1b) differs significantly from that shown in Fig. 1a. It oscillates around ~ 0.37 and these oscillations are damped. In fact, this damping is rather slow, so that the amplitude of the oscillations remains equal to about 0.15 (after ~ 200 laser pulses). For this case the dynamics of our system is regular, as seen from Fig. 1d, where the classical map of these trajectories is depicted. In practice, all of them exhibit regular behaviour.

For this case the behavior of the quantum model of our system differs significantly from that corresponding to the classical model. And thus, Fig. 1c shows sinusoidal oscillations of the mean number of photons between 0 and 1. This is the same behavior as that discussed in [6] and it corresponds to one-photon state generation (OPSG). We do not observe here any correspondence between the classical and quantum trajectories as was the case in [8]. This is a consequence of the fact that the mean number of photons is not large enough.

Fig. 2 corresponds to the same situation as that depicted in Fig. 1 but for stronger laser pulses ($\epsilon = \pi/5$). We see that $|\alpha|^2$ for the single trajectory (Fig. 1a) exhibits chaotic behavior reaching ~ 120 for times exceeding ~ 200 pulses. It is obvious that for stronger external fields regular trajectories split and chaotic behavior occurs as seen in Fig. 2d. Instead of regular trajectories we observe growing and spreading in time cloud of chaotically located points. Nevertheless, despite this chaotic picture, the "averaged" classical trajectory grows almost linearly in time

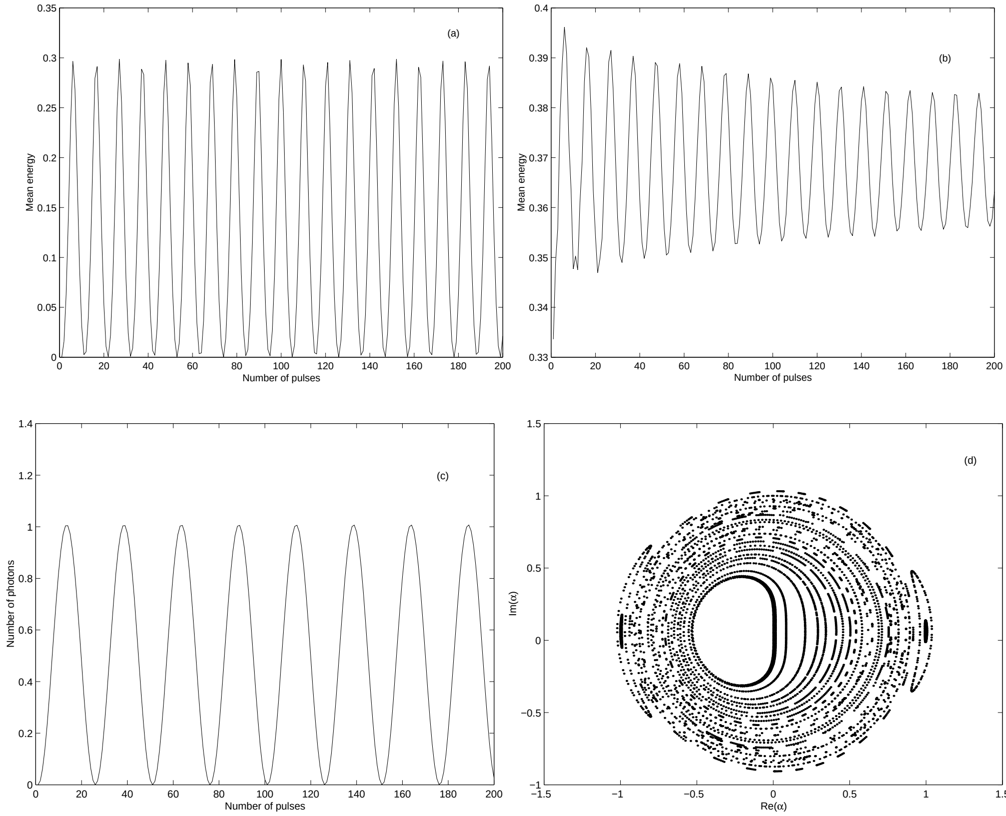

Fig. 1. Classical mean energy $|\alpha|^2$ (a), "averaged" classical energy (b), mean number of photons $\langle a^\dagger a \rangle$ (c) and stroboscopic, classical map of α (d). The strength of external pulses $\epsilon = \pi/25$ and the time between two subsequent pulse $T = \pi$.

