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A HAMILTONIAN FOR CAVITY DECAY*

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We derive from first principles the phenomenological Hamiltonian for high-Q cavity damping that is often adopted in quantum optics and show that it is a far better approximation than was previously thought. We also obtain an explicit expression for the coupling strength between the discrete cavity quasimodes and the continuum that reveals the fast-time non-Markovian nature of cavity damping and the limit of validity of the independent reservoir assumption.

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1 Introduction

A typical problem one often encounters in quantum optics is how to describe the damping of the radiation field in a cavity. Since the early days of the laser, the conventional way of dealing with this problem has been to adopt a phenomenological system-reservoir approach [1]. In an ideal situation the damping is entirely due to leakage of radiation out of the cavity. Then the small system consists of the set of quantized harmonic oscillators associated with the discrete modes the cavity would have in the absence of damping. The reservoir is another set of quantized harmonic oscillators associated with the continuum of external free-space modes. This conventional approach has been very successfully applied to real experimental situations involving cavities with a high quality factor (Q). One of the phenomenological parameters in this approach is the strength of the coupling between cavity and outside modes. It is often assumed that this coupling strength should be independent of the frequency of the free-space mode, if the output mirror reflectivity is constant within the frequency range of interest. In this paper, however, we derive this phenomenological approach from first principles and show that the coupling strength depends on the frequency even when the reflectivity is frequency-independent. The main physical consequence of this frequency dependence is that for short times compared to one cavity round-trip time, the reservoir (free-space) is no longer Markovian. Our explicit expression for this frequency-dependent coupling strength unlocks this fast time non-Markovian regime that had been previously inaccessible within the phenomenological approach.

A number of authors have looked for rigorous foundations for the conventional phenomenological approach to cavity damping. The first to do so were Lang, Scully, and Lamb in a paper

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[2] where they also introduced the modes of the “universe” approach. They showed that the conventional approach is a good approximation for high-Q cavities but eventually breaks down as the cavity Q decreases. A more thorough comparison between the phenomenological Hamiltonian and the modes of the “universe” treatment was done by Barnett and Radmore [4]. They adapted the technique of Fano diagonalization [3] to deal with dressed operators, and diagonalized the phenomenological Hamiltonian for the cavity model used by Lang, Scully, and Lamb [2] where the outcoupling mirror is described by a delta-function permittivity. Barnett and Radmore found that for this model the spatial distribution of the fields calculated from the phenomenological Hamiltonian already deviates from the exact one to first order in the transmissivity even in the high-Q regime. As a by-product of our treatment, we show that this negative result is a peculiarity of the delta-function model they have adopted for the outcoupling mirror and that the phenomenological approach can be a much better approximation of the exact damping dynamics for other types of outcoupling mirrors.

The problem of deriving the phenomenological approach from first principles was addressed by Knöll, Vogel, and Welsch [5] who derived the phenomenological Langevin equations from a modes of the “universe” treatment and by Van der Plank and Suttorp [6] who derived the phenomenological master equation. However, master and Langevin equations provide only a partial account of the phenomenological approach where the reservoir of external modes is eliminated from the description. They also involve extra approximations such as coarse-graining. A full account requires a Hamiltonian. Much of the appeal and usefulness of the conventional phenomenological approach is that it can be described by a very simple Hamiltonian with a direct physical meaning attached to each of its three terms (see Section 3). This phenomenological Hamiltonian is the starting point, for instance, of the input-output theory developed by Gardiner and Collett [7] that so successfully relates intra-cavity squeezing (what is often calculated) to squeezing of the external field (what is often measured). Refs. [5, 6] do not provide a derivation of this phenomenological Hamiltonian. Recently though, Dalton, Barnett, and Knight [8] have made some progress in this direction. They consider modes of the “universe” that are solutions of Helmholtz equation for idealized, rather than the actual, permittivity and permeability functions describing the cavity walls. The idea is that by choosing the idealized permittivity and permeability functions to *approach* those of perfect mirrors, the idealized modes of the “universe” will fall at least in two categories: the ones they call cavity quasimodes that practically vanish outside but are large inside, and what they call external quasimodes that almost vanish inside but are large outside. Then generalizing a technique introduced by Glauber and Lewenstein [9], they show that, despite not being the true modes of the system, their quasimodes can still describe the quantum dynamics of the field exactly. But unlike the Hamiltonian for the true modes of the “universe”, the new one is not diagonal: their quasimodes are coupled. This is very reminiscent of the phenomenological Hamiltonian where discrete cavity modes couple to continuous external modes. However, as the ordinary modes of the “universe”, these idealized modes of the “universe” still form a continuum. The general theory presented by Dalton, Barnett, and Knight [8] does not single out explicitly which part of this continuum constitutes what they call cavity quasimodes that approximate the discrete cavity modes, and which part constitutes the external quasimodes that approximate the external continuum. In the approach presented here, on the other hand, discrete cavity and continuum external modes are explicitly identified from the beginning. This allows us to derive the phenomenological Hamiltonian in its widely used form with cavity modes and external modes clearly separated, and also to obtain an explicit expression for the coupling strength.

