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EXTENDED QUANTUM MECHANICS

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The work can be considered as an essay on mathematical and conceptual structure of non-relativistic quantum mechanics (QM) which is related here to some other (more general, but also to more special and "approximative") theories. QM is here primarily reformulated in an equivalent form of a Poisson system on the phase space consisting of density matrices, where the "observables", as well as "symmetry generators" are represented by a specific type of real valued (densely defined) functions, namely the usual quantum expectations of corresponding selfadjoint operators. It is shown in this paper that inclusion of additional ("nonlinear") symmetry generators (i.e. "Hamiltonians") into this reformulation of (linear) QM leads to a considerable extension of the theory: two kinds of quantum "mixed states" should be distinguished, and operator – valued functions of density matrices should be used in the rôle of "nonlinear observables". A general framework for physical theories is obtained in this way: By different choices of the sets of "nonlinear observables" we obtain, as special cases, e.g. classical mechanics on homogeneous spaces of kinematical symmetry groups, standard (linear) QM, or nonlinear extensions of QM; also various "quasiclassical approximations" to QM are all subtheories of the presented extension of QM - a version of the extended quantum mechanics (EQM). A general interpretation scheme of EQM extending the usual statistical interpretation of QM is also proposed. Eventually, EQM is shown to be (included into) a C^*-algebraic (hence linear) quantum theory.

Mathematical formulation of these theories is presented. The presentation includes an analysis of problems connected with differentiation on infinite – dimensional manifolds, as well as a solution of some problems connected with the work with only densely defined unbounded real–valued functions on the (infinite dimensional) "phase space" which correspond to unbounded operators (generators) and to their nonlinear generalizations. Also "nonlinear deformations" of unitary representations of kinematical symmetry Lie groups are introduced. Possible applications are briefly discussed, and some specific examples are presented.

The text contains also brief reviews of Hamiltonian classical mechanics, as well as of QM. Mathematical appendices make the paper nearly selfcontained.

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Contents

1 Introduction

We present in this work a straightforward, and a "very natural" theoretical extension of traditional (linear) quantum mechanics (QM), providing a general framework of several physical theories. It contains QM itself, its (almost all up to now published) nonlinear modifications and extensions, and also its "semiclassical approximations", together with the Hamiltonian classical mechanics (CM). This is made formally by a geometrical reformulation of QM and by its subsequent nonlinear extension (containing the unchanged linear QM as a subtheory); an interpretation scheme for this extended theory is also proposed here. Although rather "trivial" from a certain point of view, the obtained extended quantum mechanics (EQM)[2] seems to offer new insights into conceptual foundations and also possible applications of quantum theory. It renders also alternative views to different approximations and modifications of QM like, e.g., the time dependent Hartree Fock theory, WKB approximation, or the "nonlinear Schrödinger equation", which are just subtheories of EQM.[3] The presented theory provides also a global view onto solutions of dynamical equations of many of its subtheories including a specification of ways to obtaining their solutions. Having its origin in mathematically well defined models of infinite quantum systems described by traditional (hence linear!) nonrelativistic quantum field theory (QFT), cf. [130, 31, 185, 186], no mathematical inconsistencies could be expected in the basic structure of EQM.

Next Section 1.1 contains a description of the present author's motivation, including some of his presently accepted philosophical ideas, and his mostly personal view on the history of this paper. The author is aware that motivation and history of writings can be considered either from a subjective point of view of the author, or from the point of view of more "objective" history based on a review of existing published works connected in some way with the contents of the presented work. The second point of view, if taken seriously, would need considerable historical effort of experts in the related fields, and we shall not try to present it in this work; we shall add, however, some comments and references to compensate partially this gap, cf. also Remark 1.1.1.

Many important papers relevant to the contents of the present work became known to the present author only after writing his own "independent" version of the "story".[4] It is, however, important to have in sight also independently written works on the considered subject, since alternative approaches to formulation of similar theories might provide also some alternatives for interpretation and/or application of the developed formal theory. This is even more valid taken into account that the author's formulation of the presented results was rather "indirect", obtained as a byproduct of other (a priori unrelated) investigations. We are trying to give here all the relevant citations and credits we are aware of.[5]

The Section 1.2 contains a heuristic description of the general construction of main concept-

[2] The obtained EQM provides rather "metatheoretical framework" for a broad class of physical theories than a specific theory of a given class of physical systems.

[3] Let us note here that, for general dynamical systems (resp. systems of differential equations), "(non–)linearity" is not an unambiguous specification: Any linear equation can be transformed into a nonlinear form by a change of variables and, conversely, many nonlinear equations can be rewritten into a form of linear ones, cf. e.g. a Poincaré theorem [9, Chap. 5,§22], or the "Koopmanism" in e.g., [152] and Remark 3.3.14. Linearity in QM is determined in our work with a help of structures on the projective Hilbert space $P(\mathcal{H})$.

[4] This explains also some omissions of citations of some relevant earlier published papers in the author's previous works: The present author would like to apologize to the authors of those omitted papers in this way.

[5] In spite of this, the bibliography remains probably rather incomplete, and the present author has to apologize repeatedly to authors of unnoticed relevant works.

s, mathematical structures, as well as interpretation problems, and possible applications of the presented nonlinear extension of quantum mechanics (NLQM). The Section 1.3 contains some notes on organization of the paper. We include also into this chapter sections describing briefly general structure of CM (cf. Section 1.4), as well as of QM (Section 1.5), because it provides a starting framework for forthcoming theoretical constructions.

1.1 Notes on Motivation, Background Ideas, and History

The present author is aware of problematic nature of claims about "the originality" of ideas in Science, and of the corresponding "priorities". Even if written in the author's relative isolation, the ideas might come indirectly into the author's mind, through various cultural and social manifestations, or simply by reading also scientific papers not manifestly related to the considered problem. The author will not try to do complicated introspective psychological considerations on origins of his own ideas, what would be necessary to give quite honest (but in any case subjective) answers to questions on "the originality", or at least on "the independence", of obtaining the presented results. We shall try, in the next paragraphs, to describe as honestly as possible in a brief exposition the genesis and history of ideas resulting in this paper. That might be useful also for better understanding of place of the presented theory in the framework of contemporary theoretical physics.

1.1.1. **Remark** *(On contexts and contributions of this work).* Let us mention here at least some references considered by the author as important for a sight on the present work in the broader scientific context . The presented work can be put in a connection with attempts at specific nonlinear generalizations of QM (NLQM) considered as a Hamiltonian field theory on the projective Hilbert space as the "phase space" with a specific (quantum) statistical interpretation; the present work generalizes and unifies such theories. A pioneer work in this direction was, perhaps, the short paper [147] by T.W.B. Kibble, containing a sketch of nonlinear pure–state dynamics and also suggestive motivation directed to applications and generalizations in relativistic QFT and general relativity (GR). Trials (unsuccessful) to formulate quantum statistical interpretation of such theories, as well as some dynamics of mixed states contains the proposal [273] by S. Weinberg. In the papers by R. Cirelli et al., eg. in [63, 67], the authors formulate in a mathematically clear geometrical way standard QM, and they are looking for general principles for possible generalizations of (pure state) quantum kinematics. The papers by M. Czachor et al. [70] contain also proposals for description of dynamics of density matrices in NLQM (accepting essentially the author's proposal from [24]), and also there are investigated methods to solutions of dynamical equations for some classes of generators. The author's paper [24] contains all the essentials of the here presented theory. Connections with older formulations of NLQM and with semiclassical approximations, as well as some proposals for a search for generalized (pure state) kinematics are contained in [11].

Any of the (to the present author) known published papers do not contain consistent proposals of definitions and of quantum statistical interpretation of **nonlinear observables**;[6] such a defini-

[6] This seems to be true also for the papers [78, 169] by Doebner, Goldin, Lücke et al.; their "Doebner–Goldin" NLQM (DG) appears to be non-Hamiltonian, hence it does not fully "fit" into the kind of presently analyzed theories: For testing the belonging of DG to the here analyzed class of NLQM, one should, e.g. to check, whether the r.h.s. of [78, Eq. (1.2)] can be rewritten in the form of the r.h.s. of (2.1.26), resp. in a form $\left(\hat{\alpha}_{P_\psi} + \mathbf{f}^0(P_\psi)\right)\cdot\psi$, with α a closed one–form on $P(\mathcal{H})$, and $\hat{\alpha} \in \mathcal{L}(\mathcal{H})_s = \mathfrak{T}_s^*$ being its operator form (cf. page 49).

tion and interpretation of observables is given in this work. It is given here also an inclusion of the introduced (nonlinear) EQM into a linear theory of a bigger system described in framework of algebraic QT, cf. also [31, 86]. Work with unbounded generators is proposed here in a flexible way: One can restrict attention to a certain set of submanifolds of the "quantum phase space" $\mathcal{S}_*$:= the space of all density matrices, the union of which is not necessarily dense in $\mathcal{S}_*$. Two kinds of "mixed states" are introduced, what is a natural consequence of nonlinear dynamics, cf. also [145, 70]. A unitary representation of a Lie group G is chosen here as a "parameter" serving to specify all the general elements of the theory: the domains of definitions (in $\mathcal{S}_*$) of unbounded generators, the sets of generators, of symmetries, of observables, and of states of the described system; it specifies the UG–system Σ_{UG}. This allows us to determine also the concept of a $\boldsymbol{UG_I}$**–subsystem** of a given G–system Σ_G; also a general definition of a subsystem of a physical system in NLQM was not satisfactorily established in the known literature. We shall not, however, look here for generalized kinematics (i.e. alternatives to $\mathcal{S}_*$, cf. [63, 67, 11]), neither we shall try to formulate here a solution of the "problem of measurement in QM" (understood, e.g. as a dynamical description of the "reduction of wave packet"). ♡

This work is a modified and completed version of the preprint [24].[7] The author decided to publish it now also because of recently renewed interest in nonlinear QM (NLQM) (see, e.g. [145, 49, 144, 15, 11, 67, 57], or [78, 70, 109, 169]),[8] as well as in foundational questions of connections of QM with CM, cf. e.g. [52, 197, 198, 111, 217, 286, 162, 44, 66, 193], or also [202, 98, 113, 257, 94, 233, 249].[9]

Moreover, it can be assumed that ideas contained in this work will be useful for construction of some (not only physical) models.

1.1-a On initial ideas and constructions

The idea of a natural nonlinear generalization of QM (leading to the paper [24]) appeared to the present author after an equivalent reformulation of QM in terms of CM on (infinite dimensional) symplectic manifold $P(\mathcal{H})$ in the works [26, 27]. This was, in turn, a result of trials to understand connections between QM and CM more satisfactorily than via the limits $\hbar \to 0$:[10] A part of the effort was a formalization of the Bohr's beautiful argumentation, e.g. in [21, 22], on necessity of using CM for a formulation of QM as a physical theory, combined with the author's requirement

[7]The author is deeply indebted to Vlado Bužek for his strong encouragement in the process of the author's decision to prepare and publish this new version of [24], as well as for the kind support and also for the effective help he rendered in the process of preparation of the publication of this work.

[8]The author is indebted also to (that time) PhD students, esp. to M. Gatti and E. Grešák, who helped him to make clear some technical features of the presented work, cf. [241, 244, 102, 112, 206].

[9]M. Czachor and his coworkers are acknowledged for their repeated interest in the author's work, as well as for the kind submissions of information about the progress of their work. The author expresses his dues also to S.T. Ali, P. Busch, V. Bužek, G. Chadzitaskos, R. Cirelli, V. Černý, H.-D. Doebner, G.G. Emch, M. Fecko, G.A. Goldin, K.R.W. Jones, N.P. Landsman, J.T. Lewis, E. Lieb, W. Lücke, H. Narnhofer, P. Prešnajder, E. Prugovečki, A. Rieckers, G.L. Sewell, R.F. Streater, W. Thirring, J. Tolar, T. Unnerstall, R.F. Werner, A. Zeilinger, W.H. Żurek, and other colleagues and friends for discussions, and/or for providing him with their relevant papers, and/or for giving him moral support.

[10]For a review and citations on various approaches to "quantization" and "dequantization" with their rich history beginning with the advent of QM see e.g. [79, 255, 100, 245, 97, 141, 256, 161]; some connections of CM and QM via $\hbar \to 0$ could be seen from [132, 123, 146]; for a recent trial to define the limit $\hbar \to 0$ in a mathematically correct way cf. also [206].

on "universality" of quantum theory (QT),[11] the effort possibly hopeless if taken too literally.[12]

The papers [26, 27] resulted from the recognition of quantum pure–state space $P(\mathcal{H})$ as a natural symplectic (even a Kähler) manifold; this personal "finding" was gradually reached at studying of generalized coherent states (GCS)[13] in QM, [149, 199, 17, 166, 242, 71, 200], in looking for their possible usage in describing connections between QM and CM. We benefitted also from the description of symplectic structure on (finite dimensional) complex projective spaces [7]. Works on their quantum mechanical connections/applications [16, 56, 133, 221, 212, 222] was encouraging in this effort. As the author can judge today, many important results have been obtained in the literature. Unfortunately, not all of the details of the cited works were clearly seen by him during the time when he formulated his theory: There was a variability of languages and interpretations in various papers, as well as a lack of sufficient mathematical rigor which obviously was an obstacle for a better understanding. There were also important unnoticed works containing some of the author's later results, e.g. [251, 147, 154].[14]

Conceptually important in the search of QM $\leftrightarrow$ CM connections was appearance of symmetry groups G allowing a unified theoretical description of "changes of objects with a specified identity", cf. mainly [69, 275, 281, 138], and givig a framework for description of physical quantities; we have restricted our attention to Lie groups, where distinguished one–parameter subgroups correspond to specific physical quantities (cf. Galileo, or Poincaré groups). The cited papers using sets of GCS used them either as a tool for description of some "quasiclassical approximations" to QM in various specific situations, or as a formulation of a "quantization" procedure, cf. also more recent literature, e.g. [4, 3, 161].

Generalized coherent states were usually considered as submanifolds of the Hilbert space determined either as some more or less arbitrary parametrically determined manifolds (usually finite dimensional), or as orbits of continuous unitary representations of a Lie group G. An essential rôle is ascribed to a symmetry Lie group G also in the present paper: This corresponds to the accepted (hypothetical) point of view according to which observables in physics are necessarily connected in some way with a group of symmetries.[15]

1.1.2. **Interpretation.** This "philosophy" can be substantiated by the following simple intuitive consideration: Physical situations (e.g. different states of a physical system) corresponding to different values of a "physical quantity" should be connected by some transformations which

[11] We distinguish here QM from QT, the later including also mathematically well defined parts or versions of QFT, e.g. the nonrelativistic C^*-algebraic theory of systems "with infinite number of degrees of freedom". In this understanding, QT can describe also macroscopic parameters of "large" quantal systems, composing their classical subsystems.

[12] The intention of the author was even to formulate a general model of the measurement in QM, being up to now an unsolved fundamental problem of QM (if QM is considered as a "universal theory"), [28]. This author's effort started in 1961 at Charles University and/or Czech Technical University (ČVUT) in Prague (the Faculty of Technical and Nuclear Physics – FTJF – was administratively moved between these two universities in those years), later continued also in a small seminar formed by J. Jersák, V. Petržílka, J. Stern, and the present author; in the framework of this seminar was formulated a simple (unpublished) proof of impossibility of information transmission by "reduction of wave packets" corresponding to the EPR–like quantum measurements according to the traditional (Copenhagen) formulation of QM, cf. Note 1.5.9 on page 39.

[13] The author is indebted to P.Prešnajder for turning his attention to GCS.

[14] For the citation [154], as well as for some other useful notes made during the correspondence concerning [27] the author is obliged to K.Hepp. The author obtained the citation [64] from K.R.W.Jones. About the citation [147] was the author informed by N.P.Landsman.

[15] A possible generalization of this point of view might lead to the assumption, that observables are determined by local groups, [148], or gruppoids, [161]; the Landsman's book [161] contains also other relevant ideas and techniques, as well as citations.

make possible to assure that the different values are really "values of the same quantity"; the assumption of transitivity and invertibility of these up to now unspecified transformations seems to be natural for quantities without some exceptional values in their range. This results in the hypothesis of presence of a group defining physical observables (resp. quantities); some further "physically natural" continuity requirements then end at a Lie group.[16] ♦

1.1.3. **Remark**. The presence of a Lie group G in the following considerations has, however, also a technical function: it offers us an easy possibility to work with specific unbounded observables described by not everywhere defined functions on the symplectic manifold $P(\mathcal{H})$; such observables correspond (in the linear case) to usual unbounded operators describing physical quantities in QM. The corresponding technical tool is the existence of the $\boldsymbol{C^\infty(G)}$**–domains** (e.g. the **Gårding domains**) of strongly continuous unitary representations $U(G)$ of any Lie group G.[17] ♡

The importance of Lie group representations for QM was stressed already by founders of QM, let us mention especially Weyl and Wigner [280, 275, 281, 282]; applications of Lie groups in fundations of QM was afterwards elaborated by many others, e.g., cf. [167, 168, 148, 267, 5, 104, 75]. Also Prugovečki's and Twareque Ali's papers, e.g. [213, 214, 2, 215, 216], were stimulating for the present author's work: Some intuitively convenient statistical interpretation of GCS in QT was also looked there for. The Weyl's book [275] contains, in an implicit way (as it was perceived by the present author), some of the main ideas concerning connection of QM with CM formulated in the papers [26, 27].[18]

In our presentation, orbits of coadjoint representations of G play an important rôle. They appear naturally in QM as orbits of expectation functionals corresponding to GCS, which are calculated on generators of the considered Lie group representation $U(G)$; these generators are usually interpreted in QM as distinguished sets of quantummechanical "observables". The canonical symplectic structure on these $Ad^*(G)$–orbits is described, e.g., in the monograph [148], cf. also Appendix A.4. The general coordinate–free differential geometric formalism of Ellie Cartan and its applications to CM is described, e.g. in [1, 151, 258, 61], cf. Appendix A.3.

Generalized coherent states determined by continuous unitary representations $U(G)$ of finite dimensional Lie groups G provide a "semiclassical background" to approximate descriptions of quantum theory. Points of the manifolds of coherent states can be canonically parametrized in many cases by points of an orbit of the coadjoint representation $Ad^*(G)$. In these cases, a canonical Poisson structure corresponding to that one existing on the $Ad^*(G)$–orbit can be defined on the manifold of coherent states. It is possible to determine canonically a specific "projection" of quantum mechanical (=: **quantal**) dynamics to such a "classical phase space", [27]. Some

[16] As concerns a general gnoseological approach of the present author to Theoretical knowledge, it is close in a certain feature to that of K. R. Popper, [209, 210], cf. also [127]; we accept, e.g. that each scientific assertion can be considered just as a hypothesis: There is no "final truth" in our Knowledge. Moreover, any "meaningful" assertion concerning possible empirical situations should be falsifiable by some empirical tests. Let us add, however, that one should distinguish different "degrees of certainty" of various claims: Although mathematically formulated, claims on empirical contents should undergo our identification with specific "extratheoretical" situations, and this process cannot be fully formalized.

[17] For an application of this kind of ideas cf. also the theory of "Op*-algebras", [163].

[18] The above mentioned inspiring "ideas", "stimulations", etc. are difficult to specify and formulate clearly: They were often hidden in the stylistic form of presentation of otherwise "quite simple facts" by the cited authors; e.g. the Weyl's considerations on "Quantum Kinematics" in [275, Chapter IV.D], presently known to every physics student as CCR, were perceived by the present author as very stimulating – much later than during his student's years.

satisfactory (unambiguous, and general) interpretation of these canonical "classical projections" is, however, still missing.[19]

Methods of the "time dependent Hartree–Fock description" of fermionic systems, or more generally, of the "time dependent variational principle" in QM, [154], can be reduced in many cases to specifications of the general procedure of the mentioned "classical projections". The "classical projections" of quantum dynamics to orbits of coherent states can lead, in some formally chosen cases (i.e. chosen regardless to existence of any possibility of a physical interpretation of the considered dynamics), to such a classical dynamics which has little in common with the original quantum system. This left an open question to us, in what cases "classical projections" are "close" to the projected quantum dynamics, [27].

The dynamics of an individual subsystem of the infinite quantum system in mean–field theory (MFT) is described exactly by such a kind of "classical projections", [33, 31]. In this is hidden a connection of our EQM with a (linear) QT of infinite quantum systems, cf. also Subsection 1.1-b.

We shall show, in Section 3.6, that the dynamics in NLQM (modified with respect to that of Ref. [273] for the cases of evolution of "mixed states") can also be described in this way. We obtain a mathematically correct and physically consistently interpreted standard type of quantum theory (i.e. a C^*-algebraic theory) in the case of such a mean-field reinterpretation of the "classical projections of QM". We shall describe these theories in a form of a generalized quantum mechanics of autonomous physical systems. "Observables" in the presented theory are expressible as operator–valued functions $\mathfrak{f} : \mathbf{F} \mapsto \mathfrak{f}(\mathbf{F})$ of a classical field with values $\mathbf{F}$ appearing in corresponding interpretation also in MFT. In models of MFT the "classical field" $\mathbf{F}$ can describe, e.g. collective variables describing macroscopic quantum phenomena like superconductivity, or other "global observables" describing a large quantum system. The classical field $\mathbb{F}$ (cf. Definition 2.2.17) aquiring values in $Lie(G)^* \ni \mathbf{F}$ is here present in a rôle of a "macroscopic background" of the considered quantum system. The (nonlinear) dynamics, as well as the probabilistic interpretation of the theory can be described, however, independently of any use of "background fields": The introduction of the field $\mathbb{F}$ (which is a function of the quantum states $\varrho \in \mathcal{S}_*(\mathcal{L}(\mathcal{H}))$ appears like an alternative description (or an "explanation") of the dynamics which can lead to simpler solutions of problems. We have not specified unambiguously a physical interpretation of dependence of the operators $\mathfrak{f}(\mathbf{F})$ on values $\mathbf{F}$ of the macroscopic field $\mathbb{F}$. It can be suggested, e.g. that $\mathbb{F}$ takes part in determination of "physical meaning" of the quantum observables: For each value $\mathbf{F}$ of $\mathbb{F}$, "the same" quantum observable $\mathfrak{f}$ is described by a specific operator $\mathfrak{f}(\mathbf{F})$. We have introduced, however, a standard prescription for calculation of probabilities of measured results of observables represented by the operator – valued functions $\mathfrak{f} : \mathbf{F} \mapsto \mathfrak{f}(\mathbf{F})$ which is consistent with the traditional one, cf. formulas (2.3.4), (2.3.9), and (2.3.10). We also expect that traditional foundational problems in physics like the "quantum measurement problem", or the question on "origins of irreversibility" might be fruitfully reformulated in the presented framework.

[19] We have known the "mean-field" interpretation of such quantum motions, cf. Subsection 1.1-b, and Section 3.4; physical origin of such a classical "background field" might be looked for in hypothetical, or sometimes even known, existence of some "long–range forces", representing an influence of, e.g. (let us allow some visions to ourselves) Coulomb forces with quantum correlations of distant stars to the considered microsystem, cf. [285].

1.1-b Relation to infinite systems

An important element in building the presented scheme of EQM was construction of classical quantities of an infinite quantal system. This was done in usual C^*-algebraic language [91, 76, 77, 227, 196, 42, 254], cf. also [27]. The author was especially inspired by the papers [131, 130, 16], the monographs on quantum–theoretical description of systems "with infinite number of degrees of freedom" [91, 42], some general ideas of Einstein, Bohr, Heisenberg and other thinkers expressed in many, nowadays difficult identifiable, places (as introductions to books and papers, popular and philosophical writings, quotations by other people, etc.), as well as by some other, both "technically & ideologically" composed papers, like the review [284] on "large N limits" in QM.

Let us describe briefly the obtained picture of kinematics of an infinite quantum system in which a commutative ("classical macroscopic") subalgebra $\mathcal{N}$ of observables is determined by a unitary representation $U(G)$ of a Lie group G. Let the large quantum system consists of N copies of equal systems described in separable Hilbert spaces $\mathcal{H}_m$ by algebras of observables $\mathcal{L}(\mathcal{H}_m), m = 1, 2, \dots N$. Then the algebra of observables $\mathcal{A}_N$ of the composed system is isomorphic to $\mathcal{L}(\otimes_{m=1}^{N}\mathcal{H}_m)$, and nothing essentially new is obtained: It has only one "reasonable" irreducible representation (up to unitary equivalence). The so called C^*–inductive limit for $N \to \infty$ of $\mathcal{A}_N$, cf. [227], however, is an algebra $\mathcal{A}$ of a different type: It has uncountably many mutually inequivalent faithful irreducible representations. Subsets of these representations could be parametrized by some "classical quantities", which can be themselves realized as a (commutative) C^*-algebra in the center $\mathcal{Z}$ of the double dual $\mathcal{A}^{**}$ of $\mathcal{A}$. But the center $\mathcal{Z}$ is an incredibly big algebra which cannot be, probably, used as a whole to some useful description of macroscopic properties of "the system $\mathcal{A}$". Here was used a Lie group G for obtaining a specification of a subalgebra of $\mathcal{Z}$ of a "reasonable size". The use of a Lie group G allowed also a natural introduction of a Poisson structure [274, 177, 7], and consequently classical dynamics into the "relevant part" of $\mathcal{Z}$.[20]

These constructions were motivated by some attempts to understand possible quantummechanical basis of classical description of macroscopic bodies, cf. [131, 27, 28], as well as of interaction of that bodies with microscopic systems described by QM. This effort included trials to solve the old problem of modeling the "measurement process in QM" [131, 28]. Although this questions were extensively studied during the whole history of existence of QM, cf., e.g. [189, 14, 276, 168, 50], no approach to their solution, hence no answers, are generally accepted up to now. In the process of modeling of interaction of microsystems with macroscopic bodies in QM framework, a quantum description of macroscopic bodies was a necessary preliminary step. The simplest possibility was a study of kinematics of an infinite set of equal quantum systems in the framework of C^*-algebraic theory. This is formulated in [27]. One of the most important questions was a "proper" choice of observable quantities of such a big system.[21] This was done

[20] A Poisson structure is, however, always present in any noncommutative C^*-algebra in the form of the commutator of any of its two elements, cf. also [86]. This can be used to obtain, by a certain limiting procedure, cf. [31, 32, 86], also a Poisson structure on some subsets of the commutative W^*-algebra $\mathcal{Z}$. The Poisson structure obtained in this way is identical with that one connected with a Lie group action. Lie groups are, however, useful (besides for technicalities in dealing with unbounded generators) for interpretation of abstract "observables", and for determination of "proper subsets" of the huge centre $\mathcal{Z}$.

[21] That a choice of "observable observables" is a nontrivial task also from a quite different point of view is claimed in [191].

by a choice of the kinematical Lie group G mimicking macroscopic motions of the large (composed) quantum system: The representation $U(G)$ acted equally on any "elementary subsystem" described by $\mathcal{L}(\mathcal{H}_m), m = 1, 2, \ldots \infty$.

The resulting formulation of nonlinear quantum dynamics in the presented extension of QM can be connected with the specific form of the author's formulation of dynamics of infinite quantum systems [31, 33, 264, 265, 86] with interactions of "mean–field type", having its roots in [130]. [22] Our citations of works relevant for the theory of microscopic description of macroscopic phenomena in quantum systems are incomplete; some other relevant citations can be found in [42, 43, 247, 238].

Many modifications and generalizations of the sketched description of classical quantities of infinite quantum systems, including their dynamics, are possible. Some of them will probably lead to the same "microscopic" nonlinear dynamics, as it is in the case of MFT. The presented results can be considered as just a first step in investigation of macroscopic dynamics from quantum–theoretical point of view. There were performed already some works containing more sophisticated description (than ours) and more detailed results in this direction, cf. e.g. Sewell's papers [239, 240], or some works in algebraic quantum field theory (QFT),[23] e.g. in [99].

We shall briefly return to some technicalities of the description of "macroscopic subsystems" of large quantum systems in Section 3.4.

1.1-c Questionable "subsystems"

A general interpretation of EQM considered as a "fundamental theory" is not formulated in this work. It can be, however, conjectured that a viable possibility for its interpretation is (by admitting the linear QM as "the fundamental theory of simple systems") a description of "relatively isolated systems", i.e. "ordinary" quantal systems moving in an external field which is in turn influenced by (or correlated with) these quantal systems. Let us give here some motivation and background to this rough idea.

One of the most basic concepts of contemporary physical theories, and, perhaps, of the methodology of the whole Science, is the concept of *isolated systems* the description of which is especially "simple": It is supposed, that there are specific "circumstances" under which we can deal with phenomena independently of the rest of the world. Examples are: idealized bodies "sufficiently distant from all other bodies" described in CM in framework of an "inertial coordinate system", realized, e.g., by atoms in a dilute gas during a certain time intervals. More generally, we are used to think about any specific "object" as determined "relatively independently" of other objects (except of some generally accepted "background", e.g. inertial frame, or vacuum). Mere possibility to formulate such concepts of various "isolated systems" which approximately describe some observed phenomena can be considered as one of the miracles of human existence. More detailed investigation (and specification) of any phenomenon usually shows that such a simple description is of a restricted use, and better results might be obtained by consideration of a "larger piece of the world"; the identity of the "considered (sub-)system"

[22] From the personal author's point of view, it was obtained in a sense "occasionally". The resulting dynamics of the infinite mean-field systems [27, 31] was a natural result of a simple question: *How to define a microscopic Hamiltonian dynamics on the infinite quantum system leading to a given (arbitrarily chosen) classical dynamics on the part of the centre $\mathcal{Z}$ specified with a help of the mentioned representation $U(G)$?*

[23] This is the theory formulated by Araki, Haag, Kastler, and others, cf. [122, 37, 6, 118, 81, 82, 83, 120, 38].

can be then, however, lost.[24]

An often used "first step" to describe some "influence of other systems" onto the "considered one" is an introduction of an appropriate (possibly time dependent) "external field". This procedure corresponds to the formal construction (and logic) of nonrelativistic CM: The motion of a body interacting with other ones can be expressed in CM as its motion in a time dependent "external field" (determined by a known motion of "other bodies" in the presence of "the considered one"). Subsystems in CM are, in this sense, clearly definable (they are continually described by a point in their phase sub-space), and we can consider them as *relatively isolated*: They move according to certain nonautonomous evolution laws (as if they have their own – time dependent – Hamiltonians), what can be intuitively understood as "just a (time dependent) deformation" of a background of formerly isolated system, leaving the identity of the system "essentially untouched",[25] and this has introduced a change into the dynamical law of the system.

The determination of isolated systems, as well as of subsystems in QM is much more problematic than in CM. Schrödinger equation describes, in analogy with CM, dynamics of a physical system in a given external field: Systems described in this way can be considered as "relatively isolated". This formulation was very successful in description of scattering and motion in external (macroscopic) fields, of dynamics of atoms and small molecules, as well as in approximate descriptions of a lot of phenomena in many–particle systems. QM time evolution of mutually interacting systems occurring initially in uncorrelated pure states (i.e. in a pure "product–state") leads usually in later times to an "entangled" state of the composed system.[26] The states of constituent subsystems are described in such a state just by density matrices (which are mathematical objects also used for description of "mixed states" in a common sense interpretation, i.e. in the "ignorance" interpretation which is common in classical statistical physics), and time development of these (obtained by taking the "partial trace" of the evolving pure state of the whole "isolated" system) need not be Hamiltonian (e.g. [71, 54, 252]). Since nontrivial interaction (and also entanglement) between states of charged microscopic particles and quantum states of macroscopic bodies (if considered as quantum many–particle systems) is present also in systems whose constituent subsystems are separated by cosmic distances, cf. [285], an empirically realizable definition of isolated systems in QM remains a problem. We assume that EQM provides also a possibility of an approximative Hamiltonian evolution for some of such "basically entangled" situations.

Another problem of QM connected with the problem of determination of subsystems is the classical "problem of measurement in QM", cf. [189, 14, 276, 50, 28]. It can be, perhaps, considered as an (up to now unknown) process of "entanglement" of the states of the measured microsystem with macroscopically distinguishable states of the apparatus.[27] A determination of

[24] A version of the concept of an "isolated system" necessarily appears in any kind of reproducible reflection in human thinking. Its specification, however, varies with accepted "paradigms" [156] (let us stay with a mere intuition on these ambiguous philosophical concepts), e.g. the meaning of the physical system representing a falling stone was different for Aristotle from that of Galileo, and also it was different for Einstein from that of Mach, [170].

[25] It is an analogy to "external" gravitational field in general relativity acting on a "test body": it is a "deformed" inertial frame corresponding to the background determined by massive bodies (e.g. by distant stars) – in a sense similar to that of the Mach's approach to CM [170, 270].

[26] Theoretical, as well as experimental investigation of "entangled states" in QM is quite intensive in last years, cf. e.g. [268, 165, 137, 54, 252, 55, 286, 217].

[27] Recently are quite popular "solutions" of the quantum measurement problem via "decoherence", cf. [288], resp. via "decoherent histories" approach, cf. [193, 84]; the present author considers them at most as preliminary attempts to

a clear cut between "microscopic" and "macroscopic" is missing in both of these problems. Perhaps, the only available, formally well defined formulation of the "micro–macro difference" can be found in the framework of the C^*-algebraic formulation of QT, [91, 42, 238]. In this framework, also some models for the measurement process in QM were formulated, [131, 28, 278]; the process needed there, however, an infinitely long time interval for its completion. We expect that EQM provides a way also for description of the mentioned micro–macro "entanglement".

1.1-d Some basic building blocks of EQM

Our Extended Quantum Mechanics contains many theories as exact (i.e. obtained without any "approximations" in a usual sense) subtheories. They are considered usually as different (but inequivalent) possibilities of descriptions of the same system, e.g. one of the theories is considered as an "approximation" of another one. Examples are WKB, Hartree–Fock, or classical mechanical approximations to descriptions of some problems in QM, or CM and QM themselves. All these subtheories are obtained from the general scheme of EQM by specifying three subsets (which are, however, mutually consistently interconnected) of corresponding three general building sets of theoretical objects.[28]

In classical mechanics [277, 1, 258, 7, 172] (CM) as well as in **quantum theories (QT)** [172, 74, 189, 168], three main (mutually interconnected) classes of fundamental objects (corresponding to basic concepts of the theory) are used: (i) **observables**, (ii) **states**, and (iii) **symmetries**. A one parameter subgroup of symmetries specifies a chosen **dynamics** of the system, and the corresponding parameter is called the **time**.[29] The mathematical representation of these classes and formulation of their mutual connections do not always use "physically motivated" properties only; some clarity in expression of connections between constructs of formalized theories and empirical and conceptual analysis of phenomena is often reached by a subsequent specification and interpretation of the used mathematical objects. Any fundamental theory of the process of measurement of an arbitrary mathematically defined "observable", considered as a dynamical process within QM is not known; we are not able generally decide which mathematically defined "observables" are accessible to empirical identifications; similar comment applies to "states", and also to "symmetries". This lack of "bijective correspondence" between classes of known empirical situations and objects of a theory could make the theory, on the other hand, more flexible.

We shall reformulate and extend the formalism of QM so that it will include QM and a class of its (nonlinear) generalizations. Such an EQM contains much larger variety of "observables", "states", and "symmetries" than does the traditional QM. These extended sets of fundamental objects contain different subsets representing different "subtheories" of the extended QM. Between these subtheories we shall find, in addition to ordinary (linear) QM, also, e.g. Hamiltonian CM with phase spaces being homogeneous phase spaces of Lie groups,[30] several existing formu-

attack the problem.

[28] Other conventional relations between CM and QM are "quantizations", and "dequantizations", the later understood usually as a limiting procedure denoted by "$\hbar \to 0$".

[29] In the considered specific theories the time parameter is in a sense "global", so that it is meaningful to speak about states and observables of the (total) system *at a time* $t \in \mathbb{R}$.

[30] A homogeneity requirement on phase spaces with respect to some topological group seems to us natural from an "epistemological" point of view, cf. Subsection 1.1-a, resp. Interpretation 1.1.2, and Remark 1.1.3. There would be no problem, however, to find in EQM also Hamiltonian CM on a general, not necessarily specified by a group action,

lations of nonlinear quantum dynamics, cf. Subsection 3.3-e, and [11], and also the frequently used approximations to quantum dynamics consisting of its specific restrictions to manifolds of generalized coherent states of the considered system, or also the WKB-approximation [11] are in our extension obtained as "subtheories" (without making any approximations). The mentioned specifications are obtained by corresponding choices of subsets of "observables", "states", and "generators" of symmetry groups, and are usually mainly determined (cf. Section 2.2) by a choice of a unitary representation $U(G)$ of a Lie group G in the Hilbert space $\mathcal{H}$ corresponding to the traditional quantummechanical description of the considered system:[31] E.g., QM corresponds to the choice $G := \{e\}$ (a one-point set), cf. Section 3.3 (this does not exclude a use of other group representations $V(G_1)$ in the description of "microscopic observables", and "symmetries" in QM; $V(G_1)$ will play, however, another rôle than the picked out $U(G)$ in the theory!); CM of N scalar particles is specified by the Schrödinger representation $U(G)$ of the $6N + 1$ - dimensional Weyl–Heisenberg group G, and by additional restrictions to the sets of "permitted" (or "physical") states, generators and observables, cf. point 3.1.4. Another approaches to incorporation of classical observables into an extended quantum formalism were published, e.g., in [203], cf. also our Section 3.4, Appendix B, and [31, 32, 28].

The dynamics (generally nonlinear) of EQM on the "quantum phase space $\mathcal{S}_*$" can be recovered as a *subdynamics of linear dynamics of a larger quantal system*. This can be seen from Section 2.3, where in Definition 2.3.3(ii) a C^*-algebra of observables $\mathcal{C}^G$ was introduced such, that our evolutions in EQM are (linear) automorphism groups of this C^*-algebra, cf. also Section 3.4. Looking on the obtained EQM "from a side", we could recover similarity between our transition from QM (resp. NLQM) to EQM (and its linear realization on the C^*-algebra $\mathcal{C}^G$), and the "Koopmanism" in CM (cf. Remark 3.3.14): While in the Koopman transition the CM was "linearized" by transferring the phase space $(M; \Omega)$ as a sort of "spectrum space" (cf. Appendix B, Example B.3.5) into the infinite–dimensional Hilbert space $L^2(M, \mu_\Omega)$, and its (nonlinear) dynamics into a (linear) unitary group, in our consideration of EQM (leading to nonlinear evolution on the "quantum phase space" $\mathcal{S}_*$, i.e. in a "restricted Schrödinger picture") as a C^*-algebraic theory we obtain (in the corresponding "Heisenberg picture") a linear quantum dynamics on a C^*-algebra (namely $\mathcal{C}^G$), cf. [35]. The state space of this C^*-algebra is, however, much larger than $\mathcal{S}_*$, or even than the space $\mathcal{M}_{+1}(\mathcal{S}_*)$ of probability measures on $\mathcal{S}_*$ (of which is $\mathcal{S}_*$ the subspace of Dirac measures, in a canonical way).

1.2 A Brief Description of the Contents

For better orientation of readers in the contents of the following text, we shall give here also a brief and heuristic explanation of some of the main points of the contents of this paper, as well as some of their interconnections. Some notes on the placing of different parts of the contents in the text can be also found in Section 1.3.

Let us introduce first some notation used in this paper:

1.2.0. Notation. *(i) We use usually different fonts (e.g., f, $\mathbf{f}$, f, $\mathfrak{f}$, $\tilde{f}$, $\hat{f}$, $\mathfrak{f}$) for different kinds of mathematical objects.*[32] *By $\&$ is denoted the logical conjunction "and".*

symplectic submanifold of $P(\mathcal{H})$.

[31] The group G *cannot be generally identified* with the group of symmetries of the system!

[32] Bold form of symbols will be used sometimes, mainly in their definitions, however, also for the otherwise nonbold ones, which are of the same typographic form.

(ii) The relation $A(x) \equiv B(x)$ expresses (usually) assertion, that values of the two functions A and B are mutually equal on the intersection $D(A) \cap D(B)$ $(\ni x)$ of the domains $D(A)$ (resp. $D(B)$) of definition of the functions A and B.
(iii) The symbol $f(\cdot, y)$ denotes the function $x \mapsto f(x, y)$. $\diamondsuit$

1.2.1 (QM, and NLQM). QM is traditionally formulated in terms of selfadjoint operators X on a complex Hilbert space $\mathcal{H}$ which play the double rôle of the "observables", as well as of the "generators" of symmetry groups in the theory. QM can be equivalently reformulated in terms of (infinite dimensional) classical Hamiltonian mechanics on the phase space $P(\mathcal{H})$ consisting of one-dimensional complex subspaces $\mathbf{x}, \mathbf{y}, \ldots$ of $\mathcal{H}$.[33] Linear operators $X = X^*$ on $\mathcal{H}$ then correspond to the functions $h_X : \mathbf{x} \mapsto h_X(\mathbf{x}) := Tr(P_{\mathbf{x}} X) \equiv \langle x|X|x\rangle / \langle x|x\rangle$ on $P(\mathcal{H})$, where $P_{\mathbf{x}} := P_x \quad (0 \neq x \in \mathbf{x} \subset D(h_X)$, cf. (2.2.1)) is the orthogonal projection onto $\mathbf{x}$. The Poisson bracket is

$$\{h_X, h_Y\}(\mathbf{x}) = i\, Tr(P_x[X, Y]) =: h_{i[X,Y]}(\mathbf{x}), \tag{1.2.1}$$

where $[X, Y] := XY - YX$ is the commutator. The Schrödinger equation is then equivalent to Hamiltonian equations corresponding to 1.2.1: If H is the Hamiltonian operator of a QM system, then the evolution of the "observables" h_X is described by the Heisenberg-Hamilton (resp. von Neumann-Liouville) equations

$$\frac{d}{dx} h_X(\varphi_t^H \mathbf{x}) = \{h_H, h_X\}(\varphi_t^H \mathbf{x}),\ \mathbf{x} \in P(\mathcal{H}),\ t \in \mathbb{R}. \tag{1.2.2}$$

where φ_t^H is the "Hamiltonian" (resp. "Poisson") flow on $P(\mathcal{H})$ corresponding to the unitary evolution $t \mapsto exp(-itH)x$ of vectors $x \in \mathcal{H}$, i.e. a one-parameter group of transformations of $P(\mathcal{H})$ conserving Poisson brackets which can be determined from (1.2.2). This immediate rewriting of QM differs from an "ordinary Hamiltonian CM" on $P(\mathcal{H})$ by a specific restriction of the set $\mathcal{F}(P(\mathcal{H}))$ of differentiable real valued functions used as "observables" and "generators": QM uses only those $f \in \mathcal{F}(P(\mathcal{H}))$ that have the form $f \equiv h_X (X = X^*)$. Let us call these h_X **affine functions** (or also "Kählerian functions", [63]) on $P(\mathcal{H})$: They can be considered as affine functions defined on all convex combinations $\varrho := \sum_j \lambda_j P_j \in \mathcal{S}_*$ of the pure states $P_j \in P(\mathcal{H})$; they can be characterized, however, in a "purely geometrical way" in the framework of $P(\mathcal{H})$ with a help of canonical metrics on it (cf. [63, 26, 27]): affine functions $f \in \mathcal{F}(P(\mathcal{H}))$ are exactly those f which generate Poisson flows conserving the metrics (equivalently: conserving transition probabilities, (1.2.4)), and, in that case, they are expressible by linear operators X, i.e. $f = h_X$. We shall sometimes call the affine functions f also "linear functions". Other $f \in \mathcal{F}(P(\mathcal{H}))$ will be called **nonlinear functions on $P(\mathcal{H})$**. The "equation of motion" (1.2.2) for general functions $f, h \in \mathcal{F}(P(\mathcal{H}))$ has the form

$$\frac{d}{dt} f(\varphi_t^h \mathbf{x}) = \{h, f\}(\varphi_t^h \mathbf{x}),\ \mathbf{x} \in P(\mathcal{H}),\ t \in \mathbb{R}, \tag{1.2.3}$$

where the Poisson bracket is the unique extension of (1.1) to more general real-valued functions $h, f, \ldots$ on $P(\mathcal{H})$ (cf. Sec. 2.1).

[33] Such a scheme should be supplied by an interpretation scheme extending the probabilistic interpretation of QM. Such an interpretation is given later, cf. Interpretation 2.3.11.