(Fig. 2b). The quantum evolution of the system (Fig. 2c) leads to an increase in the mean number of photons, albeit different in nature. We observe rapid oscillations that modulate slowly varying variations. The latter are very slow, so it may be too slow in practice to determine the character of the growth in $|\alpha|^2$. Figure 3 corresponds to the case of strong external excitations ($\epsilon = pi$) and short times between two subsequent pulses ($T = 0.02$). For this case, the mean energy $|\alpha|$ oscillates regularly between 0 and ~ 70. Fig. 3a shows that two pictures: single trajectory picture (circle marks) and "averaged" trajectory one (solid line), give very similar results. Moreover, as seen form Fig. 3b, the mean number of photons in the quantum model behaves almost identicaly as the mean energy in the "averaged" trajectory picture and oscillates between 0 and ~ 75. Obviously this result agrees with that for the single trajectory simulations (circle marks).

It seems obvious that for a greater number of photons the quantum description should be close to the classical one and, as a consequence, the quantum state should correspond to the classical

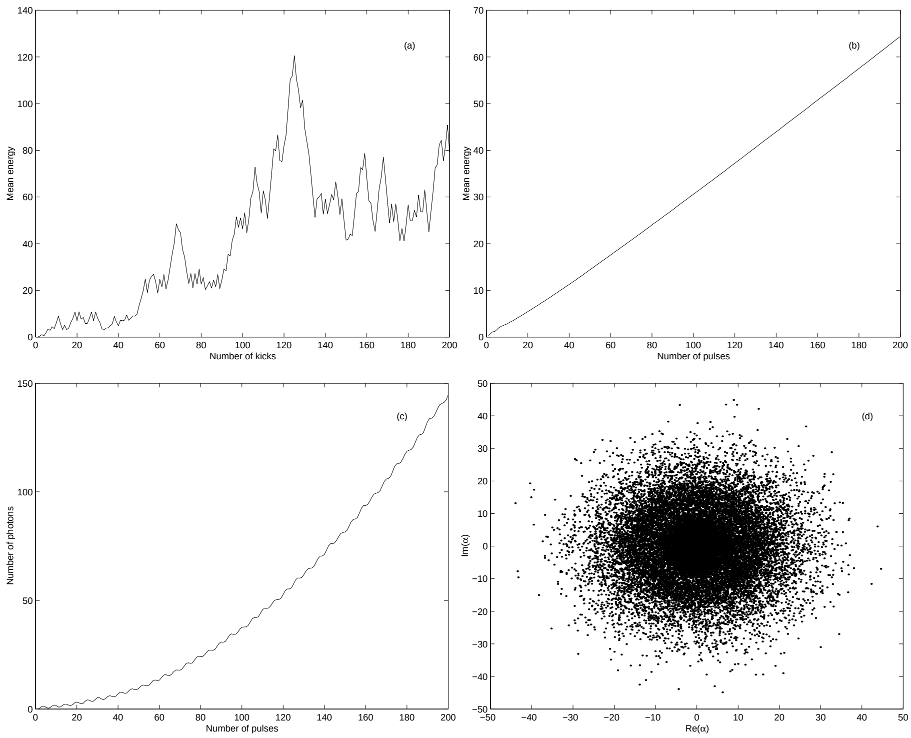

Fig. 2. The same as in Fig.1 but for stronger laser pulses ($\epsilon = \pi/5$)