In the next section, we describe the simple model of a cavity we adopt here and its exact modes of the "universe" treatment. In Section 3, we discuss the key idea behind the phenomenological Hamiltonian approach and explain why the negative result of Barnett and Radmore [4] is a peculiarity of the delta-function model they have adopted for the outcoupling mirror. Then, in Section 4, we derive the phenomenological Hamiltonian from the exact modes of the "universe" treatment as an approximation and obtain an explicit expression for the coupling strength. Finally, in Section 5, we discuss and summarize our conclusions.

2 Model

Our basic ideas should apply to a general cavity. For simplicity, however, we will restrict our considerations here to a one-dimensional model where we consider only linearly polarized electromagnetic waves propagating in the x direction. The polarization of the electric field defines the y axis and that of the magnetic field, the z axis. We re-scale the fields dividing them by the square root of the transverse area in the yz plane as in Refs. [2]. The cavity consists of a perfect plane mirror at $x = -L$ and a semitransparent plane mirror of reflectivity r and transmissivity t at $x = 0$. Unlike most previous theoretical treatments of leaky cavities, we do not use a microscopic model for the semitransparent mirror. As in Ref. [10], we only assume that the semitransparent mirror is non-absorptive so that

$$|r|^2 + |t|^2 = 1 \tag{1}$$

and

$$r^*t + t^*r = 0. \tag{2}$$

The microscopic model of a semitransparent mirror as a delta function permittivity is a special case where, apart from (1) and (2), r and t also satisfy $t = r + 1$. The advantage of not adopting a microscopic model for the semitransparent mirror will become apparent in the next section.

A long but straightforward calculation shows that the intra-cavity fields are given in terms of the modes of the "universe" by

$$\hat{E}_{\text{in}}(x) = -i\int_0^\infty dk\sqrt{\frac{\hbar ck}{\pi\epsilon_0}}e^{ikL}\mathcal{L}(k)\sin[(x+L)k]\hat{a}(k) + H.c. \tag{3}$$

and

$$\hat{B}_{\text{in}}(x) = -\frac{1}{c}\int_0^\infty dk\sqrt{\frac{\hbar ck}{\pi\epsilon_0}}e^{ikL}\mathcal{L}(k)\cos[(x+L)k]\hat{a}(k) + H.c. \tag{4}$$

The global operators $\hat{a}(k)$ satisfy the usual commutation relations for continuous annihilation and creation operators, i.e.

$$[\hat{a}(k), \hat{a}(k')] = 0 \tag{5}$$

and

$$[\hat{a}(k), \hat{a}^\dagger(k')] = \delta(k - k'). \tag{6}$$

The intra-cavity field strength for a global mode is

$$\mathcal{L}(k) = \sum_{l=0}^{\infty} t\left(-re^{i2kL}\right)^l = \frac{t}{1+r\exp(i2kL)}. \tag{7}$$