The **formal transition from QM to NLQM** consists (in our "classical–like" rewriting of QM) in the addition to affine "generators" of QM of also the nonlinear ones. Such an infinite-dimensional classical mechanics on $P(\mathcal{H})$ is developed in Sections 3.2, 3.3-a. Inclusion of these nonlinear functions between generators implies, however, a sequence of problems for quantum theory. ♠

1.2.2 (Evolutions and mixtures). The basic Wigner theorem (cf. Proposition 3.2.6) states that to any bijective transformation φ of $P(\mathcal{H})$ onto itself conserving the "transition probabilities", i.e.

$$Tr(P_{\mathbf{x}}P_{\mathbf{y}}) \equiv Tr(P_{\varphi\mathbf{x}}P_{\varphi\mathbf{y}}), \tag{1.2.4}$$

there exists a unitary or antiunitary operator U on $\mathcal{H}$ such, that

$$P_{\varphi\mathbf{x}} \equiv P_{Ux}, \text{with } 0 \neq x \in \mathbf{x},\ \forall \mathbf{x} \in P(\mathcal{H}). \tag{1.2.5}$$

The corresponding operators U_t are unitary for continuous families $t \mapsto \varphi_t, (\varphi_0 := id_{P(\mathcal{H})})$ of mappings φ satisfying (1.2.4). The unitary operators in (1.2.5) are determined essentially uniquely by φ, up to numerical factors. This means, that φ from (1.2.4) with a unitary U uniquely determines a *-automorphism α_φ of the von Neumann algebra $\mathcal{L}(\mathcal{H})$ of bounded operators on $\mathcal{H}$. Such an automorphism, in turn, determines the dual mapping α^*_φ that affinely and bijectively maps the space $\mathcal{S}_*$ of all density matrices onto itself and extends the mapping $\varphi : \alpha^*_\varphi \mathbf{x} \equiv \varphi\mathbf{x}$. On the other side, [42, Theorem 3.2.8, Corolary 3.2.13, Examples 3.2.14 and 3.2.35], if a one parameter family $\varphi(t)$ of bijections of the pure states $P(\mathcal{H})$ onto itself can be extended by a "sufficiently continuous" family $t \mapsto \alpha^*_{\varphi(t)}$ of affine bijections $\alpha^*_{\varphi(t)}$ of $\mathcal{S}_*$ onto itself, then there is a one-parameter family of *-automorphisms $\alpha_{\varphi(t)}$ of $\mathcal{L}(\mathcal{H})$ represented by unitary operators $U(t)$ such that $P_{\varphi(t)\mathbf{x}} \equiv U(t)P_{\mathbf{x}}U(t)^*$. It can be shown [63] (cf. also Proposition 3.3.1) that the transformations $\varphi := \varphi^h_t (t \in R)$ solving (1.2.3) satisfy (1.2.4) iff there is some $H = H^*$ such that $h \equiv h_H$. Hence, evolutions determined from (1.2.3) for nonlinear h necessarily violate (1.2.4), and φ^h_t cannot be (for all t) extended by affine mappings of $\mathcal{S}_*$ onto itself. This also means that φ^h_t cannot be extended into a transformation of density matrices $\varrho := \sum \lambda_j P_{\mathbf{x}_j} =: \sum \lambda_j \mathbf{x}_j$, conserving affine combinations, i.e. for any such extension $\tilde{\varphi}^h_t$ there is

$$\tilde{\varphi}^h_t \varrho \not\equiv \sum_j \lambda_j \varphi^h_t \mathbf{x}_j. \tag{1.2.6}$$

This has consequences described in Note 3.3.3, and in Interpretation 2.1.24, as well as in Subsection 2.1-e: An evolution $\tilde{\varphi}^h_t$ of density matrices cannot be expressed by "the same" evolution φ^h_t of pure components of their decompositions. This has several further consequences. ♠

1.2.3 (Emergence of nonlinear observables). The evolution φ^h_t (in the "Heisenberg picture") of affine "observables" h_X does not lead identically to affine observables, i.e. there are *no* such one parameter sets of operators $X(t)^* \equiv X(t), X(0) := X$ that $h_X(\varphi^h_t \mathbf{y}) \equiv h_{X(t)}(\mathbf{y})$, for nonlinear h. Hence, inclusion of nonlinear generators implies necessity of inclusion of also "nonlinear observables" into the theory. The probabilistic interpretation of such observables is not possible in a traditional way, cf. Interpretation 2.1.24. The interpretation inspired by "mean–field

interpretation" is described in Interpretation 2.3.11, where the expression of nonlinear functions $h \equiv h_{\mathfrak{f}}, h_{\mathfrak{f}}(\mathbf{x}) := Tr(P_{\mathbf{x}}\mathfrak{f}(\mathbb{F}(\mathbf{x})))$ with a help of conveniently chosen operator-valued functions $\mathfrak{f}$ is used (see Definition 2.2.17 for $\mathbb{F}$). A restriction of possible choices of the functions h , as well as of nonlinear generators, can be determined by a choice of the representation $U(G)$, cf. also Definitions 2.2.26, 2.3.2–2.3.5. ♠

1.2.4 (Two kinds of mixtures). Impossibility of a unique extension of φ_t^h (determined by the function h defined on $P(\mathcal{H})$ only) to a mapping φ_t^h on $\mathcal{S}_*$ leads to necessity of investigation of a natural "Poisson structure" and a consequent definition of φ_t^h for "Hamiltonian functions h" defined now on the whole $\mathcal{S}_*$, cf. Section 2.1. This provides a solution of problems arising in the earlier trials to formulate NLQM with connection of evolution of mixed states, cf. also [273, 272]. These facts lead also to necessity of distinction of two kinds of "mixed states" in nonlinear extensions of QM. These are introduced in Subsection 2.1-e, and in Definition 2.3.5. The *elementary mixtures* correspond to density matrices considered as points of the *elementary phase space* $\mathcal{S}_*$; these elementary mixtures are transformed by Poisson flows φ^h as points of $\mathcal{S}_*$, independently of their possible convex decompositions. Another kind of "mixed states" is described by probability measures μ on $\mathcal{S}_*$, which are not concentrated in one point: these are called the *genuine mixtures* (corresponding to the term "Gemenge" used in [50]). Evolution of states described by μ's is given by evolutions of points in the support of μ. This offers, e.g., a possibility to distinguish between the state described by an elementary mixture – e.g. the density matrix ϱ_I of a subsystem I (obtained as the "partial trace" [71]) of a composed system I+II being as a whole in a pure state[34] belonging to the manifold $P(\mathcal{H}_I \otimes \mathcal{H}_{II})$ on one side, and, on the other side, a state with the same barycentre [42] ϱ_I (expressed now by a probability measure μ_I on $\mathcal{S}_{I*}$) obtained after some "reduction of the wave packet", cf. [189, 276, 28]: in the last case the different states occurring in the support of the measure μ_I of the microsystem I are correlated with macroscopically distinguishable states of the measuring apparatus (usually declared as "pointer positions"); this correlation can be reflected in a description of states by genuine mixtures μ_I. ♠

1.2.5 (Unbounded generators). Another (rather "technical", at first sight) complication arising in our process of reformulation and extension of QM in geometrical terms is connected with the necessity of a use of unbounded selfadjoint operators X on the Hilbert space $\mathcal{H}$ in QM. It is a generally known mathematical theorem that such operators are defined on dense linear subsets $D(X)$ of $\mathcal{H}$ certainly different from the whole $\mathcal{H}$. Hence, our extension to nonlinear theory requires to use of also ("linear", or not) functions f on $\mathcal{S}_*$ in a rôle of generators that are not defined everywhere on the corresponding manifold of quantum states, and also are not locally bounded on $\mathcal{S}_*$; such nonlinear f's could be obtained, e.g. as some nonlinear perturbations of the (only densely defined, unbounded) function h_X corresponding to unbounded X. The main technical advantage of the use of the representation $U(G)$ is that it offers a possibility of definition of a class of nonlinear unbounded generators h generating Poisson flows φ_t^h on $\mathcal{S}_*$ that extends the set of affine (unbounded) generators h_X (the later generate projections of the common unitary flows $U(t) := exp(-itX)$). This is done in Section 2.2 with a help of the "macroscopic field $\mathbb{F}$", cf. Definitions 2.2.17 and 2.2.26. The representation $U(G)$ enters

[34] Hamiltonian evolutions of ϱ_I – linear, or not – are, however, rather rare consequences of evolutions of the composed system $I + II$; these evolutions should be rather specific in those cases.

into the determination of the set of "relevant generators h"; taking part in determination of the "considered physical system" in this way, the use of $U(G)$ has not only "technical rôle", but it has also a "physical meaning". ♠

1.2.6 (Structure of observables). Section 2.3 is devoted to definition of observables, to investigation of their algebraic properties, and of their transformation groups. It is proposed, in the geometrical setting, to describe observables by functions $\mathfrak{f} : \mathcal{S}_* \times \mathcal{S}_* \to \mathbb{R}, (\varrho; \nu) \mapsto \hat{f}(\varrho, \nu)$ *of two variables*, the first one is called *the quantum variable* and the function $\hat{f}(\cdot, \nu)$ is affine. The observables are related to the choice of $U(G)$ that determines (cf. Definition 2.2.17, Definition 2.2.26, and Proposition 2.2.32) an affine mapping $\mathbb{F} : \mathcal{D}(\mathbb{F})(\subset \mathcal{S}_*) \to \mathcal{E}_{\mathbb{F}}(\subset Lie(G)^*)$ describing a "classical field". The dependence of observables $\mathfrak{f}$ on the second "*macroscopic*" (or "classical") variable ν can be restricted to an "indirect dependence", i.e. $\hat{f}(\varrho, \nu) \equiv Tr(\varrho \mathfrak{f}(\mathbb{F}(\nu)))$ for some operator-valued function $\mathfrak{f}$ on (a subset $\mathcal{E}$ of) $Lie(G)^*$. Restriction to such a type of dependece on the quantum states $\nu \in \mathcal{S}_*$ provides a tool for dealing with the above mentioned (see 1.2.5) unbounded functions. We see that a general type of "quantum fields" $\mathfrak{f} : \mathcal{E}_{\mathbb{F}} \to \mathcal{L}(\mathcal{H})$ enters naturally into the game, cf. Definitions 2.2.26, 2.3.3, as well as Interpretation 2.3.11. ♠

1.2.7 (Possible applications). The presented theory is still in a preliminary stage: Its mathematical form is more elaborated than its possible physical interpretations. As a consequence, we restrict our attention in this work to existing theories and their incorporation into our conceptual scheme. We give here some general technical procedures to approach solutions of nonlinear dynamical (Schrödinger) equations (Section 3.5). We propose also a general mechanism for "delinearizations" of unitary group representations in Proposition 2.3.20. A general interpretation scheme of EQM is proposed, cf. e.g. Subsection 2.1-e, Interpretation 2.3.1, Definition 2.3.5, and Interpretation 2.3.11.

As concerns some proposals of *new applications* of the EQM (in addition to all ones of QM), they could be found also *without requiring a "fundamental nonlinearity"* in laws of Nature (i.e., now in QT). We consider here description of systems, which can be considered as "relatively closed" subsystems of larger (linear) QM systems. Such might be some "mesoscopic systems" of large molecules, of "trapped" Bose–Einstein condensates, etc. As concerns (non-)linearity of physical laws, it can be suspected that pervasive scientific thinking is nowadays "generally linear": Even if dealing with nonlinear equations, mappings or "effects", we express them eventually in terms of linear spaces (real numbers, additive operations = commutative groups, "linearizations" of different kinds, etc.). Linearity seems to be one of the present time "paradigms" of our thinking. As is shown in several places of this work, any of considered nonlinear theories can be extended to a linear theory "of a larger system" (generalized "Koopmanism"). Hence, conversely, we can expect nonlinear behaviour by specific restrictions of dynamics to subsystems. Possibilities of various interpretations of the presented general theoretical scheme of EQM are left open here for further development. ♠

1.2.8 (Notes on a Weinberg's proposal). In some papers, [273], S. Weinberg posed a question on a possible nonlinear modification of QM (motivated by his aim to formulate a way to testing fundamental principles of QM), and sketched a specific proposal of "nonlinear quantum mechanics" (NLQM). Trials to obtain a consistent generalization of the traditional interpretation of QM to this theory led, however, to difficulties connected mainly with the appearing lack of conservation of the "transition probabilities" under nonlinear transformations in QM. There are difficulties

with appearing possibility of superluminal communication via Einsten-Podolsky-Rosen (EPR)-type experiments, difficulties with the statistical interpretation of the formalism (as will be shown in Subsection 3.3-a) and also difficulties with description of composed systems.[35] Weinberg's description of evolution of mixed states of subsystems (it was basis dependent), as well as statistical interpretation of predictions (it was based on an approximation motivated by KAM theory) were even mathematically and conceptually ambiguous.

We shall reformulate here NLQM in the mathematically unambiguous terms of symplectic reformulation of QM discovered some time ago, cf. [251, 26, 62, 27, 63] by extending it subsequently by "nonlinear quantities". This formulation admits the interpretation suggested by a specific formulation of quantum mean-field models: the given QM system is considered as an individual subsystem of an infinite collection of equal quantum subsystems interacting mutually via a very weak, long range, and permutation invariant interaction; its dynamics can be described as a quantum dynamics of an individual subsystem moving in the time dependent "external" classical field given by actual values of intensive quantities of the infinite system. Mathematical unambiguity of this MFT ensures such property for our EQM. Since also more realistic interactions than that of MFT, e.g. the Coulomb interaction, are "of long range" and lead in specific limits to validity of a certain forms of MFT, cf. Thomas–Fermi theory [260], one can expect existence of applications of the presented theory in realistic situations.

We shall return to a reformulation of a part of Weinberg's theory in Section 3.6. ♠

Let us note that we shall not present in this work any review of mean–field theory (MFT), in spite of its (at least "ideological") importance for understanding of some constructions of the present paper, as well as of their proposed interpretation; for a brief review of MFT cf., e.g., [33], the introductory sections of [32], or in [265, 264, 263]; cf. also Section 3.4.

1.3 Remarks on the Text

The text is divided into three Chapters, including this introductory one, and of three appendices (numbered alphabetically) divided to (sub)sections. The second chapter entitled "Extended Quantum Mechanics" contains the general formal and interpretational scheme of the presented theory, the EQM. The last one: "Specifications and Applications" contains a description of some more specific theories which are included as subtheories into EQM. Chapters are divided into sections, numbered separately in each chapter. Subsections, formulas, and assertions (of all kinds, consecutively, including Definitions, Theorems, Remarks, Interpretations, some unnamed paragraphs, etc.) are numbered within each Section separately. For better orientation at reading, the end of text of Theorems, Propositions and Lemmas is denoted by ♣, end of Definitions and Notations is denoted by ♢, and that of Interpretations by ♦; Notes, Remarks, Illustrations and Examples are finished by ♡, and some other unnamed numbered paragraphs are ended by the sign ♠.

The appendices are written in a language, which is not always strictly rigorous from mathematical point of view, what is due to the author's desire to make the mathematical text easier to

[35] There are, however, works devoted to search of some observable deviations from the QM predicted by the Weinberg formulation of NLQM; in some of these works also proposals for experimental tests of predictions of this formulation of NLQM were given.

read for more readers. The contents of (sub)sections is briefly seen from the Table of contents. Phrases and formulas typed **boldface** are usually newly defined expressions. The bibliography is far from complete; this is also due to many sources and connections of EQM.

The text is written as a *physically motivated mathematical model* intended, however, to provide a framework for solution of actual physical problems. Hence, it is not quite *physically neutral* as a purely mathematical text, perhaps, should be. There are included paragraphs denoted by "Interpretation" containing some of these author's ideas and proposals, but also some (perhaps) generally accepted parts of quantum theory (QT).

We did not try to use some "up to date mathematics", and the level is "slightly graduate". Appendices might help readers to refresh some mathematical concepts and facts. They contain some technical prerequisites on topology, differential calculus on Banach spaces and differential geometry (also on Banach manifolds), on Lie groups, basic facts on C^*-algebras, and W^*-algebras, and their representations and automorphisms (i.e. symmetries), as well as a brief information on unbounded symmetric operators and their symmetric and selfadjoint extensions. The appendices can serve, together with Sections 1.4, and 1.5 presenting briefly general schemes of CM and QM, to fix notation, and also to pedagogical purposes (independently of Chapters 2, and 3).

The given scheme of EQM contains also Hamiltonian classical mechanics (CM) as a subtheory in an obvious way, as it is mentioned in the paragraph 3.1.4. In Section 3.1 also other subtheories and some invented applications of EQM are listed, and an "itinerary" of the Chapter 3 is there given. It is not mentioned there a possibility of an application to a formulation of connections of the theory of general relativity with QT, since the present author is not acquainted with the actual status of these problems.[36] Equally it is not discussed a hypothesis on possible application of methods close to the presented ones to the "algebraic quantum field theory" (QFT): Let us just mention that a "self-consistent approach" could be, perhaps, useful in dealing with such classical objects like "domains in Minkowski space" in a framework of any quantum theory.

We were not intended to criticize here in details the Weinberg's formulation of a nonlinear modification of QM; some relevant criticism was presented in published papers, e.g. in [106, 272]. The Section 3.6 is devoted to just a reformulation of our NLQM on $P(\mathcal{H})$ in terms close to those used in the Weinberg's paper [273]. This allows us to compare in mathematically clear terms the two approaches to a generalization of QM, which might be considered (up to the used interpretations) practically identical on the set of vector states, resp. on $P(\mathcal{H})$. Some useful algorithms for solution of these nonlinear Schrödinger equations might be found in Section 3.5. A reduction of solutions of a class of nonlinear Schrödinger equations connected with a group action on $P(\mathcal{H})$ to two "simpler" problems: to solutions of classical Hamilton's equations (possibly, finite dimensional), and to solution of a linear time–dependent Schrödinger equation is described in that Section 3.5.

Other theories described here as subtheories of EQM entered to NLQM as "approximate theories" to problems of linear QM: It might be rather interesting how nonlinearities enter into approximated linear theories of QM. We shall present, e.g., (partly elaborated) cases of time–dependent Hartree-Fock theory in 3.3-d, and a class of nonlinear Schrödinger equations known also from traditional attempts to formulate nonlinear modifications of QM, cf. Subsection 3.3-e.

[36] It might be assumed that works by, e.g. C.J. Isham and/or A. Ashtekar contain relevant attempts of this kind.

A connection of EQM with "quantum theory of large systems" (i.e. with a class of nonrelativistic QFT) is sketched briefly in Section 3.4. This connection seems to us crucial from the interpretational point of view, since the presented EQM appears (in a slightly different form) as a well formulated *linear QT of large quantal systems*. Such a linear QT contains also classical macroscopic observables in a natural way, as a *consequence of local quantum kinematics*, where a specific rôle of symmetry groups and a "mean–field" dynamics can be introduced, to point out those of the obtained (unnecessary huge) set of "observables" which are interpretable, hence "useful".

At the beginning of the Chapter 2, in Subsections 2.1-a, and 2.1-b of Section 2.1, the mainly "kinematical structure" of the theory is described, whereas the next two Subsections 2.1-c, and 2.1-d describe the way of constructing "dynamics", and also more general one–parameter symmetry groups. Only bounded and differentiable, hence "nice" objects are considered in details in these subsections. The following Sections 2.2, and 2.3 consist, perhaps, the most technical parts of the paper containing also important interpretation proposals. They contain both a solution (and some hints for alternatives) of the technical problem of dealing with unbounded nonlinear generators ("Hamiltonians"), as well as definitions and interpretation proposals for "observables". The Section 2.3 contains the basic definitions of a variety of described (sub–)systems, and also a description of "nonlinear realizations" of symmetry Lie groups.

Before starting with a description of tools for our generalization of QM to EQM, let us, however, present in the next two sections brief reviews of traditional CM, and also of QM, a knowledge an understanding of which is a necessary prerequisite for successful reading of Chapter 2.

1.4 A General Scheme of Hamiltonian Classical Mechanics

We present a brief review of geometric formulation of classical mechanics in this section. The presented scheme is standard [1] and represents an important part of intuitive and technical background for our subsequent constructions. The language used will be that of a simple version of global differential geometry: We want to avoid as much as possible a use of coordinates for sake of transparency and formal and conceptual simplicity; this will be our "policy" in all the following text. Some review of a necessary minimum of mathematical background is presented in the Appendix A.

1.4-a Classical phase space and dynamics

Let us first mention basic general concepts, and subsequently some examples will be given.

The space of **classical "pure states"** in a model of Hamiltonian mechanics, i.e. the **phase space** $(\boldsymbol{M;\Omega})$, is a differentiable manifold M of finite (even) dimension endowed with a symplectic (i.e. nondegenerate and closed) two-form Ω. The specification of the form Ω is equivalent to a specification of a nondegenerate Poisson structure on M, i.e. to definition of **Poisson brackets** $\{f,h\}$ on the set $\mathcal{F}(M)$ $(\ni f,h)$ of infinitely differentiable real valued functions on M.

The equivalence between Poisson and symplectic structures on a (symplectic) manifold is only the case, however, of a **nondegenerate Poisson structure** , i.e. that one satisfying all the five following defining properties:

1.4.1. Definitions (Poisson structure). *Let M be a differentiable manifold, and let a mapping $\{\cdot,\cdot\} : \mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M)$ be given. Assume the following properties of $\{\cdot,\cdot\}$:*

(i) $\{\cdot,\cdot\}$ is **antisymmetric***: $\{f,h\} \equiv -\{h,f\}$;*

(ii) $\{\cdot,\cdot\}$ is **bilinear***: $\{f, h_1 + \lambda h_2\} \equiv \{f,h_1\} + \lambda\{f,h_2\}$;*

(iii) $\{f,\cdot\}$ is, for any fixed $f \in \mathcal{F}(M)$, a **derivation***: $\{f, h_1h_2\} \equiv \{f,h_1\}h_2 + h_1\{f,h_2\}$;*

(iv) **Jacobi identity***: $\{h_1,\{h_2,h_3\}\} + \{h_3,\{h_1,h_2\}\} + \{h_2,\{h_3,h_1\}\} = 0$ is fulfilled;*

(v) $\{\cdot,\cdot\}$ is **nondegenerate***: If, for a fixed $f \in \mathcal{F}(M)$, there is $\{f,h\} \equiv 0$ for all $h \in \mathcal{F}(M)$, then $f \equiv const.$ on each connected component of M.*

If $\{\cdot,\cdot\}$ satisfies first four properties (i) - (iv), then it is called a **Poisson structure** *on M.*

A manifold M endowed with a Poisson structure is called a **Poisson manifold***, [274].*

Relation of a general Poisson manifold M to its canonically determined symplectic submanifolds is such that M decomposes uniquely to union of disjoint manifolds M_ι each of them is endowed with a uniquely defined symplectic structure $\Omega^{(\iota)}$ determined by the Poisson structure $\{\cdot,\cdot\}$, and canonically determining it on corresponding M_ι. The dimensions of the **symplectic leaves M_ι** *might be mutually different. Any $h \in \mathcal{F}(M)$ determines a unique* **Hamiltonian vector field $\mathbf{v}_h$** *on the whole M by the formula*

$$df(\mathbf{v}_h) \equiv \mathbf{v}_h(f) := \{h,f\}, \ \text{for all } f \in \mathcal{F}(M). \tag{1.4.1}$$

The same formula can be obtained for a symplectic manifold by combining (1.4.1) *with* (1.4.3)*. Corresponding Hamiltonian flows leave each the symplectic leaf M_ι invariant.* $\diamondsuit$

This allows us to ascribe to each function $h \in \mathcal{F}(M)$ a unique (local) flow φ^h on M representing solutions of Hamilton's dynamical equations

$$\frac{df_t}{dt} = \{h, f_t\}, \quad \text{with } f_t(x) := f(\varphi^h_t x), \ t \in \mathbb{R}, \ x \in M, \tag{1.4.2}$$

with the Hamiltonian function h: for the initial state $x(0) := x \in M$ the state in a time $t \in \mathbb{R}$ is expressed by $x(t) = \varphi^h_t x$. This is done in the following way: The symplectic form Ω determines the *Hamiltonian vector field* $\mathbf{v}_h$ on the phase space M corresponding to an arbitrary differentiable function $h \in \mathcal{F}(M) := C^\infty(M,\mathbb{R})$, by the formula

$$\Omega_x(\mathbf{v}_h, \mathbf{w}) := -d_x h(\mathbf{w}), \ \text{for all } x \in M, \mathbf{w} \in T_xM. \tag{1.4.3}$$

Then the **Poisson bracket** is defined by

$$\{f,h\} := \Omega(\mathbf{v}_f, \mathbf{v}_h), \ f,h, \in \mathcal{F}(M), \tag{1.4.4}$$

and the right hand side of the equation (1.4.2) is just $\mathbf{v}_h(f_t)$. The solutions $x(t) = \varphi^h_t x$ of (1.4.2) needn't exist for all times $t \in \mathbb{R}$ for any initial condition $x \in M$, and φ^h represents in general just a collection of *local flows*. If φ^h exists for all t on M, it is called the (global Hamiltonian) **flow of the vector field $\mathbf{v}_h$**. A vector field with global flow is called **complete vector field**. General criteria for deciding what Hamiltonian h on a given $(M;\Omega)$ has complete vector field $\mathbf{v}_h$ are not known, although some criteria are known for specific classes of (possibly symplectic)

manifolds; especially, on compact M all vector fields are complete. Let us note that "completeness" of a Hamiltonian vector field of h on a dense invariant subset of M is equivalent [1, 2.6.14, and 2.6.15] to essential (anti-)selfadjointness of a densely defined linear operator on the complex Hilbert space $\mathcal{H} := L^2(M, d\mu_\Omega)(\ni f)$, cf. Appendix C. Here the measure μ_Ω used in the definition of the square integrability in the Hilbert space $\mathcal{H}$ is the n-th power of Ω, cf. Appendix A.3, if $\dim(M) = 2n$:

$$\mu_\Omega(\Lambda) = \int_\Lambda \wedge^n \Omega. \tag{1.4.5}$$

The mentioned antiselfadjoint operator acts on differentiable functions $f \in \mathcal{H}$ as the differential operator determined by the vector field $\mathbf{v}_h$:

$$f \mapsto \mathbf{v}_h(f) := df(\mathbf{v}_h).$$

A **symplectic transformation** of $(M; \Omega)$ is a diffeomorphism φ of M onto itself conserving the form Ω, i.e.: $\varphi^*\Omega \equiv \Omega$. Hamiltonian flows are one-parameter groups of symplectic transformations (hence, they conserve the measure (1.4.5) - this is the **Liouville theorem** used in classical statistical mechanics). Conversely, each one-parameter group of symplectic transformations defines its (at least local – in open neighbourhoods of all points of M) Hamiltonian function generating the given flow [1, 7]. Any symplectic transformation can be considered as a (kinematical) symmetry of the considered classical system. If the dynamics is described by the Hamiltonian h with the flow φ^h, and a symmetry one-parameter group is described by the flow φ^f corresponding to its "Hamiltonian" f, and if, moreover, the Poisson bracket of the corresponding Hamiltonians vanishes: $\{f, h\} = 0$, then the two flows mutually commute:

$$\varphi^h_t \circ \varphi^f_s \equiv \varphi^f_s \circ \varphi^h_t.$$

In this case, the function f represents an **integral of motion**, resp. a **conserving quantity** of the system, cf. eq. (1.4.2). If there is a Lie group G (cf.A.4) acting on M **transitively** (i.e. for any $x, y \in M$ there is a $g \in G$ such, that its action transforms x to y) by symplectic transformations, the phase space M is called a **symplectic homogeneous space of** G.

Let us give now some simple examples:

1.4.2. **Examples**.

(i) The linear space $M := \mathbb{R}^{2n}$ of $2n-$tuples of Cartesian coordinates $(q_1, \dots q_n, p_1, \dots p_n)$ is endowed with the symplectic form $\Omega := \sum_{j=1}^n dp_j \wedge dq_j$. The Poisson bracket is in the given coordinates expressed in the standard form

$$\{f, h\} = \sum_{j=1}^n \left(\frac{\partial f}{\partial p_j}\frac{\partial h}{\partial q_j} - \frac{\partial h}{\partial p_j}\frac{\partial f}{\partial q_j} \right). \tag{1.4.6}$$

Symmetries of this space contain linear symplectic transformations described by $2n \times 2n$ matrices commuting with the matrix S with elements (in the considered "canonical" basis) $S_{j,k} \equiv 0$, except of $S_{j,j+n} \equiv -S_{j+n,j} = 1$ $(j = 1, 2, \dots n)$, but also affine transformations consisting of arbitrary parallel shifts $\varphi : x \mapsto \varphi(x) \equiv x + a$, for any fixed $a \in \mathbb{R}^{2n}$. Symmetries are, of course, all the symplectomorphism of the form φ^h_t (the above mentioned linear transformations, as well

as affine ones, are also of this form; e.g. shifts are generated by linear the $h(q,p) \equiv \sum_{j=1}^{n}(c_j q_j + d_j p_j)$; quadratic h's correspond to groups of linear transformations). Let us mention explicitly, that specific quadratic h's describe the dynamics of "harmonic oscillators", whereas those h's which contain (in their Taylor expansion, e.g.) terms of higher than the second order in the standard canonical coordinates $(p;q)$ lead to nonlinear canonical flows on M.

(ii) The complex projective space $CP^n := P(\mathbb{C}^{n+1})$ constructed from the linear space $\mathbb{C}^{n+1}$ as the factor-space consisting of its one-dimensional complex subspaces can be considered as $2n$-dimensional real manifold endowed with a canonical symplectic structure [7, Appendix 3]. This is a special case of complex projective Hilbert spaces $P(\mathcal{H})$ considered in Section 3.2, and finite dimensional examples in specific charts can be straightforwardly constructed.

(iii) Cotangent bundles: Let Q be any differentiable manifold and $M := T^*Q \equiv T^0_1 Q$ be its cotangent bundle, cf. also Appendix A.3. Hence, points of M are linear functionals $p \in T^*_q Q := (T_q Q)^*$ "attached to points" $q \in Q$; the natural projection $\tau : T^*Q \to Q$ maps $p \in T^*_q Q$ to $\tau(p) = q \in Q$. The derivative (i.e. the tangent mapping) of τ is

$$\tau_* := T\tau : TM := T(T^*Q) \to TQ.$$

The **canonical one form ϑ** on the cotangent bundle $M = T^*Q$ is defined by:

$$\begin{aligned} \vartheta &: p(\in M) \mapsto \vartheta_p \in T^*_p M, \\ \vartheta_p &: \mathbf{v}(\in T_p M) \mapsto \vartheta_p(\mathbf{v}) := p \circ \tau_*(\mathbf{v}). \end{aligned} \tag{1.4.7}$$

Then $\Omega := \mathrm{d}\vartheta$ is a symplectic form on M, the **canonical symplectic form on T^*Q**. If $\{q_1, q_2, \ldots, q_n\}$ are local coordinates on Q, then $p \in T^*Q = M$ is expressed (in the corresponding chart on M) as $p \equiv \sum_{j=1}^{n} p_j \mathrm{d}q_j$. In this coordinate neighbourhood one has

$$\vartheta = \sum_{j=1}^{n} p_j \mathrm{d}q_j \circ \tau_* \equiv \sum_{j=1}^{n} p_j \tau^* \mathrm{d}q_j,$$

and from commutativity of pull–backs with exterior differentiation d, and from the basic property $\mathrm{d} \circ \mathrm{d} \equiv 0$, we have the canonical expression for Ω in that neighbourhood :

$$\Omega := \mathrm{d}\vartheta = \sum_{j=1}^{n} \mathrm{d}p_j \wedge \tau^* \mathrm{d}q_j.$$

Hence, any cotangent bundle is a symplectic manifold in a canonical way. Taking $Q := \mathbb{R}^n$, we obtain the example (i), where the coordinates $\{q_j, p_k\} \in \mathbb{R}^{2n}$ can be chosen global (corresponding, e.g. to a trivial coordinate (linear) chart on $Q = \mathbb{R}^n$). $\heartsuit$

1.4.3. **Remark** *(On the notion of "chaos").* The Liouville theorem on noncontractibility of the phase volume, cf.(1.4.5) and the text following it, implies nonexistence of **attractors**, [1], of Hamiltonian flows. The attractors, especially so called **"strange attractors"**, [243], are usually connected with the notion of **chaos**, [10, 269, 116], in dynamical systems. This does not mean that in Hamiltonian systems does not occur a chaotic motion. The "chaoticity" of motion is characterized rather by its instability with respect to choices of initial conditions than by presence of

some attractors. Such instabilities seem to occur generically in Hamiltonian systems. This fact remained hidden for most of physicists for several decades: Mainly so called **(completely) integrable systems** were described in university textbook literature: These are, roughly speaking, systems the dynamics of which can be fully described on surfaces of given values of integrals of motion, in conveniently chosen coordinates, as systems of independent linear harmonic oscillators; parameters of the oscillators might depend on values of the integrals of motion; the "integrals-of-motion surfaces" decompose the energy submanifolds and all they are diffeomorphic to tori T^n, or to cylinders, [7, 8, 1]. This was, perhaps, due to the fact that all known explicitly solved ($\equiv$ integrated) models were of this kind.[37] It was proved [176], however, that the set of integrable systems is in a well defined sense rare in the set of all possible Hamiltonian systems. In the cited paper [176] no restrictions to dynamics coming, e.g. from observed symmetries of physical systems were considered; such restrictions could, perhaps, enlarge the "relative size" of integrable systems. But, on the other hand, some "physically realistic" systems in classical mechanics were proved to be **nonintegrable**, e.g. the three (and more) body problem in celestial mechanics (i.e. in the nonrelativistic model of planetary systems with point masses moving in $\mathbb{R}^3$ and interacting via the Newton potential) is nonintegrable, [1]. $\heartsuit$

1.4-b Observables and states in classical mechanics

Also CM can be formulated in terms familiar from QM. This formal analogy is useful for description of classical subsystems in the quantummechanical framework. Concepts introduced in this subsection are useful also in formulation of classical statistical mechanics, see e.g. [158, 262, 224, 225].

As a set of classical observables can be chosen, e.g. the C^*-algebra (without unit, if M is not compact) $C_0(M)$ of all complex–valued bounded continuous functions on the phase space M tending to zero at infinity, cf. Appendix B. This C^*-algebra can be completed by unit ($:= \mathbb{I} \equiv 1 =$identically unit function on M), and this completion will be called the $\boldsymbol{C^*-}$**algebra of classical observables**, denoted by $\mathcal{A}_{cl}$.[38] The algebraic operations are defined pointwise on M: for $f, h \in \mathcal{A}_{cl}$ one has $(f\langle)(m) \equiv f(m)h(m)$, $(f + \lambda h)(m) \equiv f(m) + \lambda h(m)$, $f^*(m) \equiv \overline{f(m)}$, and the norm is the supremal one, i.e. $\|f\| := \sup\{|f(m)| : m \in M\}$. The spectrum space of $\mathcal{A}_{cl}$ is just the one–point compactification of M. Further extensions of the algebra of observables $\mathcal{A}_{cl}$ could lead us to abelian von Neumann algebras: Let, e.g. the Borel measure μ_Ω on M be given, and consider the Banach space $L^1(M, \mu_\Omega)$ of integrable complex–valued Borel functions f on M, with the norm $\|f\|_1 := \mu_\Omega(|f|) \equiv \int |f(m)|\mu_\Omega(\mathrm{d}m)$. Its topological dual, cf. [218, 41], $L^\infty(M, \mu_\Omega)$ consisting of μ_Ω–essentially bounded Borel functions on M is a W^*–algebra containing $\mathcal{A}_{cl}$. It can be interpreted as the maximal commutative von Neumann algebra of bounded operators in $\mathcal{L}(\mathcal{H})$, namely the operators of M–pointwise multiplication by functions $f \in L^\infty(M, \mu_\Omega)$ of elements of the Hilbert space $\mathcal{H} := L^2(M, \mu_\Omega)$. The mentioned duality is realized by the sesquilinear relation

$$\langle f; h\rangle \equiv \int \overline{f(m)}h(m)\mu_\Omega(\mathrm{d}m),\ \forall f \in L^\infty(M, \mu_\Omega), h \in L^1(M, \mu_\Omega). \tag{1.4.8}$$

[37] This seems to be generally believed, cf. also [7].

[38] For the concepts and properties of C^*-algebras and von Neumann (resp. W^*-) algebras see the standard books [76, 77, 227, 228, 254, 42], and also our Appendix B.

This last definition of a (complexified, linear) set of "classical observables" as a W^***–algebra of observables** has an advantage that this C^*-algebra contains also projections in $\mathcal{L}(\mathcal{H})$ represented by multiplication operators by characteristic functions of the Borel subsets of M, by which is it generated. Hence, (also unbounded) "observables f" could be defined alternatively by **projection–valued measures** E_f (with values in multiplication projections in $\mathcal{L}\big(L^2(M,\mu_\Omega)\big)$) on Borel subsets $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$:

$$E_f : B(\in \mathcal{B}(\mathbb{R})) \mapsto E_f(B) := \chi_{f^{-1}(B)} \in \mathcal{L}(\mathcal{H}),$$

with the characteristic function χ_Λ of a Borel set $\Lambda := f^{-1}(B) \subset M$ considered as an element of $L^\infty(M,\mu_\Omega) \subset \mathcal{L}(\mathcal{H})$.

The (mathematically defined, [77, 227, 42]) **(classical) states** $\mathcal{S}(\mathcal{A}_{cl})$ on the C^*-algebra $\mathcal{A}_{cl}$ are just the probability measures $\mu \in \mathcal{M}_{+1}(M)$ on M, and the **(classical) pure states** are all the the Dirac measures $\{\delta_m : m \in M \cup \{\infty\}\}$, with $\delta_m(\Lambda) = 1 \Leftrightarrow m \in \Lambda$:

$$\mu : f(\in \mathcal{A}_{cl}) \mapsto \mu(f)(\in \mathbb{C}),\ \mu(f) := \int f(m)\mu(\mathrm{d}m).$$

If one takes, on the other side, the W^*-algebra $L^\infty(M,\mu_\Omega)$ as the C^*-algebra of observables , the set of all states on it will be "much larger" than $\mathcal{S}(\mathcal{A}_{cl})$ (which is included there as a proper subset), but the normal states on $L^\infty(M,\mu_\Omega)$ restricted to the subalgebra $\mathcal{A}_{cl}$ are just measures in $\mathcal{M}_{+1}(M)$ represented by elements of $L^1(M,\mu_\Omega)$, i.e. just the **measures absolutely continuous** with respect to μ_Ω.

1.4.4. **Interpretation.** In any case, the Dirac measures $\delta_m, m \in M$, represent "pure states", resp. in mathematical language, the extremal points of the convex set of all Borel probability measures on M. Other probability measures of this set have nontrivial, but **unique decompositions** into the extremal Dirac measures. Their physical interpretation is probabilistic, in the sense of statistical ensembles of Gibbsian statistical mechanics, [134, 115, 158]: In the ensemble described by a measure $\mu \in \mathcal{M}_{+1}(M)$, the fraction of otherwise equal physical systems having pure (="microscopic", but classical) states represented by points in the Borel subset Λ of the phase space M is equal to $\mu(\Lambda)$. This interpretation is conceptually consistent, due to the uniqueness of decomposition of μ's into the extremal points. This point hides an essential difference between CM and QM: $\mathcal{M}_{+1}(M)$ is a **simplex**, what is not the case of the state space $\mathcal{S}_*$ (or of $\mathcal{S}$) of QM. ♦

1.4-c Symplectic structure on coadjoint orbits

We shall mainly restrict our attention to such classical phase spaces M in this work, which are homogeneous spaces of a connected, simply connected Lie group G, on which the action $g : m \mapsto g{\cdot}m$ $(g \in G, m \in M)$ of G consists of symplectomorphisms:

$$f_g(m) := f(g{\cdot}m),\ \forall f, h \in C_\infty(M,\mathbb{R}) : \{f_g, h_g\} \equiv \{f,h\}_g.$$

In these cases, the phase space $(M;\Omega)$ is (locally) symplectomorphic to an orbit of the coadjoint representation (see Section A.4, and below in this subsection) either of G, or of its central extension by the additive Lie group $\mathbb{R}$, cf. [148, §15.2, Theorem 1].

Any (noncommutative) Lie group provides a canonical example of Poisson manifold. Also the most common case of the $2n$-dimensional symplectic linear space of the Example 1.4.2(i) can be considered as coming in this way from the $2n+1$-dimensional Weyl-Heisenberg group. This will be described in Subsection 3.3-b. Let us describe here the general construction.

Let G be a finite dimensional connected (for simplicity) Lie group with its Lie algebra $\mathfrak{g} := Lie(G)$, and with the exponential mapping $\exp : \mathfrak{g} \to G, \xi \mapsto \exp(\xi)$. The canonical symplectic manifolds will be found in the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$. The duality will be alternatively denoted by $F(\xi) \equiv < F; \xi >$, $F \in \mathfrak{g}^*$. The adjoint and the coadjoint representations of G on its Lie algebra $\mathfrak{g}$ (resp. on its dual $\mathfrak{g}^*$) are defined in Definition A.4.10.

Let us fix any element $F \in \mathfrak{g}^*$. Then the subset (a submanifold) $\mathcal{O}_F(G)$ of the linear space $\mathfrak{g}^*$ defined by

$$\mathcal{O}_F(G) := \{F' \in \mathfrak{g}^* : \exists g' \in G, F' = Ad^*(g')F\}$$

is called the **coadjoint orbit of** G **through F**. The space $\mathfrak{g}^*$ is decomposed into coadjoint orbits of (in general) various dimensions (as submanifolds).

Let us consider $\mathfrak{g}^*$ as differentiable manifold in which, as in any linear space, the tangent space $T_F\mathfrak{g}^*$ in any of its points F is canonically identified with the linear space $\mathfrak{g}^*$ itself. The dual space $T_F^*\mathfrak{g}^*$ then contains canonically (resp. for finite dimensional G: is identified with) the Lie algebra $\mathfrak{g}$, which is w^*–dense (i.e. $\sigma(\mathfrak{g}^{**}, \mathfrak{g}^*)$–dense) in the second dual $\mathfrak{g}^{**}$ of the Lie algebra $Lie(G)$, cf. [41, Chap.IV,§5.1]. This allows us to define canonically a Lie algebra structure on the second dual $\mathfrak{g}^{**}$. Let us denote this structure again by the bracket $[\cdot, \cdot]$. Let $f, h \in C^\infty(\mathfrak{g}^*, \mathbb{R})$. Then their differentials $\mathrm{d}_F f, \ldots$, are elements of $T_F^*\mathfrak{g}^* \sim \mathfrak{g}^{**}$, and their commutator (i.e. the canonical Lie bracket) is defined. Then we define the **Poisson structure on** $\mathfrak{g}^*$ by

$$\{f, h\}(F) := -\langle F; [\mathrm{d}_F f, \mathrm{d}_F h]\rangle, \ \forall F \in \mathfrak{g}^*, \ f, h \in C^\infty(\mathfrak{g}^*, \mathbb{R}). \tag{1.4.9}$$

The Hamiltonian vector fields $\mathbf{v}_f, \mathbf{v}_h, \ldots$, cf. (2.1.16) are then tangent to all the orbits $\mathcal{O}_F(G)$, [148, 274].

The simplest examples of functions $f \in C^\infty(\mathfrak{g}^*, \mathbb{R})$ are $f \equiv f_\xi, \xi \in \mathfrak{g}$, defined by $f_\xi(F) := F(\xi) \equiv \langle F; \xi\rangle$. Their Poisson brackets are trivially

$$\{f_\xi, f_\eta\} = -f_{[\xi,\eta]}. \tag{1.4.10}$$

The functions f_ξ generate, if used as Hamiltonian functions, the actions of one–dimensional subgroups in the $Ad^*(G)$-representation, i.e. the Hamiltonian flow of f_ξ on $\mathfrak{g}^*$ is

$$\varphi_t^{f_\xi} F \equiv Ad^*(\exp(t\xi))F, \ \forall F \in \mathfrak{g}^*, \ \xi \in \mathfrak{g}, \ t \in \mathbb{R}. \tag{1.4.11}$$

1.4.5. **Example**. Let us give a simple example of coadjoint orbits of a Lie group. Let $G := SU(2)$, the covering group of the rotation group $SO(3)$. These are 3–dimensional Lie groups with the Lie algebra generated by elements $\xi_j, j = 1, 2, 3$, corresponding to one parameter groups of rotations around tree fixed mutually orthogonal axes, and satisfying the relations (with the summation convention)

$$[\xi_j, \xi_k] = \epsilon_{jkl}\xi_l, \epsilon_{jkl} \equiv -\epsilon_{kjl} \equiv \epsilon_{klj}, \epsilon_{123} := 1.$$

The it is possible to show, that the coadjoint orbits (in the dual basis to $\{\xi_j\}$) are just all the spheres centered at origin. Hence, in this simple case, all the (symplectic) orbits $\mathcal{O}_F(G)$ are two–dimensional except of their common centre, which is a unique zero–dimensional orbit. The flows corresponding to the generators f_{ξ_j} are just rotations around the chosen axes in $\mathfrak{so}(3)^*$. ♡

1.5 Basic Concepts of Quantum Mechanics

We shall give here a review of an abstract scheme of standard quantum mechanics used for description of such systems, "classical analogs" (or "classical limits") of which are described by CM with finite–dimensional phase spaces.

The basic intuition and terminology of QM comes from CM (supplemented with a "nonclassical" statistical interpretation). This is due to the history of physics, but also, on more fundamental level, due to the intuitive necessity to express empirical statements of QM (as well as of an arbitrary theory) in terms describing macroscopic bodies of everyday life, or in terms of (again macroscopic) laboratory instruments. And states of macroscopic systems (resp. "macroscopic parameters" of physical systems) are described by classical concepts. Mathematical formalism of QM in its traditional form looks, however, rather different from that of CM. It will be shown in later sections of this work, in what aspects these two formalisms can be made almost identical, and it can be also seen, where differences are essential.

The presentation in this section will not be quite "parallel" to that of CM in Section 1.4, because we want to stress and to describe also some technicalities specific to QM.

1.5-a Pure states and dynamics in QM

The rôle played in CM by a phase space plays in QM a *normed complete (linear) space with norm determined by a scalar product* – over complex numbers, a separable Hilbert space $\mathcal{H}$. The correspondence to classical phase space is not, however, faithful enough, since there are classes of vectors in $\mathcal{H}$ corresponding to the same physical state: All vectors $\{\lambda\psi; 0 \neq \lambda \in \mathbb{C}\}$ with any chosen $0 \neq \psi \in \mathcal{H}$, correspond to the same physical state. The space of these classes is the **projective Hilbert space $P(\mathcal{H})$**; it is no more linear. Linearity seemed to be, however, important in historical development of QM, [45, 46, 232, 74, 159], and it is still important in many experimental projects due to its intuitively appealing content. We shall return briefly to this point later.[39] The points of the projective Hilbert space $P(\mathcal{H})$ are faithfully represented by one–dimensional projection operators $P_\psi, 0 \neq \psi \in \mathcal{H}, P_\psi\psi \equiv \psi$. As will be shown later, the space $P(\mathcal{H})$ is a symplectic manifold (of the real dimension $\dim_{\mathbb{R}} P(\mathcal{H}) = 2\dim_{\mathbb{C}} \mathcal{H} - 2$) in a canonical way.

1.5.1. **Interpretation (QM–CM "correspondence").** In QM–description of many phenomena, it is customary to introduce into theoretical, as well as into experimental considerations a vaguely defined concept of a **classical analogue** of the considered system described by QM, i.e. a classical–mechanical system in some way "corresponding" to the considered phenomena (resp. to QM–system). So, e.g., for a hydrogen atom described by vectors in the infinite–dimensional Hilbert space $\mathcal{H} := L^2(\mathbb{R}^6, \mathrm{d}^6 q)$, the corresponding "classical analogue" is the Hamiltonian system on the (12–dimensional) phase–space $T^*\mathbb{R}^6$, with the canonical symplectic structure (cf.

[39] It is still possible to define a "superposition of states" also in this nonlinear setting, cf. e.g. [208, 57, 67].

Examples 1.4.2(i), and (iii)) the dynamics of which is described by the Hamiltonian

$$h(q,p;Q,P) := \frac{p^2}{2m} + \frac{P^2}{2M} - \frac{e^2}{|q-Q|}; \quad q,p,Q,P \in \mathbb{R}^3.$$

The classical observables $\{q_j, p_j, Q_j, P_j,\ j = 1,2,3;\ h \in C^\infty(\mathbb{R}^{12})\}$ help to interpret the points P_ψ of infinite–dimensional symplectic "phase space" $P(\mathcal{H})$ as states of the ("real", or "genuine") QM hydrogen atom:

We associate with any of these classical functions on the phase space $\mathbb{R}^{12}$ a selfadjoint linear operator on $\mathcal{H}$ in such a way, that the "corresponding" operators $\mathfrak{X} \in \{\mathfrak{q}_j, \mathfrak{p}_j, \mathfrak{Q}_j, \mathfrak{P}_j,\ j = 1,2,3\}$ determine specific functions $h_\mathfrak{X}$ on (a dense subset of) the phase–space $P(\mathcal{H})$ (in an analogy with the observables in CM):

$$h_\mathfrak{X}(P_\psi) := Tr(P_\psi \mathfrak{X}), \quad \forall P_\psi \in P(\mathcal{H}).$$

These functions satisfy "the same" commutation relations (i.e. Poisson brackets relations) as the corresponding classical phase space variables $X \in \{q_j, p_j, Q_j, P_j,\ j = 1,2,3\}$, as we shall see later. They also form, surprisingly (cf., however, Subsection 3.3-b), an "irreducible set of variables" on the infinite–dimensional manifold $P(\mathcal{H})$ (i.e., in some sense, they generate a complete set of "coordinate functions"), if a noncommutative "$*$-product" between these functions (cf. also [97] for alternatives)

$$h_{\mathfrak{X}_1} * h_{\mathfrak{X}_2} := h_{\mathfrak{X}_1 \mathfrak{X}_2},$$

is defined.[40] In this way, the functions $h_\mathfrak{Y}$ (where $\mathfrak{Y}$ are algebraic expressions consisting of the above introduced operators $\mathfrak{X}$) form a noncommutative (infinite–dimensional) algebra.[41] Its elements are interpreted in such a way, that a "correspondence" with finite dimensional phase space $\mathbb{R}^{12}$ remains valid as a "many–to–one" mapping $\mathbb{F} : P(\mathcal{H}) \to \mathbb{R}^{12}$, defined in coordinates by

$$\mathbb{F}_X : P_\psi \mapsto h_\mathfrak{X}(P_\psi) \equiv Tr(P_\psi \mathfrak{X}) =: \mathbb{F}_X(P_\psi),\ X = q_j, p_j, Q_j, P_j,\ j = 1,2,3.$$

This mapping is then **interpreted statistically** as **expectation of "observables" $\mathfrak{X}$ in the pure states P_ψ**. Values of higher degrees (with respect to the $*$-product) of these functions are then interpreted as higher momenta of statistical distributions of these "observables X". Hence, different QM–states P_ψ with the same expectations $\mathbb{F}_X(P_\psi) = h_\mathfrak{X}(P_\psi)$ (for all $\mathfrak{X}$) differ mutually by probability distributions of some of these observables X.

