DEGENERACY OF THE LOWEST LANDAU LEVEL AND $su_q(2)$ ON THE POINCARÉ HALF PLANE

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It is shown that the presence of the quantum group symmetry $su_q(2)$ in the quantum Hall effect on the Poincare upper half plane governs the degeneracy of the lowest Landau level. It is also shown that the relation between the degree of degeneracy and the cyclic representation of $su_q(2)$ appears in accordance with q being a kth root of unity.

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Recently, many studies have been carried out for the applications of the quantum groups [1, 2] in several areas of condensed matter physics. Typically, Landau problem (i.e. charged particle with mass m moving in a constant magnetic field) [3] and fractional quantum Hall effect (FQHE) [4]. For instance, the discovery of the quantum algebra $sl_q(2)$, which can be viewed as a q-deformation of classical Lie algebra $sl(2)$, in the above stated systems [5, 6, 7, 8, 9, 10].

In the past few years, the representation theories of the quantum groups have attracted the attention of only not the mathematicians but also of the physicists, which are more interested in its application in physics. Among them, Wiegmann et al [11], Faddeev et al [12] and Hatsugai et al [13] have accomplished their remarkable works in the respect. In Ref. [14], the study of the degeneracy of the Landau levels of two dimension electron in magnetic field moving on the plane was achieved by using the cyclic representation of the quantum group $sl_q(2)$.

It is interesting to note that the fractional quantum Hall effect on the compact Riemann surfaces has captured considerable interests, and in special the upper half plane with the Pioncare metric. With this subject, Iengo et al [15] and Alimohammadi et al [9] carried out their original exploration. Certainly it is still very interesting to look for more applications of the quantum groups in this kind of the non-flat surfaces.

In this paper, studies on the degeneracy of the lowest Landau level of a particle, in presence of a constant magnetic field, moving on the Poincare upper half plane in the context of the quantum group symmetry $su_q(2)$ has been carried out. In particular, with the help of the cyclic representation of $su_q(2)$, the degree of degeneracy of the lowest Landau level is determined.

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To begin, Let us consider a spinless particle of mass m interacting with a covariantely constant magnetic field B, moving in the Poincare upper half plane $\mathbf{H} = \{z = x + iy, y > 0\}$ characterized by the following metric [16]

$$ds^2 = \frac{dx^2 + dy^2}{y^2}. \tag{1}$$

In the symmetric gauge $A_z = A_{\bar{z}} = \frac{B}{2y}$ and in the convention $m = 2$, the Hamiltonian of this system can be written as

$$H = -y^2\partial\bar{\partial} + \tfrac{iB}{2}y(\partial + \bar{\partial}) + \tfrac{B^2}{4}, \tag{2}$$

where z, $\bar{z}$ denote the position of electron in the Poincaré upper half plane $(\mathbf{H})$ and $\partial = \frac{\partial}{\partial_z}$, $\bar{\partial} = \frac{\partial}{\partial_{\bar{z}}}$ are the partial derivatives with respect to z, $\bar{z}$, respectively. It may be noted that the ground state wave functions of this Hamiltonian corresponding to the eigenvalue $\frac{B}{4}$ are given by the solutions of the following equation

$$\nabla\phi = (\partial + \frac{B}{2iy})\phi = 0. \tag{3}$$

It is convenient to introduce two operators b and c as follows [9]

$$\begin{aligned} b &= L_1, \\ c &= L_1^{-1}L_2, \end{aligned} \tag{4}$$

where the operators L_1 and L_2 take the form

$$\begin{aligned} L_1 &= \partial + \bar{\partial}, \\ L_2 &= z\partial + \bar{z}\bar{\partial}. \end{aligned} \tag{5}$$

It leads to the following relation

$$[b, c] = 1. \tag{6}$$

Now we are in position to define the symmetry operators of the above Hamiltonian in terms of the defined operators b and c. For a given vector $\varepsilon = (\varepsilon_1, \varepsilon_2)$ of the Poincaré upper half plane $(\mathbf{H})$, let us consider the following operators

$$T_\varepsilon = e^{\varepsilon_1 b + \varepsilon_2 c}; \quad \varepsilon_1, \varepsilon_2 \in C, \tag{7}$$

with the multiplication law given by

$$T_\varepsilon T_\eta = e^{\frac{1}{2}\varepsilon\times\eta}T_{\varepsilon+\eta}. \tag{8}$$

Then using Eq.(7), it is seen that these operators commute with the Hamiltonian Eq.(2), i.e.,

$$[T_\varepsilon, H] = 0. \tag{9}$$

The latter equation indicates that the system under consideration is invariant under the symmetry operators T_ε.