Analogously, the external fields are given in terms of the modes of the "universe" by

$$\hat{E}_{\rm out}(x) = \int_0^\infty dk \sqrt{\frac{\hbar ck}{4\pi\epsilon_0}} \left\{e^{-ikL} + e^{ikL}\left[r - te^{i2kL}\mathcal{L}(k)\right]\hat{a}(k)\right\} + H.c. \tag{8}$$

and

$$\hat{B}_{\rm out}(x) = \frac{1}{c}\int_0^\infty dk \sqrt{\frac{\hbar ck}{4\pi\epsilon_0}} \left\{-e^{-ikL} + e^{ikL}\left[r - te^{i2kL}\mathcal{L}(k)\right]\hat{a}(k)\right\} + H.c. \tag{9}$$

The Hamiltonian is the total energy

$$\hat{H} = \int_{-L}^{0} dx\,\hat{\mathcal{U}}_{\rm in}(x) + \lim_{\delta\to 0^+}\int_{-\delta}^{\delta} dx\,\hat{\mathcal{U}}_{\rm stm}(x) + \int_0^\infty dx\,\hat{\mathcal{U}}_{\rm out}(x), \tag{10}$$

where

$$\hat{\mathcal{U}}_\alpha(x) = \frac{\epsilon_0}{2}\left\{\hat{E}^2_\alpha(x) + c^2\hat{B}^2_\alpha(x)\right\} \tag{11}$$

with $\alpha = \text{in}$ is the energy density inside the cavity and with $\alpha = \text{out}$ is the energy density outside, and $\hat{\mathcal{U}}_{\rm stm}(x)$ is the energy density inside the semitransparent mirror. Unfortunately, we do not know the energy density inside the semitransparent mirror because we are not adopting a specific microscopic model for this mirror as the popular delta function bump in the dielectric permittivity [2]. We can, however, still find an expression for the integral of $\hat{\mathcal{U}}_{\rm stm}(x)$ over the semitransparent mirror that appears in (10) using the energy conservation relation [11]

$$\frac{\partial}{\partial x}\hat{S} + \frac{\partial}{\partial t}\hat{\mathcal{U}} = 0, \tag{12}$$

where $\hat{S}$ is the Poynting vector, which is known both inside ($\alpha = $ in) and outside ($\alpha = $ out)

$$\hat{S}_\alpha(x) = \frac{\epsilon_0 c^2}{2}\left\{\hat{E}_\alpha(x)\hat{B}_\alpha(x) + \hat{B}_\alpha(x)\hat{E}_\alpha(x)\right\}. \tag{13}$$

From (12) we find that

$$\frac{d}{dt}\left\{\lim_{\delta\to 0^+}\int_{-\delta}^{\delta} dx\,\hat{\mathcal{U}}_{\rm stm}(x)\right\} = \hat{S}_{\rm in}(0) - \hat{S}_{\rm out}(0). \tag{14}$$

Integrating in time, we obtain

$$\begin{aligned}\lim_{\delta\to 0^+}\int_{-\delta}^{\delta} dx\,\hat{\mathcal{U}}_{\rm stm}(x) &= -\frac{\hbar c}{2\pi}\int_0^\infty dk\int_0^\infty dk'\sqrt{kk'}\Big(\frac{\hat{a}(k)\hat{a}(k')}{k+k'}\Big\{\mathcal{L}(k)\mathcal{L}(k')\\ &\times \quad e^{i(k+k')L}\sin[(k+k')L]\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{i}{2}[-1+\{r-te^{i2kL}\mathcal{L}(k)\}\{r-te^{i2k'L}\mathcal{L}(k')\}]\Big\} \\
&+ \quad \frac{\hat{a}(k)\hat{a}^\dagger(k')}{k-k'}\Big\{\mathcal{L}(k)\mathcal{L}^*(k')e^{i(k-k')L}\sin[(k-k')L] \\
&+ \quad \frac{i}{2}[-1+\{r-te^{i2kL}\mathcal{L}(k)\}\{r-te^{i2k'L}\mathcal{L}(k')\}^*]\Big\}\Big) + H.c.
\end{aligned}
\tag{15}
$$

Using (11) and (15) in (10), we find that the Hamiltonian for the global operators is given by the familiar expression

$$
\hat{H} = \frac{\hbar c}{2}\int_0^\infty dk\, k\{\hat{a}^\dagger(k)\hat{a}(k) + \hat{a}(k)\hat{a}^\dagger(k)\} \tag{16}
$$

which shows that each mode of the “universe” is equivalent to an uncoupled harmonic oscillator.