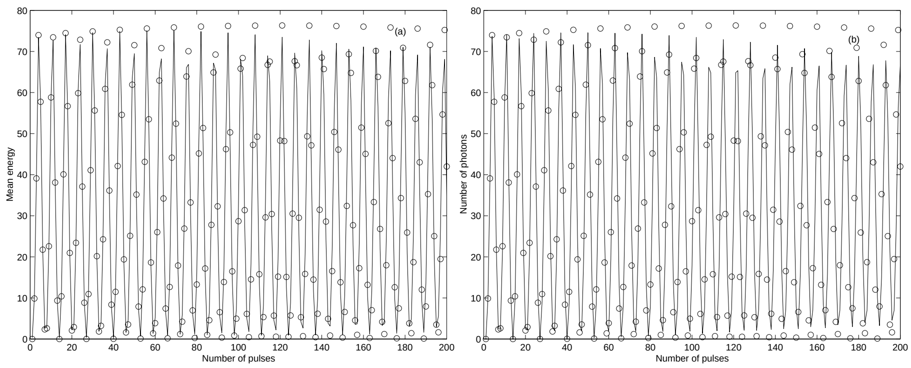

Fig. 3. Classical mean energy $|\alpha|^2$ (a,b – circle marks), "averaged" classical energy (a – solid line) and mean number of photons $\langle a^\dagger a \rangle$ (b – solid line) for strong external field $\epsilon = \pi$ and short time between two subsequent pulses $T = 0.02$.