From the defining symmetry operators Eq.(7), it is possible to give a construction of the quantum group symmetry $su_q(2)$. We begin by recalling that the generators of general $su_q(2)$ are defined by [17]

$$\begin{aligned} &[J_+, J_-] = [2J_3]_q, \\ &q^{J_3} J_\pm q^{-J_3} = q^{\pm 1} J_\pm, \end{aligned} \tag{10}$$

where the quantum symbol $[x]_q$ means $[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$.

The realization of the generators J_+, J_- and J_3 depend on two arbitrary noncolinear vectors ε and η of the Poincaré upper half plane $(\mathbf{H})$. To construct these generators, let us present the following superpositions of the symmetry operators[5-10]

$$\begin{aligned} J_+ &= \frac{T_\varepsilon - T_\eta}{q - q^{-1}}, \\ J_- &= \frac{T_{-\varepsilon} - T_{-\eta}}{q - q^{-1}}, \\ q^{+2J_3} &= T_{\varepsilon - \eta}, \\ q^{-2J_3} &= T_{\eta - \varepsilon}. \end{aligned} \tag{11}$$

Calculating the commutation relations for $J_\pm$ and J_3, we can reproduce the quantum group symmetry $su_q(2)$ Eq.(10) if we choose the q-deformed parameter to be given by the form

$$q = e^{\frac{1}{2}\varepsilon \times \eta}, \tag{12}$$

which relates q to the two arbitrary vectors ε and η as in the cases of the plane [5, 10], sphere [18] and the torus [6]. This relationship has been discussed thoroughly in ref. [9], where it is shown that there is a possible connection between the q-deformed parameter and the so-called filling factor characterizing the quantum Hall effect.

Now, let us return to the problem of degeneracy of the lowest Landau level in the system under consideration. Using the defined $J_\pm$ and J_3, we can show that these generators are three conserved quantities of the present system. After some elementary manipulations, it can be proved that

$$\begin{aligned} &[J_\pm, H] = 0, \\ &[q^{\pm J_3}, H] = 0, \end{aligned} \tag{13}$$

which lead to the conclusion that the quantum group symmetry $su_q(2)$ is the symmetry of the present system. According to the basic principle of quantum mechanics, Eqs.(10) and (13) imply that there is a degeneracy of the lowest Landau level in the present system.

In our case the magnetic flux is quantized, for this you can see for example ref.[15], on other hand it is known following ref.[19] that the quantization of the magnetic flux means that the q-deformed parameter is a root of unity, namely

$$q^k = 1. \tag{14}$$

where k is an integer value. Then from Eqs.(12) and (14), it can be seen that

$$\varepsilon \times \eta = \frac{4\pi i}{k} \tag{15}$$

It may be noted that the representation of q a kth root of unity has many important properties, like, it has cyclic representation which imply that is neither a highest weight not a lowest [20, 21] and the irreductible representation is k-dimensional.

According to quantum mechanics, two commuting operators have a common basis, it is the case for the operators $q^{\pm J_3}$ and H. For that, let us choose a set of the eigenfunctions of the Hamiltonian $|m, l>$ to be the available basis of these two operators, where $m = 0$ or 1 represents the energy of the lowest Landau level and l is a new quantum number which labels the different quantum states in the considered enery level. So, in this case, we can write

$$\begin{aligned} &H|m,l>= E_m|m,l>, \\ &q^{\pm J_3}|m,l>= q^{\pm(\lambda-l-\mu)}|m,l>, \end{aligned} \tag{16}$$

with E_m is the eigenvalue corresponding to these eigenfunctions, λ and μ are complex constants.