3 The high-Q regime and the phenomenological Hamiltonian

The key idea behind the phenomenological approach to high-Q cavity damping is the assumption that for very weak damping the cavity modes are essentially the same as if the cavity was perfect (i.e. of infinite Q), and that the only noticeable consequence of a finite Q is the appearance of a coupling between these perfect cavity modes and the continuum modes of the reservoir responsible for the damping. For the ideal case that we are considering here where the damping is entirely due to electromagnetic radiation leaking to the outside, this reservoir is the set of continuum outside modes obtained when the cavity outcoupling mirror at $x = 0$ is assumed to be a perfect reflector. The Hamiltonian that is often adopted in quantum optics to describe this phenomenological approach has the general form

$$
\begin{aligned}
\hat{H}_{\rm ph} \quad = \quad & \frac{\hbar}{2}\sum_{n=1}^{\infty}\{\hat{a}_n^\dagger\hat{a}_n + \hat{a}_n\hat{a}_n^\dagger\}\,\omega_n + \frac{\hbar c}{2}\int_0^\infty dk\;\Big\{\hat{b}^\dagger(k)\hat{b}(k) + \hat{b}(k)\hat{b}^\dagger(k)\Big\}\,k \\
+ \quad & \hbar\sum_{n=1}^{\infty}\int_0^\infty dk\;\Big\{V_n(k)\hat{a}_n^\dagger\hat{b}(k) + V_n^*(k)\hat{b}^\dagger(k)\hat{a}_n\Big\},
\end{aligned}
\tag{17}
$$

where $\hat{a}_n$ is the annihilation operator for the mode of frequency ω_n of the fictitious perfect cavity, $\hat{b}(k)$ is the annihilation operator for the external mode of frequency ck, and $V_n(k)$ is the coupling strength between the $n^{\rm th}$ cavity mode and the $k^{\rm th}$ mode of the external continuum. Each term in this very simple Hamiltonian has a direct physical meaning. The first two terms on the right hand side of (17) represent the uncoupled cavity and outside, i.e. stand for the energy stored in the cavity and external modes. The third term represents the coupling, i.e. the exchange of energy between cavity and external modes: one photon in a cavity mode can be annihilated leading to the creation of a photon in an external mode and vice-versa.

An important restriction on the coupling strength $V_n(k)$ is that it must be of first order in the output mirror transmissivity t for (17) to yield the correct electromagnetic energy loss rate from the cavity due to the leakage of radiation to the outside [1]. Now it can be shown that the coupling with the outside will introduce a shift (analogous to the Lamb shift) in the cavity resonance frequencies, at most of second order in t and, if $V_n(k)$ is broad enough, this second order shift will even vanish [4]. One way to check the validity of the phenomenological Hamiltonian (17) is to calculate exact cavity resonances. These resonances can be obtained by a simple inspection of the square modulus of the function $\mathcal{L}(k)$ that occurs in the exact modes of the "universe" expression for the cavity field (3). Using Poisson's sum formula [12] and assuming that r and t do not depend on k (this is often the case for most real cavities in the frequency range where they are designed to work), we can rewrite $|\mathcal{L}(k)|^2$ exactly as [13]

$$|\mathcal{L}(k)|^2 = \frac{1}{L}\sum_{n=-\infty}^{\infty}\frac{\gamma}{(k-k_n+\Delta)^2+\gamma^2}, \tag{18}$$