A specific feature of QM in description of such "finite systems" as the hydrogen atom is that **there are no pure states $P_\psi \in P(\mathcal{H})$ with zero dispersion of all observables in an "irreducible set"**, in our case formed by $\{q_j, p_j, Q_j, P_j,\ j = 1,2,3\}$. This means that for any $P_\psi \in P(\mathcal{H})$ there is at least one $X \in \{q_j, p_j, Q_j, P_j,\ j = 1,2,3\}$ such that for the corresponding quantum observable one has nonzero dispersion, i.e.

$$h_\mathfrak{X} * h_\mathfrak{X}(P_\psi) \neq h_\mathfrak{X}(P_\psi)^2.$$

[40] That these functions on $P(\mathcal{H})$ are not differentiable in the usual sense (they are not even everywhere defined) is not important in the considered connections: they could be replaced by some of their bounded "versions"; we can work, e.g. with bounded operators from the algebra generated by projection measures (cf. Appendices B, and C) of the (unbounded) operators $\mathfrak{X}$.

[41] For a possibility of mathematical definition of such algebras of unbounded operators see, e.g. [163].

The statistical interpretation of (even pure) states in QM differs from interpretation of states in classical statistical physics. This difference can be expressed roughly (cf. [14, 136, 172, 180]) so that in QM there is no (in some sense "natural") "phase space" (resp. a "space of elementary events" – in terminology of Kolmogorovian probability theory) consisting of points representing some (at least fictitious) dispersion–free states, such that probability measures on it would determine the quantum states. Pure states are interpreted in QM as in a sense "the most detailed possibility" of a description of states of "quantum objects" (resp. "systems").[42] In what sense, in the case of the absence of any dispersionless states, these "objects really exist" is still a discussed problem: "Object" is characterized by its state which contains just statistical predictions on possible outcomes of its interactions with other bodies at specified initial conditions, leading each time to a stable trace (i.e. a reproducibly verifiable "macroscopic change of environment" in each single case of the repeatedly obtained cases of "events of detection"); such a process, if it is correlated with values of a physical quantity, is called a "measurement in QM". The formalism of QM does not contain "single events". ♦

The quantal time evolution of vectors in $\mathcal{H}$ is supposed to be such, that it transforms, by a family of transformations

$$\phi_t \ (t \in \mathbb{R}) : P(\mathcal{H}) \to P(\mathcal{H}),\ P_\psi \mapsto \phi_t(P_\psi),$$

the classes of the vectors in $\mathcal{H}$ corresponding to the same physical interpretation, i.e. the points of $P(\mathcal{H})$, bijectively onto $P(\mathcal{H})$. Traditionally, there is another general requirement to these transformations $\phi_t(P_\psi)$: They should conserve the **transition probabilities**, i.e. the values of the nonnegative function

$$\mathrm{Pr} : P(\mathcal{H}) \times P(\mathcal{H}) \to \mathbb{R}_+, \mathrm{Pr}(P_\psi, P_\varphi) := Tr(P_\psi P_\varphi) \equiv \frac{|(\varphi, \psi)|^2}{\|\varphi\|^2 \|\psi\|^2}. \tag{1.5.1}$$

It is required:

$$Tr(\phi_t(P_\varphi)\phi_t(P_\psi)) \equiv Tr(P_\varphi P_\psi). \tag{1.5.2}$$

Considerations on possible physical interpretation of this requirement are postponed to later sections, cf. also [35].[43] According to a Wigner's theorem (cf. Proposition 3.2.6), the additional requirement of the group property of $t \mapsto \phi_t$, i.e. $\phi_{t_1+t_2} \equiv \phi_{t_1} \circ \phi_{t_2}$, and of continuity of the functions

$$t \mapsto Tr(P_\varphi \phi_t(P_\psi)),\ \forall P_\psi, P_\varphi \in P(\mathcal{H}),$$

[42] Cf., e.g. [71] for comparison of dispersions of observables in "mixed states" with those in their pure convex summands.

[43] This requirement can be connected with the **reduction postulate** of Dirac and von Neumann, [74, 189], stating that, by measuring a quantity X on a considered system, after obtaining a result x' the system suddenly "jumps" into a dispersionless state of the quantity X in which that quantity *has the the value* x'; or alternatively, that the statistical ensemble representing the system in the initial state (i.e. all members of the ensemble are initially in the same quantum state) jumps during the measurement into the statistical ensemble consisting of systems occurring in such quantum states that are all dispersionless of X with values equaling to the measurement results x'; these systems occur in the ensemble with the frequencies of the occurrence of the corresponding results x' obtained by the measurement.

suffices to imply the existence of a strongly continuous one–parameter unitary group $t \mapsto U(t)$ on $\mathcal{H}$ such, that it is

$$\phi_t(P_\psi) \equiv U(t)P_\psi U(-t) \equiv P_{U(t)\psi}.$$

Then the Stone's theorem, cf. [218, 220] and Theorem C.3.2, gives the existence of a (unique, up to an additive constant multiple of identity $I_{\mathcal{H}}$) selfadjoint operator H such, that

$$U(t) \equiv \exp(-itH). \tag{1.5.3}$$

This leads to the **Schrödinger equation** for evolution of vectors $\psi(t) \in \phi_t(P_\psi) \subset P(\mathcal{H})$:

$$\psi(t) := U(t)\psi(0) \Rightarrow i\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = H\psi(t), \quad \psi(0) \in D(H), \tag{1.5.4}$$

with $D(H)$ being the domain of the selfadjoint H, cf. Appendix C.2. Let us stress the trivial fact, that the Schrödinger equation makes no sense for "improperly chosen" initial conditions $\psi(0) \notin D(H)$.

This is the general form of time evolutions in QM. The operator H is called the Hamiltonian and it is interpreted (cf. next subsection) as an operator describing the energy observable. It should be stressed, that mere symmetry of the operator H (i.e. $(\varphi, H\psi) = (H\varphi, \psi),\ \forall \varphi, \psi \in D \subset D(H), \overline{D} = \mathcal{H}$) is not sufficient to define a one–parameter group by (1.5.3); H should be selfadjoint to generate a group, Appendix C. On the other hand, between selfadjoint H's, and strongly continuous one–parameter unitary groups $U(t)$'s there is a canonical bijection expressed by (1.5.3), cf. Theorem C.3.2.

1.5-b States and observables

States in QM (let us denote the whole set of them by $\mathcal{S}_*$) form a convex set, with "pure states" described by one dimensional projections P_ψ as its extremal (i.e. indecomposable into nontrivial convex combinations) points. Convexity of the state space can be traced back to the classical, essentially macroscopic notion of *statistical ensemble*, cf. Interpretation 1.4.4, in which expectations of all observables are expressed by the same convex combination of their expectations in **subensembles**, that intuitively correspond to "maximally specified ensembles" (in CM these "pure ensembles" are dispersion–free for all observables) .[44] It was pointed out above that in CM such a "maximal decomposition" is unique. This means, that the classical state space $\mathcal{S}(\mathcal{A}_{cl})$ forms a **simplex**, cf. [60, 182, 224, 42]. *This is not the case of QM*, what is one of its deepest differences from CM. The "shape" of $\mathcal{S}_*$ is closely connected with the set of "observables", cf. [183]. We shall not go into interesting details of these connections, but we shall rather review the standard traditional setting.

The set of **bounded quantum observables** is taken (in the theory without superselection rules, [279, 143]) to be the set of all bounded selfadjoint operators on $\mathcal{H}$, i.e. $\mathcal{L}(\mathcal{H})_s$, and as the **$C^*$-algebra of quantum observables** will be taken $\mathcal{L}(\mathcal{H})$. The **set of quantum states** will be for us here just a part of the set of all positive normalized linear functionals on $\mathcal{L}(\mathcal{H})$, namely the

[44] Importance of the convex structure of state spaces, and its relation to other theoretical concepts was stressed and analyzed, e.g. in [168, 183, 114].

normal states $\mathcal{S}_*$ consisting of functionals expressible in the (defining) faithful representation of $\mathcal{L}(\mathcal{H})$ by **density matrices** on $\mathcal{H}$, i.e. by positive operators on $\mathcal{H}$ with unit trace:

$$\varrho \in \mathcal{S}_* \Rightarrow \varrho = \sum_j \lambda_j P_{\psi_j},\ \lambda_j \geq 0,\ \sum_j \lambda_j = 1. \tag{1.5.5}$$

The expression (1.5.5) represents one of infinitely many different (if $\varrho \neq P_\psi$, for any $\psi \in \mathcal{H} \setminus \{0\}$) extremal decompositions of $\varrho \in \mathcal{S}_*$. Hence, a "statistical interpretation", like in Interpretation 1.4.4, of density matrices is questionable, cf. also [34] for a more detailed formulation.

Unbounded observables are usually given as unbounded selfadjoint operators on (a dense domain of) $\mathcal{H}$.[45] They are faithfully expressible by **projection valued measures** (PM, cf. [267, Ch.IX.4]) on the real line $\mathbb{R}$: To any selfadjoint operator $A = A^*$ corresponds a unique projection valued mapping

$$E_A : B(\in \mathcal{B}(\mathbb{R})) \mapsto E_A(B) = E_A(B)^2 = E_A(B)^*(\in \mathcal{L}(\mathcal{H}))$$

such, that for countable number of pairwise disjoint Borel sets $B_j \in \mathcal{B}(\mathbb{R})$ is E_A additive (the sums converging in the strong topology of $\mathcal{L}(\mathcal{H})$), and $E_A(\mathbb{R}) := I_\mathcal{H}$, cf. Definitions B.1.1. Such a correspondence between PM and selfadjoint operators is bijective, hence we can (and we often shall) as an observable in QM consider a PM. The standard useful formula connecting A with E_A is expressed by the strongly convergent integral, cf. Theorem B.1.3, and Proposition C.3.1:

$$A = \int_\mathbb{R} \lambda E_A(d\lambda).$$

1.5.2. **Note**. A generalization of PM leads to **positive operator valued measure** POV (or POVM), which also represents a selfadjoint operator, but it is not determined by that operator uniquely. It represents a generalization of the concept of observable given by PM. Any POVM on $\mathbb{R}$ is a positive–operator valued function

$$\Delta : B(\in \mathcal{B}(\mathbb{R})) \mapsto \Delta(B)(\in \mathcal{L}(\mathcal{H})_s),\quad 0 \leq \Delta(B) \leq I_\mathcal{H},$$

which is also countably additive (in strong topology) with respect to the additions of disjoint sets. In this case, contrary to PM, different $\Delta(B)$ $(B \in \mathcal{B}(\mathbb{R}))$ need not mutually commute. POVM can be used to modeling of imperfect measurements, reflecting nonideal sensitivity of measuring apparatuses, [71, 29, 53, 73]. We shall not go into details of this refinement of the concept of "quantummechanical observable"; see also Definition B.1.1. ♡

Let us turn now our attention to time evolution of general states (the "Schrödinger picture"), and also of observables (the "Heisenberg picture"). It is naturally defined from that of pure state space described in Subsection 1.5-a, due to linearity and/or affinity of all relevant relations. Hence, for the one–parameter unitary group $U(t) := \exp(-itH)$ describing the evolution of pure states, or also vectors in $\mathcal{H}$, the corresponding evolution of density matrices from (1.5.5) is

$$t \mapsto \varrho_t \equiv \phi_t(\varrho) := U(t)\varrho U(-t) \equiv \sum_j \lambda_j U(t) P_{\psi_j} U(-t), \tag{1.5.6}$$

[45] The forthcoming technical concepts are briefly described also in Appendices B, and C.

what is valid for all possible decompositions (1.5.5) of the density matrix ϱ. This description of time evolution corresponds to the **Schrödinger picture**.

The **Heisenberg picture** of the time evolution in QM is the dual (=transposed) transformation group ϕ^* to that of $\phi_t : \mathcal{S}_* \to \mathcal{S}_*$, since the algebra of observables $\mathcal{L}(\mathcal{H})$ is the (topological) dual space of the complex linear space spanned by density matrices and completed in the trace norm $\|\varrho\|_1 := Tr(|\varrho|)$. Since the duality is realized by the bilinear form

$$(\varrho; A)(\in \mathcal{S}_* \times \mathcal{L}(\mathcal{H})) \mapsto \langle A; \varrho\rangle \equiv Tr(\varrho A) =: \langle A\rangle_\varrho, \tag{1.5.7}$$

the time evolution of the A's in $\mathcal{L}(\mathcal{H})$, $(t; A) \mapsto A_t := \phi_t^*(A)$ is determined by the requirement

$$\langle A_t; \varrho\rangle \equiv \langle \phi_t^*(A); \varrho\rangle := \langle A; \phi_t(\varrho)\rangle,$$

and we have $\boldsymbol{A_t \equiv \phi_t^*(A) := U(-t)AU(t)}$. Let us note that, according to the introduced definition of ϕ_t^*, one has the following invariance:

$$\langle \phi_{-t}^*(A); \phi_t(\varrho)\rangle \equiv \langle A; \varrho\rangle. \tag{1.5.8}$$

Let us notice similarity of the equations (1.5.8), and (1.5.2), what will be of importance in the subsequent nonlinear extensions of QM, cf. [35].

Interpretation of states *and* observables is given by determination of a formula expressing the probability of obtaining results $\lambda \in B$ (:= a subset of the spectrum, i.e. of the set of possible values of A) by measuring of an observable A of a system occurring in the state ϱ. This probability will be denoted by $\mathrm{prob}(A \in B; \varrho)$. It can be also useful to introduce the corresponding probability measure μ_ϱ^A on the real line $\mathbb{R}$ of values of the measured quantity A:

$$\mathrm{prob}(A \in B; \varrho) \equiv \mu_\varrho^A(B) := Tr(E_A(B)\varrho). \tag{1.5.9}$$

This formula allows us essentially to express all empirically verifiable statements of QM. The expectation (mean value) is given by (1.5.7).

The assertion on nonexistence of dispersion–free states for all observables can be made precise in a form of general **Heisenberg uncertainty relations**:

1.5.3. Proposition. *Let A, B be two bounded selfadjoint operators (representing two quantal observables), and let $\langle A\rangle_\varrho := Tr(\varrho A)$ be the expectation of measured values of the arbitrary observable A in any state ϱ, $\varrho \in \mathcal{S}_*$. Let $\Delta_\varrho A := \sqrt{Tr(\varrho(A - \langle A\rangle_\varrho)^2)}$ be the dispersion of the measured values of A in the same state ϱ. Then*

$$\Delta_\varrho A \cdot \Delta_\varrho B \geq \frac{1}{2}\big|Tr(\varrho(AB - BA))\big| \equiv \frac{1}{2}\big|\langle i[A, B]\rangle_\varrho\big|. \quad \clubsuit \tag{1.5.10}$$

This proposition can be generalized to unbounded operators, with corresponding restrictions for the states $\varrho \in \mathcal{S}_*$. This shows that noncommutativity of two observables leads to nonexistence of their mutually sharp values in states with nonvanishing expectation of their commutator. Remember that for the operators $\mathfrak{Q}_j, \mathfrak{P}_j$ corresponding in QM to the classical j–th position and linear momenta coordinates, one has $[\mathfrak{Q}_j, \mathfrak{P}_k] = i\hbar\mathbb{I}_\mathcal{H}\delta_{jk}$ (on a corresponding dense domain in $\mathcal{H}$). Hence the "observables Q_j, P_j" cannot be both sharply determined in any state $\varrho \in \mathcal{S}_*$.[46] The formula (1.5.9) leads also to convenient realizations of (elements of) $\mathcal{H}$ in terms of numerical functions.

[46] For a discussion and citations on various interpretations of (1.5.10) see e.g. [51].

1.5.4. **Remark** *("Representations" in QM).* It might be useful to comment and formulate here, in some more general terms than is it usually presented, what is traditionally named "the representation theory" according to Dirac, [74]. Physicists often work with specific realizations of Hilbert space $\mathcal{H}$, according to specific physical systems to be described. Elements of the "Hilbert space of a given physical system" are often expressed as "wave functions", i.e. complex valued functions of "configuration variables" (e.g. positions of described particles). Since all infinite–dimensional separable Hilbert spaces are isomorphic, different realizations of $\mathcal{H}$ can be specified only by an additional mathematical structure. This is done by a choice of a "complete set of commuting observables", i.e. by specifying a maximal commutative von Neumann subalgebra [227, p.112][47] in $\mathcal{L}(\mathcal{H})$ generated by (mutually commuting) projection valued measures $E_A, E_B, \ldots$, of a set $A, B, \ldots$, of mutually commuting selfadjoint operators. These operators represent in QM some "simultaneously measurable observables". The von Neumann algebra $\mathcal{R}$ generated by a set

$$\mathcal{R}_0 := \{E_A(B_1), E_B(B_2), \ldots; B_1, B_2, \cdots \in \mathcal{B}(\mathbb{R})\}$$

of bounded operators (projections) in $\mathcal{H}$ containing the unit operator $I_{\mathcal{H}} \in \mathcal{L}(\mathcal{H})$ is obtained by taking the double commutant, according to famous von Neumann "bicommutant theorem", [187, 227, 254, 42], $\mathcal{R} = \mathcal{R}_0''$, in $\mathcal{L}(\mathcal{H})$.[48] Here, the commutant $\mathcal{R}_0'$ of $\mathcal{R}_0$ is given by

$$\mathcal{R}_0' := \{B \in \mathcal{L}(\mathcal{H}) : [B, A] = 0, \forall A \in \mathcal{R}_0\},$$

and $\mathcal{R}_0'' := (\mathcal{R}_0')'$, for any subset $\mathcal{R}_0 \subset \mathcal{L}(\mathcal{H})$. Any commutant in $\mathcal{L}(\mathcal{H})$ is a C^*-algebra closed in weak–operator topology of $\mathcal{L}(\mathcal{H})$, and such C^*-algebras are called **von Neumann algebras**, or W^*-algebras. The W^*-algebra $\mathcal{R}$ is maximal commutative iff $\mathcal{R} = \mathcal{R}'$, what is equivalent with the situation when the commutative W^*-algebra has a cyclic (then also separating) vector ψ_0 in $\mathcal{H}$, cf. [218]. Let $M_{\mathcal{R}}$ be the (compact Hausdorff) spectrum space (cf. Example B.3.5) of $\mathcal{R}$, hence the algebra of continuous complex valued functions $C(M_{\mathcal{R}})$ is isomorphic (denoted by $\sim$) to $\mathcal{R}$. If $\psi_0 \in \mathcal{H}$ is cyclic for $\mathcal{R} \sim C(M_{\mathcal{R}})$, then (denoting the operators in $\mathcal{R}$ by $\pi_0(f)$, for the corresponding functions $f \in C(M_{\mathcal{R}})$) the integral, i.e. the positive linear functional on $C(M_{\mathcal{R}})$ (according to the Riesz–Markov theorem, [218])

$$f\ (\in C(M_{\mathcal{R}})) \mapsto \mu^{\mathcal{R}}(f) := (\psi_0, \pi_0(f)\psi_0) \tag{1.5.11}$$

determines (if ψ_0 is normalized) a probability measure $\mu^{\mathcal{R}}$ on $M_{\mathcal{R}}$, and the mapping

$$U_{\mathcal{R}} : \pi_0(f)\psi_0\ (\in \mathcal{H}) \mapsto f \in C(M_{\mathcal{R}}) \subset L^2(M_{\mathcal{R}}, \mu^{\mathcal{R}}) \tag{1.5.12}$$

can be uniquely extended to an isomorphism of Hilbert spaces, [227]. Moreover (cf. [101, Chap. I.9]), all the functions $f \in C(M_{\mathcal{R}})$ are just all the (elements of equivalence classes of $\mu^{\mathcal{R}}$–essentially) bounded Borel functions on $M_{\mathcal{R}}$, i.e. $C(M_{\mathcal{R}}) = L^\infty(\mu^{\mathcal{R}})$.[49]

[47] A commutative algebra of bounded operators on $\mathcal{H}$ is *maximal commutative* if its arbitrary nontrivial extension by addition of an operator violates its commutativity. Such an algebra is always weakly closed in $\mathcal{L}(\mathcal{H})$, i.e. it is a W^*-algebra. cf. also Appendix B.

[48] cf. also Appendix B for technicalities.

[49] Let us note, that these functions $f \in C(M_{\mathcal{R}})$ can be considered either as elements of $\mathcal{L}(L^2(M_{\mathcal{R}}, \mu^{\mathcal{R}}))$, or as elements of the Hilbert space $L^2(M_{\mathcal{R}}, \mu^{\mathcal{R}})$ itself. Let us also note that the constant unit function $\mathbb{I}$ is an element of this Hilbert space representing a cyclic vector with respect to the maximal commutative algebra $C(M_{\mathcal{R}})$ of operators.

The spectrum space $M_{\mathcal{R}}$ of an abelian W^*-algebra $\mathcal{R}$ has a rather "wild" topology, since any W^*-algebra is generated by its projections which are, in the commutative case, continuous characteristic functions of clopen subsets of $M_{\mathcal{R}}$, which in turn form a basis of the Hausdorff topology of $M_{\mathcal{R}}$, cf. [101]. As a consequence of this **extremely disconnected** topology (cf. [254, Chap. III.1]), the function realization of $\mathcal{H}$ in (1.5.12) needn't seem to be practically convenient. If, however, there is in $\mathcal{R}$ a strongly dense unital C^*-subalgebra $\mathcal{A} \equiv \mathcal{A}_{\mathcal{R}}$ with some "nice" spectrum space M (e.g., M could be connected), then we can write for the corresponding isomorphism $U_{\mathcal{R}}$, instead of (1.5.12):

$$U_{\mathcal{R}} : \pi_0(f)\psi_0 \ (\in \mathcal{H}) \mapsto f \in C(M) \subset L^2(M, \mu^{\mathcal{A}}), \tag{1.5.13}$$

where the measure $\mu^{\mathcal{A}}$ is defined, now from $\mathcal{A}$, by the same way (i.e. via Riesz–Markov theorem) as it was done in (1.5.11) from $\mathcal{R}$, since $\mathcal{A}\psi_0\big(\mathcal{A} := \pi_0(C(M))\big)$ is again dense in $\mathcal{H}$.

Let $\mathcal{R}$ be generated by n projection measures $\{E_{A_1}, E_{A_2}, \ldots, E_{A_n}\}$, e.g. spectral measures of (possibly unbounded) selfadjoint operators $\{A_j, j = 1, \ldots n\}$; i.e. $\mathcal{R}$ is the minimal W^*-algebra containing these projections, and it is maximal commutative with a cyclic vector $\psi_0 \in \mathcal{H}$. Then the spectrum space $M = M_{\mathcal{A}}$ can be chosen homeomorphic to a compact subset of a compactification of $\mathbb{R}^n$, namely the support of the product–measure $E_{\mathcal{R}}$ (what is again a projection measure) of the spectral measures $E_{A_j}, j = 1, \ldots, n$, cf. [20, Chap. 5, §2, Theorem 6; Chap. 6, §5, Theorem 1]. We have then $L^2(M, \mu^{\mathcal{A}}) = L^2(\mathbb{R}^n, \mu^{\mathcal{A}})$, and each operator $U_{\mathcal{R}} A_j U_{\mathcal{R}}^{-1}$ acts on $L^2(\mathbb{R}^n, \mu^{\mathcal{A}})$ as "multiplication by the j–th variable":

$$U_{\mathcal{R}} A_j U_{\mathcal{R}}^{-1} \varphi(q) \equiv q_j \varphi(q), \ q \in \mathbb{R}^n, \varphi \in L^2(\mathbb{R}^n, \mu^{\mathcal{A}}).$$

We can speak now about the **$\mathcal{A}$–representation** , resp. **$\{A_j : j = 1, 2, \ldots, n\}$–representation**, of (Quantum Mechanics represented in) the Hilbert space $\mathcal{H}$.

Let us assume, that the product–measure $E_{\mathcal{R}}$ is absolutely continuous with respect to the Lebesgue measure $\mathrm{d}^n q$ on $\mathbb{R}^n$, i.e. absolutely continuous are all the probability measures

$$B(\in \mathcal{B}(\mathbb{R}^n)) \mapsto (\psi, E_{\mathcal{R}}(B)\psi), \quad \forall \psi \in \mathcal{H}, \quad \|\psi\| = 1.$$

Hence also all the probability measures

$$B(\in \mathcal{B}(\mathbb{R})) \mapsto \mu_j^{\psi}(B) := (\psi, E_{A_j}(B)\psi), \quad \forall \psi \in \mathcal{H}, \|\psi\| = 1, \quad j = 1, \ldots n,$$

are absolutely continuous with respect to $\mathrm{d}q$ on $\mathbb{R}$.[50] Since the vector $\psi_0 \in \mathcal{H}$ is cyclic and separating for $\mathcal{A}'' = \mathcal{A}' = \mathcal{R}$, the measure $\mu^{\mathcal{A}}$ is absolutely continuous with respect to the Lebesgue measure $\mathrm{d}^n q$ on $\mathbb{R}^n$. Let

$$q \ (\in \mathbb{R}^n) \mapsto f_{\psi_0}(q) := \frac{\mathrm{d}\mu^{\mathcal{A}}}{\mathrm{d}^n q}(q), \quad \frac{\mathrm{d}\mu^{\mathcal{A}}}{\mathrm{d}^n q} \in L^1(\mathbb{R}^n, \mathrm{d}^n q)$$

be a version of the Radon–Nikodym derivative (cf. [187, 218]) of $\mu^{\mathcal{A}}$ with respect to the Lebesgue measure. Then $L^2(M, \mu^{\mathcal{A}})$ can be mapped onto a subspace of $L^2(\mathbb{R}^n, \mathrm{d}^n q)$ by the unitary mapping

$$\psi(q) \mapsto \psi(q)\sqrt{f_{\psi_0}(q)}, \quad \forall \psi \in L^2(M, \mu^{\mathcal{A}}), \ q \in \mathbb{R}^n (\supset M). \tag{1.5.14}$$

[50] We do not formulate here sufficient conditions for absolute continuity of $E_{\mathcal{R}}$.

In this setting, on $\mathcal{H}$ represented by (a subspace of) $L^2(\mathbb{R}^n, \mathrm{d}^n q)$, the operators $A_j, j = 1, 2, \ldots, n$, are realized as multiplication operators by the coordinates $q_j, j = 1, 2, \ldots, n$, with $\{q_1, q_2, \ldots, q_n\} \equiv q \in \mathbb{R}^n$.[51] The probabilities (1.5.9) have now the form

$$\mathrm{prob}(\{A_j\} \in B \subset \mathbb{R}^n; P_\psi) = Tr(E_{\mathcal{R}}(B)P_\psi) = \int_B |\psi(q)|^2 \mathrm{d}^n q, \tag{1.5.15}$$

with $\|\psi\| = 1$. A special case of this situation is the usually used "position representation" of the state vectors. ♡

1.5.5. **Example** *(Position representation).* Let $\mathcal{A}$ be the subalgebra of $\mathcal{L}(\mathcal{H})$ generated by the unit operator $I_{\mathcal{H}}$ and by the operators $f(Q_1, Q_2, \ldots, Q_{3N})$, with the functions f from the Schwartz space $S(\mathbb{R}^{3N})$, where the standard position operators $Q_j, j = 1, 2, \ldots, 3N$, of the irreducible representation of G_{WH} for an N–particle system (cf. Subsection 3.3-b), were introduced. Then the spectrum space M is the one–point compactification of $\mathbb{R}^{3N}$ with the "usual" topology. The weak closure $\mathcal{R}$ of $\mathcal{A}$ in $\mathcal{L}(\mathcal{H})$ is an abelian W^*-algebra containing also projection operators belonging to the spectral decompositions of Q_j's, i.e. elements $E_{Q_j}(B)$ of their PM's. If there is a cyclic vector ψ_0 for $\mathcal{R}$ in $\mathcal{H}$, then ψ_0 is cyclic also for $\mathcal{A}$. Then we can use the unitary transformation $U_{\mathcal{R}} : \mathcal{H} \to L^2(\mathbb{R}^{3N}, \mathrm{d}^{3N} q)$ determined from (1.5.13), and (1.5.14) by $\mathcal{A}$ only. Hence $\mathcal{H} \sim L^2(\mathbb{R}^{3N}, \mathrm{d}^{3N} q)$. This is the usual "position-coordinate representation" of $\mathcal{H}$. ♡

1.5-c Symmetries and projective representations in QM

The time evolution described in Subsection 1.5-a was an example of a continuous transformation group in QM. It can be considered as a representation of a specific group ($G := \mathbb{R}$) of symmetries of a physical system, namely a representation of the observed (or postulated) **homogeneity of time**: This symmetry, described by formulas expressing fundamental laws of physics independent of the time variable, can be considered as just an expression of possibility of formulation of such laws. The invariance is encoded in the group property of the set of time–evolution operators, what corresponds to time independence of its generator (the Hamiltonian): "Dynamics" is time–independent, and differences in various possible (or observed) evolutions of the system in its "various occurrences" are ascribed to differences in "initial conditions", [281, 138], resp. in "boundary conditions" (including also "external fields").

The relevance of symmetries in physics was probably (at least) intuitively clear since the advent of any considerations which now we call "physical". Their formalization came, however, much later: Although importance of symmetries for human activities was claimed already by Leonardo da Vinci (according [281, 275]), clear understanding of their importance for formulation of geometry and laws of nature came only at about the beginning of 20th century, e.g. in works of F. Klein [150], G. Hamel [125], H. Poincaré [204], E. Mach [170, 171], P. Curie [69], A. Einstein [89, 88], and others.[52]

Their importance is clearly seen, e.g. in formulation of classical – mechanical problems on integrability (connected with the question of stability of Solar system), in Einstein discovering of relativity theories, in Gibbs formulation of statistical physics [158], etc. Clear mathematical

[51] If the spectrum of some A_j is not the whole $\mathbb{R}$, then $\mathcal{H}$ is represented by a proper subspace $L^2(\mathrm{supp}(E_{\mathcal{R}}), \mathrm{d}^n q) \subset L^2(\mathbb{R}^n, \mathrm{d}^n q)$.

[52] Many historical notes on symmetries can be found in [179].

connection of variational equations with symmetries and with integrals of motion was formulated also due to the theorems by Emmy Noether [192]. Nowadays is generally accepted the connection between Lie group invariance of "action integrals" (or/and Lagrangians) of classical physics, cf. ,e.g., [157, 160, 1], with conservation of some nontrivial functions on phase space with respect to the time evolution determined by the corresponding variational problem. These **integrals of motion** determine submanifolds in phase space left by the time evolution invariant. This leads to practical advantage of "lowering dimensions" of solved problems. Intuitively, this also allows a better specification of the (self)identity of moving physical systems.

A "quantum–field–rephrasing" of the mentioned principles was one of the leading tools in formulations of (heuristic, but successful) quantum theories of elementary particles, with quantum electrodynamics as their prototype. Also in foundations of mathematically clear (but, up to now not very successful) "axiomatic" algebraic formulation of quantum field theory (QFT), cf., e.g. [250, 120, 38], symmetry principles play a key rôle.

1.5.6. **Note**. We can suspect even more general meaning of "invariances" with respect to some group of transformations in physics: They help us to determine physically (hence operationally) meaning of "physical quantities"; very pictorially expressed, symmetry means that **some mutually different things** (states, observed values of something,. . .) **are** in a certain sense **equal**, [69], what might help us to specify how to measure them. A very fruitful principle in physics is, as is generally known, the requirement of invariance with respect to Galileo, resp. Poincaré groups, cf. also Interpretation 1.1.2. ♡

The symmetry considerations in QM are even more important and useful than in classical physics. This is, perhaps, due to the "more mathematical" and less intuitive nature of quantum theories. The a priori linear formulation of QM offered a natural application of (linear) representation theory of groups to solution of specific classes of problems in QM, esp. in classification of "elementary systems" (these might be "elementary particles", but also molecules), of their spectra and interactions, in scattering theory etc., cf. [275, 280, 281, 282, 250, 118]. One can say that symmetry considerations are lying now somewhere "in the heart" of QM. They belong, e.g., to the main tools in the search for new fundamental interactions of elementary particles.

We shall restrict now our attention to a rather specific technical question connected with appearance of symmetry considerations in mathematical formulations in QM. Symmetry groups are usually specified either from observations of specific motions of macroscopic bodies (e.g. translations and rotations of "rigid" bodies), or by some theoretical hypotheses coming from an interplay of presently accepted theoretical scheme and observations connected with it (e.g. the isospin group, and other symmetries of elementary particle theories). Groups appear then in formalisms of physical theories in a form of their "realizations", cf. [148], i.e. in a form of their actions on spaces of physically relevant theoretical objects like "states", "observable quantities", "state vectors", etc. In traditional formulations of QM, symmetries are formalized as transformations of Hilbert space vectors. It is important in some considerations to understand connections of the transformations of vectors in $\mathcal{H}$ with corresponding transformations of quantal states.

The usually required general restrictions to the set of symmetry transformations of the states of a QM–system are the same as for ϕ_t in Subsection 1.5-a, esp. in (1.5.2). There is, however, an additional complication for general (more than one–dimensional) continuous groups G of

transformations, $g(\in G) \mapsto \alpha_g : \mathcal{S}_* \to \mathcal{S}_*$. Let us assume that (1.5.2) is again fulfilled:

$$Tr(\alpha_g(P_\varphi)\alpha_g(P_\psi)) \equiv Tr(P_\varphi P_\psi). \tag{1.5.16}$$

Then a trivial adaptation of arguments following (1.5.2) (by the assumptions of the group property and continuity, as above) leads to the conclusion (cf. also [267, 148]) that a continuous family

$$g(\in G) \mapsto U(g)(\in \mathcal{L}(\mathcal{H}))$$

of unitary operators exists representing the mapping $g \mapsto \alpha_g$ as

$$\alpha_g(P_\psi) \equiv U(g)P_\psi U(g)^*. \tag{1.5.17}$$

This determines, however, the unitary operators up to phase factors, and we obtain (for details see [267, Chaps.IX, and X, esp. Theorem 10.5])

$$U(g_1 \cdot g_2) \equiv m(g_1, g_2)U(g_1)U(g_2), \tag{1.5.18}$$

where $m : G \times G \to S^1 \subset \mathbb{C}$ is a **multiplier** for the group G satisfying the following identities implied by associativity of group multiplication:

$$\begin{aligned} &m(g_1, g_2 \cdot g_3)m(g_2, g_3) = m(g_1 \cdot g_2, g_3)m(g_1, g_2), \quad \forall g_j \in G, \\ &m(g, e) = m(e, g) = 1, \quad \forall g \in G, \quad e \cdot g \equiv g. \end{aligned} \tag{1.5.19}$$

Multipliers for G form a commutative group (by pointwise multiplication; cf. (3.3.10) for additive notation) with the unit element $\mathbb{I}(g, h) \equiv 1$. If the multiplier can be removed by multiplying $U(g) \mapsto a(g)U(g)$ by some "phase factors" $a(g) \in S^1 :=$*the complex numbers of unit modulus*, then it is *similar to* $\mathbb{I}$, or **exact**. Two multipliers m_1, m_2 are mutually similar, if the multiplier $m_1 \cdot m_2^{-1}$ is similar to $\mathbb{I}$. The unitary family satisfying (1.5.18) is called a **projective representation** of G with the multiplier m. All projective representations of G obtained from the same α_G have mutually similar multipliers, and, to any projective representation U with a multiplier m, and to each multiplier m' similar to m, there is a projective representation U' with the multiplier m' leading to the same α_G according to (1.5.17) as U.

Hence, if the multiplier in (1.5.18) is exact, it is possible to choose a unitary representation (i.e. with $m \equiv 1$) corresponding to the α_G. Otherwise, it is possible to find another group G_m containing G as a normal subgroup, a **central extension** of G by the commutative group S^1 corresponding to the multiplier m, and such that the formula (1.5.18) determines its unitary representation: Elements of G_m are couples $(g; \lambda) \in G \times S^1$ with the group multiplication

$$(g_1; \lambda_1) \cdot (g_2; \lambda_2) = (g_1 \cdot g_2; m(g_1, g_2)\lambda_1\lambda_2). \tag{1.5.20}$$

The corresponding unitary representation $\tilde{U}(G_m)$ is

$$\tilde{U}(g; \lambda) := \lambda^{-1}U(g), \quad \forall g \in G, \lambda \in S^1.$$

The check that $\tilde{U}(g; \lambda)$ leads (for all $\lambda \in S^1$) to the same symmetry transformation α_g of the states than the element $U(g)$ of the projective representation $U(G)$ is straightforward.

1.5.7. **Examples**. (i) Only projective representations of the inhomogeneous Galileo group with nontrivial multipliers can be interpreted, [267], in the usual interpretation schemes of QM, as transformations of states of systems in QM representing the corresponding (relative) motions of macroscopic background. The unitary representations of this group are all "unphysical".
(ii) The most basic application of group representations in QM is, perhaps, the case of canonical commutation relations (CCR). These relations determine a Lie algebra structure in a set of basis elements (i.e. of "elementary observables" completed by the "trivial element") in a "Hamiltonian system on $\mathbb{R}^{2n}$"– both quantum and classical. These relations are expressed in CM by Poisson brackets between canonical position and momenta coordinates, and in QM they are commutators between "corresponding" selfadjoint operators (representing in some way also physical position and momenta observables). The connection with group representations is, that these operators are generators of a projective representation of the *commutative group of translations in the classical flat phase space* $\mathbb{R}^{2n}$, or they are generators (together with a unit operator) of a *unitary representation* of a one–dimensional central extension of this commutative group, i.e. of the $2n+1$–dimensional (noncommutative) Weyl–Heisenberg group G_{WH}.[53] All such (nontrivial, i.e. more than one–dimensional) irreducible projective representations are parametrized (up to unitary equivalence) by all nonzero reals, [287, 148]. Remarkable **physical feature of CCR** is, that they can correspond, to reach agreement of theoretical predictions with experiment, just to one of the infinite number of mutually inequivalent representations of classical shifts in phase space, and namely the "correct" choice of the representation **fixes the value of Planck constant** $\hbar$; cf. also Section 3.3-b for corresponding technicalities.
(iii) The (covering group of the) connected component of the Poincaré group is **rigid**, i.e. it has no nontrivial multipliers, [267, 283, 148]. It follows that any projective representation of the (connected) Poincaré group can be obtained from the corresponding unitary representation of its covering group. ♡

1.5-d On the causality problem in QM

With discussions on Einstein causality in NLQM, cf. [106, 169], or also Interpretation 2.1.24, it is interesting to pose such a question also in frameworks of *linear* QT. In a renewal of such a discussion [128, 48] (initiated probably by Fermi in 1932 [96]), there was discussed a simple mathematical theorem with impressive consequences for possibilities on "instantaneous spreading of wave packets" in QM. It can be formulated as follows:

1.5.8. Theorem (Long distance action in QM). *Let a selfadjoint lower bounded operator* $H = H^* \geq \varepsilon_0 \mathbb{I}_{\mathcal{H}}$ *on a Hilbert space* $\mathcal{H}$ *be given. Assume that, for a bounded operator* B: $0 \leq B \in \mathcal{L}(\mathcal{H})$, *and for a vector* $0 \neq \psi \in \mathcal{H}$, *there is:* $Tr(P_\psi B) = 0$. *Let us define* $\psi(t) := \exp(-itH)\psi, \forall t \in \mathbb{R}$. *Then either*
(i) $Tr(P_{\psi(t)}B) \equiv 0$ *for all* $t \in \mathbb{R}$, *or*
(ii) $Tr(P_{\psi(t)}B) \neq 0$ *for all* $t \in \mathcal{T}_{\psi,B} \subset \mathbb{R}$, *with* $\mathcal{T}_{\psi,B}$ *open and dense in* $\mathbb{R}$, *and of the total Lebesgue measure* m: $m(\mathbb{R} \setminus \mathcal{T}_{\psi,B}) = 0$. ♣

Let, e.g. P_ψ be a state of a composed system $I + II$ (say, consisting of two mutually spatially distant atoms I and II), and let $B = B^* = B^2 \neq 0$ be a projection on a subspace of $\mathcal{H}$. Assume

[53] Remember that commutative groups have only one–dimensional irreducible *unitary* representations.

that the vectors of $B\mathcal{H}$ correspond to those states of $I+II$ in which the atom II is in its excited state (we assume a possibility of determination of such states of $I+II$). Then $Tr(P_{\psi(t)}B) \neq 0$ can be interpreted as excitation in time $t > 0$ of the formerly not exited atom II "due to an influence of the atom I". The theorem can be then interpreted so that if there will be some influence at all (sometimes, in the mentioned sense), then it is **always immediate**, i.e. there is nonzero probability that it is realized instantaneously!

Above considerations show that in QM, in the described sense, the **Einstein causality is never fulfilled**. This result is a general consequence of the assumptions of the positivity of the generator H, of the interpretation of projection operators B as observables measuring of arbitrary "properties" of described systems in QM, as well as due to occurring of arbitrary projections between the observables; all these assumptions might seem to belong to general assumptions of an arbitrary quantum theory (QT).

One can now ask whether Einstein causality is fulfilled in relativistic QFT. It is argued in [48] that it is so in the algebraic formulation of QT (e.g. [118, 120, 38, 140]), as a consequence of the relativistic covariance of/and local structure of algebras of observables. This can be seen, roughly, due to consequent specific structure of algebras of localized observables (cf. Note B.4.1), as well as due to the Reeh–Schlieder theorem, cf. [140, Theorem 3.1]. This theorem implies that in "most of interesting states" of "sufficiently" localized subsystems (e.g. in the states extendable to states of the total system with restricted total energy, if the space–time region of the localization has the space–like complement with nonvoid interior) any localized positive observable has nonzero expectation, cf. also [231, 122, 120]. Hence the above assumption $Tr(P_\psi B) = 0$ cannot be fulfilled for such systems, states, and observables. Moreover, the assumed locality together with Einstein covariance lead to positive result on Einstein causality, [48].

1.5.9. **Note** *(Impossible signals due to measurements).* It might be useful to recall here that it is impossible in QM to send signals in the process of quantum measurements even if one accepts the instantaneous "reduction of wave packets", cf. Footnote 43:

Let two (mutually spatially distant) quantal subsystems I and II be "EPR–like" correlated (cf. Interpretation 2.1.24) in a given state Ψ of the composed system $I+II$. The only available "information" which could be transferred (=signalled) from II to I, as a result of the mere measurement of a quantity A of the subsystem II, might be the choice (and its possible changes) of the quantity A, resp. of its eigenbasis (i.e. its PM E_A) $\{\Phi_k\} \subset \mathcal{H}_{II}$. The only way, on the other side, of perceiving of the signal by I might be the measurement of the state ϱ of I, what is, however, independent on the choice of A. The point is, that QM is a statistical theory not containing in its formalism any objects corresponding to our intuition on a "single system" (possibly, as an element of some "ensemble of equally prepared systems") resulting in a "single event" at a measurement; cf. also [229]. ♡

2 Extended Quantum Mechanics

2.1 Elementary Quantum Phase Space

This chapter contains a description of technical features, as well as of the proposed interpretation of the theoretical construction called here extended quantum mechanics (EQM). We can emphasize here several types of problems posed and solved in this chapter; let us call them: (i) kinematical, (ii) dynamical, (iii) analytical, and (iv) interpretational.
Questions in (i) include topics that could be named "the geometry of phase space", into (ii) can be included questions connected with dynamics, as well as with continuous actions of symmetry groups on the "phase space", under (iii) we shall understand mainly technical problems connected with infinite dimensionality of the "phase space", with unboundedness of generators of the group actions etc.; interpretation in (iv) is understood as a series of notes and proposals concerning a *general scheme for interpretation* of the theory; however, many questions on possible specific empirically verifiable applications of EQM are left open here.

These sets of questions are mutually interconnected, e.g. in dealing with "geometry of phase space" we cannot avoid some technical problems connected with its infinite–dimensionality, including different topological and differential–geometrical technicalities. Similarly by dealing with "symmetry group actions" one has to deal simultaneously also with some "algebraic", or "structural" questions, and with problems connected with the (only) densely defined generators of these actions and their domains of definition. Hence, it is impossible to distinguish clearly the forthcoming sections according to the sort of problems solved in their scope. Keeping this in mind, we shall try to characterize at least roughly the contents of the sections in the present Chapter.

Section 2.1 is mainly devoted to a description of the "geometrical features" (i), describing the canonical manifold and Poisson structures on the space $\mathcal{S}_*$ of all density matrices of conventional QM. Also a preliminary description of Hamiltonian vector fields and corresponding induced dynamics is included into that section. Also in this case, as in finite dimensional ones, the Poisson "manifold" $\mathcal{S}_*$ decomposes into "symplectic leaves" left by all Hamiltonian flows invariant. All these leaves are homogeneous spaces (i.e. orbits) of the unitary group $\mathfrak{U}$ of the Hilbert space $\mathcal{H}$ with respect to its natural coadjoint action. There are, however two kinds of these orbits (leaves): The "finite dimensional" ones consist of density matrices of finite range (i.e. only finite number of their eigenvalues are positive), and the induced symplectic structure is "strongly nondegenerate", the tangent spaces having a canonical Hilbert space structure; these properties make these symplectic leaves in some sense similar to finite dimensional symplectic manifolds. The "infinite dimensional" leaves consisting of density matrices with infinite numbers of nonzero eigenvalues are only "weakly symplectic", and the naturally defined "tangent spaces" are not closed in their (again "naturally chosen") topology. The set of "finite dimensional" leaves is, fortunately, dense in the whole $\mathcal{S}_*$, so that we can restrict, for many purposes, our attention onto them.