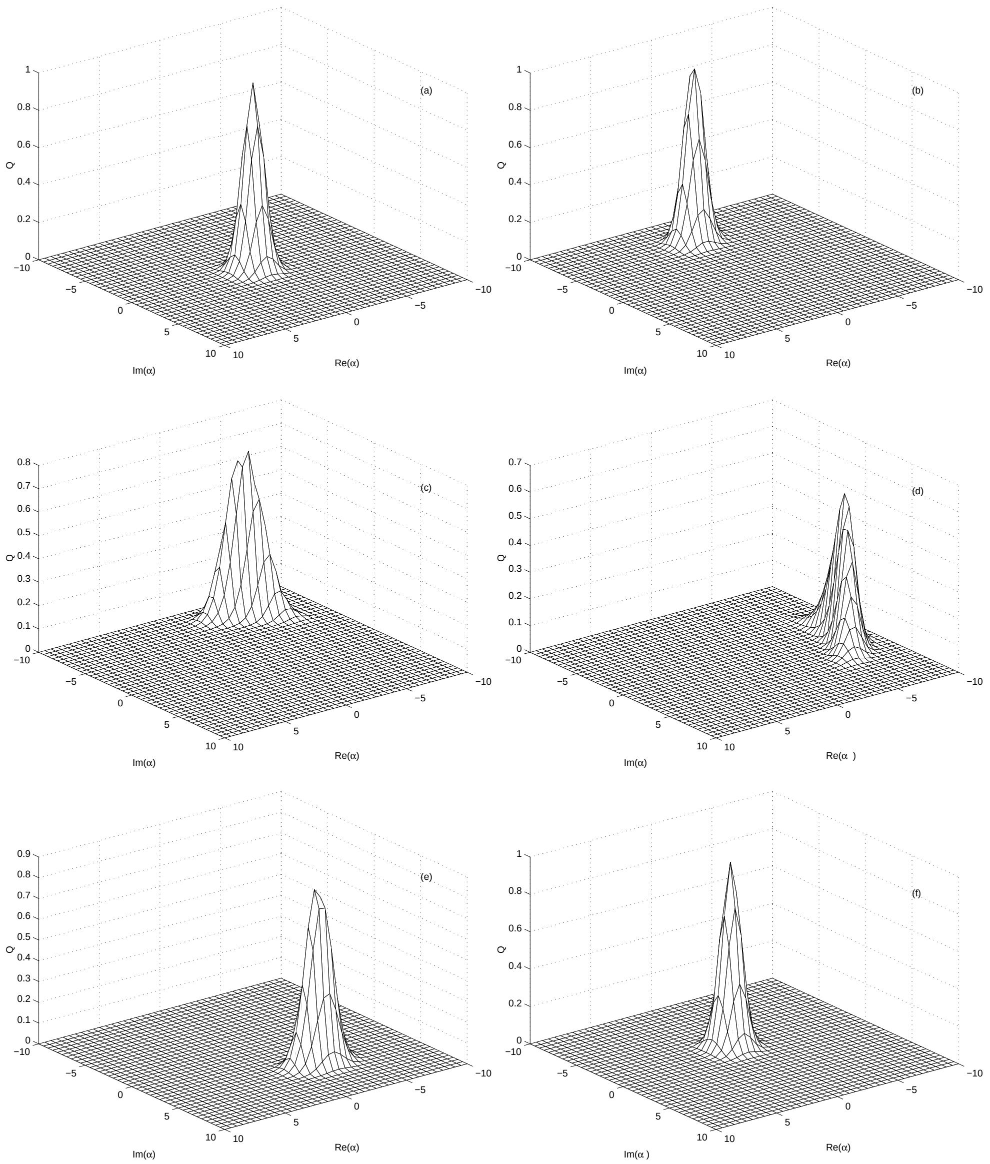

Fig. 4. Husimi Q-function for the time $t = 0$ (a) and after 3*th*, 4-*th*, 5-*th*, 6-*th* and 8-*th* pulse (b, c, d, e, f respectively). The parameters are the same as for Fig. 3.

one. However, for the case discussed here, quatnum states still manifests its quantum nature. Thus, Fig. 4 depicts the evolutrion of Husimi Q-function of the quantum state corresponding to the same parameters as in Fig. 3. We see that for the time $t = 0$ the Q-function corresponds to the vacuum state and is located at the origin of the system of coordinates. Then, after 3 pulses this function is translated on the complex plane of the parameter α toward negative values of Im(α). After the next pulse the Q-function changes its character from that remaining coherent state to that of crescent type similar to that corresponding to the displaced Kerr state [3]. This state is of the quantum nature and its properites are far from the classical picture. Next, up to 6-*th* pulse, Q-function is rotated around the center of the system and after 8-*th* pulse Q-function is shifted down toward the point of the initial position. This behaviour can be explained similarly as for the states discussed in [9]. As the mean number of photons increases sufficiently the unitary operator U_{NL} [Eq. (4)] starts to play a significant role in the whole evolution operator U [Eq. (6)] thanks to the fact that U_{NL} contains a factor proportional to n^2 and becomes dominant when we compare it with the factors proportional to $\sqrt{n}$ and n. Consequently, the nonlinear evolution has the leading role and Q-function changes its shape to the crescent type and is rotated around the center of the system.

4 Conclusions

We have discussed the system comprising the nonlinear oscillator irradiated by the series of ultra-short laser pulses. The two models were considered: classical and quantum ones. For these models we have found the time evolution for the mean energy and average number of photons, respectively, that have been treated as classical and quantum trajectories. Due to the fact that for the quantum model the initial state is determined with some incertainty the problem of simulation of this fact in the classical picture forced us to introduce “averaged” classical trajectories as in [8]. Consequently, in this paper we compare the results obtained within the three models: single classical trajectory, “averaged” one and quantum description model. We have shown that for the time between two subsequent pulses $T = \pi$ those three models give us different results. For shorter times $T = 0.02$ and for stronger external excitations $\epsilon = \pi$ the mean energies and mean number of photons evolve in a similar way within all models discussed. Nonetheless, despite this result, the clasical and quantum evolutions lead to the states that are different in nature. It is evident as we examine the Husimi Q-function corresponding to this case. The Q-function exhibits specific quantum properties of the state obtained in the quantum model, contrary to the classical states discussed in the classical ones.

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