where $\gamma = -\ln|r|/(2L)$ is the cavity loss rate due to the leakage of radiation to the outside, $k_n = (\pi/L)n$ is the perfect cavity n^{th} resonance, and $\Delta = [\arg(r) - \pi]/(2L)$ is a detuning. This detuning is of first order in t for the delta-function permittivity model [2] that is often adopted for the semitransparent mirror [4]. In this case the phenomenological Hamiltonian (17) strictly fails because the key idea behind it cannot be realized: the only way to get the correct shift is to assume that the cavity modes are not perfect modes so that their resonance frequencies are already shifted. This shift of first order of t, however, is a peculiarity of the delta-function model for the semitransparent mirror. It is a consequence of the extra condition, $t = r + 1$, that such delta-function mirrors have to satisfy. Real mirrors are not restricted by this extra condition at all and that is why we refrain from using a microscopic model for the semitransparent output mirror here and adopt, as in Ref. [10], just the two general conditions (1) and (2) that every non-absorptive mirror has to obey.

4 Deriving the phenomenological Hamiltonian

We have seen in the previous section that the coupling strength $V_n(k)$ in (17) must be of first order in t. If this coupling with the outside is to be the only first-order correction due to non-vanishing transmissivity, i.e. if the cavity modes are to be identical to perfect modes for small transmissivity, the reflectivity must remain that of a perfect reflector at least up to second order in t. If the reflectivity were to differ from that of a perfect reflector already in first order in t (the case of the delta-function model for the semitransparent mirror), the cavity modes would become different from perfect modes already in first order in t which is the same order of the coupling strength $V_n(k)$ in (17), thus violating the key idea behind the phenomenological approach.

To make sure that the cavity modes deviate from the perfect-cavity ones only to second order in t, we choose frequency-independent r and t with $r = -1 + O(t^2)$. Conditions (1) and (2) then give

$$r = -\sqrt{1-\varepsilon^2} \tag{19}$$

and

$$t = i\varepsilon, \tag{20}$$

where ε is a real positive number much smaller than one.

So up to first order in ε, we must be able to describe the cavity field in terms of perfect cavity modes and the usual expressions

$$\hat{E}_{\rm in}(x) = -i \sum_{n=1}^{\infty} \sqrt{\frac{\hbar c k_n}{\epsilon_0 L}} \left(\hat{a}_n - \hat{a}_n^\dagger\right) \sin\left[(x+L)k_n\right] \tag{21}$$

and

$$\hat{B}_{\rm in}(x) = - \sum_{n=1}^{\infty} \sqrt{\frac{\hbar k_n}{c \epsilon_0 L}} \left(\hat{a}_n + \hat{a}_n^\dagger\right) \cos\left[(x+L)k_n\right] \tag{22}$$

for the electric and magnetic fields in a perfect cavity must be equivalent to (3) up to first order in ε. Eqs. (21) and (22) then define the discrete operators $\hat{a}_n$ in terms of the global annihilation operators $\hat{a}(k)$. Substituting (21) and (22) in (3) and (4), we find

$$\hat{a}_n = \int_0^\infty dk \, \left\{\alpha_{n1}^*(k)\hat{a}(k) - \alpha_{n2}(k)\hat{a}^\dagger(k)\right\}, \tag{23}$$

where

$$\alpha_{n1}(k) = \frac{1}{\sqrt{\pi L}} \sqrt{\frac{k}{k_n}} \frac{\sin\left[(k-k_n)L\right]}{k-k_n} e^{-ikL} \mathcal{L}^*(k) \tag{24}$$

and

$$\alpha_{n2}(k) = -\frac{1}{\sqrt{\pi L}} \sqrt{\frac{k}{k_n}} \frac{\sin\left[(k+k_n)L\right]}{k+k_n} e^{-ikL} \mathcal{L}^*(k) \tag{25}$$

We can repeat the same procedure for the external fields. Up to first order in ε, these fields must correspond to the fields in free-space with a perfect mirror at the origin,

$$\hat{E}_{\rm out}(x) = -i \int_0^\infty dk \, \sqrt{\frac{\hbar c k}{\pi \epsilon_0}} \left\{\hat{b}(k) - \hat{b}^\dagger(k)\right\} \sin(kx) \tag{26}$$

and

$$\hat{E}_{\rm out}(x) = - \int_0^\infty dk \, \sqrt{\frac{\hbar k}{\pi c \epsilon_0}} \left\{\hat{b}(k) + \hat{b}^\dagger(k)\right\} \cos(kx). \tag{27}$$