Section 2.2 contains an analysis of questions connected with unboundedness of generators (i.e. "Hamiltonians") of group actions corresponding to linear, as well as to nonlinear cases. The unbounded generators always appear in any description of "nontrivial" actions of noncompact Lie groups, and cannot be avoided in the considered framework. The domain problems and description of the induced dynamics (flows) are solved in the cases when the (nonlinear) generators are constructed in a certain way from a continuous unitary representation of a Lie group G. For

more general cases, we formulate at least some proposals.

In the last Section 2.3, interconnections between all the sets (i) – (iv) of problems are especially obvious. Introduction of "nonlinear observables" is a consequence of the nonlinear dynamics. The interpretation of such observables extending the usual one leads (in the scheme proposed in this work) to introduction of observables as numerical functions of *two variables from* $\mathcal{S}_*$. It is also presented a (preliminary) classification of theories according to the choice of a Lie group G determining (sub)sets of *observables*, *generators*, and *states* of the considered (abstract) "physical system". For a given G, a further classification of generators, observables, and states is proposed. A general scheme of constructions of nonlinear generators and a description of their flows is given. The chapter ends with a description of "nonlinear" actions of Lie groups obtained from linear ones by (mathematically perhaps trivial) "symplectic deformations", with inclusion of EQM into a (linear) C^*-algebraic scheme and with a description of its general symmetries. The section contains also a description of some interpretation proposals, cf. Interpretations 2.3.1, 2.3.11, 2.3.15, and 2.3.18.

States of a "considered physical system" in QM are described (under a natural continuity requirement) by density matrices $\varrho \in \mathcal{S}_* := \mathfrak{T}_{+1} \subset \mathfrak{T}$ on the corresponding Hilbert space $\mathcal{H}$ (cf. next Subsection 2.1-a), but linearity of QM allows often to reduce the theoretical work to the work with vector states described by one–dimensional density matrices $\varrho = P_\psi$, $0 \neq \psi \in \mathcal{H}$ (i.e. pure states, if superselections are missing). Any state of that quantum-mechanically described system is expressible with a help of these "elementary vector states". In nonlinear versions of QM, operations like symmetry transformations, and specifically time evolutions, are nonlinear, resp. nonaffine; hence, if they are performed on states described by density matrices, these operations are not reducible to those on vector states. In EQM, the whole set $\mathcal{S}_*$ of density matrices will play a rôle of the set of "elementary states" in such an intuitive sense, where each density matrix $\varrho \in \mathcal{S}_*$ is considered as an analogue of a point of phase space of CM, irrespective of dimension of the range of the operator ϱ. This implies, e.g., that the time evolution of density matrix states ϱ of a "relatively isolated system" in EQM can be determined only with a help of determination of corresponding Hamiltonian flow in a neighbourhood of ϱ, that can be independent of determination of the flow in neighbourhoods of the vector states into which ϱ can be formally decomposed.

2.1-a Basic mathematical concepts and notation

Let $\mathcal{H}$ be a separable complex Hilbert space with scalar product (x, y) $(x, y \in \mathcal{H})$ chosen linear in the second factor *y*. Let $\mathfrak{F} \subset \mathfrak{T} \subset \mathfrak{H} \subset \mathfrak{C} \subset \mathcal{L}(\mathcal{H})$ be the subsets of linear operators in $\mathcal{H}$ consisting of the all finite–rank, trace–class, Hilbert–Schmidt, compact, and bounded operators respectively. All these subsets are considered as complex associative *-subalgebras (in fact ideals) of the algebra $\mathcal{L}(\mathcal{H})$, i.e. they are also invariant with respect to the involution $a \mapsto a^*$ defined as the operator adjoint mapping. The algebras $\mathfrak{C}$ and $\mathcal{L}(\mathcal{H})$ are C^*-algebras ($\mathcal{L}(\mathcal{H})$ is in fact a W^*-algebra), and all of them except of $\mathfrak{F}$ are Banach spaces ($\equiv$ B–spaces) if endowed with proper norms: $\mathfrak{T}$ is endowed by the trace–norm $\|a\|_1 := Tr|a|$ with $|a| := (a^*a)^{\frac{1}{2}}$, $\mathfrak{H}$ is endowed by the Hilbert–Schmidt norm $\|a\|_2 := \sqrt{(a,a)_2}$ corresponding to the Hilbert–Schmidt scalar product $(a, b)_2 := Tr(a^*b)$ of operators $a, b \in \mathfrak{H}$, whereas $\mathfrak{C}$ and $\mathcal{L}(\mathcal{H})$ are endowed with the usual operator norm (which is equal to the spectral radius for selfadjoint operators) denoted by $\|a\|$. Here Tr denotes the trace of the operators in $\mathfrak{T}$. Note also that $\mathfrak{T}$ contains all products of

at-least-two Hilbert–Schmidt operators, and each element of $\mathfrak{T}$ is of this form; the last statement follows from the polar decomposition of closed densely defined (hence also bounded) operators in $\mathcal{H}$ [187].

The linear space $\mathfrak{F}$ is dense in the Banach spaces $\mathfrak{T}$, $\mathfrak{H}$, and $\mathfrak{C}$, and it is dense also in $\mathcal{L}(\mathcal{H})$ in its σ-strong operator topology. The Banach space $\mathfrak{T}$ will be considered also as the topological dual space to the C^*-subalgebra $\mathfrak{C}$ of $\mathcal{L}(\mathcal{H})$, the duality being given by the bilinear form $(\varrho;\mathrm{a}) \mapsto \varrho(\mathrm{a}) := Tr(\varrho\mathrm{a}) \equiv \langle\varrho;\mathrm{a}\rangle$ on $\mathfrak{C} \times \mathcal{L}(\mathcal{H})$ ($\subset \mathfrak{T} \times \mathcal{L}(\mathcal{H})$); the same bilinear form describes the duality between $\mathfrak{T}$ and $\mathcal{L}(\mathcal{H}) \equiv \mathfrak{T}^*$.

Let us introduce also the **projective Hilbert space** $P(\mathcal{H})$ which is obtained from $\mathcal{H}$ as the factor-space consisting of all its one-dimensional complex subspaces $\mathbf{x} := \{y \in \mathcal{H} : y = \lambda x, \lambda \in \mathbb{C}\}\ 0 \neq x \in \mathcal{H}$, with the factor-topology induced by the norm of $\mathcal{H}$; it can be identified with the subset of $\mathfrak{T}$ consisting of all one-dimensional orthogonal projections P_x (projecting $\mathcal{H}$ onto $\mathbf{x}$, $0 \neq x \in \mathcal{H}$) endowed with the relative topology of the trace-norm topology, or with $\sigma(\mathfrak{T},\mathfrak{C})$-topology (these topologies are equivalent on $P(\mathcal{H})$, [42]).

We shall also use some elementary concepts of differential geometry on (also infinite-dimensional) manifolds, [1, 39, 61, 151, 40], see also our Appendix A, as well as some concepts of the theory of C^*-algebras [196, 42, 76, 77, 254], cf. also our Appendix B, in this paper.

Let us denote by $\mathcal{S}_* \subset \mathfrak{T}$ the state space of $\mathfrak{C}$; it can be canonically identified with the convex set of all normalized normal positive linear functionals on $\mathcal{L}(\mathcal{H})$: $\mathcal{S}_* := \mathcal{S}_*(\mathcal{L}(\mathcal{H})) = \mathcal{S}(\mathfrak{C})$. The general (not necesseraly normal) states $\mathcal{S}(\mathcal{L}(\mathcal{H}))$ of $\mathcal{L}(\mathcal{H})$ will be denoted by $\mathcal{S}$. Let $\mathfrak{U}$ denote the unitary group $\mathcal{U}(\mathcal{H})$ of $\mathcal{L}(\mathcal{H})$: $\mathrm{u} \in \mathfrak{U} \Leftrightarrow \{\mathrm{u} \in \mathcal{L}(\mathcal{H})\ \&\ \mathrm{uu}^* = \mathrm{u}^*\mathrm{u} = I\} \Leftrightarrow \mathrm{u} \in \mathcal{U}(\mathcal{H})$, where $I \in \mathcal{L}(\mathcal{H})$ is the identity operator. Let $\tilde{\mathfrak{A}} := \mathfrak{U}/\mathfrak{J}$ be the factor-group of $\mathfrak{U}$ with respect to the central subgroup $\mathfrak{J} := \{\mathrm{u} := \lambda I : |\lambda| = 1, \lambda \in \mathbb{C}\}$. Since all *-automorphisms of $\mathcal{L}(\mathcal{H})$ are inner [196], $\tilde{\mathfrak{A}}$ is isomorphic to the group of all *-automorphisms (cf. also [267, Vol.I]): $\alpha \in {}^*\text{-Aut}\,\mathcal{L}(\mathcal{H}) \Rightarrow \exists \mathrm{u} \in \mathfrak{U} : \alpha(\mathrm{b}) = \mathrm{ubu}^*(\forall \mathrm{b} \in \mathcal{L}(\mathcal{H}))$, and if also $\mathrm{v} \in \mathfrak{U}$ represents α in this sense, then $\mathrm{u}^*\mathrm{v} \in \mathfrak{J}$. Let $\gamma : \mathrm{u} \mapsto \gamma_\mathrm{u} \in \tilde{\mathfrak{A}} = {}^*\text{-Aut}\,\mathcal{L}(\mathcal{H})$ be the corresponding representation of $\mathfrak{U}$, $\gamma_\mathrm{u}(\mathrm{b}) := \mathrm{ubu}^*$; the kernel of γ is $\mathfrak{J}$. Moreover, $\mathfrak{U}$ (and $\mathfrak{A}$) is a (infinite–dimensional for $\dim\mathcal{H} = \infty$) Lie group; the Lie algebra $Lie(\mathfrak{U})$ of $\mathfrak{U}$ is the real subspace $\mathcal{L}(\mathcal{H})_a := \{\mathrm{x} \in \mathcal{L}(\mathcal{H}) : \mathrm{x}^* = -\mathrm{x}\}$ of antihermitean elements of $\mathcal{L}(\mathcal{H})$ [39]. Let $[\mathrm{a},\mathrm{b}]$ $(:= \mathrm{ab} - \mathrm{ba})$ be the commutator in $\mathcal{L}(\mathcal{H})$. We shall use the selfadjoint generators $\mathrm{x}^* = \mathrm{x} \in \mathcal{L}(\mathcal{H})_s := i\mathcal{L}(\mathcal{H})_a$ to represent the Lie algebra elements $i\mathrm{x} \in \mathcal{L}(\mathcal{H})_a$. The Lie bracket will be defined on $\mathcal{L}(\mathcal{H})_s$ as $(\mathrm{x};\mathrm{y}) \mapsto i[\mathrm{x},\mathrm{y}],\ \mathrm{x},\mathrm{y} \in \mathcal{L}(\mathcal{H})_s$ what corresponds to the commutator $[i\mathrm{x}, i\mathrm{y}]$ in $\mathcal{L}(\mathcal{H})_a$: $[i\mathrm{x}, i\mathrm{y}] =: i\mathrm{z} \Rightarrow \mathrm{z} = i[\mathrm{x},\mathrm{y}]$. The Lie algebra of $\mathfrak{A}$ is $Lie(\mathfrak{U}/\mathfrak{J}) = \mathcal{L}(\mathcal{H})_a/\{\mathbb{R}I\}$ - the factoralgebra by the central ideal of real multiples of identity. Let $Ad(\mathfrak{U})$ be the adjoint representation of $\mathfrak{U}$ on $Lie(\mathfrak{U}) = i\mathcal{L}(\mathcal{H})_s$, i.e. $Ad(\mathrm{u})$ is the restriction of γ_u to $\mathcal{L}(\mathcal{H})_s$:

$$Ad(\mathrm{u})\mathrm{b} \equiv Ad(\mathrm{u})(\mathrm{b}) := \mathrm{ubu}^*,\ \mathrm{b} \in \mathcal{L}(\mathcal{H})_s,\ \mathrm{u} \in \mathfrak{U}. \tag{2.1.1}$$

The (topological) dual of $\mathcal{L}(\mathcal{H})_s$ is the real subspace $\mathcal{L}(\mathcal{H})_s^*$ of $\mathcal{L}(\mathcal{H})^*$ consisting of symmetric bounded linear functionals on $\mathcal{L}(\mathcal{H})$, i.e. $\nu \in \mathcal{L}(\mathcal{H})_s^* \Rightarrow \nu(\mathrm{b}^*\mathrm{b}) \in \mathbb{R}\ (\forall \mathrm{b} \in \mathcal{L}(\mathcal{H}))$, where $\langle\nu;\mathrm{y}\rangle \equiv \nu(\mathrm{y})$ denotes the value of $\nu \in \mathcal{L}(\mathcal{H})^*$ on the element $\mathrm{y} \in \mathcal{L}(\mathcal{H})$. The state space $\mathcal{S} := \mathcal{S}(\mathcal{L}(\mathcal{H}))$ is a compact convex subset of $\mathcal{L}(\mathcal{H})_s^*$, if it is endowed with the w^*-topology, i.e. with the $\sigma(\mathcal{L}(\mathcal{H})^*, \mathcal{L}(\mathcal{H}))$-topology [218, Theorem IV.20]. Let $Ad^*(\mathfrak{U})$ be the coadjoint representation of $\mathfrak{U}$ $(\ni \mathrm{u})$ on $\mathcal{L}(\mathcal{H})_s^*$:

$$[Ad^*(\mathrm{u})\nu](\mathrm{b}) := \nu(Ad(\mathrm{u}^{-1})\mathrm{b}),\ \nu \in \mathcal{L}(\mathcal{H})_s^*,\ \mathrm{b} \in \mathcal{L}(\mathcal{H})_s. \tag{2.1.2}$$

It is clear [42, Chap. 3.2] that the state-spaces $\mathcal{S}$ and $\mathcal{S}_*$ are both $Ad^*(\mathfrak{U})$-invariant subsets of $\mathcal{L}(\mathcal{H})_s^*$.

Let $\mathcal{O}_\nu(\mathfrak{U}) := Ad^*(\mathfrak{U})\nu := \{\omega \in \mathcal{L}(\mathcal{H})_s^* : \omega = Ad^*(\mathrm{u})\nu, \mathrm{u} \in \mathfrak{U}\}$ be the $Ad^*(\mathfrak{U})$-orbit of ν. The state space $\mathcal{S}$ decomposes into union of $Ad^*(\mathfrak{U})$-orbits. Let $\varrho \in \mathfrak{T}_s := \mathfrak{T} \cap \mathcal{L}(\mathcal{H})_s^*$ be a density matrix (i.e. $\varrho \geq 0,\ Tr\ \varrho = 1$) describing equally denoted state $\varrho \in \mathcal{S}_* : \varrho(\mathrm{b}) := Tr(\varrho \mathrm{b}) \equiv \langle \varrho; \mathrm{b} \rangle,\ \mathrm{b} \in \mathcal{L}(\mathcal{H})$.

We shall use for the density matrices spectral decomposition in the form

$$\varrho = \sum_{j \geq 1} \lambda_j E_j, \tag{2.1.3}$$

where we choose the ordering of the eigenvalues $\lambda_j > \lambda_{j+1} > 0$, and the spectral projections E_j are all finite dimensional. Let us denote $E_0 := I - \sum_{j \geq 1} E_j$.

2.1.1. Lemma. *Each orbit $\mathcal{O}_\varrho := Ad^*(\mathfrak{U})\varrho$ ($\varrho \in \mathcal{S}_*$) consists of all the density matrices which have the same set of eigenvalues (including multiplicities). Hence, the state space $\mathcal{S}_*$ is $Ad^*(\mathfrak{U})$-invariant.* ♣

Proof. The $Ad^*(\mathrm{u})$-mapping is a unitary mapping conserving spectral invariants, i.e. spectrum and spectral multiplicities, cf. [124]. Hence all the elements of the orbit $\mathcal{O}_\varrho$ are density matrices with the same spectra and multiplicities. The spectral resolution of any density matrix ϱ' of the same spectral invariants as ϱ in (2.1.3) has the form:

$$\varrho' = \sum_{j \geq 1} \lambda_j E'_j,$$

with equal dimensions of E'_j and $E_j, \forall j$. Then there is a unitary operator u mapping all the E_j's onto the corresponding E'_j's for all $j \geq 0$. E.g., one can choose orthonormal bases $\{x_m\}$, resp. $\{y_n\}$ in $\mathcal{H}$ containing subbases, for all $j \geq 0$, of $E_j\mathcal{H}$, and $E'_j\mathcal{H}$, respectively, to order them in accordance with orderings of E_j's, i.e. so that $E_j x_k = x_k \Leftrightarrow E'_j y_k = y_k$ and define u by the formula

$$\mathrm{u}x_k := y_k, \quad \text{for all } k.$$

Then $\varrho' = Ad^*(\mathrm{u})\varrho$, what proves the lemma. □

Hence, the projective space $P(\mathcal{H})$ coincides with the orbit $\mathcal{O}_\varrho(\mathfrak{U})$ with $\varrho^2 = \varrho$, what characterizes one-dimensional projections ϱ in $\mathcal{H}$.

2.1-b The manifold structure of $\mathcal{S}_*$

We shall now introduce a natural manifold structure on the orbit $\mathcal{O}_\varrho$ ($\varrho \in \mathcal{S}_*$). Let $\mathfrak{U}_\varrho \subset \mathfrak{U}$ be the stability subgroup for the point $\varrho \in \mathcal{S}_*$ at $Ad^*(\mathfrak{U})$-representation. Let us note that $\mathfrak{U}_\varrho$ is the unitary group of the W^* -algebra $\boldsymbol{\{\varrho\}'}$:= (**the commutant of the density matrix ϱ in $\mathcal{L}(\mathcal{H})$**), hence it is a Lie group, and its Lie algebra $Lie(\mathfrak{U}) =: i\mathfrak{M}_\varrho$ consists [39, Chap.3, §3.10] of antisymmetric elements of the commutant $\{\varrho\}'$. The proof of the following simple lemma exemplifies methods used here in dealing with infinite-dimensional spaces.

2.1.2. Lemma. *The stability subgroup $\mathfrak{U}_\varrho$ is a Lie subgroup of $\mathfrak{U}$, [39, 40].* ♣

Proof. We shall prove that the Banach subspace $\mathfrak{M}_\varrho \subset \mathcal{L}(\mathcal{H})_s$ has a topological complement [41] in $\mathcal{L}(\mathcal{H})_s$, i.e. $\mathcal{L}(\mathcal{H})_s = \mathfrak{M}_\varrho \oplus \mathfrak{N}_\varrho \equiv$ the topological direct sum with a Banach subspace $\mathfrak{N}_\varrho$ of $\mathcal{L}(\mathcal{H})_s$. Let ϱ be expressed in the form (2.1.3). We shall use also the projection E_0 corresponding to the eigenvalue λ= 0, hence always $\sum_{j\geq 0} E_j = I$. Let $\mathbf{p_\varrho : y \mapsto p_\varrho(y) := \sum_{j\geq 0} E_j y E_j}$ be a projection of $\mathcal{L}(\mathcal{H})_s$ onto $\mathfrak{M}_\varrho$ defined by the strongly convergent series. One has

$$\|\mathrm{p}_\varrho(\mathrm{y})\| \leq \sup_j \|E_j \mathrm{y} E_j\| \leq \|\mathrm{y}\|, \quad \forall \mathrm{y} \in \mathcal{L}(\mathcal{H})_s,$$

hence the projection p_ϱ is continuous, what implies [41, Chap.I. §1.8 Proposition 12] the complementability of $\mathfrak{M}_\varrho$. The Lie group $\mathfrak{U}$ can be modeled (as a manifold) by its Lie algebra $i\mathcal{L}(\mathcal{H})_s$ via the inverse of the exponential mapping [39, Chap. III.6.4. Theorem 4], and the subgroup $\mathfrak{U}_\varrho$ is modeled via the same mapping by the complementable subspace $i\mathfrak{M}_\varrho \subset \mathcal{L}(\mathcal{H})_a$. This gives the result ([39, Chap.III.§1.3], [40, 5.8.3]). □

2.1.3. Definitions. *Let $\varrho = \sum \lambda_j E_j$ be a density matrix, $\sum_{j\geq 0} E_j = I$, as above.*

(i) Let $\mathbf{q_\varrho : \mathcal{L}(\mathcal{H})_s \to \mathfrak{N}_\varrho}$ be the **complementary projection** *to p_ϱ, $\mathrm{q}_\varrho(\mathrm{b}) = \mathrm{b} - \mathrm{p}_\varrho(\mathrm{b})$ (cf. proof of Lemma 2.1.2):*

$$\mathrm{q}_\varrho(\mathrm{b}) := \sum_{j\neq k} E_j \mathrm{b} E_k \quad \textit{for } \mathrm{b} \in \mathcal{L}(\mathcal{H})_s, \tag{2.1.4}$$

which leaves $\mathfrak{T}_s$ invariant; the sum is here strongly (resp., in $\mathfrak{T} \ni \mathrm{b}$, trace-norm-) convergent. We shall define $\mathrm{q}_\varrho(X)$ also for unbounded $X = X^*$ by the formula (2.1.4) with $\mathrm{b} := X$ for those ϱ for which it is unambiguously defined (i.e. the expressions in the sum and its strong limit exist).

(ii) Let $\mathrm{ad}^* : \mathcal{L}(\mathcal{H})_s \to \mathcal{L}(\mathcal{L}(\mathcal{H})_s^*)$ be defined by $\mathbf{ad^*(y) : \nu \mapsto ad^*(y)\nu}$:

$$[\mathrm{ad}^*(\mathrm{y})\nu](\mathrm{z}) := i\nu([\mathrm{y},\mathrm{z}]), \ (\forall \mathrm{y},\mathrm{z} \in \mathcal{L}(\mathcal{H})_s, \nu \in \mathcal{L}(\mathcal{H})_s^*). \tag{2.1.5}$$

We can see that the space $\mathfrak{T}_s$ is invariant with respect to all operators $\mathrm{ad}^*(\mathrm{y})$, $\mathrm{y} \in \mathcal{L}(\mathcal{H})_s$, [196, Proposition 3.6.2].

(iii) Let $\varrho \in \mathcal{O}_\nu(\mathfrak{U})$. Let us denote $\boldsymbol{T_\varrho\mathcal{O}(\mathfrak{U}) := T_\varrho\mathcal{O}_\nu(\mathfrak{U})} :=$ $\{\mathrm{c} \in \mathfrak{T}_s : \mathrm{c} = i[\varrho,\mathrm{b}], \mathrm{b} \in \mathcal{L}(\mathcal{H})_s\}$ *the set of vectors in $\mathfrak{T}_s$ tangent to the curves $c_\mathrm{b} : t \mapsto c_\mathrm{b}(t) := Ad^*(\exp(-it\mathrm{b}))\varrho$ at ϱ, $\mathrm{b} \in \mathcal{L}(\mathcal{H})_s$; these curves cover a neighbourhood of ϱ on the orbit $\mathcal{O}_\varrho(\mathfrak{U})$=$\mathcal{O}_\nu(\mathfrak{U}) \subset \mathfrak{T}_s$. We shall also denote $\mathrm{ad}^* : \mathcal{L}(\mathcal{H})_s \to T_\varrho\mathcal{O}(\mathfrak{U})$, $\mathrm{b} \mapsto \mathbf{ad^*_\varrho(b) := ad^*(b)\varrho} \equiv i[\varrho,\mathrm{b}] \in \mathfrak{T}_s$ for $\varrho \in \mathfrak{T}_s$. One can easily check that q_ϱ leaves $T_\varrho\mathcal{O}(\mathfrak{U})$ pointwise invariant, i.e. $T_\varrho\mathcal{O}(\mathfrak{U}) \subset \mathfrak{N}_\varrho$.*

(iv) For an arbitrary $\mathrm{c} \in \mathcal{L}(\mathcal{H})_s$, and $n \in \mathbb{Z}_+ \setminus \{0\}$, let

$$\beta_\varrho^{(n)}(\mathrm{c}) := i \sum_{j\neq k}^{\max(j;k)\leq n} E_j \mathrm{c} E_k (\lambda_k - \lambda_j)^{-1}, \tag{2.1.6}$$

where in the summation are included also the values $j = 0, k = 0$ of the indices.

Let $\beta_\varrho : T_\varrho\mathcal{O}(\mathfrak{U}) \to \mathfrak{N}_\varrho$ be the mapping

$$\beta_\varrho(\mathrm{c}) := i\sum_{j\neq k} E_j \mathrm{c} E_k(\lambda_k - \lambda_j)^{-1}. \tag{2.1.7}$$

The **limits $\boldsymbol{\beta_\varrho(c)}$ of the strongly convergent sequences** $\left\{\boldsymbol{\beta_\varrho^{(n)}(\mathrm{c}) : n \geq 1}\right\}$ *define the mapping β_ϱ. We shall define β_ϱ(*c*) in this way also for those $\mathrm{c} \in \mathcal{L}(\mathcal{H})$, as well as for those unbounded operators* c*, for which this sequence is defined and converges strongly in $\mathcal{L}(\mathcal{H})$.*

(v) Let $\|\mathrm{c}\|_\varrho := \|\beta_\varrho(\mathrm{c})\|$, where $\|\mathrm{b}\|$ denotes the operator norm of $\mathrm{b} \in \mathfrak{N}_\varrho$ in $\mathcal{L}(\mathcal{H})_s$.

(vi) Let $\mathcal{O}_\varrho(\mathfrak{U})$ be endowed with the canonical [39, Chap.III,§1.6] analytic manifold structure of the homogeneous space $\mathfrak{U}/\mathfrak{U}_\varrho$. We shall call this structure **the canonical manifold structure** *on $\mathcal{O}_\varrho(\mathfrak{U})$, and the notion of the* **manifold $\boldsymbol{\mathcal{O}_\varrho(\mathfrak{U})}$** *will mean namely the set $\mathcal{O}_\varrho(\mathfrak{U})$ endowed with this structure.* $\diamondsuit$

2.1.4. **Notes**.

(i) A direct inspection shows that β_ϱ is a linear bijection of $T_\varrho\mathcal{O}(\mathfrak{U})$ onto $\mathfrak{N}_\varrho = \mathrm{q}_\varrho(\mathcal{L}(\mathcal{H})_s)$: if $\mathrm{c} := i[\varrho, \mathrm{b}]$, with $\mathrm{b} \in \mathfrak{N}_\varrho$, then $\beta_\varrho(\mathrm{c}) = \mathrm{b}$. It is the inverse mapping to the mapping $\mathrm{ad}_\varrho^* : \mathfrak{N}_\varrho \to T_\varrho\mathcal{O}(\mathfrak{U})$; let us note that ad_ϱ^* is $\|\cdot\| \mapsto \|\cdot\|_1$- continuous, hence also $\|\cdot\| \mapsto \|\cdot\|$-continuous: $\|[\varrho, \mathrm{b}]\| \leq \|[\varrho, \mathrm{b}]\|_1 \leq 2\|\varrho\|_1\|\mathrm{b}\|$.

(ii) It is clear that $\mathrm{c} \mapsto \|\mathrm{c}\|_\varrho$ is a norm on $T_\varrho\mathcal{O}(\mathfrak{U})$. The mapping ad_ϱ^* is $\|\cdot\| \mapsto \|\cdot\|_\varrho$ isometric; the corresponding "$\|\cdot\|_\varrho$-topology" of $\mathcal{O}_\varrho(\mathfrak{U})$ is finer than the "$\|\cdot\|_1$-topology" induced by the trace-norm topology of $\mathfrak{T}_s$. $\heartsuit$

The following proposition specifies the manifold properties of the orbits $\mathcal{O}_\varrho(\mathfrak{U})$ in $\mathfrak{T}_s$ (endowed with its $\|\cdot\|_1$-topology).

2.1.5. Proposition. *Let us consider $T_\varrho\mathcal{O}(\mathfrak{U})$ as the normed space with the norm $\|\cdot\|_\varrho$. Then $T_\varrho\mathcal{O}(\mathfrak{U})$ is a B-space, and β_ϱ is a Banach space isomorphism. This B-space structure on $T_\varrho\mathcal{O}(\mathfrak{U})$ coincides with the one induced by the canonical manifold structure of $\mathcal{O}(\mathfrak{U})$ on its (equally denoted) tangent spaces $T_\varrho\mathcal{O}(\mathfrak{U})$. Furthermore, the following four statements (i) – (iv) are then equivalent:*

(i) $\varrho \in \mathcal{S}_*$ is finite-dimensional, i.e. $\varrho \in \mathfrak{F}$; we shall write also $\dim(\varrho) < \infty$ in this case.

(ii) The range $\mathfrak{N}_\varrho$ of the mapping q_ϱ coincides with $T_\varrho\mathcal{O}(\mathfrak{U})$ (considered now as a linear subspace of $\mathfrak{T}_s$).

(iii) The set $T_\varrho\mathcal{O}(\mathfrak{U})$ is a closed subspace of $\mathfrak{T}_s$.

(iv) $\mathcal{O}_\varrho(\mathfrak{U})$ is a regularly embedded [61] submanifold of $\mathfrak{T}_s$.

Moreover, one has:

(v) For $\varrho \in \mathfrak{F}$, the subspace $T_\varrho\mathcal{O}(\mathfrak{U})$ of $\mathfrak{T}_s$ is reflexive.

(vi) For any $\varrho \in \mathcal{S}_*$, $T_\varrho\mathcal{O}(\mathfrak{U})$ is dense (in the strong topology of $\mathcal{L}(\mathcal{H})_s$) in $\mathfrak{N}_\varrho := \mathrm{q}_\varrho(\mathcal{L}(\mathcal{H})_s)$. $\clubsuit$

Proof. $\mathfrak{N}_\varrho$ is a B-subspace of $\mathcal{L}(\mathcal{H})_s$, and β_ϱ is a linear isometry (hence homeomorphism) of $T_\varrho\mathcal{O}(\mathfrak{U})$ (with the norm $\|\cdot\|_\varrho$) onto $\mathfrak{N}_\varrho$, what follows directly from definitions, cf. Notes 2.1.4.

This gives the first assertion. The second one follows because of complementability of the space $\mathfrak{M}_\varrho = i\cdot Lie(\mathfrak{U}_\varrho)$, $\mathfrak{M}_\varrho \subset \mathcal{L}(\mathcal{H})_s = \mathfrak{M}_\varrho \oplus \mathfrak{N}_\varrho$, and the inverse mapping of the mapping $Ad^*(\exp(-i(\cdot)))\varrho : \mathfrak{N}_\varrho \to \mathcal{O}_\varrho(\mathfrak{U}),\ \mathrm{a} \mapsto Ad^*(\exp(-i\mathrm{a}))\varrho$, if restricted to an open neighbourhood of the zero point of $\mathfrak{N}_\varrho$, can be chosen as a chart of the manifold $\mathcal{O}_\varrho(\mathfrak{U})$.

(i)⇒(ii): If $\varrho \in \mathfrak{F}$, then (2.1.4) shows that also $\mathrm{q}_\varrho(\mathrm{a}) \in \mathfrak{F}$ for any $\mathrm{a} \in \mathcal{L}(\mathcal{H})_s$, since $\mathfrak{F}$ is an ideal in $\mathcal{L}(\mathcal{H})$. The application of the formula (2.1.7) to $\mathrm{c} := \mathrm{q}_\varrho(\mathrm{a})$ shows that $\mathrm{q}_\varrho(\mathrm{a}) = i[\varrho, \beta_\varrho(\mathrm{q}_\varrho(\mathrm{a}))] \in T_\varrho\mathcal{O}(\mathfrak{U})$. Also, q_ϱ leaves $T_\varrho\mathcal{O}(\mathfrak{U}) \subset \mathcal{L}(\mathcal{H})_s$ pointwise invariant. Hence $\mathfrak{N}_\varrho = T_\varrho\mathcal{O}(\mathfrak{U})$.

(i)⇒(iii): It follows now that for $\varrho \in \mathfrak{F}$ the set $\mathfrak{N}_\varrho$ is a subset of $\mathfrak{T}$. Since $\mathfrak{N}_\varrho$ is closed in the norm-topology of $\mathcal{L}(\mathcal{H})_s$ and on the subset $\mathfrak{T}_s \subset \mathcal{L}(\mathcal{H})_s$ the trace-topology determined by $\|\cdot\|_1$ is finer than the topology of $\mathcal{L}(\mathcal{H})_s$ determined by $\|\cdot\|$, $\|\mathrm{x}\| \leq \|\mathrm{x}\|_1$ $(\forall \mathrm{x} \in \mathfrak{T})$, it follows that $\mathfrak{N}_\varrho = T_\varrho\mathcal{O}(\mathfrak{U})$ is closed also in trace-topology, i.e. (iii).

(ii)⇒(i): Let $\varrho \notin \mathfrak{F}$. Let $e_j \equiv |e_j\rangle (j \geq 1)$ be an infinite orthonormal set in $\mathcal{H}$ such that $E_j e_j = e_j (\forall j)$, cf.(2.1.3). Let us define $\mathrm{a} \in \mathcal{L}(\mathcal{H})_s$ by the formula (in the Dirac notation, [74])

$$\mathrm{a} := \sum_{j\geq 1} \alpha_j (|e_{2j}\rangle\langle e_{2j+1}| + |e_{2j+1}\rangle\langle e_{2j}|), \quad \|\mathrm{a}\| < M < \infty. \tag{2.1.8}$$

We have $\mathrm{a} = \mathrm{q}_\varrho(\mathrm{a}) \in \mathfrak{N}_\varrho$ for any bounded real sequence $\{\alpha_j\}$, but for some choices of $\{\alpha_j\}$ (e.g.$\alpha_j \equiv 1$) one has $\mathrm{a} \notin \mathfrak{T}_s \supset T_\varrho\mathcal{O}(\mathfrak{U})$. This proves that $\mathfrak{N}_\varrho \neq T_\varrho\mathcal{O}(\mathfrak{U})$.

Let us make now a technical remark providing an alternative proof of the last statement, as well as a device to further work:

2.1.6. **Remark**. Let us chose in (2.1.8) $\alpha_j := \gamma_j(\lambda_{2j} - \lambda_{2j+1})$, where $0 < \gamma_j \to \infty$ for $j \to \infty$, but still $\sum_{j\geq 1} \gamma_j(\lambda_{2j} - \lambda_{2j+1}) < \infty$. Such a choice of strictly positive divergent sequence $\{\gamma_j\}$, for a given $\lambda_j > 0, \sum_{j\geq 1} \lambda_j = 1$, is always possible. Then $\mathrm{a} \in \mathfrak{T}_s$. Let us now calculate $\beta_\varrho^{(n)}(\mathrm{a})$ according to (2.1.6):

$$\beta_\varrho^{(2n+1)}(\mathrm{a}) = i\sum_{j\geq 1}^{n} \gamma_j \big(|e_{2j+1}\rangle\langle e_{2j}| - |e_{2j}\rangle\langle e_{2j+1}|\big). \tag{2.1.9}$$

Due to divergence of $\{\gamma_j\}$, it is clear that the result "$\beta_\varrho(\mathrm{a})$" diverges for $n \to \infty$, i.e. we can obtain in this way at the best an unbounded operator. This shows that our $\mathrm{a} \notin T_\varrho\mathcal{O}(\mathfrak{U})$, although it is still in $\mathfrak{T}_s$. This is another proof of the inequality $\mathfrak{N}_\varrho \neq T_\varrho\mathcal{O}(\mathfrak{U})$, because $\beta_\varrho : T_\varrho\mathcal{O}(\mathfrak{U}) \to \mathfrak{N}_\varrho$ is $\|\cdot\|_1 \to \|\cdot\|$–continuous. ♡

(vi): Since the sequence $\{i[\varrho, \beta_\varrho^{(n)}(\mathrm{a})] : n \geq 1\} \subset T_\varrho\mathcal{O}(\mathfrak{U})$ converges strongly to $\mathrm{a} \equiv \mathrm{q}_\varrho(\mathrm{a})$ $(\forall \mathrm{a} \in \mathfrak{N}_\varrho := \mathrm{q}_\varrho(\mathcal{L}(\mathcal{H})_s))$, it is seen that $T_\varrho\mathcal{O}(\mathfrak{U})$ (considered as a subspace of $\mathfrak{T}_s \subset \mathcal{L}(\mathcal{H})_s$) is strongly dense in $\mathfrak{N}_\varrho$. This proves (vi).

(iii)⇒(i): Let us chose $\mathrm{a} \in \mathfrak{N}_\varrho \cap \mathfrak{T}_s \setminus T_\varrho\mathcal{O}(\mathfrak{U})$. The preceding considerations also show that $T_\varrho\mathcal{O}(\mathfrak{U})$ is not closed in $\mathfrak{T}_s$ if $\varrho \notin \mathfrak{F}$; namely, according to the Remark 2.1.6, and the formula (2.1.9), one can choose $\mathrm{a} \in \mathfrak{T}_s \cap \mathfrak{N}_\varrho$ such that the sequence $\{\|i[\varrho, \beta_\varrho^{(n)}(\mathrm{a})] - \mathrm{a}\|_1 : n \geq 1\}$ converges to zero. This means that the sequence $i[\varrho, \beta_\varrho^{(n)}(\mathrm{a})] \in T_\varrho\mathcal{O}(\mathfrak{U})$ converges to $\mathrm{a} \notin T_\varrho\mathcal{O}(\mathfrak{U})$.

(iv)⇔(iii): The restriction of the projection $\mathrm{q}_\varrho : \mathfrak{T}_s \to \mathfrak{T}_s$ is continuous also in the trace-norm topology, what follows from continuity of p_ϱ in that topology: For positive operators $\mathrm{c} \in \mathfrak{T}$, all $E_j \mathrm{c} E_j \geq 0$, hence

$$\|\mathrm{p}_\varrho(\mathrm{c})\|_1 = Tr(\sum_j E_j \mathrm{c} E_j) = \sum_j Tr(E_j \mathrm{c} E_j) = \sum_j Tr(E_j \mathrm{c}) = Tr(\mathrm{c}) = \|\mathrm{c}\|_1,$$

and the continuity of p_ϱ follows. The equivalence of the norms $\|\cdot\|_\varrho$ and $\|\cdot\|_1$ on $T_\varrho\mathcal{O}(\mathfrak{U})$ in the case of $\varrho \in \mathfrak{F}$ can be shown as follows: Let $\mathrm{c} := i[\varrho, \mathrm{b}] \in T_\varrho\mathcal{O}(\mathfrak{U}) \Rightarrow \mathrm{q}_\varrho(\mathrm{b}) = \beta_\varrho(\mathrm{c})$, and from (2.1.7) and the definition of the norm $\|\cdot\|_\varrho$ we obtain

$$\|\mathrm{c}\|_\varrho \leq \left(\sum_{j \neq k} |\lambda_j - \lambda_k|^{-1} \right) \|\mathrm{c}\|_1, \tag{2.1.10}$$

where the sum is taken over a finite index set. The opposite inequality is obtained by the known property of the trace-norm:

$$\|\mathrm{c}\|_1 \equiv \|[\varrho, \mathrm{q}_\varrho(\mathrm{b})]\|_1 \leq 2\|\mathrm{q}_\varrho(\mathrm{b})\| \|\varrho\|_1 = 2\|\mathrm{c}\|_\varrho,$$

since $\mathrm{q}_\varrho(\mathrm{b}) = \beta_\varrho(\mathrm{c})$, and $\|\varrho\|_1 = 1$. This fact, and the derived implications of finite dimensionality of $\varrho \in \mathfrak{F}$ give the validity of the assertion (iv). It is clear that (iv) cannot be true if (iii) were not valid.

(v) If $\varrho \in \mathfrak{F}$, then the B-space $T_\varrho\mathcal{O}(\mathfrak{U})$ is a Hilbert space, cf. Theorem 2.1.19, hence $T_\varrho\mathcal{O}(\mathfrak{U})$ is reflexive. □

The proved Proposition 2.1.5 shows, that only finite-dimensional density matrices ϱ's generate $\mathrm{Ad}^*(\mathfrak{U})$ orbits with mathematically convenient properties: Their tangent spaces are in the $\mathfrak{T}_s$-induced topology closed and reflexive. This has important consequences for the following theoretical implications. Hence, we ask the question, whether it would be possible to restrict our attention, in some appropriate sense, to these "finite dimensional orbits", and simultaneously not to loose the control on the whole space $\mathcal{S}_*$. The next lemma indicates, that it might be possible.

2.1.7. Lemma. *The set-union of the orbits $\{\mathcal{O}_\varrho(\mathfrak{U}) : \varrho \in \mathfrak{F}\}$ is a dense subset of $\mathcal{S}_* \subset \mathfrak{T}_s$, in the norm-topology of $\mathfrak{T}_s$.* ♣

Proof. Any density matrix $\varrho \in \mathcal{S}_*$ is approximated in $\|\cdot\|_1$ by finite dimensional ones, what is seen, e.g. from its spectral resolution:

$$\varrho = \sum_j \lambda_j E_j = \|\cdot\|_1 - \lim_{n\to\infty} \kappa_n \sum_{j=1}^{n} \lambda_j E_j, \text{ with } \kappa_n := \left(\sum_{k=1}^{n} \lambda_k \dim(E_k) \right)^{-1}.$$

□

2.1-c Poisson structure on quantum state-space

We shall consider the set $\mathcal{S}_*$ ($\subset \mathcal{L}(\mathcal{H})^*$) as the set of relevant physical states in the following considerations, i.e. **the quantum phase space** will mean for us the set of **normal states**.[54]

Let us now introduce a Poisson structure [177, 274], [7, Appendix 13] on the linear space $\mathfrak{T}_s$ containing $\mathcal{S}_*$ as a bounded convex subset. The Poisson structure will allow us to ascribe (Poisson-) Hamiltonian vector fields (on $\mathfrak{F}$, at least) with the corresponding flows leaving the state space $\mathcal{S}_*$ invariant.

It will be useful to use, in the following mathematical formulations, the standard differential calculus on Banach manifolds [234, 40, 61] based on the Fréchet differential calculus of mappings between (linear) Banach spaces [58, 235, 61].[55]
If the Fréchet derivative of a function $f : \mathfrak{T} \to \mathbb{R}$ exists, then there exists also directional (so called Gateaux) derivative:

$$D_\nu f(\omega) = \lim_{t\to 0} \frac{1}{t} \left[f(\nu + t\omega) - f(\nu) \right], \ \forall \omega \in \mathfrak{T}. \tag{2.1.11}$$

Conversely, if the Gateaux derivative (2.1.11) exists in a neighbourhood U of a point $\nu \in \mathfrak{T}$, and *if it is continuous linear, continuously depending on* $\nu \in U$, $D.f : U \to \mathcal{L}(\mathfrak{T}, \mathbb{R})$, then also the Fréchet derivative (A.2.1a) exists [58].

We shall be mainly interested, in the following text, in the $Ad^*(\mathcal{U})$-invariant subset $\mathcal{S}_*$ consisting of normal states on $\mathcal{L}(\mathcal{H})$. Let $\mathcal{F} := \mathcal{F}(\mathfrak{T}_s)$ be the set of infinitely norm-differentiable real functions on $\mathfrak{T}_s$, with its trace-norm $\|\cdot\|_1$. Let us denote by $\mathcal{F}(\mathcal{B})$ the set of the restrictions of $f \in \mathcal{F}$ to some subset $\mathcal{B}$ of $\mathfrak{T}_s$.

2.1.8. **Remark**. Noncompactnes of $\mathcal{S}_*$ allows, e.g. that $\mathcal{F}(\mathcal{S}_*)$ contains also unbounded functions on $\mathcal{S}_*$, e.g. any $f \in \mathcal{F}$ with the restriction $f : \varrho \mapsto f(\varrho) := \ln(Tr(\varrho^2))$ for $\varrho \in \mathcal{S}_* \subset \mathfrak{T}_s$ is unbounded. Put, e.g., with orthonormal e_j's, $\varrho_N := \sum_{j=1}^N \frac{1}{N} |e_j\rangle\langle e_j|$, whence $Tr\varrho_N^2 = \frac{1}{N}$, $\lim_{N\to\infty} \ln Tr(\varrho_N^2) = -\infty$. ♡

The definition of the F-derivative and its expression (2.1.11) also apply to $f \in \mathcal{F}$, and the notation $D_\nu f$ will not lead to any ambiguity for $f \in \mathcal{F}$.

We shall often work with infinite–dimensional manifolds modelled by Banach (specifically, e.g., in the case of pure state manifold $P(\mathcal{H})$, or of any $\mathcal{O}_\varrho(\mathfrak{U})$ with $\dim \varrho < \infty$, by Hilbert) spaces, cf. Appendix A.3. The main ideas, and many of general constructions and theorems work in that cases similarly as in the case of more common finite dimensional manifolds. We shall point out differences in specific cases, if it will be needed. In the case of the linear manifold $\mathfrak{T}_s$, and for a differentiable function $f \in \mathcal{F}$, the derivative $D_\nu f$ belongs to the cotangent space

[54] It might be mathematically interesting, and, perhaps, also physically useful, to formulate analogies of the following constructions on the space $\mathcal{S}$ of all positive normalized functionals on $\mathcal{L}(\mathcal{H})$. This leads to technical complications and, for purposes of our physical interpretations, it would be unnecessary. cf. also [24], where a (heuristic) trial for such a formulation was presented. A nice and useful property of $\mathcal{S}$ is its compactness in the w^*-topology, what is not the case of $\mathcal{S}_*$.

[55] Let us note, for a preliminary information, that in this infinite–dimensional differential calculus "most" of the usual differential operations in finite–dimensional spaces remain formally, under certain conditions, unchanged: the differential is the "linear part of difference", where should be used the Banach-norm limit for its definition. The rules for writing the Taylor expansion, differential of composed maps, for calculation of derivatives of "products" etc. have the same formal expressions as in finite–dimensional case, see also the Appendix A.2.

$T^*_\nu(\mathfrak{T}_s)= \mathcal{L}(\mathcal{H})$, and we shall deal with it also as with an operator in the sense of this canonical isomorphism.

2.1.9. Definitions.

(i) Let $\mathcal{F}_\varrho$ denote the algebra $C^\infty(\mathcal{O}_\varrho(\mathfrak{U}), \mathbb{R})$ of functions on the manifold $\mathcal{O}_\varrho(\mathfrak{U})$. The restrictions of functions from $\mathcal{F}$ to $\mathcal{O}_\varrho(\mathfrak{U})$ belong to $\mathcal{F}_\varrho$, because the topology on the manifold $\mathcal{O}_\varrho(\mathfrak{U})$ is finer than the relative topology coming from $\mathfrak{T}_s$ (cf. proof of Proposi- tion 2.1.5).