Following the representation theory of the quantum algebras at root of unity [22], the action of the $su_q(2)$ generators of on the eigenfuctions $|m, l>$ can be written as follows

$$\begin{aligned} &J_+|m,l>= [\lambda-\mu-l+1]|m,l-1>, \\ &(1 \leq l \leq k-1), \\ &J_+|m,0>= \xi^{-1}[\lambda-\mu+1]|m,k-1>, \\ &J_-|m,l>= |m,l+1>, \\ &(0 \leq l \leq k-2), \\ &J_-|m,k-1>= \xi|m,0>, \\ &q^{\pm J_3}|m,l>= q^{\pm(\lambda-l-\mu)}|m,l>, \end{aligned} \tag{17}$$

where λ, ξ and μ are three complex constants which can be determined by the cyclic properties of the representation. The dimension of the irreductible representation is just k. Therefore, it is clear from Eq. (17) that the degree of degeneracy of the lowest Landau level is also k.

In this paper it has been disscussed that the degeneracy of the lowest Landau level of an electron interacting with a magnetic field B moving on the Poincaré upper half plane is due to the quantum group symmetry $su_q(2)$. With the help of the irreductible cyclic representation of $su_q(2)$ when the q-deformed parameter is a root of unity, it is found that the degree of such degeneracy coincides perfectly with the dimension of the $su_q(2)$ representation and equals to k.

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References

[1] V. Drinfeld: *Proc. ICM*-86, Berkeley, 1986, pp. 798-820; M. Jimbo: *Lett. Math. Phys.* **10** (1985) 63; **11** (1986) 247; *Commun. Math. Phys.* **102** (1986) 537.

[2] P.P. Kulish, N. Yu Reshetikhein: *J. Sov. Math.* **23** (1983) 2345; E.K. Sklyanin: *Usp. Math. Nauk.* **40** (1985) 214.

[3] L.D. Landau, E.M. Lifshitz: *Quantum Mechanics*, 3rd ed. (Pergamon, Oxford, 1986) Ch. XV, Section 112

[4] P.E. Prange, S.M. Girvin: *The Quantum Hall Effect* (Springer, New York, 1990).

[5] I.I. Kogan: *Int. J. Mod. Phys.* **A9** (1994) 3889.

[6] H.-T. Sato: *Mod. Phys. Lett.* **A9** (1994) 451.

[7] D.F. Wang, C. Gruber: *Phys. Lett.* **A201**, (1995) 257.

[8] C.-L. Ho: *Mod. Phys. Lett.* **A10** (1995) 2665.

[9] M. Alimohammadi, A. Shafei Deb Abad: *J. Phys.* **A29** (1996) 551; M. Alimohammadi, H. Mohseni Sadjadi: *J. Phys.* **A29** (1996) 5551.

[10] A. Jellal, *Int. J. Theor. Phys.* **38** (1999) 1905.

[11] P.B. Weigmann, A.V. Zabrodin: *cond-mat/9310017; connd-mat/9312088.*

[12] L.D. Faddeev, R.M. Kashaev: *Commun. Math. Phys.*, **169** (1995) 181.

[13] Y. Hatsugai, M. Khomto, Y.S. Wu,: *Phys. Rev. Lett.* **73** (1994) 1134.

[14] G.-H. Chen, L.-M. Kuang, M.L. Ge: *Phys. Lett.* **A213** (1996) 231.

[15] R. Iengo, D. Li: *Nucl. Phys.* **B413** (1994) 735.

[16] A. Comtet: *Ann. Phys.* **173** (1987) 185.

[17] C.N. Yang, M.L. Ge: *Quantum Group*, Nankai University Proc. and references therein.

[18] N. Aizawa, S. Sacha, H.-T. Sato: *Mod. Phys. Lett.* **A10** (1995) 853.

[19] G.-H. Chen, M.L. Ge: *cond-mat/9509092*

[20] E. Date, M. Jimbo, K. Miki, T. Miwa: *Phys. Lett.* **A148** (1990) 45.

[21] H.-C. Fu: *Ph. D. dissertation*, Nankai University (1994).

[22] H.P. Su, M.L. Ge: *J. Phys.* **A24** (1991) L959; 3265.