From (8), (9), (26), and (27), we obtain the following expression for the $k^{\rm th}$ external mode annihilation operator

$$\hat{b}(k) = \int_0^\infty dk' \, \left\{\beta_1^*(k',k)\hat{a}(k') - \beta_2(k',k)\hat{a}^\dagger(k')\right\}, \tag{28}$$

where

$$\beta_1(k,k') = \delta(k-k') + \frac{1}{\pi} \left|\frac{1+r}{t}\right| \sqrt{\frac{k}{k'}} e^{-ikL} \cos(kL) \mathcal{L}^*(k) \lim_{\delta \to 0^+} \frac{1}{k'-k+i\delta} \tag{29}$$

and

$$\beta_2(k,k') = \frac{1}{\pi}\left|\frac{1+r}{t}\right|\sqrt{\frac{k}{k'}}e^{-ikL}\cos(kL)\mathcal{L}^*(k)\lim_{\delta\to 0^+}\frac{1}{k'+k-i\delta}. \tag{30}$$

It follows from the orthogonality relations for the perfect cavity and external modes and from the well-known equal-time commutator between the electric and magnetic fields that $\hat{a}_n$, $\hat{a}_n^\dagger$, $\hat{b}(k)$, and $\hat{b}^\dagger(k)$ obey the usual commutation relations taken for granted in the phenomenological approach, namely

$$[\hat{a}_n, \hat{a}_{n'}^\dagger] = \delta_{nn'}, \tag{31}$$

$$[\hat{a}_n, \hat{a}_{n'}] = [\hat{b}(k), \hat{b}(k')] = [\hat{a}_n, \hat{b}(k)] = [\hat{a}_n^\dagger, \hat{b}(k)] = 0, \tag{32}$$

and

$$[\hat{b}(k), \hat{b}^\dagger(k')] = \delta(k-k'). \tag{33}$$

Now, as the perfect intra-cavity modes and the perfect external modes used in the field expansions (21), (22), (26), and (27) must together be sufficient to describe any spatial configuration of the fields correctly up to first order in ε, we must be able to reconstruct the global "universe" operators by making a proper linear combination of the cavity and external mode operators. To be more precise, the inverse of relations (23) and (28),

$$\hat{a}(k) = \sum_{n=1}^{\infty}\left\{\alpha_{n1}(k)\hat{a}_n + \alpha_{n2}(k)\hat{a}_n^\dagger\right\} + \int_0^\infty dk'\left\{\beta_1(k,k')\hat{b}(k') + \beta_2(k,k')\hat{b}^\dagger(k')\right\}, \tag{34}$$

has to hold up to first order in ε, with $\alpha_{n1}(k)$, $\alpha_{n2}(k)$, $\beta_1(k,k')$, and $\beta_2(k,k')$ given by (24), (25), (29), and (30) respectively [14]. This can be verified by calculating the commutation relations of $\hat{a}(k)$ and $\hat{a}^\dagger(k)$ using eqs. (23) to (25) and (28) to (34). This straightforward but rather long calculation shows that eqs. (5) and (6) are recovered up to first order in ε so that (34) is indeed valid up to this order.

Now, if we substitute (34) in the Hamiltonian (16), we obtain, after another long but straightforward calculation, an approximate Hamiltonian for small ε involving the perfect cavity and outside mode annihilation and creation operators rather than the global modes of the "universe" operators. This approximate Hamiltonian, which is valid as long as (34) holds (i.e. up to first order in ε) has the same form as the conventional phenomenological Hamiltonian (17) with $\omega_n = ck_n$ and

$$V_n(k) = -\frac{\varepsilon}{2\hbar\sqrt{\pi L}}e^{-ikL}\frac{\sin[(k-k_n)L]}{(k-k_n)L}. \tag{35}$$