(ii) The mapping from $\mathcal{F} \times \mathcal{F}$ to $\mathcal{F} : (f; h) \mapsto \{f, h\}$, where

$$\{f, h\}(\nu) := \nu(i[D_\nu f, D_\nu h]), \quad \nu \in \mathfrak{T}_s, \tag{2.1.12}$$

will be called the **Poisson structure** *on $\mathfrak{T}_s$. The function $\{f, h\} \in \mathcal{F}$ is the* **Poisson bracket** *of the functions f and h from $\mathcal{F}$.*

(iii) The functions $h_{\mathrm{y}} \in \mathcal{F}$ ($\mathrm{y} \in \mathcal{L}(\mathcal{H})_s$) are defined by $h_{\mathrm{y}}(\nu) := \nu(\mathrm{y}) \equiv Tr(\nu\mathrm{y}),\ \forall \nu \in \mathfrak{T}_s$. Then $D_\nu h_{\mathrm{y}} = \mathrm{y}$, the second derivative $D^2_\nu h_{\mathrm{y}} = 0$, and the Poisson bracket of two such functions is

$$\{h_{\mathrm{x}}, h_{\mathrm{y}}\}(\nu) = i\,\nu([\mathrm{x}, \mathrm{y}]) \equiv h_{i[\mathrm{x},\mathrm{y}]}(\nu). \tag{2.1.13}$$

From this we obtain Poisson brackets for polynomials in functions h_{x} ($\mathrm{x} \in \mathcal{L}(\mathcal{H})_s$) with a help of derivation property (cf. Proposition 2.1.10), in accordance with (2.1.12). $\diamondsuit$

The space $\mathfrak{T}_s$ can be considered as an infinite–dimensional manifold with the atlas consisting of one chart determined by the identity mapping on $\mathfrak{T}_s$. Then the tangent space $T_\nu\mathfrak{T}_s$ to $\mathfrak{T}_s$ at each point ν will be canonically identified with the vector space $\mathfrak{T}_s$ itself. The space $\mathcal{L}(\mathcal{H})_s$ is then canonically identified with $T^*_\nu\mathfrak{T}_s$. In this interpretation, we can also consider the derivative (cf. Appendix A.2) $D_\nu f \in T^*_\nu\mathfrak{T}_s$ as differential of $f \in \mathcal{F}$ on the manifold $\mathfrak{T}_s$, as it is used in differential geometry. The usual symbol $d_\nu f$ will be used, however, to stress the restriction of the differentiation to some "smaller" manifold in $\mathfrak{T}_s$. For a real function f continuously differentiable as a function on the manifold $\mathcal{O}_\varrho(\mathfrak{U})$ we shall denote by $d_\nu f$ the differential of f in the point ν on the orbit $\mathcal{O}_\varrho(\mathfrak{U})$ ($\ni \nu$). We shall also identify $d_\varrho f := \mathrm{q}_\varrho(D_\varrho f) \in \mathfrak{N}_\varrho \subset \mathcal{L}(\mathcal{H})_s$ considered as an element of the cotangent space $T^*_\varrho\mathcal{O}(\mathfrak{U}) := (T_\varrho\mathcal{O}_\nu(\mathfrak{U}))^* := T^*_\varrho\mathcal{O}_\nu(\mathfrak{U})$; this identification (resp. representation) of the cotangent space is possible due to the identities:

$$\begin{aligned} &d_\varrho f(c) := Tr(c\mathrm{q}_\varrho(D_\varrho f)) = i\,Tr([\varrho, \beta_\varrho(c)]\mathrm{q}_\varrho(D_\varrho f)) = \\ &i\,Tr(\beta_\varrho(c)[\mathrm{q}_\varrho(D_\varrho f), \varrho]) = i\,Tr(\beta_\varrho(c)[D_\varrho f, \varrho]) = \\ &i\,Tr([\varrho, \beta_\varrho(c)]D_\varrho f) = Tr(cD_\varrho f), \text{ for all } c \in T_\varrho\mathcal{O}(\mathfrak{U}). \end{aligned} \tag{2.1.14}$$

The operator $d_\varrho f$ represents the pull-back of $D_\varrho f \in T^*_\varrho\mathfrak{T}_s$ with respect to the embedding of $\mathcal{O}_\varrho(\mathfrak{U})$ into $\mathfrak{T}_s$, if $\varrho \in \mathfrak{F}$. Now we can write the Poisson bracket in the form:

$$\{f, h\}(\nu) = i\nu([d_\nu f, d_\nu h]). \tag{2.1.15}$$

The form (2.1.15) shows, that the value of the Poisson bracket (2.1.12) in a point $\nu \in \mathfrak{T}_s$ depends on the restrictions of the functions $f, h \in \mathcal{F}$ onto the orbit $\mathcal{O}_\nu(\mathfrak{U})$ only. This is due to the fact, that the orbits $\mathcal{O}_\varrho(\mathfrak{U})$ are the "symplectic leaves" of the Poisson manifold $\mathfrak{T}_s$, [274], as will be seen from the following. The orbits are **Poisson submanifolds** [274] of the Poisson manifold $\mathfrak{T}_s$. We shall now prove that (2.1.12) really determines a structure of a **Poisson manifold** on the Banach manifold $\mathfrak{T}_s$:

2.1.10. Proposition. *The Poisson bracket from* (2.1.12) *has all the general properties of the Poisson structure [274, 7] (coinciding with that of Hamiltonian classical mechanics, except of nondegeneracy), i.e. for all $f, h, k \in \mathcal{F}$, and all $\lambda \in \mathbb{R}$ one has:*

$$\begin{array}{lll} (i) & \{f,h\} = -\{h,f\}; & \textit{(antisymmetry)} \\ (ii) & \{f,h+\lambda k\} = \{f,h\} + \lambda\{f,k\}; & ((ii)\&(i) \Rightarrow \textit{bilinearity}) \\ (iii) & \{f,hk\} = \{f,h\}k + h\{f,k\}; & \textit{(derivation property)} \\ (iv) & \{f,\{h,k\}\} + \{h,\{k,f\}\} + \{k,\{fh\}\} = 0; & \textit{(Jacobi identity)} \ \clubsuit \end{array}$$

Proof. The first three properties are immediate consequences of Definitions 2.1.9, cf. also (2.1.16). The validity of (iv) follows immediately from (2.1.13) and from the properties (i) - (iii) for such functions f, h, k which have form of polynomials in the specific type of functions $h_{\mathrm{a}} \in \mathcal{F},\ \mathrm{a} \in \mathcal{L}(\mathcal{H})$, (2.1.13). For general f, h, k one can prove (iv) directly as follows:
Let us first express $D_\nu\{h,k\} \in \mathcal{L}(\mathcal{H})_s$ according to (2.1.11), (2.1.12),

$$\begin{aligned} \omega(D_\nu\{h,k\}) &= \left.\frac{d}{dt}\right|_{t=0} (\nu + t\omega)(i\,[D_{\nu+t\omega}h, D_{\nu+t\omega}k]) \\ &= \omega(i\,[D_\nu h, D_\nu k]) + \nu(i\,[D_\nu^2 h(\omega,\cdot), D_\nu k]) + \nu(i\,[D_\nu h, D_\nu^2 k(\omega,\cdot)]), \end{aligned}$$

where the second derivatives in any point ν are symmetric bilinear $\|\cdot\|_1$-continuous functions on $\mathfrak{T}_s$. Hence, the linear mapping $D_\nu^2 k(\omega,\cdot) : \varrho \mapsto D_\nu^2 k(\varrho,\omega) = D_\nu^2 k(\omega,\varrho) \equiv \varrho(D_\nu^2 k(\omega,\cdot))$ can be (and is here) considered as an element of $\mathcal{L}(\mathcal{H})_s$. We need to calculate $\{f,\{h,k\}\}(\nu) := i\,\nu([D_\nu f, D_\nu\{h,k\}])$. With a help of the notation (2.1.5) and of the above derived formula for $D_\nu\{h,k\}$ we obtain

$$\begin{aligned} \{f,\{h,k\}\}(\nu) &= [\mathrm{ad}^*(D_\nu f)\nu](D_\nu\{h,k\}) \\ &= -\nu([D_\nu f,[D_\nu h, D_\nu k]]) - [\mathrm{ad}^*(D_\nu k)\nu](D_\nu^2 h(\mathrm{ad}^*(D_\nu f)\nu,\cdot)) + \\ &\quad [\mathrm{ad}^*(D_\nu h)\nu](D_\nu^2 k(\mathrm{ad}^*(D_\nu f)\nu,\cdot)) \\ &= -\nu([D_\nu f,[D_\nu h, D_\nu k]]) - D_\nu^2 h(\mathrm{ad}^*(D_\nu f)\nu, \mathrm{ad}^*(D_\nu k)\nu) + \\ &\quad D_\nu^2 k(\mathrm{ad}^*(D_\nu f)\nu, \mathrm{ad}^*(D_\nu h)\nu). \end{aligned}$$

From the symmetry of second derivatives, and from validity of Jacobi identity for commutators of operators in $\mathcal{L}(\mathcal{H})$, we obtain the result. □

2.1-d Hamiltonian vector fields and flows

In the case of a finite–dimensional Poisson manifold M, the Poisson structure determines a vector field $\mathbf{v}_f$ to each differentiable function f on M, so called **Hamiltonian vector field** corresponding to the **Hamiltonian function** f:

$$\pounds_{\mathbf{v}_f}(h) \equiv dh(\mathbf{v}_f) := \{f,h\}, \tag{2.1.16}$$

where $\pounds_{\mathbf{v}}$ denotes the **Lie derivative** (uniquely extendable to a derivation of any tensor field on M, [151, 40]) with respect to the vector field **v**: The Poisson bracket $\{f,h\}(\nu)$ at fixed f and ν is a differential operator on real valued functions differentiable at ν, which determines unique - in the case of finite–dimensional M - vector $\mathbf{v}_f \in T_\nu\mathrm{M}$.

In the case of infinite–dimensional manifolds, the relation between (first order) differential operators and tangent vectors is not always an isomorphism of normed spaces, [61]. The following lemma is, however, valid, [61, Chapter VII.A.1]:

2.1.11. Lemma. *Let M be a manifold modeled by a Banach space E, hence the tangent spaces $T_mM,\ m \in M$, are isomorphic to E. Let us assume that E is reflexive: E = E** (:= the double topological dual of E). Let a first order differential operator (i.e. satisfying the Leibniz rule) $\Delta : \mathcal{F}(U) \to \mathcal{F}(U),\ f \mapsto \Delta f,\ U \subset M$ (with domain U of a chart $(U;\varphi;E)$ containing $m \in M$), satisfy the following inequality for a $K > 0$:*

$$|\Delta f(m)| \leq K \, \|D_{\varphi(m)}(f \circ \varphi^{-1})\|_{E^*}. \tag{2.1.17}$$

Then the operator $\hat{\Delta}_m : f \mapsto \Delta f(m)$ can be identified with the vector $\Delta_m \in T_mM \cong E^{**}$: $\Delta_m(d_m f) := \Delta f(m)$. ♣

Proof. The equation (2.1.17) shows that the kernel of the operator $\hat{\Delta}_m$ contains the kernel of $d_m f$, and also is bounded. Hence, it is defined as bounded linear functional on T^*_mM ($\ni d_m f \ \forall f \in \mathcal{F}(U)$), i.e. as an element of $(T^*_mM)^* \cong E^{**} = E$. □

Let us check validity of (2.1.17) for the Poisson bracket $\hat{\Delta}_\nu(\cdot) := \{h, \cdot\}(\nu)$:

$$|\{h, f\}(\nu)| \leq 2\|\nu\|_1 \, \|D_\varrho h\|_{\mathcal{L}(\mathcal{H})} \, \|D_\varrho f\|_{\mathcal{L}(\mathcal{H})}.$$

Reflexivity of the tangent spaces $T_\nu\mathcal{O}(\mathfrak{U})$ is the case for "finite–dimensional" orbits $\mathcal{O}_\nu(\mathfrak{U})$, cf. Proposition 2.1.5(v).

2.1.12. **Remark**. The considerations preceding (2.1.15) show, that the Poisson bracket $\{h, f\}$ in a point $\nu \in \mathfrak{F} \cap \mathcal{S}_*$ can be calculated with a help of restrictions $h\rceil_{\mathcal{O}_\nu(\mathfrak{U})}, f\rceil_{\mathcal{O}_\nu(\mathfrak{U})}$ only, cf. (2.1.15), i.e. the orbits $\mathcal{O}_\varrho(\mathfrak{U})$ are themselves Poisson manifolds regularly embedded into $\mathfrak{T}_s$, and this embedding is a Poisson morphism [274]. ♡

Let us restrict our attention, for a while, to "finite–dimensional" orbits $\mathcal{O}_\nu(\mathfrak{U})$. From the Lemma 2.1.11 and the above mentioned facts we can see that to any $f \in \mathcal{F}$ there is associated, for any $\nu \in \mathfrak{F}$, the **Hamiltonian vector field** $\mathbf{v}_f$ on $\mathcal{O}_\nu(\mathfrak{U})$, $\mathbf{v}_f(\varrho) \in T_\varrho\mathcal{O}_\nu(\mathfrak{U})$, expressed by

$$\mathbf{v}_f(\varrho) = \mathrm{ad}^*_\varrho(d_\varrho f) = \mathrm{ad}^*_\varrho(D_\varrho f). \tag{2.1.18}$$

Note that $d_\varrho f = \mathrm{q}_\varrho(D_\varrho f) \in \mathfrak{F}$ for all $\varrho \in \mathcal{O}_\nu(\mathfrak{U})$, and that $D_\nu f \in \mathcal{L}(\mathcal{H})_s$, hence the unitary group

$$\mathrm{u}_{f,\nu} : t \mapsto \mathrm{u}_{f,\nu}(t) := \exp(-itD_\nu f)$$

generates a curve on $\mathcal{O}_\nu(\mathfrak{U}) = \mathcal{O}_\varrho(\mathfrak{U})$ determining $\mathbf{v}_f(\nu)$:

$$d_\nu h(\mathbf{v}_f) = \left.\frac{d}{dt}\right|_{t=0} h(Ad^*(\mathrm{u}_{f,\nu}(t))\nu). \tag{2.1.19}$$

This again indicates the "usual" (i.e. as in finite–dimensions) connection between differentiable curves and tangent vectors $\mathbf{v}_f(\nu) \in T_\nu\mathcal{O}_\varrho(\mathfrak{U})$.

2.1.13. **Notes**.

(i) Each element of $\mathcal{L}(\mathcal{H})_s$ can be written in the form $D_\nu f$ for some smooth real-valued function $f \in \mathcal{F}$: For a given $\mathrm{b} \in \mathcal{L}(\mathcal{H})_s$ one can chose $f(\nu) := Tr(\mathrm{b}\nu)$; then $D_\nu f = \mathrm{b}$.

(ii) The reflexivity of $T_\varrho\mathcal{O}(\mathfrak{U})$, for $\varrho \in \mathfrak{F}$, implies that each vector $\mathbf{v} \in T_\varrho\mathcal{O}(\mathfrak{U})$ is of the form (2.1.18) for some $D_\varrho f \in \mathcal{L}(\mathcal{H})_s$.

(iii) Although the Hamiltonian vector fields were defined on orbits $\mathcal{O}_\nu(\mathfrak{U})$ for $\nu \in \mathfrak{F}$ only, they are extendable by (2.1.18) to the whole space $\mathfrak{T}_s$:

$$\mathbf{v}_f : \mathfrak{T}_s \to \mathfrak{T}_s,\ \nu \mapsto \mathbf{v}_f(\nu) := \mathrm{ad}^*_\nu(D_\nu f). \tag{2.1.20}$$

Since

$$\|\mathbf{v}_f(\nu)\|_1 = \|[\nu, D_\nu f]\|_1 \leq 2\,\|\nu\|_1\,\|D_\nu f\|,$$

and the function $\nu \mapsto D_\nu f$ is infinitely (continuously) differentiable, the uniqueness of the extension of $\mathbf{v}_f$ to $\mathfrak{T}_s$ follows from the density of $\mathfrak{F}$ in $\mathfrak{T}$. $\heartsuit$

2.1.14. Definition. *Let $f \in \mathcal{F},\ \nu \in \mathfrak{T}_s$, and let $\mathbf{v}_f(\nu) \in T_\nu\mathcal{O}_\nu(\mathfrak{U}) \subset T_\nu\mathfrak{T}_s$ be determined by equation* (2.1.18)*. The smooth vector field $\nu \mapsto \mathbf{v}_f(\nu)$ is called the* **Hamiltonian vector field** *on $\mathfrak{T}_s$.* $\diamondsuit$

Now we could proceed further also with the Hamiltonian vector fields $\mathbf{v}_f$ restricted to "finite–dimensional" orbits $\mathcal{O}_\varrho(\mathfrak{U})$ being the Hamiltonian vector fields on Poisson manifolds $\mathcal{O}_\varrho(\mathfrak{U})$, $\varrho \in \mathfrak{F}$.

Each $\mathbf{v}_f$ from (2.1.18) ($f \in \mathcal{F}$) determines a differential equation [40] on the infinite–dimensional Banach manifold $\mathfrak{T}_s$ with a **maximal solution $\tilde{\varphi}^f$**, $\tilde{\varphi}^f(t, \varrho) \in \mathfrak{T}_s$, defined on an open domain in $\mathbb{R} \times \mathfrak{T}_s \ni (t; \varrho)$ containing $\{0\} \times \mathfrak{T}_s$. For values of t_j's for which the objects entering into (2.1.21) are defined, the formula

$$\tilde{\varphi}^f(t_1 + t_2, \varrho) \equiv \tilde{\varphi}^f(t_2, \tilde{\varphi}^f(t_1, \varrho)) \tag{2.1.21}$$

is satisfied. If the domain is the whole $\mathbb{R} \times \mathfrak{T}_s$, what means that the **vector field $\mathbf{v}_f$ on $\mathfrak{T}_s$ is complete**, we obtain a one-parameter group of diffeomorphisms $\tilde{\varphi}^f_t$ ($t \in \mathbb{R}$) of $\mathfrak{T}_s$:

$$\tilde{\varphi}^f_t(\varrho) := \tilde{\varphi}^f(t, \varrho) \text{ for all } t \in \mathbb{R}, \varrho \in \mathfrak{T}_s. \tag{2.1.22}$$

We shall now express the (local) flow $\tilde{\varphi}^f_t$, i.e. the solution of Hamilton's equations (obtained by combining (2.1.16) and (2.1.19), or (2.1.12)), in a form of Schrödinger (resp. Dyson) equation.

2.1.15. Proposition. *Let $\nu \in \mathfrak{T}_s,\ f \in \mathcal{F}, \nu(t) := \tilde{\varphi}^f_t(\nu)$ for t in an open interval $J_\nu \subset \mathbb{R}$ containing zero. Let we represent the differentials $D_\nu f$, resp. $d_\nu f$ of $f \in \mathcal{F}$ by operators in $\mathcal{L}(\mathcal{H})$, e.g., as above: $d_\nu f := \mathrm{q}_\nu(D_\nu f)$. Let us consider the equation*

$$i\,\frac{d}{dt}\mathrm{u}_f(t, \nu) = d_{\nu(t)}f \cdot \mathrm{u}_f(t, \nu), \tag{2.1.23}$$

where $d_{\nu(t)}f\cdot$ denotes the (left) multiplication in the algebra $\mathcal{L}(\mathcal{H})_s$. The equation (2.1.23)*, with the initial condition $\mathrm{u}_f(0,\nu)\equiv I_{\mathcal{H}}$, has a unique (unitary) solution $t\mapsto \mathrm{u}_f(t,\nu)\in\mathcal{L}(\mathcal{H}),\ t\in J_\nu,\ \nu\in\mathfrak{T}_s$. This solution satisfies the "cocycle identity"*

$$\mathrm{u}_f(t+s,\nu)=\mathrm{u}_f(s,\tilde\varphi_t^f\nu)\mathrm{u}_f(t,\nu) \tag{2.1.24}$$

for those $t,s\in J_\nu$, for which both sides of (2.1.24) *are defined. One has, moreover,*

$$\tilde\varphi_t^f\nu:=\tilde\varphi_t^f(\nu)=Ad^*(\mathrm{u}_f(t,\nu))\nu, \tag{2.1.25}$$

and this, together with (2.1.24) *shows fulfillment of* (2.1.21). ♣

Proof. Unique solvability of (2.1.23) on each interval $J'_\nu\subset J_\nu$ on which the function $t\to\|d_{\nu(t)}f\|$ is uniformly bounded follows from general theory of differential equations in Banach spaces, cf. [235, Chap.V.§2.Theorem 4]. Unitarity and the property (2.1.24) can be proved, e.g. by the method of the proof of [218, Theorem X.69] using the Dyson expansion, since $t\mapsto d_{\nu(t)}f$ is norm-continuous. Finally, (2.1.25) can be verified by differentiation and by the uniqueness of the local flow $\tilde\varphi^f$ of the vector field $\mathbf{v}_f$. □

2.1.16. **Notes**.

(i) The equation (2.1.23) is a generalized form of the Dyson equation known from QM, which in turn is a time-dependent generalization of Schrödinger equation. For $f(\nu)\equiv h_{\mathrm{H}}(\nu):=Tr(\nu\mathrm{H})$, with $\mathrm{H}\in\mathcal{L}(\mathcal{H})_s$, and with $\nu\in P(\mathcal{H})$, the equation reduces to the Schrödinger equation with the Hamiltonian H.

(ii) The substitution $\nu(t):=Ad^*(\mathrm{u}_f(t,\nu))\nu$ into (2.1.23) makes that equation for $\mathrm{u}_f(t,\nu)$ manifestly nonlinear. We shall see in Section 3.6 that the equation (2.1.23) can be equivalently rewritten, in the case $\nu\in P(\mathcal{H})$, into the form of the nonlinear version of QM proposed in [273], and also into the more traditional versions of "nonlinear Schrödinger equations", cf. Subsection 3.3-e.

(iii) The equation (2.1.25) shows, that the obtained form of Hamiltonian flows on "quantum phase space" $\mathfrak{T}_s$ can be expressed with a help of coadjoint action of the unitary group $\mathfrak{U}$ of the algebra $\mathcal{L}(\mathcal{H})$, hence it leaves invariant the orbits $\mathcal{O}_\nu(\mathfrak{U})$. This gives the invariance of the quantum state space $\mathcal{S}_*$, as it is formulated in the following theorem. ♡

2.1.17. Theorem. *Let $f\in\mathcal{F},\ \varrho\in\mathcal{S}_*$. Then $\mathcal{O}_\varrho(\mathfrak{U})$ is $\tilde\varphi^f$-invariant. Hence also $\mathcal{S}_*$ is $\tilde\varphi^f$-invariant.* ♣

Proof. The result follows from the relation (2.1.25) showing that $\tilde\varphi_t^f$ can be realized by the $Ad^*(\mathfrak{U})$-action, and $\mathcal{S}_*$ consists of the $Ad^*(\mathfrak{U})$-orbits $\mathcal{O}_\varrho(\mathfrak{U})$, $\varrho\in\mathcal{S}_*$. □

Let us specify non-uniqueness of cocycles (2.1.24) satisfying (2.1.25). We obtain "physically equivalent" evolution equations connected by a "gauge transformation", cf. also Section 3.6, Remark 2.1.18, and Proposition 2.3.23.

2.1.18. **Remark**. The cocycle u_f satisfying (2.1.25) is nonunique. The same evolution $\tilde\varphi^f$ is obtained also from the solutions u'_f of the equations resulting after the insertion $d_\nu f$ + $\mathbf{f}^0(\nu)$ in the place of $d_\nu f$ into (2.1.23), where $\mathbf{f}^0:\nu\mapsto\mathbf{f}^0(\nu)$ is a norm-continuous function from $\mathcal{S}_*$ (or

from the whole $\mathfrak{T}_s$) to $\mathcal{L}(\mathcal{H})_s$ with values in $\mathfrak{M}_\nu = i\cdot Lie(\mathfrak{U}_\nu)$, i.e., as an operator in $\mathcal{L}(\mathcal{H})$, the value $\mathbf{f}^0(\nu)$ commutes with the operator ν for any ν:

$$i\,\frac{d}{dt}\mathrm{u}_f(t,\nu) = \left[d_{\nu(t)}f + \mathbf{f}^0(\nu(t))\right]\cdot \mathrm{u}_f(t,\nu). \tag{2.1.26}$$

Specifically, one can use $D_\nu f = \mathrm{p}_\nu(D_\nu f) + \mathrm{q}_\nu(D_\nu f)$ instead of $d_\nu f := \mathrm{q}_\nu(D_\nu f)$ in (2.1.23). Let us mention, moreover, that the continuity requirement to the function $t \mapsto d_{\nu(t)}f + \mathbf{f}^0(\nu(t))$ in the assumptions of the Proposition 2.1.15 can be weakened: For validity of the conclusions as well as of the proof of the proposition it suffices to assume strong-operator continuity of this "time-dependent Hamiltonian" together with its locally uniform (in the parameter t) boundedness. ♡

Now we shall investigate the geometry of manifolds $\mathcal{O}_\varrho(\mathfrak{U})$ for "finite–dimensional" $\varrho \in \mathfrak{F}$, especially a naturally determined metric and symplectic structures on them. It will be seen in the Section 3.2 that the obtained structure leads to the standard symplectic, and also metric (known as the "Fubini-Study metric") structures on the space of pure quantum states $P(\mathcal{H})$, this both structures connected by complex structure coming from that in the underlying Hilbert space $\mathcal{H}$ (this is called a Kählerian structure):

2.1.19. Theorem. *Let $\dim \varrho < \infty$. Let us define a complex-valued tensor field $\Psi : \varrho \mapsto \Psi_\varrho \equiv \Gamma_\varrho - i\,\Omega_\varrho$ on the manifold $\mathcal{O}_\varrho(\mathfrak{U})$, where Γ_ϱ and Ω_ϱ are real two-covariant tensors on $T_\varrho\mathcal{O}(\mathfrak{U})$ ($\ni \mathbf{v}, \mathbf{w}$):*

$$\Psi_\varrho(\mathbf{v},\mathbf{w}) := \Gamma_\varrho(\mathbf{v},\mathbf{w}) - i\,\Omega_\varrho(\mathbf{v},\mathbf{w}) := 2Tr\left(\varrho\,\beta_\varrho(\mathbf{v})\beta_\varrho(\mathbf{w})\right). \tag{2.1.27a}$$

Then the B-space $T_\varrho\mathcal{O}(\mathfrak{U})$ is a real Hilbert space with scalar product Γ_ϱ endowed also with the two-form Ω_ϱ (here $[\cdot,\cdot]_-$ is the commutator, and $[\cdot,\cdot]_+$ is the anticommutator in $\mathcal{L}(\mathcal{H})$, and β_ϱ is as in (2.1.7)):

$$\Gamma_\varrho(\mathbf{v},\mathbf{w}) \equiv Tr(\varrho[\beta_\varrho(\mathbf{v}),\beta_\varrho(\mathbf{w})]_+),\ \ \Omega_\varrho(\mathbf{v},\mathbf{w}) \equiv i\,Tr(\varrho[\beta_\varrho(\mathbf{v}),\beta_\varrho(\mathbf{w})]_-). \tag{2.1.27b}$$

Γ is a Riemannian metrics, and Ω is a symplectic form on $\mathcal{O}_\varrho(\mathfrak{U})$, both are strongly nondegenerate, [61]. The symplectic form Ω ascribes to each $f \in \mathcal{F}_\varrho$, 2.1.9, the vector field $\mathbf{v}_f$:

$$\Omega_\nu(\mathbf{v}_f,\mathbf{w}) \equiv -d_\nu f(\mathbf{w}), \tag{2.1.28}$$

coinciding with $\mathbf{v}_f$ from (2.1.18) *for $f \in \mathcal{F}(\mathcal{O}_\varrho(\mathfrak{U}))$, and the corresponding Poisson bracket*

$$\{f,h\} \equiv \Omega(\mathbf{v}_f,\mathbf{v}_h) \tag{2.1.29}$$

coincides with the one defined in (2.1.12) *and* (2.1.15).

Moreover, the following norms are all mutually equivalent on $T_\varrho\mathcal{O}(\mathfrak{U})$: $\|\cdot\|$, $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\varrho$, and $\|\cdot\|_\Gamma := \Gamma(\cdot,\cdot)^{\frac{1}{2}}$. ♣

Proof. The equivalence of the norms $\|\cdot\|_1$, and $\|\cdot\|_\varrho$, as well as the completeness of $T_\varrho\mathcal{O}(\mathfrak{U})$ was proved in Proposition 2.1.5. To prove equivalence of norms $\|\cdot\|$, and $\|\cdot\|_1$, let us write $\varrho = \sum_{j\geq 1}\lambda_j E_j$, with $\sum_{j\geq 0}E_j = I,\ \lambda_1 > \lambda_2 > \cdots > \lambda_N,\ \lambda_0 := 0$, as before. Let $\mathrm{c} = \mathrm{q}_\varrho(\mathrm{c}) := i\,[\varrho,\mathrm{a}] = i\,[\varrho,\mathrm{q}_\varrho(\mathrm{a})] \in T_\varrho\mathcal{O}(\mathfrak{U}),\ \forall \mathrm{a} \in \mathcal{L}(\mathcal{H})_s$. Then $\|\mathrm{c}\| \leq \|\mathrm{c}\|_1 \leq \sum_{j\neq k}\|E_j\mathrm{c}E_k\|_1 =$

$2\sum_{j>k}\|E_j\mathrm{c}E_k\|_1 \le 2\sum_{j>k}\|E_j\|_1\|\mathrm{c}E_k\| \le 2\|\mathrm{c}\|\sum_{k\ge 0}\sum_{j(>k)}\|E_j\|_1$, where the degeneracy of λ_j equals $\|E_j\|_1 < \infty$ for $j \neq 0$, and the number $N+1$ of mutually different eigenvalues λ_j of ϱ is finite. This proves equivalence of the norm $\|\cdot\|_1$ with $\|\cdot\|$, hence also their equivalence with $\|\cdot\|_2$, since always $\|\mathrm{c}\| \le \|\mathrm{c}\|_2 \le \|\mathrm{c}\|_1$. We have further $\frac{1}{2}\|\mathrm{c}\|_\Gamma^2 = Tr(\varrho\beta_\varrho(\mathrm{c})^2) = Tr(\varrho\mathrm{q}_\varrho(\mathrm{a})^2) = \|\varrho\mathrm{q}_\varrho(\mathrm{a})^2\|_1 \le \|\varrho\|_1\|\mathrm{q}_\varrho(\mathrm{a})^2\| = \|\mathrm{q}_\varrho(\mathrm{a})^2\| \equiv \|\mathrm{c}\|_\varrho^2$. On the other hand, since $0 \le \lambda_j \le 1$, one has

$$\begin{aligned}\varrho(\beta_\varrho(\mathrm{c})^2) &= \sum_{k\neq j}\lambda_j Tr(E_j\mathrm{a}E_k\mathrm{a}E_j) \ge \sum_{k\neq j}\lambda_j^2 Tr(E_j\mathrm{a}E_k\mathrm{a}E_j)\\ &\ge \sum_{j>0}\sum_{k(\neq j)}\lambda_j^2 Tr(E_j\mathrm{a}E_k\mathrm{a}) - \sum_{k\neq j}\lambda_j\lambda_k Tr(E_j\mathrm{a}E_k\mathrm{a})\\ &= \frac{1}{2}Tr([\varrho,\mathrm{q}_\varrho(\mathrm{a})][\mathrm{q}_\varrho(\mathrm{a}),\varrho]) = \frac{1}{2}\|\mathrm{c}\|_2^2.\end{aligned}$$

These inequalities together with the previously proved equivalences show also the desired equivalence of $\|\cdot\|_\Gamma$. This proves also nondegeneracy of Γ; its analytic dependence on the point ϱ of the orbit $\mathcal{O}_\nu(\mathfrak{U})$ can be proved from its dependence on elements of the group $\mathfrak{U}$ acting on $\mathcal{O}_\nu(\mathfrak{U})$. The explicit form of Ω

$$\Omega_\varrho(\mathbf{v},\mathbf{w}) \equiv i\,\varrho([\beta_\varrho(\mathbf{v}),\beta_\varrho(\mathbf{w})]) \tag{2.1.30}$$

shows, after inserting into it $\mathbf{v} := \mathrm{ad}^*_\varrho(d_\varrho f)$, and $\mathbf{w} := \mathrm{ad}^*_\varrho(d_\varrho h)$, that it can be expressed by our Poisson bracket (2.1.12): we obtain (2.1.29), according to (2.1.18). The closedness $d\Omega = 0$ follows from the proved Jacobi identity for the Poisson brackets (Proposition 2.1.10). The mapping $d_\varrho f(\in T^*_\varrho\mathcal{O}(\mathfrak{U})) \mapsto \mathbf{v}_f(\varrho) := \mathrm{ad}^*_\varrho(d_\varrho f) \in T_\varrho\mathcal{O}(\mathfrak{U})$ $(f \in \mathcal{F}_\varrho)$ is an isomorphism, what is a consequence of the proved equivalence of topologies on $T_\varrho\mathcal{O}(\mathfrak{U})$, of the surjective property of the mapping $\mathrm{ad}^*_\varrho : \mathfrak{N}_\varrho \to T_\varrho\mathcal{O}(\mathfrak{U})$, $d_\varrho f \mapsto \mathrm{ad}^*_\varrho(d_\varrho f)$, as well as of the reflexivity of the Hilbert space $(T_\varrho\mathcal{O}(\mathfrak{U}); \|\cdot\|_\Gamma)$. This proves that Ω is strongly nondegenerate. □

2.1.20. **Note *(Symplectic and Poisson structures)*.** Existence of symplectic form Ω is useful to easy introduction of a canonical (induced) Poisson structure on submanifolds of $M = \mathcal{O}_\varrho(\mathfrak{U})$ determined, e.g. by actions of symmetry groups: The pull back by embeddings is well defined for covariant tensor fields (i.e. for elements of $\mathcal{T}^0_n(M)$, whereby $\mathcal{T}^0_1(M)$ are one-forms on M), what is not the case of Poisson bracket (remember that the Poisson structure is determined by a two-contravariant antisymmetric tensor field, i.e. the element of $\mathcal{T}^2_0(M)$, cf. also (2.1.15), [177, 274]).

One could try to introduce a symplectic form $\tilde{\Omega}$ on the whole space $\mathfrak{T}_s$ in such a way, that the forms Ω_ϱ on $\mathcal{O}_\varrho(\mathfrak{U})$'s ($\varrho \in \mathcal{S}_*$) are its restrictions by embeddings $\iota_\varrho : \mathcal{O}_\varrho(\mathfrak{U}) \to \mathfrak{T}_s$, i.e. $\Omega_\varrho \equiv \iota^*_\varrho\tilde{\Omega}$. This cannot be done by a naive "extension" of the formula (2.1.30) to the whole $\mathfrak{T}_s$; e.g., for $\dim\varrho = \infty$, the mapping β_ϱ has not a "natural" extension to $\mathfrak{T}_s$, cf. (2.1.9). We shall not investigate this possibility here (it can be connected with considerations in Remarks 3.2.1). ♡

Let us note, that $\mathbf{v}_f \equiv 0$ for a Hamiltonian vector field $\mathbf{v}_f$ does not mean $f(\cdot) \equiv const.$ on connected components of a considered Poisson manifold M, as it is valid for a nondegenerate Poisson structure (of Hamiltonian classical mechanics, e.g.), cf. Definition 1.4.1. The vanishing of $\mathbf{v}_f$ only implies constancy of restrictions of f to connected components of symplectic leaves of M, e.g. the leaves $\mathcal{O}_\varrho(\mathfrak{U})$ of $\mathcal{S}_*$, resp. of $\mathfrak{T}_s$.

2.1-e On interpretation: Subsystems and two types of mixed states

The space $\mathcal{S}_*$ with the introduced Poisson structure will play in EQM a rôle similar to the phase space of classical mechanics. It contains pure states of standard QM described by points ν of the orbit $\mathcal{O}_\varrho(\mathfrak{U}) = P(\mathcal{H})$ with $\varrho = \varrho^2$, i.e. consisting of one-dimensional orthogonal projections on $\mathcal{H}$, as well as the states described by density matrices $\varrho \neq \varrho^2$ traditionally called "mixtures". This type of mixture can always be obtained (cf., e.g. [105, 71, 34]) by the restriction

$$\mathbf{p}_I : \mathcal{S}_*^{I+II} \to \mathcal{S}_* := \mathcal{S}_*^I,$$

(the "partial trace",[71, 50], i.e. $\mathbf{p}_I \equiv Tr_{II}$) of a pure state $\varrho_{I+II} = (\varrho_{I+II})^2 \in \mathcal{S}_*^{I+II}$ of a composed system "$I + II$" (described with a help of the Hilbert space $\mathcal{H}_{I+II} := \mathcal{H}_I \otimes \mathcal{H}_{II}$, with $\mathcal{H}_I := \mathcal{H}$) to a given state $\varrho_I := \varrho \in \mathcal{S}_*$ of the considered subsystem, $\varrho = \mathbf{p}_I(\varrho_{I+II})$.[56]

Work with EQM requires introduction of two different types of "mixed states":[57]

2.1.21. Definition. *Let the states described by density matrices be called* **elementary states** *(also* **elementary mixtures** *to stress possibility of $\varrho \neq \varrho^2$). The topological space $\mathcal{S}_*$ endowed with the Poisson structure will be then called the* **elementary phase space** *for QM.*

Another type of states (let us call them **genuine mixtures***) are described by probability measures μ on the set $\mathcal{S}_*$ of normal states on $\mathcal{L}(\mathcal{H})$ endowed with a Borel structure, cf. also [34]. The set of elementary mixtures can be considered as the subset of the set of genuine mixtures consisting of the Dirac measures (each concentrated on its own one–point subset of $\mathcal{S}_*$).* $\diamondsuit$

2.1.22. **Remark**. We shall not investigate in details, in this paper, various possible convenient Borel structures on $\mathcal{S}_*$, i.e. a σ-algebras of subsets of $\mathcal{S}_*$ generated by open subsets in a topology; we shall not need it in our general considerations. From the point of view of measure theory, cf. [60, 42], it is convenient to work on locally compact spaces. There are two ways how to introduce a "relatively compact" topology on $\mathcal{S}_*$, coming as the relative topology from its compactification in a natural way:

(i) The space $\mathcal{S}_*$ is a subset of $\mathcal{S}$ – the set of all states on $\mathcal{L}(\mathcal{H})$ which is compact in $\sigma(\mathcal{L}(\mathcal{H})^*, \mathcal{L}(\mathcal{H}))$ topology. The induced topology from this w^*-topology coincides on $\mathcal{S}_*$ with the (topology induced from the) natural norm topology on $\mathcal{L}(\mathcal{H})^*$, [42, Proposition 2.6.15]. Observe that the restriction of the norm of $\mathcal{L}(\mathcal{H})^*$ to $\mathcal{S}_*$ coincides with the trace-norm $\|\cdot\|_1$ of $\mathfrak{T}_s$. Moreover, $\mathcal{S}_*$ is w^*-dense in $\mathcal{S}$, [42, Example 4.1.35]. Hence, $\mathcal{S}$ is a natural compactification of $\mathcal{S}_*$.

(ii) Another way of introduction of a "relatively compact" topology in $\mathcal{S}_*$ is (a priori different than that in (i)) w^*–topology coming from the duality $\mathfrak{C}(\mathcal{H})^* = \mathfrak{T}(\mathcal{H})$, i.e. the $\sigma(\mathfrak{T},\mathfrak{C})$-topology, where the duality is expressed by the formula $\langle \varrho; \mathrm{c} \rangle \equiv Tr(\varrho \mathrm{c})$. By the same argument as in (i), [42, Proposition 2.6.15], the w^*–topology on $\mathcal{S}_*$ coincides with the norm-topology of $\mathfrak{C}^* = \mathfrak{T}$, hence again with the trace-norm topology. $\heartsuit$

This way of introduction of (the same, as we see) topology on $\mathcal{S}_*$ leads us to another compact set (let us denote it $\langle \mathcal{S}_* \rangle$), a subset of which is $\mathcal{S}_*$:

[56] A more general definition of "subsystems" can be found in Definition 2.3.8.

[57] The concept of "states" will be reconsidered and generalized after introduction of "the observables" of the considered systems in Section 2.3.

2.1.23. Definition. *The set $\langle \mathcal{S}_* \rangle$ is the (w^*-compact) convex span of $\mathcal{S}_*$ and of the zero element of $\mathfrak{C}^*$. The compact $\langle \mathcal{S}_* \rangle$ is sometimes called [196] the* **quasi state space** *of the C^*-algebra $\mathfrak{C}$.* $\diamondsuit$

Let us return now to description of the genuine mixtures. Let $f \in \mathcal{F}$ be an "observable" (cf., however, Definitions 2.3.3, and Interpretation 2.3.11, for more elaborated concepts). If we interpret, in accordance with the standard interpretation of formalism of QM, its value $f(\varrho)$ as "the expectation value $\langle f \rangle_\varrho$ of f in the state ϱ", then the expectation in a (genuine mixture–) state μ would be naturally determined by the formula

$$\mu(f) := \langle f \rangle_\mu := \int f(\varrho)\mu(d\,\varrho). \tag{2.1.31}$$

If f is an **affine function**, i.e. $f := h_{\mathrm{a}}$ for some $\mathrm{a} \in \mathcal{L}(\mathcal{H})$ (later the denotation "affine" will be used also for functions f which are not everywhere defined and which correspond to unbounded operators X, $f \equiv h_X$, cf. Sec. 2.2), and if $\mathfrak{b}(\mu) \in \mathcal{S}_*$ is the **barycentre** (also **resultant**, resp. intuitively the "center of mass") of μ [42], then

$$\mu(h_{\mathrm{a}}) = h_{\mathrm{a}}(\mathfrak{b}(\mu)) = \mathfrak{b}(\mu)(\mathrm{a}),\ \forall \mathrm{a} \in \mathcal{L}(\mathcal{H}). \tag{2.1.32}$$

This shows, that there is no observable difference between the genuine mixture μ and the corresponding elementary mixture $\mathfrak{b}(\mu) \in \mathcal{S}_*$ in the case, if only affine functions can be observed. For other continuous f (i.e. for $f \not\equiv h_{\mathrm{a}}$ for any $\mathrm{a} \in \mathcal{L}(\mathcal{H})$, let us call such functions (bounded) **nonlinear functions**; they will appear as a new kind of **observables**, resp. **generators**, cf. Definitions 2.3.2, 2.3.3) one has $\mu(f) \neq f(\mathfrak{b}(\mu))$ for a general μ (identity $\mu(f) \equiv f(\mathfrak{b}(\mu))$ for all μ would lead to $f = h_{\mathrm{a}}$ for some $\mathrm{a} \in \mathcal{L}(\mathcal{H})$). Moreover, if the time evolution $\tilde{\varphi}^f$ is generated by the Hamiltonian vector field $\mathbf{v}_f$ corresponding to a nonlinear f, then, even for affine h_{a}, one has, contrary to the case of affine generators f, $\mu_t(h_{\mathrm{a}}) \not\equiv h_{\mathrm{a}}(\tilde{\varphi}^f_t\,\mathfrak{b}(\mu))$, where $\mu_t := \mu \circ \tilde{\varphi}^f_{-t}$ (cf. also Note 3.3.3). This shows some reasons for making distinctions between two kinds of "mixtures" in the presence of nonlinear observables (and nonlinear evolution generators). If we accept a sufficiently large class of nonlinear "observables" f, e.g. $f \in \mathcal{F}_b(\mathcal{S}_*) \equiv$ **uniformly bounded infinitely differentiable functions** on $\mathcal{S}_*$, then a genuine mixture μ coincides with an elementary mixture ϱ iff $\mu = \delta_\varrho :=$ *the Dirac measure concentrated on the one-point set* $\{\varrho\} \subset \mathcal{S}_*$.

Mutually different genuine mixtures $\mu \neq \mu'$ "corresponding" to a given elementary state $\varrho = \mathfrak{b}(\mu) = \mathfrak{b}(\mu')$ can be interpreted as different extensions of a given state of the "considered microsystem" (the observables of which are described in the traditional way - exclusively by the affine observables) to mutually different states of a larger system (say, a macrosystem, cf. Section 3.4, and also [31, Section II.C]) described by a larger set of observables, see Definition 2.3.3. In this sense, the formalism described in this work, and describing (many – also "most" of the earlier known – versions of) nonlinear dynamics in QM can be shown as a restriction to a subsystem of a linear evolution of some larger (say **macroscopic**) quantal system, cf. also [35].

2.1.24. **Interpretation.**

(i) Let us consider a density matrix $\varrho \in \mathfrak{T}_{+1}(\mathcal{H}_I)$ of a "system I", and a normalized vector $\Psi \in \mathcal{H}_I \otimes \mathcal{H}_{II}$ of a "composed system I+II" such, that its restriction to the "subsystem I" (i.e.

the partial trace with respect to the "system II") gives a density matrix ϱ:

$$Tr\big(\mathbf{p}_I(P_\Psi)\mathrm{a}\big) := Tr\big((\mathrm{a}\otimes\mathbb{I}_{II})\cdot P_\Psi\big) = Tr(\varrho\cdot\mathrm{a}), \quad \forall \mathrm{a}\in\mathcal{L}(\mathcal{H}).$$

Such a "system II", and a vector–state P_Ψ (resp. vector Ψ), always exist for any given ϱ. Let $\{\Phi_k : k\in K\}$ be an orthonormal basis in $\mathcal{H}_{II}$: $\sum_{k\in K} P_{\Phi_k} = \mathbb{I}_{II}$, and let $\{\varphi_j : j\in J\}$ be an arbitrary basis in the Hilbert space of the "considered system" $\mathcal{H}_I$. Then the set of vectors $\{\varphi_j\otimes\Phi_k : j\in J, k\in K\}\subset\mathcal{H}_I\otimes\mathcal{H}_{II}$ forms a basis in the Hilbert space of the composed system, and there is a unique decomposition

$$\Psi = \sum_{j\in J,k\in K} c_{jk}\varphi_j\otimes\Phi_k = \sum_{k\in K}\left(\sum_{j\in J} c_{jk}\varphi_j\right)\otimes\Phi_k.$$

Let us define the nonnegative numbers

$$\lambda_k := \|\sum_{j\in J} c_{jk}\varphi_j\|^2,$$

for which the normalization property of Ψ gives $\sum_{k\in K}\lambda_k = 1$, and let us define also the unit vectors (in general mutually *nonorthogonal*)

$$\psi_k := \frac{1}{\sqrt{\lambda_k}}\sum_{j\in J} c_{jk}\varphi_j$$

in the Hilbert space $\mathcal{H}_I$. Then we obtain for the given density matrix ϱ the expression:[58]

$$\varrho = \sum_{k\in K}\lambda_k P_{\psi_k}. \tag{2.1.33a}$$

This decomposition does not depend on a choice of the basis $\{\varphi_j : j\in J\}\subset\mathcal{H}_I$. We see that an arbitrary orthonormal basis $\{\Phi_k : k\in K\}$ in $\mathcal{H}_{II}$ determines a unique decomposition of ϱ. The vector Ψ is here considered fixed, and it is written in the form:

$$\Psi = \sum_{k\in K}\sqrt{\lambda_k}\psi_k\otimes\Phi_k.$$

Let us assume now, that an observable is "measured" on the composed system I+II such, that it just **performs a filtering** of the subsystem II according to the chosen basis $\{\Phi_k : k\in K\}$, corresponding (in a sense of the classical "reduction postulate" [189], cf. Footnote 43) to the **measurement of the quantity** $\boldsymbol{A := \sum_{k\in K}\alpha_k P_{\Phi_k}}$, where $\alpha_j (j\in K)$ are arbitrary, mutually distinct real numbers. One can imagine a situation similar to that in the Bohm version of the Einstein-Podolsky-Rosen (EPR) "gedanken experiment", [90, 14, 276], that the systems I and II are in the instant of measurement (being in the state P_Ψ in that time) mutually very distant and noninteracting, so that the measurement of the quantity $A\in\mathcal{L}_s(\mathcal{H}_{II})$ (or, what is the same in

[58] It might happen also $P_{\psi_k} = P_{\psi_m}$ for some $k\neq m$.

QM, of the quantity $\mathbb{I}_I \otimes A \in \mathcal{L}_s(\mathcal{H}_I \otimes \mathcal{H}_{II})$ of the composed system) "does not affect" the state of the subsystem I. After the measurement, according to the "reduction postulate", the state of the composed system is

$$\varrho^A_\Psi = \sum_{k\in K} \lambda_k P_{\psi_k} \otimes P_{\Phi_k},$$

and the reduced density matrix, if calculated *after the measurement*, is again the same ϱ, as in (2.1.33a). Hence, the state (i.e. the reduced density matrix) of the subsystem I does not depend on a choice of the measured quantity A of the subsystem II, but its decomposition (2.1.33a) is dependent on this choice.[59]

The situation can be generalized to the measurement of a quantity A with degenerate discrete spectrum: $\boldsymbol{A = \sum_l \alpha_l E_l}$, where E_l are orthogonal projections in $\mathcal{H}_{II}$ commuting with all P_{Φ_k}, and $\sum_l E_l = \mathbb{I}_{II}$. Then the state of the composed system *after the reduction* is

$$\varrho^A_\Psi := \sum_l (\mathbb{I}_I \otimes E_l)\cdot P_\Psi \cdot (\mathbb{I}_I \otimes E_l) = \sum_l \kappa_l \, P_{\Psi_l},$$

with $\Psi_l \propto (\mathbb{I}_I \otimes E_l)\Psi$, $\kappa_l := \sum_{k\in(l)} \lambda_k$, and $(l) := \{k \in K : P_{\Phi_k} E_l = P_{\Phi_k}\}$. The above decomposition of the reduced density matrix ϱ corresponding to this alternative measurement situation is

$$\varrho = \sum_l \kappa_l \, \varrho_l, \quad Tr(\varrho_l \cdot \mathrm{a}) := Tr\big((\mathrm{a} \otimes \mathbb{I}_{II})\cdot P_{\Psi_l}\big) \ (\forall \mathrm{a} \in \mathcal{L}(\mathcal{H}_I)). \tag{2.1.33b}$$

Now we can try, however, to interpret the density matrix ϱ obtained by the restriction to the subsystem I *after the measurement* of the quantity A of the subsystem II *not as an indecomposable entity*, i.e. as an elementary state, but **we are going to interpret its different decompositions (2.1.33) as different genuine mixtures**. Hence we shall assume that the process of measurement of A on the correlated subsystem II **transforms** the elementary mixture ϱ (what is an empirically indecomposable quantity) into the corresponding genuine mixture determined by the (empirically identifiable) decomposition (2.1.33a) into the elementary components P_{ψ_k}, resp. by the decomposition (2.1.33b) into the elementary components ϱ_l, with the same barycentre ϱ. This is an important difference in interpretations for nonlinear dynamics: If the evolution of the subsystem I after the measurement on the subsystem II is nonlinear, its state ϱ will evolve, generally, in course of some time after the measurement, into different states, in dependence of what quantity was measured on the distant (and noninteracting) but correlated system II. We see now that **if we accept instantaneous "reduction of the wave packet" of the composed system**, and, moreover, **if we qualify the obtained decomposition (2.1.33a) as the genuine mixture of the components $\boldsymbol{P_{\psi_k}}$ (resp. the decomposition (2.1.33b) as the genuine mixture of the components $\boldsymbol{\varrho_l}$)**, then the subsequent different evolutions of the mixtures with the same initial barycentre (obtained at different choices of the measured quantity A) can lead to distinguishable states before a light signal coming from the distant system II can bring any information about the quantities A measured on the system II, cf. also [106].[60]

[59] The state P_{ψ_k} can be called, in accordance with [93], the *relative state* of I with respect to the state P_{Φ_k} of II, if the state of $I + II$ is Ψ. We shall not discuss here, however, consequences of EQM on possible mutual influence of different "branches" in the *many world interpretation* of QM, cf. [106].

[60] This is, perhaps, a different situation from that one discussed in [169], where a sudden "localized" change of a nonlinear evolution generator led to instantaneous change of time evolution "at distant places".

(ii) Let us try to give at least a vague, intuitive formulation of an (hypothetical) alternative for the above described *transformation of an elementary mixture into a genuine one*, by which the "action at a distance" is avoided:

In our understanding, a quantum measurement is a physical process by which a quantum interaction of the micro-object with an "apparatus" leads to specific macroscopic changes of the apparatus states, by which statistical distribution of eigenstates of the micro-object corresponding to its measured observable in its given quantal state is "copied" into a corresponding classical statistical distribution of mutually classically different (i.e. mutually noninterfering) "pointer positions". A generally accepted description of such a dynamical process is still missing, [276, 14, 28]. Let us assume, however, that we have some description of this process in a framework of QT. Let us consider the combined quantal system $I+II+III$, where we added to our combined system $I+II$ also an apparatus III. Then, during the (many times repeated) measurement of A on II, the apparatus states (let us denote them $\tilde{\Psi}_k$) corresponding to the classes of states of II with sharp values α_k of A (with their vectors lying in the subspaces $E_k\mathcal{H}_{II} \subset \mathcal{H}_{II}$) become (in some short but nonzero time) eventually correlated with the states ϱ_k of the "distant" subsystem I (that was left undisturbed during the measurement). This correlation with "pointer positions" $\tilde{\Psi}_k$ correspods to the "reduction of the wave packet" and it has no observable influence on the system I. According to our present (rather provisional) hypothesis, the presumed process of transformation of ϱ into the genuine mixture $\sum \kappa_k\, \varrho_k$ begins either *after the measurement*, or already *at installing of the apparatus*. This corresponds to two (not mutually exclusive) eventualities:

(first) Since different pointer positions $\tilde{\Psi}_k$ represent different macroscopic states of "the environment" for the system I (we need not be any more interested in the future fate of the measured subsystem II), these macroscopic states might have different "influences" (as different values of an external potential, or a "field") on the correlated states ϱ_k. These "influences" might be very weak, just to provide a possibility to distinguish between different states ϱ_k in the mixed state (2.1.33b).

(second) The environment of the system I was changed by installation of an apparatus for measurement of A on II, and this change (providing information about the set $\{E_k\}$ of projections characterizing A) performs an "influence" on I transforming ϱ into the status of the genuine mixture from (2.1.33b).

We expect, however, that this "influences" will be spread in both the cases *with finite velocity*. Hence, in a presently badly understood way, the elementary mixture ϱ changes into the corresponding genuine mixture (given by the decomposition of ϱ specified by the measured quantity A of II) in a finite time, avoiding the above described "nonlinearity reason" for a superluminal communication between I and apparatuses measuring different observables A of II; other possible "sources of noncausality" mentioned in Subsection 1.5-d, or in the Footnote 60, needn't be improved by such a "mechanism". Let us note finally that these considerations, to lead to a consistently formulated part of QT, should be reconsidered in frameworks of Einstein relativistic theories, cf. remarks and citations on page 38. ♦

Let us note that earlier attempts [47] to introduce nonlinearities into QM were connected with trials to make drastic changes in interpretation of the formalism of QM and, contrary to the here presented theory, they did not include the traditional "linear" theory as a specific "subtheory".

2.2 Unbounded Generators

We have introduced the Poisson structure on the elementary state space of QM with the help of the group $\mathfrak{U}$ (resp. $\tilde{\mathfrak{A}}$) which can be considered as a "maximal possible symmetry group" of described systems in the sense, that each orbit $\mathcal{O}_\varrho(\mathfrak{U})$ is a homogeneous space of its action in $\mathfrak{T}_s$, whereas any "physically acceptable" (unitary) operation leaves all the orbits $\mathcal{O}_\varrho(\mathfrak{U})$ ($\varrho \in \mathfrak{T}_s$) invariant. In the setting of preceding sections, generators of all there described transformations (including time evolution) were functions $f \in \mathcal{F}$; the corresponding "linear" generators $f = h_{\mathrm{a}}$ correspond to *bounded* selfadjoint operators $\mathrm{a} \in \mathcal{L}(\mathcal{H})$ only. The "realistic models" describing particles and fields which are, e.g. invariant with respect to Poincaré, or Galilei symmetries have, however unbounded Hamiltonians, and the generators of many symmetry subgroups are also unbounded. These symmetry groups are usually finite–dimensional noncompact Lee groups, hence there are no "interesting" unitary representations with all the generators bounded. Such a "more realistic" situation cannot be described by the formalism developed up to now: To keep general ideas of our (nonlinear) extension of quantum theory untouched, mathematically correct description requires more sophisticated considerations: It leads to "Hamiltonian functions" only defined on dense subsets of $\mathfrak{T}_s$, and these Hamiltonians are not even locally bounded.[61] We shall proceed stepwise, starting with the linear theory.

2.2-a Some probabilistic aspects of selfadjoint operators

To obtain structures useful to effective description of measurable quantities of a specific considered system, as well as to obtain their empirical interpretation, one has to specify symmetry groups G "smaller" than $\mathfrak{U}$. These groups are related to the quantal system by their continuous (in some topologies) representations in $\mathfrak{A}$, resp. by their projective representations $U(G)$ in $\mathfrak{U}$, cf. [267]. Such realizations of G leave the structure of the elementary (quantum) phase space invariant. These representations may not be analytic, and their weaker continuity properties are connected with existence of unbounded generators. Then we are faced with the problem of description of locally unbounded functions on $\mathcal{S}_*$, playing the rôle of "observables" or "generators" $f \notin \mathcal{F}$, corresponding to the unbounded operators. These functions are not defined on any nonempty open subset of $\mathcal{S}_*$, nevertheless they could generate (in a specific way) one parameter subgroups of transformations of $\mathcal{S}_*$. This functions appear usually in the form $f := h_X$, where X is an unbounded selfadjoint operator generating the unitary group $U^X : t \mapsto \exp(-itX)$, and

$$h_X(\varrho) := i \left.\frac{d}{dt}\right|_{t=0} \varrho(\exp(-itX)) \tag{2.2.1}$$

for such ϱ, for which the derivative exists; this set of $\varrho \in \mathcal{S}_* := \mathcal{S}_*(\mathcal{L}(\mathcal{H}))$ **will be denoted by $\boldsymbol{\mathcal{D}(h_X)}$**. Let $D(X) \subset \mathcal{H}$ be the domain of X, and let $\boldsymbol{D(h_X) :=} \{x \in \mathcal{H} : P_x \in \mathcal{D}(h_X)\}$. Clearly, $D(X) \subset D(h_X)$, and $D(h_X)$ is U^X –invariant.

One of the main problems considered in this section will be the question of possibility of generalization of the developed Poisson formalism to locally unbounded (not everywhere defined)

[61] The difference from the infinite dimensional Lie group $\mathfrak{U}$ of all unitaries in $\mathcal{L}(\mathcal{H})$ consists in discontinuity of the relevant unitary representations $U(G)$ of noncompact finite–dimensional Lie groups G: The one–parameter subgroups $t \mapsto U(\exp(t\xi)) \in \mathfrak{U}$, $\xi \in Lie(G)$, of $U(G) \subset \mathfrak{U}$ representing Lie subgroups of G are not all *Lie subgroups* of $\mathfrak{U}$: Some of them are discontinuous in norm–topology of $\mathcal{L}(\mathcal{H})$, what is the topology with respect to which $\mathfrak{U}$ is endowed with a Lie group structure.

nonlinear generators of transformation groups, e.g. to some nonlinear perturbations of unbounded affine generators h_X. A partial solution of this problem will be reached with a help of group representations.

2.2.1. *Some other characterizations of $\mathcal{D}(h_X)$ are relevant also from the point of view of possible interpretations of the presented formalism. Let $\mathbf{E}_X$ denote the projection–valued (spectral) measure of a selfadjoint operator X. Let μ_ϱ^X be the probability measure on the spectrum of X: sp(X) $\subset \mathbb{R}$, $\mu_\varrho^X(B) := Tr(\varrho \mathbf{E}_X(B))$, corresponding to any $\varrho \in \mathcal{S}_*$. The characteristic function of μ_ϱ^X is $t \mapsto Tr(\varrho \exp(itX))$. The domain $\mathcal{D}(h_X)$ consists of all such points $\varrho \in \mathcal{S}_*$, for which the following limit exists and is continuous in the real parameter t, [95]:*

$$\exp(ith_X(\varrho)) := \lim_{n\to\infty} \left(Tr(\varrho \exp\left(i\frac{t}{n}X\right)) \right)^n . \tag{2.2.2a}$$

The probability measure corresponding to the characteristic function $t \mapsto \exp(ith_X(\varrho))$ is the Dirac measure δ_λ on $\mathbb{R}$ concentrated at $\lambda = h_X(\varrho)$. [It can be shown, that this λ can be interpreted as "a sharp value of a macroscopic observable X_Π" in a quantum theory of infinitely large systems, cf. [31, 24], cf. also Section 3.4.] ♡

2.2.2. Let us mention still another (probabilistic) characterization of the domain $\mathcal{D}(h_X)$, [95, Chap.XVII,§2.a, and Chap.XV,§4]: Let χ_n be the characteristic function (indicator) of the interval $(-n; n) \subset \mathbb{R}$, let $id_\mathbb{R}$ denote the identity function $\lambda \mapsto \lambda$ on $\mathbb{R}$, and let $\mathbb{I}$ denote the function identically equal to 1 on $\mathbb{R}$. Let $\mu(f)$ denote the value of the integral of the function f with respect to a measure μ. Then $\mathcal{D}(h_X)$ consists of those $\varrho \in \mathcal{S}_*(\mathcal{L}(\mathcal{H}))$ for which the sequence of integrals $\mu_\varrho^X(\chi_n\, id_\mathbb{R})$ (cf.2.2.1) has a finite limit for $n \to \infty$, and for which simultaneously

$$\lim_{n\to\infty} \mu_\varrho^X(n(\mathbb{I} - \chi_n)) = 0. \tag{2.2.2b}$$

We have in that case

$$\lim_{n\to\infty} \mu_\varrho^X(\chi_n id_\mathbb{R}) = h_X(\varrho). \tag{2.2.2c}$$

This shows that the existence of the first momentum $\mu_\varrho^X(id_\mathbb{R})$ of the probability measure μ_ϱ^X : $id_\mathbb{R} \in L^1(\mathbb{R}, \mu_\varrho^X)$ (i.e. the existence of the expectation of X in the state ϱ, i.e. the integrability of the absolute value $|id_\mathbb{R}|$) implies $\varrho \in \mathcal{D}(h_X)$. ♡

2.2.3. *Similar considerations show, that $\varrho \in \mathcal{D}_r(X)$ (where $\mathcal{D}_r(X)$ is specified in Definition 2.2.4 below) is equivalent to the existence of the second momentum: $\mu_\varrho^X((id_\mathbb{R})^2) < \infty$ for $\varrho \in \mathfrak{F}_s \cap \mathcal{S}_*$. Since the existence of second momentum of a probability measure on $\mathbb{R}$ implies the existence of the first one, we have $\mathcal{D}_r(X) \subset \mathcal{D}(h_X)$.* ♡

Defined according to (2.2.1), h_X uniquely determines X, which in turn uniquely determines the one parameter unitary group $U^X(t)$. We intend to determine the flow $\tilde{\varphi}_t^X \varrho := Ad^*(\exp(-itX))\ \varrho$ from the (densely defined) generator h_X, or rather from its "differential" dh_X, as a Poisson flow corresponding unambiguously to "the Hamiltonian h_X", and we shall generalize such a determination of flows to nonlinear unbounded generators.

2.2-b Unbounded "linear" generators

Let us now start an investigation of possible generating of Hamiltonian flows by real–valued functions defined on a dense set of $\mathcal{S}_*$, and locally unbounded. It is clear that this will be only possible for a restricted class of functions, especially if chosen from the "nonlinear" ones. We shall consider now the most simple and basic case of a "linear" function, namely the function h_X corresponding to an unbounded selfadjoint operator X defined in the subsection 2.2-a. We shall need to choose some subsets of the domain $\mathcal{D}(h_X)$ where h_X will be in a convenient sense "differentiable", so that we shall be able to define on sufficiently large subset of $\mathcal{S}_*$ the corresponding vector field, and subsequently its flow, so that this flow will be coincident with the canonical unitary flow generated by X.

Let us restrict our attention to subsets of "finite dimensional" density matrices $\varrho \in \mathfrak{F}$ only, what is motivated by technical consequences of Proposition 2.1.5.

2.2.4. Definitions (Domains).

(i) The **domain of the selfadjoint operator** X *on the Hilbert space* $\mathcal{H}$ *will be denoted* $D(X) \subset \mathcal{H}$*; the* **subdomain of its analytic vectors** *is denoted by* $D_a(X) := D^\omega(X) \subset D(X)$*. The space of* **infinitely differentiable vectors**

$$D^\infty(X) := \left\{ x \in \mathcal{H} : \left. \frac{d^n}{dt^n} \right|_{t=0} \exp(itX)x \in \mathcal{H}, \quad \forall n \in \mathbb{N} \right\} \tag{2.2.3a}$$

will also be denoted by $D_d(X) \subset D(X)$*. Clearly* $D_a(X) \subset D_d(X) \subset D(X)$*.*

(ii) The **domain of the generator** δ_X *of the group* $t\ (\in \mathbb{R}) \mapsto Ad^*(\exp(-itX))\varrho$*,* $\forall \varrho \in \mathfrak{T}_s$*, of the B-space automorphisms of* $\mathfrak{T}_s$ *will be denoted by* $\mathcal{D}(\delta_X)$*:*

$$\varrho \in \mathcal{D}(\delta_X) \Leftrightarrow \left. \frac{d}{dt} \right|_{t=0} \left(\exp(-itX)\varrho \exp(itX) \right) \in \mathfrak{T}_s,\ \forall \varrho \in \mathfrak{T}_s. \tag{2.2.3b}$$

(iii) The **restricted domain** *of the generator* δ_X *is*

$$\mathcal{D}_r(\delta_X) := \mathcal{D}(\delta_X) \cap \mathfrak{F}_s \cap \mathcal{S}_*. \tag{2.2.3c}$$

(iv) $\mathcal{D}_r(X)$ *will denote the set of all finite real–linear combinations of one–dimensional projections* $P_x,\ x \in D(X)$*, i.e the set of all selfadjoint finite rank operators with range in* $D(X)$*.* $\boldsymbol{\mathcal{D}_r(X)}$ *will be called the* **restricted domain of** X*.*

(v) The subset of $\mathcal{D}_r(X)$ *consisting of operators with their range in the set of analytic vectors of* X *will be denoted by* $\mathcal{D}_{ra}(X)$*, and called the* **restricted analytic domain of** X*. The operators in* $\mathcal{D}_r(X)$ *with range in* $D_d(X)$ *will be denoted by* $\mathcal{D}_{rd}(X)$*.*

(vi) Let

$$\mathcal{D}_{ra}(\delta_X) := \mathcal{D}_r(\delta_X) \cap \mathcal{D}_{ra}(X).$$

This is the **restricted analytic domain of** δ_X*.* $\diamondsuit$

The following lemma expresses some important properties of the domain $\mathcal{D}_r(\delta_X)$.

2.2.5. Lemma. *For any selfadjoint operator X on $\mathcal{H}$ one has:*

(i) The domain $\mathcal{D}(\delta_X)$ of the generator δ_X contains exactly those $\varrho \in \mathfrak{T}_s$ for which the following two conditions are fulfilled:

a. The operator $\varrho \in \mathcal{L}(\mathcal{H})$ leaves the domain $D(X) \subset \mathcal{H}$ of X invariant.

b. The operator $i\,[\varrho, X]$ (a priori defined, in the case of validity of (a), on the domain $D(X)$) is uniquely extendable to an operator lying in $\mathfrak{T}_s \subset \mathcal{L}(\mathcal{H})$.

(ii) The inclusion $\mathcal{D}_r(\delta_X) \subset \mathcal{D}(h_X)$ is valid.

(iii) For all $\varrho \in \mathcal{D}_r(\delta_X)$ it is $\varrho X \in \mathcal{D}_r(X) \subset \mathfrak{F}$ & $X\varrho \in \mathfrak{F}$ (the products are considered here as unique continuous extensions of the operators initially defined on $D(X)$).

(iv) For $\varrho \in \mathcal{D}_{ra}(\delta_X)$ we have also $X\varrho \in \mathcal{D}_{ra}(X)$. ♣

Proof. (i) is proved in [71, Lemma 5.1 of Chap.5]. It implies, that $\varrho \in \mathcal{D}_r(\delta_X) \Rightarrow \varrho X \in \mathfrak{F}$ & $X\varrho \in \mathfrak{F}$, where the products with X are considered as the corresponding (unique) bounded extensions in $\mathcal{L}(\mathcal{H})$. From these facts we see, that, for the considered ϱ, the expectation $\mu^X_\varrho(id_{\mathbb{R}}) = h_X(\varrho)$ exists, cf.(2.2.2), what in turn implies $\varrho \in \mathcal{D}(h_X)$, i.e. (ii). With ϱ as in (iii), X and ϱ are defined on the domain $D(X)$, and the range of ϱ is in $D(X)$; hence, both products are densely defined finite–range operators, the first one in $\mathcal{D}_r(X)$. The last statement (iv) is valid due the fact that the set of analytic vectors of X is invariant also with respect to the action of the operator X. □

It will be useful to introduce the following

2.2.6. Notation. *Let us denote $\mathcal{D}_{r*}(\delta_X)$, resp. $\mathcal{D}_{r*}(X)$, resp. $D_*(X)$ the variable symbols with possibilities $* \in \{\circ, d, a\}$, where $D_\circ(X) := D(X)$, e.g.. An assertion containing the symbol $*$ (in the described contexts) will be valid for all choices of the alternatives (with the same value chosen in all places of the assertion simultaneously), if something else will not be specified for it; the assertion might be expressed by a sequence of sentences. That assertion might be also numbered by attached $*$ corresponding to any of the choices.* ♢

Let us formulate several useful simple implications of these facts in the following

2.2.7. Lemma*.

*(*i) The domain $\mathcal{D}_{r*}(\delta_X)$ consists of all finite convex combinations of one–dimensional projections $P_x,\ x \in D_*(X) \subset \mathcal{H}$, i.e. $\mathcal{D}_{r*}(\delta_X) \subset \mathcal{D}_{r*}(X)$. All domains $\mathcal{D}_{r*}(\delta_X)$ (for $* = \circ, d, a$) are dense in $\mathcal{S}_*$, resp. the domains $\mathcal{D}_{r*}(X)$ are dense in $\mathfrak{T}_s$, in the topology induced by $\|\cdot\|_1$ of $\mathfrak{T}_s$.*

*(*ii) For $\varrho \in \mathcal{D}_{r*}(\delta_X)$, one has*

$$\delta_X(\varrho) = i\,[\varrho, X] = i\,[\varrho, \mathrm{q}_\varrho(X)] \in T_\varrho\mathcal{O}(\mathfrak{U}) \subset \mathfrak{T}_s(\mathcal{H}); \tag{2.2.4a}$$

$$\beta_\varrho(\delta_X(\varrho)) = \mathrm{q}_\varrho(X) \in \mathfrak{N}_\varrho \subset \mathfrak{F} \subset \mathcal{L}(\mathcal{H}). \tag{2.2.4b}$$

*(*iii) The sets of vectors $\{i\,[\varrho, \mathrm{b}] : \varrho \in \mathcal{D}_{r*}(\delta_X),\ \mathrm{b} \in \mathcal{D}_{r*}(X)\}$ are all dense in $T_\varrho\mathcal{O}_\varrho(\mathfrak{U})$, $\forall \varrho \in \mathcal{D}_{r*}(\delta_X)$ in its topology given by any of the equivalent norms mentioned in Theorem 2.1.19.* ♣

Proof. (*i) From Lemma 2.2.5(i), and the definition in 2.2.4(iii), as well as from the corresponding definitions of $\mathcal{D}_{r*}(\delta_X) := \mathcal{D}_r(\delta_X) \cap \mathcal{D}_{r*}(X)$, with the help of spectral decompositions of $\varrho \in \mathcal{D}_{r*}(\delta_X)$, the first assertion of (*i) follows immediately. It is sufficient to prove the density for $* = a$. Density of the set $\mathcal{D}_{ra}(X)$ in $\mathfrak{T}_s$ will be proved from its density in $\mathfrak{F}$ in $\|\cdot\|_1$–topology, because $\mathfrak{F}$ is dense in $\mathfrak{T}_s$ in this topology. But it suffices to prove arbitrary close approximatebility of one–dimensional projections by such projections from $\mathcal{D}_{ra}(X)$, i.e. by $\{P_x : x \in D_a(X)\}$. Since $D_a(X)$ is linear and dense in $\mathcal{H}$, unit vectors in $D_a(X)$ are dense in unit sphere of $\mathcal{H}$ (by triangle inequality). Then, for two unit vectors $x, y \in \mathcal{H}$, we can use:

$$\|x - y\|^2 = 2(1 - \mathrm{Re}(x, y)) > 1 - |(x, y)|^2 = \frac{1}{4}\|P_x - P_y\|_1^2,$$

where the second equation is proved by calculation of eigenvalues of $\mathrm{d}_{xy} := P_x - P_y; \mathrm{d}_{xy}$ is selfadjoint with trace zero, and range two–dimensional, hence its two eigenvalues are opposite reals $\pm\lambda$; then, by calculating $Tr(\mathrm{d}_{xy}^2) = 2(1 - Tr(P_x P_y)) = 2\lambda^2$ one obtains the desired equation. This easily leads to a proof of density of $\mathcal{D}_{ra}(X)$ in $\mathfrak{T}_s$. The density of $\mathcal{D}_{ra}(\delta_X)$ in $\mathcal{S}_*$ follows then by a use of convexity of both sets.

(*ii) This is a consequence of Lemma 2.2.5(i), as well as of our constructions in Section 2.1-b, see esp. Definitions 2.1.3.

(*iii) For any $\varrho \in \mathcal{D}_{r*}(\delta_X)$, it is $\{i\,[\varrho, \mathrm{b}] :\ \mathrm{b} \in \mathcal{D}_{r*}(X)\} \subset T_\varrho \mathcal{O}_\varrho(\mathfrak{U})$. Due to inequality

$$\|[\varrho, \mathrm{b}]\|_1 \leq 2\|\varrho\|_1 \|\mathrm{b}\|,\ \forall \varrho \in \mathfrak{T}_\mathrm{s}, \mathrm{b} \in \mathcal{L}(\mathcal{H})_\mathrm{s},$$

we know, that the linear mapping $\mathrm{b} \mapsto i\,[\varrho, \mathrm{b}]$ is continuous and can be uniquely extended to the whole $\mathcal{L}(\mathcal{H})_s$ ($\ni \mathrm{b}$), the range of the extended mapping being the whole $T_\varrho \mathcal{O}_\varrho(\mathfrak{U})$. This leads eventually to validity of the statement. □

The following assertion is important for our subsequent constructions.

2.2.8. Proposition. *Let* $\varrho \in \mathcal{D}_{r*}(\delta_X),\ \mathrm{b} \in \mathcal{D}_{r*}(X)$. *Then* $Ad^*(\exp(-it\mathrm{b}))(\varrho)$ $(\equiv \exp(-it\mathrm{b})\varrho\exp(it\mathrm{b})) \in \mathcal{D}_{r*}(\delta_X)$, *i.e.* $\mathcal{D}_{r*}(\delta_X)$ *is invariant with respect to the unitary flows generated by* $\mathrm{b} \in \mathcal{D}_{r*}(X)$. ♣

Proof. There is a projection $P_\mathrm{b} \in \mathcal{D}_{r*}(X)$ such that $\mathrm{b} = \mathrm{b}P_\mathrm{b}$ (P_b might be chosen to be the range projection of b). Then $\exp(i\mathrm{b}) = \exp(i\mathrm{b})P_\mathrm{b} + I - P_\mathrm{b}$, hence

$$\begin{aligned}
&Ad^*(\exp(-it\mathrm{b}))(\varrho) = \\
&P_\mathrm{b}\exp(-it\mathrm{b})\varrho\exp(it\mathrm{b})P_\mathrm{b} + \varrho - \varrho P_\mathrm{b} - P_\mathrm{b}\varrho + P_\mathrm{b}\varrho P_\mathrm{b} - \\
&P_\mathrm{b}\exp(-it\mathrm{b})\varrho + \varrho\exp(it\mathrm{b})P_\mathrm{b} - P_\mathrm{b}\varrho\exp(it\mathrm{b})P_\mathrm{b} - P_\mathrm{b}\exp(-it\mathrm{b})\varrho P_\mathrm{b}.
\end{aligned}$$

The expression consists of a sum of elements of $\mathcal{D}_{r*}(X)$ with ranges contained in the Hilbert subspace determined by the orthogonal projection $(\sum_{j\geq 1} E_j) \vee P_\mathrm{b} \in \mathcal{D}_{r*}(X)$, where we used the spectral projections E_j of ϱ. Hence $Ad^*(\exp(-it\mathrm{b}))\varrho \in \mathcal{D}_{r*}(X)$. Due to unitarity of the transformation of ϱ, we have also $Ad^*(\exp(-it\mathrm{b}))\varrho \in \mathcal{D}_{r*}(\delta_X)$. This proves the assertion. □

Let us now define $d_\varrho h_X \in T_\varrho^* \mathcal{O}_\nu(\mathfrak{U})$ for $\varrho \in \mathcal{D}_r(\delta_X) \subset \mathcal{D}(h_X)$. For these ϱ's, we can write $h_X(\varrho) = Tr(\varrho X)$. According to the Proposition 2.2.8, we can write for $b \in \mathcal{D}_r(X)$:

$$d_\varrho h_X(i[\varrho,\mathrm{b}]) = \tfrac{d}{dt}\big|_{t=0} h_X(\exp(-it\mathrm{b})\varrho\exp(it\mathrm{b})) = \\ Tr(i\,[\varrho,\mathrm{b}]X) = i\,Tr(\mathrm{b}[X,\varrho]) = i\,Tr([\varrho,\mathrm{b}]\mathrm{q}_\varrho(X)), \tag{2.2.5}$$

so that $d_\varrho h_X$ is represented by the operator $\mathrm{q}_\varrho(X)$. In the calculations in (2.2.5), there was used (iii) and (iv) of Lemma 2.2.5, as well as Lemma 2.2.7. In this way, we arrived to the

2.2.9. Definition. *Let $\varrho \in \mathcal{D}_{r*}(\delta_X)$. Then the* **generalized differential of $\boldsymbol{h_X}$, $\boldsymbol{d_\varrho h_X}$,** *is the element of $T^*_\varrho\mathcal{O}_\varrho(\mathfrak{U})$ represented by $\mathrm{q}_\varrho(X) \in \mathfrak{N}_\varrho$, according to the correspondence*

$$i\cdot[\varrho,\mathrm{b}]\ (\in T_\varrho\mathcal{O}_\varrho(\mathfrak{U})) \mapsto i\,Tr([\varrho,\mathrm{b}]\mathrm{q}_\varrho(X)),\ \mathrm{b}\in\mathcal{L}(\mathcal{H})_s,$$

as explained above, cf.(2.2.5). $\diamondsuit$

The definition can be abbreviated as

$$d_\varrho h_X(i\,[\varrho,\mathrm{b}]) = \mathrm{ad}^*_\varrho(\mathrm{b})(d_\varrho h_X) = \mathrm{ad}^*_\varrho(\mathrm{b})(\mathrm{q}_\varrho(X)). \tag{2.2.6}$$

Such a "differential" $d_\varrho h_X$ is defined till now in points $\varrho \in \mathcal{D}_{r*}(\delta_X)$ as a linear functional on vectors $i\,[\varrho,\mathrm{b}] \in T_\varrho\mathcal{O}_\varrho(\mathfrak{U})$ for $\mathrm{b}\in\mathcal{D}_{r*}(X)$ only. But these vectors are dense in $T_\varrho\mathcal{O}_\varrho(\mathfrak{U})$ (in any of the equivalent norms mentioned in Theorem 2.1.19), because $\mathcal{D}_r(X)$ is dense in $\mathcal{L}(\mathcal{H})_s$, and $\mathfrak{F}$ is dense in $\mathfrak{T}_s$, cf. Lemma 2.2.7. Consequently, we can uniquely extend $d_\varrho h_X$ to a bounded linear functional, $d_\varrho h_X = \mathrm{q}_\varrho(X) \in T^*_\varrho\mathcal{O}_\varrho(\mathfrak{U}) \subset \mathcal{L}(\mathcal{H})$.

We shall turn now to the question, whether and how the "differential" dh_X defined just on a subset $\varrho \in \mathcal{D}_{r*}(\delta_X)$ of $\mathcal{S}_*$ can determine the "unitary flow" $Ad^*(\exp(-itX))$ on the whole state space $\mathcal{S}_*$ in a "geometric way". We define the "Hamiltonian vector field" $\mathbf{v}_X(\varrho)$ corresponding to the function h_X via its "differential" $d_\varrho h_X$ in the point $\varrho \in \mathcal{D}_{r*}(\delta_X)$ with a help of Poisson brackets according to (2.1.15) and (2.1.16), i.e. in the representation of tangent vectors in $T_\varrho\mathcal{O}_\varrho(\mathfrak{U})$ used above, we have

$$\mathbf{v}_X(\varrho) = i\,[\varrho,\mathrm{q}_\varrho(X)] \equiv \mathrm{ad}^*_\varrho(d_\varrho h_X),\ \forall\varrho\in\mathcal{D}_{r*}(\delta_X), \tag{2.2.7}$$

in accordance with equation (2.1.18). It is clear, that vectors $\mathbf{v}_X(\varrho)$ are tangent to curves $t \mapsto Ad^*(\exp(-itX))(\varrho)$ in each point $\varrho \in \mathcal{D}_{r*}(\delta_X)$ of their definition. These curves are all lying in the domain $\mathcal{D}_{r*}(\delta_X)$, since the unitary flow $Ad^*(\exp(-itX))$ leaves $\mathcal{D}_{r*}(\delta_X)$ invariant. But the closure of $\mathcal{D}_{r*}(\delta_X)$ in $\|\cdot\|_1$ –topology is the whole $\mathcal{S}_*$. Moreover, the functions $\varrho \mapsto Ad^*(\exp(-itX))(\varrho),\ \forall t\in\mathbb{R}$ are continuous in $\|\cdot\|_1$, hence could be uniquely extended by continuity from $\mathcal{D}_r(\delta_X)$ on the whole $\mathcal{S}_*$. In this way, we have seen that a complete flow on $\mathcal{S}_*$ is uniquely determined by the "Hamiltonian vector field" (2.2.7) defined on a dense subset $\mathcal{D}_{r*}(\delta_X)$ of $\mathcal{S}_*$ only. It remained, however, partially open the question here, how to determine the flow "from the function h_X alone", i.e. without an explicit use of the operator X, with having given the function h_X and its "directional (Gateaux, (2.1.11)), or partial derivatives" on the corresponding domains only. The known properties of the linear operator $X^* = X$ might serve to us as a hint to look for relevant properties of h_X only. A description of the resulting dynamics might be given as follows:

2.2.10. *The flow $\varrho \mapsto \tilde{\varphi}_t^X \varrho$ on $\varrho \in \mathcal{D}_{ra}(X)$ corresponding to the vector field* (2.2.7) *can be described by unitary cocycles (what are just unitary groups in these cases), according to eq.* (2.1.26) *(with interchanged $\nu \leftrightarrow \varrho,\ f \leftrightarrow h$).* ♣

We want to generalize the described situation to "Hamiltonian functions" generating Poisson (or Hamiltonian) flows, also not being of the form h_X for any selfadjoint X and, moreover, are also only densely definable in $\mathcal{S}_*$. The most simple generalization is, probably, the generator $h(\varrho) := f(h_X(\varrho))$, where f is a sufficiently differentiable real valued function on $\mathbb{R}$. We shall go further: We shall generalize and investigate the preceding constructions to functions $h(\varrho) := f(h_{X_1}(\varrho), h_{X_2}(\varrho), \dots h_{X_k}(\varrho))$, $f \in C^\infty_{\mathbb{R}}(\mathbb{R}^k)$, for "conveniently chosen" sets of (in general noncommuting) selfadjoint operators X_j $(j = 1, 2, \dots k)$ on $\mathcal{H}$. Before that, however, a more general framework will be sketched.

2.2-c On unbounded nonlinear generators

As we saw in the example of selfadjoint operators and the corresponding "linear" generators – locally unbounded Hamiltonian functions h_X, the definition of a (Poisson) flow from such a function h_X might be possible, if we determine from it a densely (in $\mathcal{S}_*$) defined vector field $\mathbf{v}_f(\cdot)$ having integral curves (lying, of course, in its domain), in an agreement with (2.1.18). Hence, the domain $\{\nu \in \mathcal{S}_* : \mathbf{v}_f(\nu) \text{ exists}\}$ should consist of (at least) one dimensional differentiable (C^1 –) submanifolds of (sufficiently many of) $\mathcal{O}_\varrho(\mathfrak{U})$'s (we shall again consider $\varrho \in \mathfrak{F}_s \cap \mathcal{S}_*$ only).

2.2.11. **Remark** *(Speculating on "integral" submanifolds).* To make possible a use of the Poisson structure at construction of smooth vector fields on *smooth manifolds*, as well as their integral curves from only densely defined functions on $\mathcal{O}_\varrho(\mathfrak{U}),\ \varrho \in \mathfrak{F}_s$, and also to have possibility to define Poisson brackets for several such densely defined functions, we would need algorithms to construct some "convenient" more than one–dimensional smooth submanifolds in domain of definition of our densely defined objects, and this seems to be a nontrivial question in a general case. A solution will be found in subsequent sections for a specific class of densely defined generators and vector–fields determined by Lie group representations: A given continuous unitary representation of a Lie group determines in the state space $\mathcal{S}_*$ smooth submanifolds (orbits of GCS). Hamiltonian vector fields on these submanifolds can be defined from given "nonlinear" real–valued functions with a help of the existing "Kählerian" structure Ψ $\big($cf. (2.1.27); let us note that this structure is Kählerian only if restricted to $P(\mathcal{H})\big)$. Existence of such apriori defined domains of definition is typical also for some standard approaches to not–everywhere defined vector fields and/or Hamiltonian functions, cf. [59, 178]. Let us speculate a little now on alternative possibilities for construction of some smooth submanifolds of $\mathcal{O}_\varrho(\mathfrak{U})$'s, determined by some apriori given objects, e.g. by an (only densely defined) vector field $\mathbf{v}_f(\nu)$.[62] The rough idea consists in looking for possibility of construction of some submanifolds in $\mathcal{S}_*$ of more than one dimension from such a "relatively poor" object as a vector field. These submanifolds might become a "playground" for definition of other vector fields and they integral curves.

Let us formulate here just some "toy examples" how to define, to a given (possibly not everywhere defined) vector field $\mathbf{v}_f(\varrho)$, other vector fields such that they both together (perhaps) span a symplectic submanifold of $\mathcal{O}_\varrho(\mathfrak{U})$. Our proposals might be useful as hints for a search

[62] The vectors $\mathbf{v}_f(\nu)$ needn't belong to a (possibly Hamiltonian, in some sense) vector field determined by a function f; the letter "f" might be here just a label.

of alternatives to cases described in literature, if the assumptions required there are not fulfilled. This new vector field will be constructed via the symplectic and metric structures on $\mathcal{O}_\varrho(\mathfrak{U})$ given by (2.1.27), i.e. by Ω_ϱ and Γ_ϱ respectively. Note, that these structures are invariant with respect to "unitary automorphisms" of $\mathcal{O}_\varrho(\mathfrak{U})$, i.e. for a given unitary operator $\mathrm{u} \in \mathfrak{U} := \mathcal{U}(\mathcal{H})$ the corresponding mapping $Ad^*(\mathrm{u}) : \mathcal{O}_\varrho(\mathfrak{U}) \to \mathcal{O}_\varrho(\mathfrak{U})$ leaves invariant not only the symplectic form, but also the metric; the **push–forward** $(Ad^*(\mathrm{u}))_*\mathbf{v}$ of a vector field $\varrho \mapsto \mathbf{v}(\varrho) \equiv i\,[\varrho, \mathrm{b}_\mathbf{v}(\varrho)] \in T_\varrho\mathcal{O}_\varrho(\mathfrak{U})$ is $\frac{d}{dt}\big|_{t=0}\,\mathrm{u}\exp(-it\mathrm{b}_\mathbf{v}(\varrho))\varrho\exp(it\mathrm{b}_\mathbf{v}(\varrho))\mathrm{u}^* = i\,[\mathrm{u}\varrho\mathrm{u}^*, \mathrm{u}\mathrm{b}_\mathbf{v}(\varrho)\mathrm{u}^*]$, hence the **pull–back** of the bilinear form Ψ_ϱ by the same mapping is

$$\begin{aligned}\big((Ad^*(\mathrm{u}))^*\Psi\big)_\nu(\mathbf{v}(\nu),\mathbf{w}(\nu)) &= \Psi_{\mathrm{u}\nu\mathrm{u}^*}\big(\big(Ad^*(\mathrm{u})\big)_*\mathbf{v}(\nu), \big(Ad^*(\mathrm{u})\big)_*\mathbf{w}(\nu)\big)\\ &= 2(\mathrm{u}\nu\mathrm{u}^*)\big(\beta_{\mathrm{u}\nu\mathrm{u}^*}((Ad^*(\mathrm{u}))_*\mathbf{v})\beta_{\mathrm{u}\nu\mathrm{u}^*}((Ad^*(\mathrm{u}))_*\mathbf{w})\big)\\ &= 2\,Tr\big(\mathrm{u}\nu\mathrm{b}_\mathbf{v}\mathrm{b}_\mathbf{w}\mathrm{u}^*\big) = 2\,Tr\big(\nu\mathrm{b}_\mathbf{v}\mathrm{b}_\mathbf{w}\big) = \Psi_\nu(\mathbf{v}(\nu),\mathbf{w}(\nu)).\end{aligned}$$

We shall present here two possibilities of construction of linear independent vector fields from a given one. We do not, however, even formulate precisely a question of their "integrability" to some integral submanifolds containing these vector fields as sections of their tangent bundles, e.g. in a sense of the Frobenius theorem, cf., e.g. [1, 61, 9]. The integrability questions would need more specific assumptions on the (domain of the) vector field $\mathbf{v}_f$.

(i): Let us fix a point $\nu \in \mathcal{O}_\varrho(\mathfrak{U})$, and a vector $\mathbf{v}_f(\nu) \in T_\nu\mathcal{O}_\varrho(\mathfrak{U})$. We shall construct another vector $\check{\mathbf{v}}_f(\nu)$ forming with it a "canonical pair" (with respect to the form Ω). Let, for any subset $N \subset T_\nu\mathcal{O}_\varrho(\mathfrak{U})$, its orthogonal complement (in sense of the real Hilbert space structure given by Γ) be denoted by $N^\perp$, and the skew–orthogonal complement by $N^\angle := \{\mathbf{v} \in T_\nu\mathcal{O}_\varrho(\mathfrak{U}) : \Omega_\nu(\mathbf{v},\mathbf{w}) = 0\ \forall \mathbf{w} \in N\}$. It is clear that $N^\angle$ is a closed linear subspace of $T_\nu\mathcal{O}_\varrho(\mathfrak{U})$, and that $N^{\angle\angle\angle} = N^\angle$, resp. also $N^{\angle\angle} = N$ for a closed linear subspace N, similarly as it is valid for orthogonal complements. For any nonzero $\mathbf{v} \in T_\nu\mathcal{O}_\varrho(\mathfrak{U})$ the space $[\mathbf{v}]^\angle$ is of codimension one. Hence $\big[[\mathbf{v}]^\angle\big]^\perp$ is one–dimensional, the nonzero vectors of which have nonzero "skew-product" with $\mathbf{v}$, and are orthogonal to it. Let us choose for any $\nu \in \mathcal{D}(\mathbf{v}_f) \equiv$ the domain of $\mathbf{v}_f$:

$$\check{\mathbf{v}}_f(\nu) \in \big[[\mathbf{v}_f(\nu)]^\angle\big]^\perp,\ \ \Omega_\nu(\mathbf{v}_f(\nu), \check{\mathbf{v}}_f(\nu)) := 1,\ \ \Gamma_\nu(\mathbf{v}_f(\nu), \check{\mathbf{v}}_f(\nu)) = 0. \tag{2.2.8a}$$

We can ascribe, in this way, to any vector field $\mathbf{v}_f(\nu)$ a "canonically conjugated" vector field $\check{\mathbf{v}}_f(\nu)$.