The Hamiltonian (17) with (35) can be used to study the decay of the cavity. One readily obtains the result

$$\frac{d}{dt}\langle\hat{a}_n^\dagger\hat{a}_n\rangle = -\Gamma\langle\hat{a}_n^\dagger\hat{a}_n\rangle, \tag{36}$$

with $\Gamma = |\epsilon|^2/(2cL)$. However, it is more interesting to study the properties of the outside field. The explicit Hamiltonian allows us to study the quantum properties of the radiation field leaking out of a cavity containing a non-classical field initially. Moreover, the derivation indicates that for low-Q cavities, the operators for the cavity field and the outside can no longer be expected to commute. The modes needed to describe all possible fields inside, including the correct boundary conditions, must be expected to be non-orthogonal.

5 Conclusions

So we have derived the phenomenological Hamiltonian (17) from first principles and shown that it is an approximation of the exact Hamiltonian up to first order in the transmissivity. We have established a definite criterion for the validity of this Hamiltonian. We have shown that the approximation it entails, where the cavity modes remain perfect modes with the finite transmissivity only introducing a coupling to the continuum of external modes, is possible only when the reflectivity of the outcoupling mirror remains 100% up to first order in the transmissivity. When the outcoupling mirror reflectivity already differs from 100% to first order in the transmissivity, the cavity modes will differ from perfect cavity modes already in that order and the effect of a finite transmissivity will not be limited to the introduction of a coupling between perfect cavity and external modes: the modes themselves will have to be corrected. This latter situation is the case of the delta-function model of the semitransparent mirror investigated by Barnett and Radmore [4] and others [2] before. This explains why they have found that the phenomenological resonance frequencies had a small shift of first order in the transmissivity when compared with the actual resonance frequencies for such outcoupling mirror.

We have also obtained an explicit expression for the coupling strength $V_n(k)$ in (17). This expression shows that contrary to the usual assumption, frequency-independent reflectivity and transmissivity coefficients do not lead to a frequency-independent coupling strength. There are two physical consequences of this frequency dependence. First, it shows that the reservoir of external modes is intrinsically non-Markovian. However, as the sinc function in (35) has a width of the order of π/L, the non-Markovian dynamics can only be resolved on the fast-time scale of $\pi/(Lc)$, i.e. when the round-trip time is also resolved. This fast-time scale is inaccessible in master equation and white-noise Langevin treatments because it is outside the regime where the coarse-graining approximation is valid. The second consequence is that because sinc functions corresponding to different cavity modes overlap, the effective reservoirs associated with each cavity mode are not completely independent. Since the early days of laser physics, whenever there is a need to account for more than one cavity mode, it is often assumed that each cavity mode has its own independent reservoir responsible for its damping [15]. Our approach shows that there is only one huge reservoir for all the cavity modes, the reservoir formed by the continuum of external modes. However, as the overlap between sinc functions of successive cavity modes is small, occurring only at their wings, the independent reservoir assumption works in most practical situations. Any departure from this assumption can only be noticed in the fast-time non-Markovian regime, where the slowly varying k dependence of $|V_n(k)|^2$ can be resolved.

A similar phenomenological Hamiltonian arises in solid state physics, in the one-dimensional problem of tunneling of electrons through a finite narrow barrier that divides an infinite potential well into two symmetrical regions. It was introduced by Bardeen [16]. Analogously to (17), Bardeen's Hamiltonian is the sum of three Hamiltonians, one describing the electrons on the left-hand side of the barrier, another describing those on the right-hand side of the barrier, and a third one describing the tunneling. Prange [17] has shown how to derive Bardeen's Hamiltonian from first principles using a perturbation treatment. It would be quite convenient if Prange's methods could be applied to the cavity problem. Unfortunately, they cannot. The main reason for this is that Prange relies on the existence of a first quantized Hamiltonian where both the infinite potential well and the tunelling barrier are defined and from which the second quantized form is derived in the standard way. For the electromagnetic radiation field, however, there is no first quantized Hamiltonian to start from. Another important difference between the cavity problem and Prange's is that in Prange's case there is no dissipation. Dissipation only arises when one of the regions is infinite as in the cavity problem.

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