(ii): An alternative way to construct another vector field $\nu \mapsto \check{\mathbf{v}}_f(\nu)$ to a given $\nu \mapsto \mathbf{v}_f(\nu)$ might be as follows:

$$\check{\mathbf{v}}_f(\nu) \propto [\nu, [\nu, \beta_\nu(\mathbf{v}_f(\nu))]],\ \mathbf{v}_f(\nu) := i\,[\nu, \mathrm{b}(\nu)]. \tag{2.2.8b}$$

This proposal allows us to construct also more than two–dimensional subspaces of $T_\nu\mathcal{O}_\nu(\mathfrak{U})$ ($\nu \in \mathcal{D}_{r*}(\delta_X),\ \mathrm{b}(\nu) \in \mathcal{D}_{r*}(X)$) containing a given field $\mathbf{v}_f^{(1)}(\nu) := \mathbf{v}_f(\nu)$ together with the vector field $\mathbf{v}_f^{(2)}(\nu) \propto \check{\mathbf{v}}_f(\nu)$. In terms of our operator representations of $T_\nu\mathcal{O}_\nu(\mathfrak{U})$ we can construct a sequence of (a finite number of linearly independent) vector fields by the formula:

$$\mathbf{v}_f^{(n)}(\nu) := i\cdot[\nu, \mathbf{v}_f^{(n-1)}(\nu)] := i^n\cdot[\nu, \mathrm{b}(\nu)]^{(n)}, \tag{2.2.8c}$$

where $[\nu,\mathrm{b}]^{(n+1)} := [\nu,[\nu,\mathrm{b}]^{(n)}]$, $[\nu,\mathrm{b}]^{(1)} := [\nu,\mathrm{b}] := \nu\mathrm{b}-\mathrm{b}\nu$. Let us mention some properties of these vector–fields with respect to the bilinear form Ψ_ν, cf. Theorem 2.1.19; they are derivable from simple properties of the commutators and traces:

$$\Psi_\nu(\mathbf{v}_f^{(n)},\mathbf{v}_f^{(m)}) = (-1)^k\cdot\Psi_\nu(\mathbf{v}_f^{(n-k)},\mathbf{v}_f^{(m+k)}) \tag{2.2.8d}$$

$$= (-1)^{n-m}\cdot\overline{\Psi_\nu(\mathbf{v}_f^{(n)},\mathbf{v}_f^{(m)})}. \tag{2.2.8e}$$

Since the symplectic form $-\Omega_\nu$ is the imaginary part of Ψ_ν, and the metric Γ_ν is the real part, we see that the fields $\mathbf{v}_f^{(n)}$ and $\mathbf{v}_f^{(n+1)}$ are pointwise mutually orthogonal, whereas $\mathbf{v}_f^{(n)}$ and $\mathbf{v}_f^{(n+2)}$ are mutually skew–orthogonal ($\forall n\in\mathbf{N}$). Observe also, that all these fields have, in a given point ν, nonzero values simultaneously: this is due to the fact, that for $\nu\in\mathcal{D}_{r*}(\delta_X)$ the mapping β_ν is an isomorphism (resp. it can be considered as an automorphism, after a natural identification, cf. Notes 2.1.4, and Proposition 2.1.5) of $\mathfrak{N}_\nu$ and $T_\nu\mathcal{O}_\nu(\mathfrak{U})$:

$$\beta_\nu(i\,[\nu,\mathrm{q}_\nu(\mathrm{b})]) = \mathrm{q}_\nu(\mathrm{b}) = \mathrm{q}_\nu(\mathrm{q}_\nu(\mathrm{b})). \tag{2.2.8f}$$

This allows us to extend the sequence of vector fields $\mathbf{v}_f^{(n)}$, $(n=1,2,\dots)$ to all integers $n\in\mathbb{Z}$. We shall assume here that $\mathrm{b}(\nu)\in\mathfrak{N}_\nu$ $(\forall\nu\in\mathcal{D}(\mathbf{v}_f))$. We define:

$$\mathbf{v}_f^{(0)}(\nu) := \beta_\nu(\mathbf{v}_f^{(1)}(\nu)) \equiv \mathrm{b}(\nu),\ \mathbf{v}_f^{(-n)}(\nu) := \beta_\nu^n\big(\mathrm{b}(\nu)\big),\ \forall n\in\mathbb{Z}. \tag{2.2.8g}$$

Since the ranges of ν and $\mathrm{b}(\nu)$ are finite–dimensional, only a finite number of elements of $\{\mathbf{v}_f^{(n)}: n\in\mathbb{Z}\}$ are linearly independent. It is also easily seen that the bilinear form Ψ_ϱ is nonzero on any pair of these vectors, what follows from (2.2.8d) and from:

$$\begin{aligned}\Psi_\varrho(\mathbf{v}_f^{(n+1)},\mathbf{v}_f^{(n+2)}) &= Tr\big(\varrho\,[\beta_\varrho(\mathbf{v}_f^{(n+1)}),\beta_\varrho(\mathbf{v}_f^{(n+2)})]\big) = Tr\big(\varrho\,[\mathbf{v}_f^{(n)}(\varrho),\mathbf{v}_f^{(n+1)}(\varrho)]\big)\\ &= i\cdot Tr\big(\varrho\,[\mathbf{v}_f^{(n)}(\varrho),[\varrho\,,\mathbf{v}_f^{(n)}(\varrho)]]\big)\\ &= i\cdot Tr\big([\varrho\,,\mathbf{v}_f^{(n)}(\varrho)]^2\big)\neq 0,\end{aligned} \tag{2.2.8h}$$

since all the $\mathbf{v}_f^{(n)}(\varrho)$'s are represented by selfadjoint trace class operators on $\mathcal{H}$. $\heartsuit$

We shall proceed, also in nonlinear generalizations, in the framework of Hilbert space $\mathcal{H}$, since this allows us to use some usual techniques with linear mappings and scalar product, as well as intuition and/or interpretation from the standard QM. We believe, however, that the developed ideas can be used also in a "purely geometrical" transcription (and possible modifications), [67, 11], of the theory developed in this paper.

2.2.12. Notation (Domains). *Let us assume, that a norm–dense linear subset $\mathbf{D}$ of $\mathcal{H}$ is given. This means also, that any finite linear combination $\sum_{\alpha=1}^k c_\alpha x_\alpha$ of vectors $x_\alpha\in\mathbf{D}$ also belongs to $\mathbf{D}$, hence finite–dimensional subspaces generated by such vectors are subspaces of $\mathbf{D}$. Let us denote by $\boldsymbol{\mathcal{D}}_r$ the set of all finite real–linear combinations of finite dimensional projections to subspaces of $\mathbf{D}$, $\boldsymbol{\mathcal{D}}_r\subset\mathfrak{F}_s$. In the general scheme constructed here in an analogy with preceding subsection, the set $\boldsymbol{\mathcal{D}}_r$ is here the object corresponding to $\mathcal{D}_{r*}(X)$ in Subsection 2.2-b. Let $\boldsymbol{\mathcal{D}}_{r+}^1 := \boldsymbol{\mathcal{D}}_r\cap\mathcal{S}_*$ be the object corresponding to $\mathcal{D}_{r*}(\delta_X)$ in Subsection 2.2-b. $\boldsymbol{\mathcal{D}}_{r+}^1$ is dense in $\mathcal{S}_*$, in the $\|\cdot\|_1$–norm topology.* $\diamondsuit$

2.2.13. Definitions (Generalized fields and integrability).

(i) Let $h : \boldsymbol{\mathcal{D}}^1_{r+} \to \mathbb{R}$ be such that there exist

$$d_\varrho h(i[\varrho, \mathrm{b}]) := \frac{d}{dt}\Big|_{t=0} h(\exp(-it\mathrm{b})\varrho\exp(it\mathrm{b})),\ \forall \varrho \in \boldsymbol{\mathcal{D}}^1_{r+}, \mathrm{b} \in \boldsymbol{\mathcal{D}}_r, \tag{2.2.9a}$$

and that it is bounded linear in the variable $i\,[\varrho, \mathrm{b}],\ \mathrm{b} \in \boldsymbol{\mathcal{D}}_r$; let its unique bounded linear extension is expressed by the operator $\mathrm{q}_\varrho(\boldsymbol{\mathcal{D}}_r h) := d_\varrho h \in \mathfrak{N}_\varrho \subset \boldsymbol{\mathcal{D}}_r \subset \mathcal{L}(\mathcal{H})_s$:

$$d_\varrho h(i[\varrho, \mathrm{b}]) = i\,Tr\left(\mathrm{q}_\varrho(\boldsymbol{\mathcal{D}}_r h)\cdot[\varrho, \mathrm{b}]\right),\ \forall \mathrm{b} \in \mathcal{L}(\mathcal{H})_s. \tag{2.2.9b}$$

*This densely defined function $d.h$: $\varrho \mapsto \mathrm{q}_\varrho(\boldsymbol{\mathcal{D}}_r h) \in \mathfrak{N}_\varrho,\ \varrho \in \boldsymbol{\mathcal{D}}^1_{r+}$, will be called the $\boldsymbol{\mathcal{D}}_r$–***generalized differential*** *of h.*

(ii) The corresponding (densely defined in $\mathcal{S}_*$) **(generalized $\boldsymbol{\mathcal{D}}_r$–Hamiltonian vector field** *is:*

$$\mathbf{v}_h(\varrho) := \mathrm{ad}^*_\varrho(\mathrm{q}_\varrho(\boldsymbol{\mathcal{D}}_r h)) \in T_\varrho\mathcal{O}_\varrho(\mathfrak{U}),\ \varrho \in \boldsymbol{\mathcal{D}}^1_{r+}. \tag{2.2.9c}$$

Let us stress that values of this vector field also belong to $\boldsymbol{\mathcal{D}}_r \subset \mathfrak{T}_s$.

(iii) Let us assume that $\boldsymbol{\mathcal{D}}_r$ contains the set $\mathcal{V}$ of mutually disjoint submanifolds $\mathcal{V}_\iota$, $\mathcal{V} := \{\mathcal{V}_\iota : \iota \in \Upsilon := $ an index set$\}$, such that their union $\cup\mathcal{V} := \cup_{\iota\in\Upsilon}\mathcal{V}_\iota$ is dense in $\boldsymbol{\mathcal{D}}_r$. Further assume that for a given $h : \boldsymbol{\mathcal{D}}_r \to \mathbb{R}$ with $\boldsymbol{\mathcal{D}}_r$–generalized differential its $\boldsymbol{\mathcal{D}}_r$–Hamiltonian vector field is tangent to $\mathcal{V}_\iota$ in any point $\nu \in \mathcal{V}_\iota,\ \forall\iota \in \Upsilon$, so that the restrictions of $\mathbf{v}_h(\nu)$ to $\mathcal{V}_\iota \ni \nu$ are smooth vector fields on the all $\mathcal{V}_\iota$'s. Then we call the $\boldsymbol{\mathcal{D}}_r$–generalized differential of h to be $\mathcal{V}$**–integrable**.

(iv) Consider the situation from (iii) above, and let the differential $\mathrm{q}_\nu(\boldsymbol{\mathcal{D}}_r h)$ be $\mathcal{V}$–integrable. Let us assume that the local flows $\tilde\varphi^h_t$ of these vector fields on $\mathcal{V}$ continuously depend on initial conditions, i.e. the functions

$$(\nu; t) \mapsto \tilde\varphi^h_t(\nu),\ \forall(\nu; t) \in \mathcal{D}_\Upsilon \subset \cup\mathcal{V} \times \mathbb{R}\ (\mathcal{D}_\Upsilon \supset \cup\mathcal{V} \times \{0\}), \tag{2.2.9d}$$

*are all continuous on the union $\cup\mathcal{V}$ in the topology induced from $\|\cdot\|_1$. Here $\mathcal{D}_\Upsilon$ is the domain of the definition of the local flows, and it is $\mathcal{D}_\Upsilon = \cup\mathcal{V} \times \mathbb{R}$ if the flows are complete (i.e. defined for all $t \in \mathbb{R}$). In this case the flows on leaves of $\mathcal{V}$ can be uniquely extended to a flow on $\mathcal{S}_*$. Then we call the $\boldsymbol{\mathcal{D}}_r$–generalized differential to be $\mathcal{S}_*$–***integrable***.*[63] ♢

We shall look now, for a moment, back to the "linear cases" to show that they are contained in our present generalized scheme:

2.2.14. Proposition (Differentials for "linear" generators). *Let X be a selfadjoint operator on $\mathcal{H}$, let $\boldsymbol{\mathcal{D}}^1_{r+}$:= $\mathcal{D}_{r*}(\delta_X)$, $\boldsymbol{\mathcal{D}}_r$:=$\mathcal{D}_{r*}(X)$. Then the $\boldsymbol{\mathcal{D}}_r$–generalized differential dh_X of h_X, $h_X(\varrho) := Tr(\varrho X)$, exists. The differential dh_X is $\mathcal{V}$–integrable for $\mathcal{V} := \big\{\mathcal{V}_\nu :\ \ \mathcal{V}_\nu := \{\exp(-itX)\nu\exp(itX) : t \in \mathbb{R}\}, \nu \in \mathcal{D}_{ra}(\delta_X)\big\}$.* ♣

Proof. The proof is contained in the text following the Definition 2.2.9. □

[63] Some variations on these definitions allowing more refined classification of flows, what are extendable to submanifolds of $\mathcal{S}_*$ only, are sketched in [24].

2.2.15. **Notes**. We could choose in the Proposition 2.2.14 more than one–dimensional $\mathcal{V}_\nu$ as submanifolds with smooth Hamiltonian vector field (2.2.7) constructed with a help of Proposition 2.2.8. Our simplest choice was, however, enough to demonstrate a consistency feature of the theory. ♡

The "Schrödinger equation" for the unitary cocycles describing the Hamiltonian flow of the $\boldsymbol{\mathcal{D}}_r$–Hamiltonian vector field $\mathbf{v}_h$ can be written as in (2.1.23), resp. (2.1.26):

$$i\,\frac{d}{dt}\mathrm{u}_h(t,\varrho(0)) = \left[q_{\varrho(t)}(\boldsymbol{\mathcal{D}}_r h) + \mathrm{h}^0(\varrho(t))\right]\mathrm{u}_h(t,\varrho(0)), \quad \mathrm{u}_h(0,\varrho(0)) := I_{\mathcal{H}}, \qquad (2.2.10)$$

where $\varrho(t) \equiv \mathrm{u}_h(t,\varrho(0))\varrho(0)\mathrm{u}_h^{-1}(t,\varrho(0)),\ \forall\varrho(0) \in \boldsymbol{\mathcal{D}}_r$. *The equation* (2.2.10) *is an expression of general form of dynamical (nonlinear Schrödinger) equations.* We intend to discuss various specifications of this equation in subsequent parts of this work. If the function h^0 on $\boldsymbol{\mathcal{D}}_r$ is chosen "sufficiently nice" (e.g. sufficiently continuous, with values in $\mathfrak{M}_\varrho \cap \boldsymbol{\mathcal{D}}_r$), the objects in this equation are well defined on the dense domain $\boldsymbol{\mathcal{D}}_r$. In specific cases, the equation (2.2.10) can be considered as a nonautonomous (i.e. time dependent) linear Schrödinger–Dyson equation provided the dependence $t \mapsto q_{\varrho(t)}(\boldsymbol{\mathcal{D}}_r h)$ is known; this "time–dependence of Hamiltonian" can be sometimes obtained in an independent way, without solving this nonlinear equation. Such a possibility of "elimination of nonlinearity" will arise in specific applications investigated in Section 3.5.

2.2-d Nonlinear generators from group representations

We have sketched in Subsection 2.2-c a formulation of the problem of construction of some "convenient" submanifolds in $\mathcal{O}_\varrho(\mathfrak{U})$, with $\varrho \in \mathcal{D}_{r*}(\delta_X)$, on which some (on $\mathcal{O}_\varrho(\mathfrak{U})$ only) densely defined vector fields could be determined as smooth vector fields in the corresponding tangent subbundles. This was the case, e.g., of densely defined "nonlinear" Hamiltonian vector fields from Definitions 2.2.13, but also the case of the "linear" Hamiltonian function h_X, if we wanted to proceed in the determination of the corresponding Hamiltonian flow in a geometric way (i.e. without a return to the functional analysis connected with the selfadjoint operator X on $\mathcal{H}$). The proposals outlined in Remark 2.2.11 were left in a very preliminary form. Analogical theory of that one for generators in "linear case" would be, e.g. some hypothetical nonlinear generalization of the von Neumann theory of symmetric and selfadjoint operators (**"deficiency–indices" theory**, cf. [218], and also Appendix C.2);[64] we are not aware of existence of such a theory.[65] We have worked above with a "large" domain $\mathcal{D}_{r*}(\delta_X)$, containing one–dimensional solutions of the equation (2.2.10). Rigorous and systematic methods for solving that equation were, however, missing.[66] Now we shall use Lie group representations to allow us rigorous work with nonlinear unbounded generators of specific kind; its specification to solution of (2.2.10) is described in Section 3.5.

[64] It is known that, e.g. completeness of locally Hamiltonian vector fields is (up to subsets of measure zero) equivalent to essential selfadjointness of their generators in the "Koopman version" of CM; this follows from a Povzner theorem, cf. [211], [1, Theorem 2.6.15 and Proposition 2.6.14].

[65] An exception might be a theory of unbounded derivations on C^*-algebras, cf. [228]; this could be used in our case after an "embedding" of our nonlinear system into a larger linear one, cf. also [27, 31].

[66] Cf., however, [59, §4.1], where the concept of "manifold domain" was introduced; this can be applied, in the case of single selfadjoint generator X, to its domain $D(X) \subset \mathcal{H}$ endowed with the graph–norm, cf. also (C.2.2).

Let G be a real Lie group [39], and let $U(G)$ be its **strongly continuous unitary or projective representation** in $\mathcal{H}$, hence $U : g(\in G) \mapsto U(g)(\in \mathfrak{U}),\ g \mapsto Tr(\varrho U(g))$ being continuous on G for all $\varrho \in \mathcal{S}_*$. Assume that $U(G)$ has a $U(G)$–invariant dense set $D^\omega(G) \subset \mathcal{H}$ of analytic vectors, i.e. $x \in D^\omega(G) \Leftrightarrow$ the function $g \mapsto U(g)x$ is real analytic in a neighbourhood of the identity $e \in G$. This is the case [13] of each strongly continuous $U(G)$ of any finite dimensional Lie group G, as well as of an analytic representation U of an arbitrary Lie group, e.g. the defining representation of the unitary group $\mathfrak{U} := \mathcal{U}(\mathcal{H})$ in$\mathcal{H}$. Let $\boldsymbol{\mathcal{D}^\omega(G)}$ be the (norm–dense) $Ad^*(U(G))$–invariant set of analytic elements $\nu \in \mathcal{S}_*$, i.e. the functions $g \mapsto Ad^*(U(g))\nu$ from G to $\mathfrak{T}_s$ are real analytic around $e \in G$. Let us write also $g \cdot \nu := Ad^*(U(g))\nu$. Let $Lie(G) \equiv \mathfrak{g}$ denote the **Lie algebra of G**, and let $\exp : Lie(G) \to G$ be the exponential mapping. Then we have $U(\exp(t\xi)) =: \exp(-itX_\xi), \xi \in Lie(G)$, for a selfadjoint (in general unbounded) operators X_ξ on $\mathcal{H}$. The mapping $\boldsymbol{\xi \mapsto X(\xi) := X_\xi}$ is a Lie algebra morphism: It is linear, and on a dense ($U(G)$–, and also $X(Lie(G))$–)invariant domain (common for all $X_\xi, \xi \in Lie(G)$), e.g. on $D^\omega(G)$, satisfies the relation, [13]:

$$[X_\xi, X_\eta] := X_\xi X_\eta - X_\eta X_\xi = i\, X_{[\xi,\eta]}. \tag{2.2.11}$$

Here $[\xi, \eta] \in Lie(G)$ denotes the Lie bracket. Let $\mathcal{O}_\varrho(G) \subset \mathcal{O}_\varrho(\mathfrak{U}) \cap \mathcal{D}^\omega(G)$ be the $Ad^*(U(g))$–orbit of the G–action on $\mathfrak{T}_s$ through ϱ, $\mathcal{O}_\varrho(G) := \{U(g)\varrho U(g)^* : g \in G\}$. Let

$$h_{X(\xi)}(\nu) := \nu(X_\xi) := i\ \frac{d}{dt}\Big|_{t=0} \nu(\exp(-itX_\xi)), \tag{2.2.12}$$

for $\nu \in \mathcal{D}(h_{X(\xi)})$, cf.(2.2.1), and (2.2.2) with $X := X(\xi)$. Let us denote $\boldsymbol{G_\nu := \{g \in G : U(g) \in \mathfrak{U}_\nu\}}$ the stability subgroup of G at $\nu \in \mathcal{S}_*$ with respect to the action $Ad^*(U(\cdot)) : (g; \nu) \mapsto g \cdot \nu$. The following lemma shows that the set of nice (i.e. "analytic finite dimensional") orbits of the action of G on $\mathcal{S}_*$ satisfy not only conditions on $\boldsymbol{\mathcal{D}}_r$ stated in Definition 2.2.13, but these orbits also can be used in the rôle of the submanifolds mentioned in the Remark 2.2.11. Let us first introduce notation

$$\mathcal{D}_r^\omega(G) := \cap\{\mathcal{D}_{ra}(X_\xi); \xi \in Lie(G)\}, \quad \dim G < \infty, \tag{2.2.13}$$

i.e. the $Ad^*(U(G)$–invariant set $\boldsymbol{\mathcal{D}_r^\omega(G) \subset \mathcal{D}^\omega(G)}$ consists of finite dimensional density matrices with ranges in $D^\omega(G)$.

2.2.16. Lemma. *Let G be a finite–dimensional Lie group, and let $\nu \in \mathcal{D}^\omega(G)$. Then $\mathcal{O}_\nu(G)$ is an embedded submanifold [61] of $\mathfrak{T}_s$ lying in $\mathcal{S}_*$. If $\varrho \in \mathcal{D}_r^\omega(G)$, then $\varrho \in \mathcal{D}_{ra}(X_\xi)$, and $d_\varrho h_{X(\xi)} \in \mathfrak{N}_\varrho$, for all $\xi \in \mathfrak{g}$. The vectors $\mathbf{v}_{X(\xi)}(\varrho) := \mathrm{ad}^*_\varrho(d_\varrho h_{X(\xi)})(\xi \in \mathfrak{g} \equiv Lie(G))$ form the linear space $T_\varrho \mathcal{O}_\varrho(G)$. The union of the submanifolds $\mathcal{O}_\nu(G)$ ($\nu \in \mathcal{D}_r^\omega(G)$) composes a norm–dense subset of $\mathcal{S}_*$. The vectors $\mathbf{v}_{X(\xi)}(\varrho),\ \varrho \in \mathcal{D}_r^\omega(G)$, compose generalized vector fields $\mathbf{v}_{X(\xi)}(\cdot)$ ($\xi \in Lie(G)$) on $\mathcal{S}_*$ generating the flows $(t; \varrho) \mapsto \tilde{\varphi}_t^\xi(\varrho) := Ad^*(U(\exp(t\xi)))\varrho$.* ♣

Proof. Due to the continuity of $U(G)$, and because $\mathfrak{T}$ is a Hausdorff space, G_ν is a closed (hence Lie) subgroup of G. This implies that $Ad^*(U(\cdot))\nu$ can be considered as a bijective mapping of the analytic manifold G/G_ν onto the orbit $\mathcal{O}_\nu(G)$. This mapping is analytic, and its differential (i.e. the tangent map) maps the tangent space $T_e(G/G_\nu)$ onto a finite–dimensional subspace of $T_\nu \mathfrak{T}_s$, which is complementable. This fact together with the $Ad^*(U(\cdot))$–invariance

of $\mathcal{D}^{\omega}(G)$ implies, [40], that $\mathcal{O}_{\nu}(G)$ is an embedded submanifold of $\mathfrak{T}_s$. The second, and the third assertions are implied by the considerations developed in the Subsection 2.2-b, since the vector–fields $\mathbf{v}_{X(\xi)}(\nu)$:= $\mathrm{ad}^{*}_{\varrho}(d_{\varrho}h_{X(\xi)})$ generate the flows $\tilde{\varphi}^{\xi}$ which were used to formation of the orbit $\mathcal{O}_{\varrho}(G)$. The existence of a dense subset of $\mathcal{S}_{*}$ of analytic elements lying in $\mathcal{D}^{\omega}_{r}(G)$ with respect to the norm–topology of $\mathfrak{T}_s$in $\mathcal{S}_{*}$ implies the fourth assertion. Differentiation of these flows demonstrates also validity of the last statement. □

Let us extend now our definition of Poisson brackets (2.1.12) to densely defined functions $h_{X(\xi)}$ ($\xi \in Lie(G)$) defined on a dense subset of $\mathcal{S}_{*}$ consisting of orbits $\mathcal{O}_{\nu}(G)$. According to the construction of orbits $\mathcal{O}_{\nu}(G)$ from the "flows of $\mathbf{v}_{X(\xi)}(\cdot)$ generated by $h_{X(\xi)}$", it is clear that the vector fields $\mathbf{v}_{X(\xi)}(\cdot)$ are tangent to those orbits everywhere where they are defined. Let $\nu \in \mathcal{D}^{\omega}_{r}(G)$. Since $\mathrm{q}_{\nu}(X_{\xi}) = d_{\nu}h_{X(\xi)} \in \mathfrak{N}_{\nu}$ ($\xi \in Lie(G)$), and also $\nu X_{\xi} \in \mathfrak{F}$, we can define the commutator $i\,[d_{\nu}h_{X(\xi)}, d_{\nu}h_{X(\eta)}] \in \mathcal{L}(\mathcal{H})_{s}$, and the Poisson bracket according to the relation (2.1.15), cf. also Definitions 2.1.3:

$$\{h_{X_{\xi}}, h_{X_{\eta}}\}(\nu) := i\,\nu([d_{\nu}h_{X_{\xi}}, d_{\nu}h_{X_{\eta}}]) = \mathrm{ad}^{*}_{\nu}(\mathrm{q}_{\nu}(X_{\xi}))(\mathrm{q}_{\nu}(X_{\eta})). \tag{2.2.14a}$$

On the other hand, according to (2.2.11), one also has

$$h_{X_{[\xi,\eta]}}(\nu) = Tr(\nu X_{[\xi,\eta]}) = -i\,Tr(\nu[X_{\xi}, X_{\eta}]) = -i\,\nu([\mathrm{q}_{\nu}(X_{\xi}), \mathrm{q}_{\nu}(X_{\eta})]), \tag{2.2.14b}$$

what gives the result:

$$\{h_{X_{\xi}}, h_{X_{\eta}}\}(\nu) = -h_{X_{[\xi,\eta]}}(\nu). \tag{2.2.14c}$$

We shall consider this relation as the definition of the Poisson bracket in the Lie algebra of functions $h_{X(\xi)}$ ($\xi \in Lie(G)$) defined on their common domain

$$\boldsymbol{\mathcal{D}}(\mathbb{F}) := \{\nu \in \mathcal{S}_{*} : \text{the Fréchet differential of } g \mapsto \nu(U(g)) \text{ exists}\}, \tag{2.2.15a}$$

what implies[67] that[68]

$$\boldsymbol{\mathcal{D}}(\mathbb{F}) \subset \cap\{\mathcal{D}(h_{X(\xi)}) : \xi \in Lie(G)\}. \tag{2.2.15b}$$

The intersection $\cap\{\mathcal{D}(h_{X(\xi)}) : \xi \in Lie(G)\}$ is the domain consisting of those $\nu \in \mathcal{S}_{*}$ for which the function $g \mapsto \nu(U(g))$ is Gateaux differentiable. If $\dim G < \infty$, then the (continuous) Gateaux differentiability implies Fréchet differentiability, cf. [234, Lemma 1.15], hence

$$\boldsymbol{\mathcal{D}}(\mathbb{F}) = \cap\{\mathcal{D}(h_{X(\xi)}) : \xi \in Lie(G)\}, \text{ for } \dim G < \infty. \tag{2.2.15c}$$

The derivation property of Poisson brackets (Proposition 2.1.10) allows us to extend definition of this Poisson bracket to polynomials in variables $h_{X_{\xi}}$ ($\xi \in Lie(G)$) on the domain $\boldsymbol{\mathcal{D}}(\mathbb{F})$. The derivation property for the Poisson bracket of our not everywhere defined functions follows from the derivation property of commutators (also of unbounded operators on common invariant domains) via the equations (2.2.14) valid on $\boldsymbol{\mathcal{D}}(\mathbb{F})$. If we want to use polynomials in the variables

[67] Here the Fréchet differential can be understood as the differential of a mapping defined on the Banach manifold G, cf. [40, 234, 61].

[68] For explanation of the notation $\boldsymbol{\mathcal{D}}(\mathbb{F})$ see Definition 2.2.17 below.

h_{X_ξ} as generators of evolution of our generalized quantummechanical system determined by the described Poisson structure on $\mathcal{S}_*$, we have to define also Poisson brackets of these polynomials with differentiable (locally bounded) functions $f \in \mathcal{F}$. These are naturally determined for $\varrho \in \mathcal{D}_r^\omega(G) \subset \boldsymbol{\mathcal{D}}(\mathbb{F})$ by the formula:

$$\{h_{X_\xi}, f\}(\varrho) := i\,\varrho([\mathrm{q}_\varrho(X_\xi), d_\varrho f]). \tag{2.2.16}$$

This relation determines the vector fields $\mathbf{v}_{X_\xi}(\cdot)$ on $\mathcal{O}_\varrho(G)$ in accordance with Lemma 2.2.16.

Now we shall define the mapping $\mathbb{F}$, what appears to be one of the most useful objects for our subsequent considerations:

2.2.17. Definitions (Domains and momentum mapping $\mathbb{F}$). *Let $Lie(G)^* \equiv \mathfrak{g}^*$ denote the dual space to the Lie algebra of G (recall that $Lie(G)$ is a normable topological algebra also for infinite–dimensional G). Define also the* **restricted domain** $\boldsymbol{\mathcal{D}}_r(\mathbb{F}) := \mathcal{D}_r^\omega(G) \subset \boldsymbol{\mathcal{D}}(\mathbb{F})$, *cf. (2.2.13), and (2.2.15), of the mapping $\mathbb{F}$ (the* **Momentum mapping***), cf. [7, 1], which is defined on the domain $\boldsymbol{\mathcal{D}}(\mathbb{F})$ as follows:*

$$\mathbb{F} :\ \boldsymbol{\mathcal{D}}(\mathbb{F}) \to Lie(G)^*,\ \varrho \mapsto \mathbb{F}(\varrho) := F_\varrho, \tag{2.2.17a}$$
$$\text{with}\ \ F_\xi(\varrho) \equiv F_\varrho(\xi) := h_{X(\xi)}(\varrho).$$

Let us denote also by $f_\xi : Lie(G)^* \to \mathbb{R}$ the functions $f_\xi(F) := F(\xi) :=$ (the value of $F \in Lie(G)^*$ on the vector $\xi \in Lie(G)$). The **domain of** $\mathbb{F}$, *i.e. the set $\boldsymbol{\mathcal{D}}(\mathbb{F}) := \cap\{\mathcal{D}(h_{X(\xi)}) : \xi \in Lie(G)\} \subset \mathcal{S}_*(\mathcal{L}(\mathcal{H})_s)$ is $Ad^*(U(G))$ –invariant.*[69]

One can prove immediately validity of the following equivariance property:

$$F_{g\cdot\varrho} := \mathbb{F}(Ad^*(U(g))\varrho) = Ad^*(g) \circ \mathbb{F}(\varrho),\ \text{for all}\ \varrho \in \boldsymbol{\mathcal{D}}(\mathbb{F}),\ \text{and all}\ g \in G, \tag{2.2.17b}$$

since $U(g)X_\xi U(g)^* = X_{Ad(g)\xi}$ for all $\xi \in Lie(G)$; here $Ad^*(G)$ is the **coadjoint representation** *of G in $Lie(G)^*$, i.e. the dual representation to the adjoint representation $Ad(G)$, cf. Definition A.4.10,*

$$Ad(g)\xi := \left.\frac{d}{dt}\right|_{t=0} g \cdot \exp(t\xi) \cdot g^{-1}. \tag{2.2.17c}$$

Let $\mathbb{F}(\varrho)$ be called the (value of the) **U(G)-field** $\mathbb{F}$ *corresponding to the microscopic state ϱ.* ♢

2.2.18. **Remark**. The continuity of of the mapping $\mathbb{F}(\varrho) : \xi \mapsto \mathbb{F}(\varrho)(\xi)$ for $\varrho \in \boldsymbol{\mathcal{D}}(\mathbb{F})$ is trivial for finite dimensional G, since each finite dimensional linear function is continuous (in the unique lc–topology); in the case of a general Lie group representation (we restrict our attention to the representations with a dense analytic domain $\mathcal{D}_r^\omega(G) \subset \boldsymbol{\mathcal{D}}(\mathbb{F})$) the continuity for $\varrho \in \boldsymbol{\mathcal{D}}(\mathbb{F})$ is implied by the definition of points $\varrho \in \boldsymbol{\mathcal{D}}(\mathbb{F})$: Fréchet differentiability means linearity and continuity of the obtained mapping

$$\xi \mapsto h_{X_\xi} \equiv Tr(\varrho X_\xi) = i\,d_{g=e}[\varrho(U(g))](\xi).$$

We shall usually consider in the following, however, finite–dimensional Lie groups G. ♡

[69] For a general definition, and also for various applications of momentum mappings cf., e.g. [1, 179].

2.2.19. **Remark**. Let us note that the states $\varrho \in \boldsymbol{\mathcal{D}}(\mathbb{F})$ are exactly those normal states of a constituent microsystem of a macroscopic one (in the description of infinite quantal systems composed of equal "microscopic constituents", cf. Section 3.4) in infinite (symmetric) tensor products ω_ϱ of which the "macroscopic observables" $X_{\xi\Pi}$ ($\xi \in Lie(G)$) are defined:

$$\omega_\varrho := \bigotimes_{p\in\Pi} \varrho_p \in \mathcal{S}_*(\mathcal{A}^{**}) \ (\varrho_p \equiv \varrho), \tag{2.2.18a}$$

$$\omega_\varrho(X_{\xi\Pi}) = \mathbb{F}(\varrho)(\xi) \equiv F_\varrho(\xi) := h_{X_\xi}(\varrho), \tag{2.2.18b}$$

$$X_{\xi\Pi} := (\text{"}w''\text{)} - \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \sum_{p\in\Lambda} X_p, \tag{2.2.18c}$$

where $p \in \Pi$ distinguishes copies of the "microscopic constituents", Λ is a finite subset of these copies, and X_p are "equal observables" for distinguished copies $p \in \Pi$. The limit in the formula above is taken in a specific weak ("w") topology (we shall not specify it here, see, e.g. [31]). In this connection, the introduced function $\mathbb{F}(\varrho)$ is called also the $\boldsymbol{U(G)}$**–macroscopic field** corresponding to the "microscopic state" ϱ.
Observe also, that the value of the $\boldsymbol{\mathfrak{U}}$**–macroscopic field** corresponding to $\varrho \in \mathcal{S}_*$ (for the defining representation $\mathfrak{U} \to \mathfrak{U}$ of the unitary group of $\mathcal{H}$) is ϱ itself: The dual space to the $Lie(\mathfrak{U}) := i\,\mathcal{L}(\mathcal{H})_s$ can be identified with $\mathcal{L}(\mathcal{H})_s^*$ containing the (normal) state space $\mathcal{S}_*$ as an $Ad^*(\mathfrak{U})$–invariant subset. This is in a sense maximal "classical macroscopic phase space" $\mathcal{S}_* \ : \mathbb{F}(\nu) \equiv \mathbb{F}_\mathfrak{U}(\nu) = \nu \ (\forall \nu \in \mathcal{S}_*)$. Such a "macroscopic field" separates points of the elementary quantum phase space, i.e. the macroscopic field $\mathbb{F}_\mathfrak{U}$ determines corresponding microscopic states. **This can be considered as a formalization of the conventional belief of QM that a macroscopically determined "preparation procedure" determines the corresponding microscopic state of a considered quantummechanical system uniquely.** ♡

We could temporarily take the point of view that only "macroscopic properties" of the system (in the sense of Remark 2.2.19) described by the values of $\mathbb{F}$ are interesting for us. Then it would be interesting to know in what extent the values $\mathbb{F}(\nu)$ separate the points ν of an orbit $\mathcal{O}_\varrho(G)$.

2.2.20. Lemma. *Let $\varrho \in \boldsymbol{\mathcal{D}}_r(\mathbb{F})$, $\xi, \eta \in Lie(G)$. Then*

$$\frac{d}{dt} F_{\exp(t\eta)\cdot\varrho}(\xi) = F_\varrho([Ad(\exp(-t\eta))\xi, \eta]), \tag{2.2.19}$$

for all $t \in \mathbb{R}$. In particular, if we have a fixed $\eta \in Lie(G)$ such that the derivative in (2.2.19) *vanishes for all $\xi \in Lie(G)$ at one value of $t \in \mathbb{R}$, then it vanishes for all ξ at all values of $t \in \mathbb{R}$.* ♣

Proof. By a use of the identity

$$U(g) X_\xi U(g^{-1}) = X_{Ad(g)\xi},$$

as well as of the relation

$$\left.\frac{d}{dt}\right|_{t=0} F_{\exp(t\eta)\cdot\varrho}(\xi) = F_\varrho([\xi, \eta]), \ \forall \xi, \eta \in \mathfrak{g}, \tag{2.2.20}$$

cf.(2.2.11), and (2.2.17), we obtain

$$
\begin{aligned}
F_{\exp(t\eta)\cdot\varrho}([\xi,\eta]) &= Tr\Big(U(\exp(t\eta))\varrho U(\exp(-t\eta))X_{[\xi,\eta]}\Big) && \text{(2.2.21a)}\\
&= -i\,Tr\Big(\varrho\, U(\exp(-t\eta))[X_\xi, X_\eta]U(\exp(t\eta))\Big) && \text{(2.2.21b)}\\
&= -i\,Tr\Big(\varrho\,[U(\exp(-t\eta))X_\xi U(\exp(t\eta)), X_\eta]\Big) && \text{(2.2.21c)}\\
&= -i\,Tr\Big(\varrho\,[X_{Ad(\exp(-t\eta))\xi}, X_\eta]\Big) && \text{(2.2.21d)}\\
&= F_\varrho([Ad(\exp(-t\eta))\xi,\eta]). && \text{(2.2.21e)}
\end{aligned}
$$

After a subsequent application of (2.2.20) with $\varrho \mapsto \exp(t\eta)\cdot\varrho$, the preceding calculation gives the result. □

This lemma gives an answer to the question on separation properties of $\mathbb{F}$ on $\mathcal{O}_\varrho(G)$:

Let $\eta \in Lie(G)$ be such that $F_\varrho([\xi,\eta]) = 0, \forall \xi \in Lie(G)$. Then $\mathbb{F}(\exp(t\eta)\cdot\varrho) = \mathbb{F}(\varrho)$ for all $t \in \mathbb{R}$, hence the points $exp(t\eta)\cdot\varrho \in \mathcal{O}_\varrho(G)$ for different values of t cannot be distinguished by the values of the field $\mathbb{F}$. The vectors η form the Lie algebra of the **stability subgroup of** G **at the point** $\mathbb{F}(\varrho)$ with respect to the action of the $Ad^*(G)$–representation denoted by $\boldsymbol{G}_{\mathbb{F}(\boldsymbol{\varrho})}$. Clearly, it is valid

2.2.21. Lemma. *Let $G_\varrho \subset G$ be the stability subgroup of the $Ad^*(U(G))$–action of G on $\mathcal{S}_*$, at the point $\varrho \in \mathcal{S}_*$. Then $G_\varrho \subset G_{\mathbb{F}(\varrho)}$, and the equality $G_\varrho = G_{\mathbb{F}(\varrho)}$ is valid iff the restriction of the mapping $\mathbb{F}$ to $\mathcal{O}_\varrho(G)$ is a bijection onto an $Ad^*(G)$–orbit in $Lie(G)^*$.* ♣

2.2.22. **Remark**. A definition of Poisson bracket on $\mathcal{O}_\varrho(G)$, with $\varrho \in \mathcal{D}^\omega_r(G)$, equivalent to that in (2.2.14), can be given with a help of the (strongly) symplectic structure Ω_ϱ by definition of a closed two–form $\boldsymbol{\iota}^*_\varrho\Omega_\varrho$ – the pull back of the "overlying" form Ω by the embedding $\boldsymbol{\iota}$ of the manifold $\mathcal{O}_\varrho(G)$ into $\mathcal{O}_\varrho(\mathfrak{U})$, in the case if the obtained two–form on the submanifold $\mathcal{O}_\varrho(G)$ is nondegenerate. If the restricted symplectic structure $\boldsymbol{\iota}^*_\varrho\Omega_\varrho$ is degenerate, we can obtain a symplectic manifold by factorization of $\mathcal{O}_\varrho(G)$ according to the orbits of stability subgroups $G_{\mathbb{F}(\nu)}$ leaving the values $\mathbb{F}(\nu) \in \mathfrak{g}^*,\ \nu \in \mathcal{O}_\varrho(G)$ invariant, [26, 27]. ♡

One can construct examples of representations $U(G)$ with both even– and odd–dimensional orbits $\mathcal{O}_\varrho(G)$ ($\varrho \in \boldsymbol{\mathcal{D}}_r(\mathbb{F})$) (for finite–dimensional G [27], cf. also our Subsection 3.3-c). Orbits of the $Ad^*(G)$–representation are always "even–dimensional": They are endowed with a canonical Kirillov–Kostant symplectic structure corresponding to the standard Poisson structure (called also *Berezin brackets*) on $Lie(G)^* \equiv \mathfrak{g}^*$:[70]

$$\{f_\xi, f_\eta\}(F) = -F([\xi,\eta]) := -f_{[\xi,\eta]}(F). \tag{2.2.22}$$

If $\mathbf{v}_\xi(F) \in T_F(\mathfrak{g}^*)$ ($\xi \in Lie(G)$) are the vectors tangent at $F \in \mathfrak{g}^* \equiv Lie(G)^*$ to the flows $(t;F) \mapsto Ad^*(\exp(t\xi))F$, then the Kirillov–Kostant symplectic form Ω^K can be expressed as

$$\Omega^K_F(\mathbf{v}_\xi, \mathbf{v}_\eta) = -F([\xi,\eta]). \tag{2.2.23}$$

[70] These considerations might also be valid for infinite–dimensional Lie groups, cf. [7, Appendix 13].

Comparison of the relation (2.2.22) with (2.2.14) shows, that the mapping $\mathbb{F}$ is a **Poisson morphism**, [274]: The functions

$$\mathbb{F}^* f_\xi := f_\xi \circ \mathbb{F} = h_{X(\xi)} =: F_\xi \equiv \mathrm{f}_\xi$$

on $\boldsymbol{\mathcal{D}}_r(\mathbb{F})$ satisfy (2.2.14), what leads to a definition of Poisson brackets for all functions f on $\mathcal{O}_\varrho(G)$ ($\varrho \in \boldsymbol{\mathcal{D}}(\mathbb{F})$) which are expressible in the form [71]

$$\mathrm{f} := \mathbb{F}^* f := f \circ \mathbb{F}, \quad f \in C^\infty(\mathfrak{g}^*, \mathbb{R}). \tag{2.2.24}$$

2.2.23. **Remark**. In the case of infinite–dimensional groups, we cannot expect reflexivity of $\mathfrak{g}$: For $\mathfrak{g} := \mathcal{L}(\mathcal{H})_s = Lie(\mathfrak{U}) = \mathfrak{T}_s^*$ and infinite–dimensional Hilbert space $\mathcal{H}$ one has $\mathfrak{g}^* = \mathcal{L}(\mathcal{H})_s^* \neq \mathfrak{T}_s$, and $\mathfrak{g}^{**}$ is strictly larger than $\mathfrak{g}$. Then we have to be careful in reading (2.2.24): If the differentiation of $f \in C^\infty$ is taken in the canonical norm–topology of $\mathfrak{g}^*$, then the first differentials of f's belong generally to $\mathcal{L}(\mathcal{H})^{**}$, and needn't be expressible as bounded operators on $\mathcal{H}$. The space $\mathcal{L}(\mathcal{H})^{**}$ is, however a von Neumann algebra in a canonical way, [227, 254, 76, 77, 42], hence also endowed with a canonical Poisson–commutator structure, which is unique extension of that of $\mathcal{L}(\mathcal{H})$. Another possibility would be to take derivatives on $\mathfrak{g}^*$ in the weak*–topology (in some sense, cf. [155] for a theory of differentiation on locally convex spaces), in which case we could stay in $\mathfrak{g}$ ($\ni df$); in this case we would work with a restricted set of functions f differentiable in a weaker than norm–sense. We shall consider norm differentiability, if another possibility is not mentioned explicitly. Most of formulas can be considered, however, also in another interpretation. ♡

The functions $\mathrm{f} : \mathcal{O}_\varrho(G) \to \mathbb{R}$ of the form (2.2.24) will play a rôle of (nonlinear, unbounded – in general) generators of transformation groups (e.g. of time evolution) in our theory, cf. Proposition 2.3.20. Their mutual Poisson brackets are defined in accordance with (2.2.14) in the following way:

$$\{\mathbb{F}^* f, \mathbb{F}^* h\}(\nu) := \mathbb{F}^*\{f, h\}(\nu)\ \forall \nu \in \boldsymbol{\mathcal{D}}(\mathbb{F}), \forall f, h \in C^\infty(Lie(G)^*, \mathbb{R}), \tag{2.2.25a}$$

where the bracket on the right side of the relation is the Berezin bracket. The equation (2.2.25a) shows that the mapping $\mathbb{F}$ of $\boldsymbol{\mathcal{D}}(\mathbb{F})$ onto its image in $\mathfrak{g}^*$ is a **Poisson morphism (**resp. **mapping)**, cf. [274]. It follows, that trajectories of the Hamiltonian flow corresponding to Hamiltonian function $\mathrm{h} := h \circ \mathbb{F}$ on $\boldsymbol{\mathcal{D}}(\mathbb{F})$ are projected onto trajectories of the Hamiltonian flow corresponding to the Hamiltonian function h on coadjoint orbits of G. We shall find later also a possibility of determination of flows on $\boldsymbol{\mathcal{D}}(\mathbb{F})$ from given Hamiltonian flows on $\mathfrak{g}^*$. For $\nu \in \boldsymbol{\mathcal{D}}_r(\mathbb{F}), \mathrm{f} \in \mathcal{F}$, and $h \in C^\infty(Lie(G)^*, \mathbb{R})$, we shall extend our definitions of the Poisson brackets as follows:

$$\{\mathbb{F}^* h, \mathrm{f}\}(\nu) := d_{\mathbb{F}(\nu)} h \circ \{\mathbb{F}, \mathrm{f}\}(\nu), \tag{2.2.25b}$$

where $d_{\mathbb{F}(\nu)} h \in \mathcal{L}(Lie(G)^*, \mathbb{R})$ ($= Lie(G)$, in the case of weak differentiability, cf. e.g. Remark 2.2.23) is the differential of h in the point $\mathbb{F}(\nu) \in Lie(G)^*$, $\{\mathbb{F}, f\}(\nu) \in Lie(G)^*$ is defined by its values $\{\mathbb{F}_\xi, \mathrm{f}\}(\nu) := \{h_{X(\xi)}, \mathrm{f}\}(\nu) \in \mathbb{R}$ on the elements $\xi \in Lie(G)$, and $\{h_{X(\xi)}, \mathrm{f}\}$ is defined in (2.2.16).

[71] We shall usually distinguish typographically, in the following text, real valued functions f, h defined on the dual of the Lie algebra, $\mathfrak{g}^* \equiv Lie(G)^*$, from the "corresponding" functions $\mathrm{f} := \mathbb{F}^* f, \mathrm{h} := \mathbb{F}^* h$ defined on domains lying in $\mathcal{S}_*$. To stress the difference of domains, we shall write also f, e.g. for arbitrary functions $\mathrm{f} \in \mathcal{F}(\mathcal{S}_*)$.

Let us note also, that

$$d_\varrho(\mathbb{F}^* f) = \mathrm{q}_\varrho(X(d_{\mathbb{F}(\varrho)} f)),\ for\ \varrho \in \boldsymbol{\mathcal{D}}_r(\mathbb{F}).$$

Let $\{\xi_j : j = 1, 2, \ldots \dim(G) < \infty\}$ be a basis of $\mathfrak{g} = Lie(G)$ and let $F_j := F(\xi_j)$ be coordinates of $F \in \mathfrak{g}^*$ in the dual basis. Then the Poisson bracket (2.2.25b) can be expressed as

$$\{\mathbb{F}^* h, \mathrm{f}\}(\nu) = \sum_j \partial_j h((\nu)) \{\mathrm{h}_{X(\xi_j)}, \mathrm{f}\}(\nu), \tag{2.2.25c}$$

and the Poisson bracket (2.2.25a) can also be written in the form:

$$\{\mathbb{F}^* f, \mathbb{F}^* h\}(\nu) = \sum_{j,k} \partial_j f(\mathbb{F}(\nu)) \partial_k h(\mathbb{F}(\nu)) \{F_j, F_k\}(\mathbb{F}(\nu)). \tag{2.2.25d}$$

Observe that (cf. Theorem 2.1.19) the restriction to the submanifold $\mathcal{O}_\varrho(G)$ of the symplectic form Ω defined in (2.1.27) on $\mathcal{O}_\varrho(\mathfrak{U})$ (i.e. the pull–back of Ω by the embedding of $\mathcal{O}_\varrho(G)$ into $\mathcal{O}_\varrho(\mathfrak{U})$) coincides with the pull–back of the Kirillov–Kostant form Ω^K by the mapping $\mathbb{F}$:

$$\left(\mathbb{F}^* \Omega^K\right)_\nu (\mathbf{v}, \mathbf{w}) = \Omega_\nu(\mathbf{v}, \mathbf{w}), \tag{2.2.25e}$$

for $\nu \in \mathcal{O}_\varrho(G),\ \mathbf{v}, \mathbf{w} \in T_\nu \mathcal{O}_\varrho(G)$, and $\varrho \in \boldsymbol{\mathcal{D}}_r(\mathbb{F})$.

The formulas (2.2.25b), (2.2.25c) show that the function $\mathrm{Q} := \mathbb{F}^* Q,\ Q \in C^\infty(\mathfrak{g}^*, \mathbb{R})$, generates a generalized (densely defined) vector field $\mathbf{v}_\mathrm{Q}$ on $\mathcal{S}_*$ with values (for $\dim G < \infty$):

$$\mathbf{v}_\mathrm{Q}(\nu) = \sum_j \partial_j Q(\mathbb{F}(\nu))\ \mathbf{v}_{X(\xi_j)}(\nu). \tag{2.2.26a}$$

For an arbitrary G, and such Q that $d_{\mathbb{F}(\nu)} Q \in \mathfrak{g} \subset \mathfrak{g}^{**}$ one has:

$$\mathbf{v}_\mathrm{Q}(\nu) = \mathrm{ad}^*_\nu(\mathrm{q}_\nu(X(d_{\mathbb{F}(\nu)} Q))). \tag{2.2.26b}$$

This describes a class of Hamiltonian (resp. Poisson) generalized vector fields generating the flows $\tilde{\varphi}^t_Q$ leaving the corresponding $U(G)$–orbits in the state space $\mathcal{S}_*$ invariant. One can see that the generating Hamiltonian functions Q are constant on the orbits of the action $Ad^*(U(G_{\mathbb{F}(\nu)}))$, i.e.

$$\mathrm{Q}(Ad^*(U(\exp(t\eta))\nu)) \equiv \mathrm{Q}(\nu),\ \ \eta \in Lie(G_{\mathbb{F}(\nu)}).$$

This suggests an idea how to restrict the Poisson actions of other generators to the orbits $\mathcal{O}_\varrho(G)$, cf.2.2.26. We shall also introduce

2.2.24. Definition (Poisson structure on submanifolds of $\mathcal{O}_\varrho(\mathfrak{U})$). *Let $\mathcal{N}$ be a submanifold of $\mathcal{O}_\varrho(\mathfrak{U})$, and $\Omega_\nu,\ \nu \in \mathcal{O}_\varrho(\mathfrak{U})$ be the symplectic form from* (2.1.27)*. Let the restriction of Ω to $\mathcal{N}$, i.e. the pull back with respect to embedding $\iota_\mathcal{N} : \mathcal{N} \to \mathcal{O}_\varrho(\mathfrak{U})$, $\Omega^\mathcal{N} := \iota^*_\mathcal{N} \Omega$ be nondegenerate. Then the symplectic structure $\Omega^\mathcal{N}$ on $\mathcal{N}$ will be also called* **the restriction of the Poisson structure on $\mathcal{S}_*$ to $\mathcal{N}$.** $\diamondsuit$

Let us formulate now a theorem containing some results and consequences of the preceding considerations:

2.2.25. Theorem. *Let $Q \in C^\infty(\mathfrak{g}^*, \mathbb{R}),\ \mathrm{Q} := Q \circ \mathbb{F}$, hence $\mathrm{Q} \in C^\infty(\mathcal{O}_\nu(G), \mathbb{R}),\ \forall \nu \in \boldsymbol{\mathcal{D}}_r(\mathbb{F})$. Assume that $d_{\mathbb{F}(\varrho)} Q \in \mathfrak{g}$ for some $\varrho \in \boldsymbol{\mathcal{D}}_r(\mathbb{F})$ (this assumption might be nontrivial for infinite–dimensional G). Then $\mathbf{v}_Q(\nu)$ from* (2.2.26b) *is a Hamiltonian vector field on $\mathcal{O}_\varrho(G)$ (hence, it is tangent to $\mathcal{O}_\varrho(G)$, everywhere on $\mathcal{O}_\varrho(G)$) corresponding to the Poisson structure on $\mathcal{O}_\varrho(\mathfrak{U})$ determined by the pull–back of $\mathbb{F}$,* (2.2.25a)*, or, equivalently, to the "original" Poisson structure on $\mathcal{S}_*$ restricted to the ($\|\cdot\|_1$–dense) collection of orbits $\mathcal{O}_\nu(G)$ lying in $\boldsymbol{\mathcal{D}}_r(\mathbb{F})$. Then the (local) flow $\tilde{\varphi}^t_Q$ leaves the orbits $\mathcal{O}_\nu(G)$ invariant.* ♣

We shall now formulate concepts describing Hamiltonian dynamics and symmetries on "allowed" submanifolds of $\mathcal{O}_\varrho(\mathfrak{U})$.

2.2.26. Definitions (Classical and restricted G–dynamics).

(i) Let Ran$(\mathbb{F}) \subset \mathfrak{g}^*$ *denote the image of $\boldsymbol{\mathcal{D}}(\mathbb{F})$ under $\mathbb{F}$. We shall consider $\mathfrak{g}^*$ either with its canonical (coming from that of $\mathfrak{g}$) norm–topology, or with its w^*–topology (again with respect to the canonical norm–topology of $\mathfrak{g}$, [39]; this will be different from the norm–topology for infinite–dimensional G). Let $\boldsymbol{\mathcal{E}}_\mathbb{F}$ denote the closure of* Ran$(\mathbb{F})$ *in that topology. The space $\mathcal{E}_\mathbb{F}$ will be also called the* ***G*–classical (***alternatively***: G–macroscopic) phase space** *of the system. Let by $C^\infty(\mathcal{M}, \mathbb{R}),\ \mathcal{M} \subset \mathfrak{g}^*$, be denoted the set of all infinitely differentiable functions on an (arbitrary) open neighbourhood of $\mathcal{M}$ in the corresponding topology (we shall not specify here the way of differentiation on nonnormable lc–spaces, cf. however [155]).*

(ii) If the $\boldsymbol{\mathcal{D}}_r$–generalized differential of f *is $\mathcal{S}_*$–integrable we say that* f **generates the Poisson flow $\tilde{\varphi}^{\mathrm{f}}$ on $\mathcal{S}_*$.**

(iii) Let a densely defined real function $\mathrm{f} : \boldsymbol{\mathcal{D}}_r \to \mathbb{R}$ generate a Poisson flow on $\mathcal{S}_*$, and let there is a differentiable function f on (an open – in the corresponding topology – neighbourhood of) $\mathcal{E}_\mathbb{F}$, $f \in C^\infty(\mathcal{E}_\mathbb{F}, \mathbb{R})$ such, that $\mathrm{f} \equiv \mathbb{F}^* f := f \circ \mathbb{F}$ on $\boldsymbol{\mathcal{D}}_r$. Then f **is a G–classical generator**. [72]

(iv) Let f *generate a Poisson flow on $\mathcal{O}_\varrho(\mathfrak{U})$ (the submanifold $\mathcal{O}_\varrho(\mathfrak{U}) \subset \mathcal{S}_*$ can be substituted for $\mathcal{S}_*$ in obvious modifications of preceding definitions). Let ν be such that $\mathcal{O}_\nu(G) \subset \boldsymbol{\mathcal{D}}(\mathbb{F}) \cap \mathcal{O}_\varrho(\mathfrak{U}) \cap \boldsymbol{\mathcal{D}}_r$, and let the restriction f^ν of* f *to $\mathcal{O}_\nu(G)$ can be expressed in the form*

$$f^\nu(\mathbb{F}(\nu')) \equiv \mathrm{f}^\nu(\nu') := \mathrm{f}(\nu'),\ \textit{for}\ \nu' \in \mathcal{O}_\nu(G), \tag{2.2.27}$$

with some $f^\nu \in C^\infty(Ad^*(G)\mathbb{F}(\nu), \mathbb{R})$, hence $\mathrm{f}^\nu = \mathbb{F}^* f^\nu$. Then the function f *will be called a νG–***classical generator***. (Hence, the same function* f *can be a νG–classical generator for several different orbits $\mathcal{O}_\nu(G)$.)*

(v) Let f *be a νG–classical generator. Its flow $\tilde{\varphi}^{\mathrm{f}}$ needn't leave the orbit $\mathcal{O}_\nu(G)$ ($\subset \mathcal{O}_\varrho(\mathfrak{U})$) invariant.*[73] *Let $\tilde{\varphi}^{\nu,\mathrm{f}}$ be the (Poisson) flow on the orbit $\mathcal{O}_\nu(G)$ corresponding to the vector field on $\mathcal{O}_\nu(G)$ generated by f^ν according to* (2.2.25b) *and* (2.2.26) *(with h, resp. Q replaced by f^ν). The flow $\tilde{\varphi}^{\nu,\mathrm{f}}$ will be called the νG–***restriction of the flow** $\tilde{\varphi}^{\mathrm{f}}$ *to the orbit $\mathcal{O}_\nu(G)$.* ♢

[72] More sophisticated and more distinctive (and also more complicated) work with domains was presented in [24]; the corresponding modifications of concepts connected with domains presented in this paper can be, however, seen without being explicitly formulated here.

[73] This is a difference with respect to G–classical generators, cf. also Proposition 2.3.10.

Let us present now, without detailed explanation (hence without an analysis and proofs), some examples of νG–classical generators.

2.2.27. **Examples**. Let a representation $U(G)$ be given as above, and let $\mathbb{F}$ be the corresponding momentum mapping. Let Y be a selfadjoint operator on $\mathcal{H}$, and let h_Y be the corresponding (densely defined, generalized) generator. Let $\mathcal{O}_\nu(G) \subset \boldsymbol{\mathcal{D}}_r(\mathbb{F}) \cap \mathcal{D}_{ra}(\delta_Y)$, with $\mathcal{D}_{ra}(\delta_Y)$ denoting the set of analytic elements of δ_Y belonging to $\mathfrak{F}_s$. Then h_Y is a νG–classical generator, e.g., in any of the following cases:

(i) $Y := X_\xi$ for some $\xi \in \mathfrak{g}$.

(ii) $Y := i^N[X(\xi_1), [X(\xi_2), [\dots [X(\xi_N), A]\dots]]]$, where $\xi_j \in \mathfrak{g}$, $(j = 1, 2, \dots N)$, and A is such a selfadjoint operator on $\mathcal{H}$ that h_A is a νG–classical generator. The commutators can be considered here in a generalized sense, [27], so that it ensures existence of h_Y in the points $\varrho \in \mathcal{O}_\nu(G)$ in the sense of (2.2.2). This can lead to νG–classical generator h_Y even in some cases, when the above expression does not determine a well defined linear operator.

(iii) All stability subgroups $G_{\mathbb{F}(\omega)}$ $(\omega \in \mathcal{O}_\nu(G))$ of points $\mathbb{F}(\omega) \in \mathbb{F}(\mathcal{O}_\nu(G)) \equiv Ad^*(G)\mathbb{F}(\nu)$ are symmetry groups of the operator Y:

$$U(g)YU(g^{-1}) = Y,\ \forall g \in \cup\{G_{\mathbb{F}(\omega)} : \omega \in \mathcal{O}_\nu(G)\},$$

and, moreover, $h^\nu_Y \in C^\infty(\mathcal{O}_\nu(G), \mathbb{R})$.

(iv) The orbit $\mathcal{O}_\nu(G)$ is such, that $G_{\mathbb{F}(\omega)} = G_\omega$, $\forall \omega \in \mathcal{O}_\nu(G)$. The subgroups $G_\omega \subset G$ are stability subgroups of the points ω of the orbit $\mathcal{O}_\nu(G)$ for the considered action: $g \mapsto Ad^*(U(g))\omega = \omega \Leftrightarrow g \in G_\omega$. $\heartsuit$

Restrictions of "true quantum–mechanical dynamics" to various submanifolds of "coherent states" (i.e. to orbits $\mathcal{O}_\nu(G)$) are often considered [149, 221, 222] as approximations (sometimes called "quasiclassical") to the "true dynamics".[74] This is not, however, a "good approximation" for a general (linear) quantum dynamics, and what are conditions for well controlled validity (i.e. a relevance) of such approximations is not yet, as far as the present author knows, generally established.

2.2.28. **Remark**. The $U(G)$–restriction $\tilde{\varphi}^{\nu,\mathrm{f}}$ of the quantal flow $\tilde{\varphi}^{\mathrm{f}}$ needn't be "close" to $\tilde{\varphi}^{\mathrm{f}}$ for a general νG–classical generator f, not even for "classical" (or "macroscopic") quantities described by expectations of a distinguished subset of selfadjoint operators. One can compare, e.g., the evolution of the expectations $h_{X(\xi)}$ of quantum "observables" X_ξ under $\tilde{\varphi}^{\mathrm{f}}$, i.e. the function

$$(t; \xi) \mapsto h_{X(\xi)}(\tilde{\varphi}^{\mathrm{f}}_t\nu) \equiv \mathbb{F}_\xi(\tilde{\varphi}^{\mathrm{f}}_t\nu) \in \mathfrak{g}^*, \tag{2.2.28}$$

with the restricted evolution $\mathbb{F}_\xi(\tilde{\varphi}^{\nu,\mathrm{f}}_t(\nu))$, for the same initial conditions. $\heartsuit$

Let us illustrate this remark by a simple example, cf. also [27, 4.1.10].

2.2.29. **Illustration** *(Restricted and "global" flows might be "very" different).*
Let us take $\mathcal{H} := L^2(\mathbb{R}, dq)$, and let

$$\psi \in \mathcal{H},\ \psi(q) := \pi^{-\frac{1}{4}} \exp\left(-\frac{1}{2}q^2\right);$$

[74] These "restrictions" were called in [27] **"classical projections"** of quantummechanical evolutions.

let us set $\psi_z := U_z\psi$, with $z := q - ip \in \mathbb{C}$, and $U_z := \exp(i(pQ - qP))$. Here Q and P are the Schrödinger operators of position and linear momentum in QM:

$$Q\chi(q) \equiv q\chi(q),\ P\chi(q) \equiv -i\frac{\partial}{\partial q}\chi(q), \quad \chi \in \mathcal{H}.$$

Let the (artificial) "generator of time evolution" be $H := \alpha \cdot P_\psi,\ \alpha \in \mathbb{R}$, i.e. it is proportional to a one–dimensional projection. We shall consider the restriction of the corresponding flow to the orbit $\mathcal{O}_{\boldsymbol{\psi}}(G_{WH})$ of the 3–dimensional Weyl–Heisenberg group G_{WH} (cf. also Subsection 3.3-b) defined by the injective mapping of the "classical phase space" $\mathbb{C} \ni z$ into the projective Hilbert space: $z\ (\in \mathbb{C}) \mapsto P^z_\psi := U_z P_\psi U_z^* \in P(\mathcal{H})$. If we parameterize points of the orbit by $z \in \mathbb{C}$, then the restriction h^ψ_H of the corresponding Poisson generator h_H to the orbit is:

$$\mathrm{h}^\psi_H(z) \equiv Tr(P^z_\psi H) = \alpha \exp\left(-\frac{1}{2}\overline{z}z\right), \tag{2.2.29a}$$

with $z \mapsto \overline{z}$ being the complex conjugation. The restricted flow is identical (by the identification $z \longleftrightarrow P^z_\psi$) to the Hamiltonian flow

$$\tilde{\varphi}^{\psi,H}_t z \equiv \exp(-it\mathrm{h}^\psi_H(z))z, \tag{2.2.29b}$$

generated by the Hamiltonian function (2.2.29a) on the classical phase space $\mathbb{R}^2$ with the symplectic form $\Omega \equiv dp \wedge dq$. The "true quantal flow" with the same initial condition $z = q - ip$ is

$$\begin{aligned}\tilde{\varphi}^H_t z &:= Tr(\exp(-itH)P^\psi_z \exp(itH)(Q - iP)) \\ &\equiv (1 - \alpha^{-1}\mathrm{h}^\psi_H(z))z + \alpha^{-1}\mathrm{h}^\psi_H(z)\exp(-it\alpha)z.\end{aligned} \tag{2.2.29c}$$

By comparing these two evolutions of "the same classical quantities", i.e. the two motions in $\mathbb{C}$, we see two uniform motions on mutually tangent circles with different radii and different frequencies. This shows that, for a general Hamiltonians, the "classical projections" needn't be any approximations to the "true quantum dynamics". ♡

The next assertion shows in what sense the restricted generators also are of relevance for the (unrestricted) quantum theory.

2.2.30. Proposition. *Let* f *be a* νG*–classical generator and let* f^ν *be its restriction to* $\mathcal{O}_\nu(G)$*. Then, by considering the definitions* (2.2.25) *of Poisson brackets on* $\mathcal{O}_\nu(G)$*, for all* $\nu' \in \mathcal{O}_\nu(G)$*, and for any* $Q \in C^\infty(\mathfrak{g}^*, \mathbb{R})$*, the following relations are valid:*

$$\{\mathrm{Q}, \mathrm{f}\}(\nu') = \{\mathrm{Q}, \mathrm{f}^\nu\}(\nu') = \mathbb{F}^*\{Q, f^\nu\}(\nu'), \tag{2.2.30}$$

where $\mathrm{Q} := \mathbb{F}^*Q$*, and* $\mathrm{f}^\nu =: \mathbb{F}^* f^\nu$*, and where the first bracket is defined according to* (2.2.25b) *(or, equivalently, by the formula* (2.1.15) *with a help of generalized differentials of* Q *and* f*).* ♣

Proof. The second equation in (2.2.30) is just the first equation of (2.2.25). The unrestricted Poisson bracket on $\mathcal{S}_*$ occurring on the left side of (2.2.30) is equal, according to (2.2.14b), to the derivative of f along to the vector field (2.2.26) at each point $\varrho \in \mathcal{O}_\nu(G)$. This implies,

that the derivative $d_{\nu'}\mathrm{f}(\mathbf{v}_\mathrm{Q}) = \{\mathrm{Q},\mathrm{f}\}(\nu')$ in any point $\nu' \in \mathcal{O}_\nu(G)$ of an arbitrary function only depends on its restriction f^ν to $\mathcal{O}_\nu(G)$. One has $d_{\nu'}\mathrm{f}(\mathbf{v}_\mathrm{Q}) \equiv d_{\nu'}\mathrm{f}^\nu(\mathbf{v}_\mathrm{Q})$ on $\mathcal{O}_\nu(G)$, and the last derivative is expressed by the Poisson bracket (2.2.25) on $\mathcal{O}_\nu(G)$. This proves the first equation. □

2.2.31. **Remark**. The definitions (2.2.14) of the Poisson bracket *on an* $Ad^*(U(G))$*–orbit* $\mathcal{O}_\nu(G)$ were formulated with a help of selfadjoint operators on (dense domains of) $\mathcal{H}$, so that our construction of the Poisson structure on $\mathcal{O}_\nu(G)$ is not an "intrinsic construction" on the orbit alone: It uses the values of the differentials of the functions $h_{X(\xi)}$ and f as elements of infinite–dimensional spaces $T_\varrho\mathcal{O}_\nu(\mathfrak{U})$ for points of a G–orbit, $\varrho \in \mathcal{O}_\nu(G) \subset \mathcal{O}_\varrho(\mathfrak{U})$ (which is finite–dimensional in the case $\dim G < \infty$). The differential $d_\varrho\mathrm{f}$ cannot be calculated in general cases from the restriction of $\mathrm{f} \in \mathcal{F}$ to the orbit $\mathcal{O}_\nu(G)$ only. If $\mathcal{O}_\nu(G)$ is a symplectic manifold with the symplectic structure obtained by pull–back of the Kirillov–Kostant form Ω^K on $\mathbb{F}(\mathcal{O}_\nu(G)) = Ad^*(G)\mathbb{F}(\nu)$, or equivalently, if the restriction of the bilinear forms $\Omega_\varrho,\ \varrho \in \mathcal{O}_\nu(G)$ to $\mathcal{O}_\nu(G)$ (i.e. to $T_\varrho\mathcal{O}_\nu(G) \times T_\varrho\mathcal{O}_\nu(G),\ \forall\varrho \in \mathcal{O}_\nu(G)$) is nondegenerate, then we have defined on $\mathcal{O}_\nu(G)$ the necessary isomorphism (at least for $\dim G < \infty$) between $T_\varrho\mathcal{O}_\nu(G)$ and $T^*_\varrho\mathcal{O}_\nu(G)$ ($\varrho \in \mathcal{O}_\nu(G)$). In this special case, we can calculate restrictions $\tilde{\varphi}^{\nu,\mathrm{f}}$ of the flows $\tilde{\varphi}^\mathrm{f}$ to the orbit $\mathcal{O}_\nu(G)$ with a help of the restrictions f^ν, cf. Definition 2.2.26(iv), only. ♡

Let us look now on some properties of the "classical phase space" $\mathcal{E}_\mathbb{F} \subset Lie(G)^*$. Let $\mathrm{conv}_0(B)$ be the convex hull of a subset B of some locally convex space, and let $\mathrm{conv}(B)$ be its closure. Let $\mathcal{E}^0_\mathbb{F} := P(\mathcal{H}) \cap \mathbb{F}(\mathcal{D}^\omega(G))$. Then we have:

2.2.32. Proposition. *The range of* $\mathbb{F}$, $\mathrm{Ran}(\mathbb{F})$*, is a convex,* $Ad^*(G)$*–invariant subset of* $Lie(G)^*$ *containing* $\mathrm{conv}_0(\mathcal{E}^0_\mathbb{F}) = \mathbb{F}(\boldsymbol{\mathcal{D}}_r(\mathbb{F}))$*. If* $\dim(G) < \infty$*, then* $\mathrm{Ran}(\mathbb{F}) = \mathcal{E}_\mathbb{F}$*, i.e. it is a closed subset of* $Lie(G)^*$*.* ♣

Proof. The mapping $\mathbb{F} : \boldsymbol{\mathcal{D}}(\mathbb{F}) \to Lie(G)^*$ is affine, and $\boldsymbol{\mathcal{D}}(\mathbb{F})$ is convex, since $\mathcal{D}(h_{X(\xi)})$ is convex and $h_{X(\xi)}$ is affine. Hence $\mathrm{Ran}(\mathbb{F})$ is convex, and $\mathrm{conv}_0(\mathcal{E}^0_\mathbb{F}) \subset \mathrm{Ran}(\mathbb{F})$. One can see from the definitions that $\mathrm{conv}_0 = \mathbb{F}(\boldsymbol{\mathcal{D}}_r(\mathbb{F}))$, and that $\boldsymbol{\mathcal{D}}_r(\mathbb{F})$ is norm–dense in $\boldsymbol{\mathcal{D}}(\mathbb{F}) \subset \mathcal{S}_*$. The $Ad^*(G)$–invariance follows from (2.2.17), and from the $Ad^*(U(G))$–invariance of $\boldsymbol{\mathcal{D}}(\mathbb{F})$.

Let $\dim G < \infty$. The closedness of $\mathrm{Ran}(\mathbb{F})$ can be proved by construction of a projection–valued measure on $\mathfrak{g}^*$ representing the commutative group (linear space) $\mathfrak{g}$, resp. the commutative algebra of "classical observables" $C^\infty(\mathfrak{g}^*,\mathbb{R})$ (generated by the functions $f_\xi(F) := F(\xi),\ \xi \in \mathfrak{g}, F \in \mathfrak{g}^*$), [27, 31]. The support of this measure is identical with $\mathrm{Ran}(\mathbb{F})$, hence closed. □

2.3 Symmetries, Dynamics and Observables

It was shown in Sections 2.1, and 2.2, how real–valued functions $\mathrm{f} :\ \nu \mapsto \mathrm{f}(\nu),\ (\nu \in \mathcal{S}_*)$ can be used in the rôle of generators of the one–parameter families $\tilde{\varphi}^\mathrm{f}$ of transformations of elementary states. Differentiable functions on phase spaces of CM are used in double rôle: as the generators, as well as "observables", i.e. as a certain objects ascribing (numerical) values of possible results of specific measurements to states ν to which the measurements are applied. Selfadjoint operators X represent both these objects in QM: they are generators of the unitary groups $\exp(-itX)$ on $\mathcal{H}$, and also observables with probability distributions μ^X_ν, cf. (1.5.9), and point 2.2.1, of their (real) values measured in the state ν. We include into the presented scheme

also such a double rôle and the standard interpretation for the functions h_X: Besides of being generators according to assertion 2.2.10, they also could be considered as observable quantities with n–th momenta $\nu(X^n)$ (if they exist) of the probability measures μ_ν^X calculated directly from h_X, as it is indicated in formula (3.3.5) of Subsection 3.3-a, cf. also [63]. Difficulties arise, however, in trials to interpret a nonlinear function f defined on (a subset of) $\mathcal{S}_*$ in a rôle of an observable in the traditional way, as it will be shown in Interpretation 2.3.15, in Note 3.3.3, as well as in Interpretation 3.3.4.

Now we shall show that, on the other hand, the use of nonlinear generators of transformation groups in QM implies also necessity of introduction of some nonlinear "observables" together with the affine ones.

Let us assume that we have a flow $\tilde{\varphi}^f$ generated by a nonlinear generator f according to Section 2.1, and let u_f be the corresponding solution of (2.1.23). For any "observable" h_a ($\mathrm{a} \in \mathcal{L}(\mathcal{H})_s$), one has "a natural time–evolved form":

$$h_\mathrm{a}^t(\varrho) := h_\mathrm{a}(\tilde{\varphi}_t^\mathrm{f}\varrho) \equiv Tr(\varrho \mathrm{u_f}(t,\varrho)^* \,\mathrm{a}\, \mathrm{u_f}(t,\varrho)),$$

and the functions h_a^t are not generally of the form $h_{\mathrm{a}(t)}$, i.e.

$$\varrho \mapsto Tr(\varrho \mathrm{u_f}(t,\varrho)\, \mathrm{a}\, \mathrm{u_f}(t,\varrho))$$

are not affine functions of ϱ for all $t \in \mathbb{R}$; this can be seen, e.g., from [31, Proposition 4.3].

2.3.1. **Interpretation.** We propose an interpretation scheme, in which a numerical–valued function f on $\mathcal{S}_*$ can have several different interpretations as "observables" in EQM. The "appropriate choice" of the class of observables of the system depends also on the chosen symmetry group G entering into the description of the considered system. From our point of view, the specified symmetry group G could be interpreted as a group of motions of (a relevant part of) the macroscopic background determining physical meaning of the "observables", i.e. quantities used for description of empirical specification of states of a given physical system. We can interpret the genuine mixtures (cf. Subsection 2.1-e) as describing states of a "microscopic subsystem" of a composed system consisting of the "microscopic subsystem" (i.e. the considered one) and a "macroscopic background". This "background" can interact with the considered quantum system also without being influenced by it; it can be represented, e.g. by an infinite number of copies of the "considered quantum system" interacting mutually by a type of quantum mean–field interaction, [130, 31, 32, 33, 263, 264, 265, 266]. The genuine mixture of the "microsystem" corresponds to a nontrivial statistical distribution of values of macroscopic observables $X_\Pi(\xi)$): The values of some "macroscopic observables" of this "macrosystem" (describable in classical terms) are correlated with the states of the "microsystem" entering into the support of the measure determining the genuine mixture, cf. also Remark 2.2.19. ♦

We shall introduce now a (in a certain sense minimal) set of nonlinear functions representing observables and containing all the usually used "linear observables" of QM which is invariant with respect to a sufficiently large class of (nonlinear) dynamics and also with respect to the symmetry group specified by the representation $U(G)$, as it was introduced in Section 2.2-d. We shall introduce also other concepts (generators of different kinds, e.g.) forming with the chosen set of observables a consistent closed theory. This set of concepts specifies a method of determination a subtheory from the overwhelmingly large set of possible (mathematically

admissible) "generators", and "observables" of possible formally extended quantum theories. The usefulness of the (representation of the) group G is here (at least) twofold, interpretational, and technical:

(interpretation) The group G, if interpreted a priori in terms of some "macroscopic variables", cf. Remark 2.2.19, can serve as a theoretical tool for specification of interpretation of mathematically specified "observables", as well as symmetry transformations generated by a distinguished class of "generators".

(technicality) The strongly continuous unitary representation $U(G)$ is an effective device to select the dense set $\boldsymbol{\mathcal{D}}(\mathbb{F})$ of points, as well as of submanifolds $\mathcal{O}_\varrho(G)$, where the differential–geometrical objects as "differentials", or "vector fields" can be defined from a specified (by the same representation) set of generators, which are locally unbounded for many physically relevant cases ("generically" for physically relevant noncompact group representations).

2.3.2. Definitions (G–generators).

(i) Let $\mathcal{G}^G_{cl}$ denote the **Poisson algebra of G–classical generators***: $\mathrm{f} \in \mathcal{G}^G_{cl} \Leftrightarrow \mathrm{f} = \mathbb{F}^* f := f \circ \mathbb{F}$ for some $f \in C^\infty(\mathcal{E}_\mathbb{F}, \mathbb{R})$. Let $\tilde{\mathcal{E}}_\mathbb{F}(f) \subset Lie(G)^*$ be some (for each f separately chosen) open neighbourhood of $\mathcal{E}_\mathbb{F}$ in $Lie(G)^*$ endowed with one of the canonical topologies, cf. Definitions 2.2.26. The Poisson structure on $\mathcal{G}^G_{cl}$ is expressed by* (2.2.25)*.*

(ii) Let f *be densely defined real–valued function on $\mathcal{S}_*$ such that its $\boldsymbol{\mathcal{D}}_r$–generalized differential exists and it is $\mathcal{S}_*$–integrable, cf. Definition 2.2.13. Let $\nu \mapsto \mathbf{v}_\mathrm{f}(\nu)$ be the corresponding $\boldsymbol{\mathcal{D}}_r$–Hamiltonian vector field and assume, that its flow $\tilde{\varphi}^\mathrm{f}$ is complete, and leaving $\boldsymbol{\mathcal{D}}(\mathbb{F})$ invariant. Let, moreover, the flow can be described by $\mathrm{u}_\mathrm{f}(\cdot,\cdot) : \mathbb{R} \times \mathcal{S}_* \mapsto \mathfrak{U}$ satisfying* (2.1.24), (2.1.25), *and also* (2.1.23) *on a "sufficiently large" subdomain of $\boldsymbol{\mathcal{D}}_r(\mathbb{F})$ (cf. Definition 2.2.13). Then* f *will be called a* **(quantum) G–generator**.

(iii) Let, for the quantum G–generator f *of the above definition (ii), $\mathbb{F}(\tilde{\varphi}^\mathrm{f}_t\nu) = \mathbb{F}(\tilde{\varphi}^\mathrm{f}_t\nu')$ for all $\nu' \in \mathbb{F}^{-1}[\mathbb{F}(\nu)]$, for any $(t;\nu) \in \mathbb{R} \times \boldsymbol{\mathcal{D}}(\mathbb{F})$, the G–generator* f *is called a* **G–(classically) deterministic generator**. *In this case, we shall denote*

$$\varphi^\mathrm{f}_t[\mathbb{F}(\nu)] \equiv \mathbb{F}(\tilde{\varphi}^\mathrm{f}_t\nu);$$

this relation determines a flow φ^f on $\mathcal{E}_\mathbb{F}$.

(iv) A quantum G–generator which is not G–(classically) deterministic is called a **G–(classically) stochastic generator***: The quantum flow $\tilde{\varphi}^\mathrm{f}$ does not determine a classical flow, and a "corresponding" classical evolution might be considered (?) as a stochastic process.*

(v) Let a G–(classically) deterministic generator f *be such that one can choose*

$$\mathrm{u}_\mathrm{f}(t,\nu) \equiv \mathrm{u}_\mathrm{f}(t,\nu'),\ \forall \nu' \in \mathbb{F}^{-1}[\mathbb{F}(\nu)],\ (\forall (t;\nu) \in \mathbb{R} \times \boldsymbol{\mathcal{D}}(\mathbb{F})); \tag{2.3.1}$$

then we can (and we shall) write $\mathrm{u}_\mathrm{f}(t,F) := \mathrm{u}_\mathrm{f}(t,\nu')$ for $\nu' \in \mathbb{F}^{-1}[F],\ F \in \mathcal{E}_\mathbb{F}$. Let the mappings (cf. Definitions 2.3.3 for $\mathcal{C}^G$)

$$\tau^\mathrm{f}_t : \mathcal{C}^G \to \mathcal{C}^G,\ \mathfrak{h} \mapsto \mathfrak{h}_t,\ \mathfrak{h}_t(F) := \mathrm{u}_\mathrm{f}(t,F)^{-1}\mathfrak{h}(\varphi^\mathrm{f}_t F)\mathrm{u}_\mathrm{f}(t,F) \tag{2.3.2}$$

be C^*-automorphisms of $\mathcal{C}^G$ for all $t \in \mathbb{R}$. Then $\mathfrak{f}$ *is called a* **G–symmetry generator**. *The set of all G–symmetry generators will be denoted $\mathcal{G}^G$. It is $\mathcal{G}^G_{cl} \subset \mathcal{G}^G$, as will be shown in Proposition 2.3.10, and Theorem 2.3.16.* $\diamondsuit$

These definitions of different types of generators (of evolutions, or symmetry groups) specify also their relations to the corresponding transformations induced in the set of "classical variables" determined by the chosen (unitary representation $U(G)$ of the) group G. "Observables" in EQM are not sufficiently determined by real–valued functions on $\mathcal{S}_*$; the quantummechanical interpretation needs possibility of determination of probability distributions in any point $\varrho \in \mathcal{S}_*$ for general observable quantities. The following definitions of observables respect also the requirement of their invariance with respect to "Heisenberg–picture–transformations", into which nonlinearities bring modifications with respect to the linear case: One has to distinguish between transformations of elementary states (described by density matrices) and corresponding transformations of observables (described, e.g., by operator valued functions of density matrices). This distinction ensures "conservation of transition probabilities" also in nonlinear QM.

2.3.3. Definitions (G–observables).

(i) Let the $s^*(\mathcal{L}(\mathcal{H}), \boldsymbol{\mathcal{D}}(\mathbb{F}))$–topology on $\mathcal{L}(\mathcal{H})$ be given by the family of seminorms p_ν, p^*_ν ($\nu \in \boldsymbol{\mathcal{D}}(\mathbb{F})$) determined by their values $p_\nu(x) := \nu(x^*x)^{1/2}$, and $p^*_\nu(x) := \nu(xx^*)^{1/2}$ on $x \in \mathcal{L}(\mathcal{H})$. Let $\mathfrak{f}$ and $\mathfrak{h}$ be uniformly bounded operator–valued functions on $\mathcal{E}_\mathbb{F}$, $\mathfrak{f}$: $\mathcal{E}_\mathbb{F} \to \mathcal{L}(\mathcal{H})$, $F \mapsto \mathfrak{f}(F)$, $\|\mathfrak{f}\| := \sup\{\|\mathfrak{f}(F)\| : F \in \mathcal{E}_\mathbb{F}\} < \infty$, which are $s^*(\mathcal{L}(\mathcal{H}), \boldsymbol{\mathcal{D}}(\mathbb{F}))$ – continuous.[75] *Let $\mathcal{C}_{bs}$ be the set of all such functions endowed with (pointwise) operations: $(\mathfrak{f} + \lambda\mathfrak{h})(F) := \mathfrak{f}(F) + \lambda\mathfrak{h}(F)$, $(\mathfrak{f}\mathfrak{h})(F) := \mathfrak{f}(F)\mathfrak{h}(F)$, and $\mathfrak{f}^*(F) := \mathfrak{f}(F)^*$, $\lambda \in \mathbb{C}$. It can be shown [27] that $\mathcal{C}_{bs}$ with these algebraic operations and the norm is a C^*-algebra. The elements of $\mathcal{C}_{bs}$ are* **unrestricted bounded G–observables**.

(ii) Let $\mathfrak{B}_U := U(G)''$ be the von Neumann subalgebra of $\mathcal{L}(\mathcal{H})$ generated by U(G). Let $\boldsymbol{\mathcal{C}^G}$ (resp. $\boldsymbol{\mathcal{C}^G_U}$) be the C^*-subalgebra, [27], of $\mathcal{C}_{bs}$ generated by the uniformly bounded operator–valued functions

$$\mathfrak{h}_{\mathrm{x},\gamma,f} : F(\in \mathcal{E}_\mathbb{F}) \mapsto \mathfrak{h}_{\mathrm{x},\gamma,f}(F) := U(\gamma(F))^* \,\mathrm{x}\, U(\gamma(F)) f(F),$$

for all $\mathrm{x} \in \mathcal{L}(\mathcal{H})$ (resp. $\forall \mathrm{x} \in \mathfrak{B}_U$), $\gamma \in C(\mathcal{E}_\mathbb{F}, G)$, $f \in C_b(\mathcal{E}_\mathbb{F}, \mathbb{R})$; elements of $\mathcal{C}^G$ will be considered also as operator–valued functions on $\boldsymbol{\mathcal{D}}(\mathbb{F})$ obtained by pull–back by $\mathbb{F}$:

$$\mathfrak{f} \in \mathcal{C}^G \Rightarrow \mathfrak{f} : \varrho(\in \boldsymbol{\mathcal{D}}(\mathbb{F})) \mapsto \mathfrak{f}(\mathbb{F}(\varrho)).$$

The set $\mathcal{C}^G$ (resp. $\mathcal{C}^G_U$) is called the **C^*–algebra of (quantum, bounded) G–observables** *(resp. the* **C^*–algebra of UG–observables***) of the system. Any $\mathfrak{f} = \mathfrak{f}^* \in \mathcal{C}^G$ will be called a* **G–observable**. *Elements $\mathrm{x} \in \mathcal{L}(\mathcal{H})$ are considered as elements of $\mathcal{C}^G$ for any $U(G)$: They are identified with the constant functions $\mathfrak{h}_\mathrm{x} : F \mapsto \mathfrak{h}_\mathrm{x}(F) := \mathrm{x}$ on $\mathcal{E}_\mathbb{F}$. Elements $\mathfrak{h}_\mathrm{x}$ $(\mathrm{x} \in \mathcal{L}(\mathcal{H})_s)$ generate a (complex)* **subalgebra of elementary quantum observables** *denoted by $\boldsymbol{\mathcal{C}^G_q}$ which is a subset of $\mathcal{C}^G$ isomorphic to $\mathcal{L}(\mathcal{H})$ (for any choice of $U(G)$). Any uniformly bounded element $\mathfrak{f} = \mathbb{F}^* f \in \mathcal{G}^G_{cl}$ will be considered also as the element of $\mathcal{C}^G_U$ ($\subset \mathcal{C}^G$) described by the scalar–valued function $\mathfrak{h}_f : F \mapsto \mathfrak{h}_f(F) := \mathbb{I} f(F)$ on $\mathcal{E}_\mathbb{F}$. The G–observables of this form will be called the (bounded)* **G–classical observables**. *They belong to $\mathcal{C}^G_{cl} := \mathbb{I} \cdot C(\mathcal{E}_\mathbb{F}, \mathbb{C}) \subset \mathcal{C}^G_U$, $\mathbb{I} := I_\mathcal{H}$.*

[75] Remember that a topology on $\mathfrak{g}^*$ is here understood to be one of the two canonical topologies, which are mutually equal for finite–dimensional group G, cf. definition (i) in 2.2.26.

(iii) The **unbounded G–observables** *(resp.* **UG–observables***) are functions $Y : F \mapsto Y(F)$ on $\mathcal{E}_{\mathbb{F}}$ with values in unbounded selfadjoint operators $Y(F)$ on $\mathcal{H}$, with the spectral measures $E_{Y(F)}$ such that the functions $E_{Y(\cdot)}(B) : F \mapsto E_{Y(F)}(B) \in \mathfrak{P}(\mathcal{L}(\mathcal{H}))$ (cf. Note B.4.1) belong to $\mathcal{C}^G$ (resp. to $\mathcal{C}^G_U$) for any Borel set $B \subset \mathbb{R}$. Note that we needn't specify the domains of the operators $Y(F)$ here.* $\diamondsuit$

2.3.4. Definitions (Function representation of observables).

(i) Let us denote $h_{\mathfrak{f}} : \nu \mapsto h_{\mathfrak{f}}(\nu) := \nu(\mathfrak{f}[\mathbb{F}(\nu)])$. The mapping $\mathfrak{f}(\in \mathcal{C}^G) \mapsto h_{\mathfrak{f}}$ is not injective. Let us introduce the functions $\hat{h}_{\mathfrak{f}}(\cdot,\cdot)$ of two variables $(\varrho;\nu) \in \mathcal{S}_* \times \boldsymbol{\mathcal{D}}(\mathbb{F})$, $(\varrho;\nu) \mapsto \hat{h}_{\mathfrak{f}}(\varrho,\nu) := \varrho(\mathfrak{f}[\mathbb{F}(\nu)])$. Then $h_{\mathfrak{f}} \equiv \hat{h}_{\mathfrak{f}}(\nu,\nu)$, $\nu \in \boldsymbol{\mathcal{D}}(\mathbb{F})$. The mapping $\mathfrak{f} \mapsto \hat{h}_{\mathfrak{f}}(\cdot,\cdot)$ is an injection into the set $\hat{\mathcal{C}}^G$ of real-valued functions $\hat{\mathfrak{f}}$ defined on the product $\mathcal{S}_* \times \boldsymbol{\mathcal{D}}(\mathbb{F})$ such that the dependence $\varrho \mapsto \hat{\mathfrak{f}}(\varrho,\nu)$ is affine bounded continuous for each fixed ν, and $\hat{\mathfrak{f}}(\varrho,\nu) \equiv \hat{\mathfrak{f}}(\varrho,\nu')$ for all $\nu' \in \mathbb{F}^{-1}[\mathbb{F}(\nu)]$ = **(a level set of the mapping** $\mathbb{F}$*), for each fixed $\varrho \in \mathcal{S}_*$. Continuity properties of the functions $\nu \mapsto \hat{h}_{\mathfrak{f}}(\varrho,\nu)$ are determined by properties of $\mathbb{F}$ and by the continuity of $F \mapsto \mathfrak{f}(F)$. The element $\hat{h}_{\mathfrak{f}} \in \hat{\mathcal{C}}^G$ will be called the* **function representative of the (bounded) G–observable** $\mathfrak{f}$ *of the system; elements of $\hat{\mathcal{C}}^G$ will also be called the* **G–observables***. The first variable $\varrho \in \mathcal{S}_*$ in $\hat{\mathfrak{f}}(\varrho,\nu)$ will be called the* **quantum variable***, and the second one, $\nu \in \boldsymbol{\mathcal{D}}(\mathbb{F})$, will be called the* **G–classical variable** *(cf. Section 3.4 for motivation of such terminology) of the (function representative of the) observable $\hat{\mathfrak{f}} \in \hat{\mathcal{C}}^G$. The function $h_{\mathfrak{f}} : \nu \mapsto h_{\mathfrak{f}}(\nu) := \nu(\mathfrak{f}[\mathbb{F}(\nu)])$ will be called the* **reduced function representative** *of $\mathfrak{f} \in \mathcal{C}^G$.*

(ii) Functions h_Y, and $\hat{h}_Y$, for unbounded observables Y, can be introduced as (not everywhere defined) **function representatives of unbounded observables***, in analogy with the case (i) of bounded observables, i.e. $h_Y(\varrho,\nu) \equiv Tr\big(\varrho Y\big(\mathbb{F}(\nu)\big)\big)$ on a corresponding domain in $\mathcal{S}_* \times \boldsymbol{\mathcal{D}}(\mathbb{F})$ (the domain specification would be here, generally, difficult).* $\diamondsuit$

We shall next introduce states (as linear functionals on an algebra of "observables") corresponding to the general concept of "genuine mixtures" introduced in the Subsection 2.1-e. They will be "suited" also to the just introduced constructions determined by the representation $U(G)$.

2.3.5. Definitions (G–states).

(i) Let $\mathcal{M}^G$ be the set of regular Borel probability measures on $\boldsymbol{\mathcal{D}}(\mathbb{F})$ (with its Borel structure coming from the metric topology of $\mathcal{S}_*$). The genuine mixtures $\mu \in \mathcal{M}^G$ determine the set $\mathcal{S}^{cl}_G$ of the **G–classical states $\boldsymbol{\omega_\mu}$** *of the considered system: The elements $\omega_\mu \in \mathcal{S}(\mathcal{C}^G) := (\mathcal{C}^G)^*_{+1}$ (:= the state space of the C^*-algebra $\mathcal{C}^G$) are determined by their values $\omega_\mu(\mathfrak{f})$, $\mathfrak{f} \in \mathcal{C}^G$ expressed by the integrals:*

$$\omega_\mu(\mathfrak{f}) := \mu(h_{\mathfrak{f}}) \equiv \int \nu(\mathfrak{f}\big(\mathbb{F}(\nu)\big))\, \mu(d\nu),\ \forall \mathfrak{f} \in \mathcal{C}^G. \tag{2.3.3}$$

Elementary states $\varrho \in \boldsymbol{\mathcal{D}}(\mathbb{F})$ are represented by Dirac measures δ_ϱ concentrated at ϱ.

In these states, the values of the quantum variable ϱ of $\hat{h}_{\mathfrak{f}}(\varrho,\nu)$ copy those of the classical variable. If the "microscopic state" described by the quantum variable is not connected with the classical variable in this way, one arrives at definition of more general states:

(ii) Let $\hat{\varrho} : \boldsymbol{\mathcal{D}}(\mathbb{F}) \to \mathcal{S}_*,\ \nu \mapsto \hat{\varrho}(\nu)$ be a Borel function. Let the state $\omega_{\mu,\hat{\varrho}} \in \mathcal{S}(\mathcal{C}^G)$ be defined by

$$\omega_{\mu,\hat{\varrho}}(\mathfrak{f}) := \int \hat{\varrho}(\nu)(\mathfrak{f}(\mathbb{F}(\nu)))\,\mu(d\nu) \equiv \int Tr\Big(\hat{\varrho}(\nu)\mathfrak{f}(\mathbb{F}(\nu))\Big)\,\mu(d\nu). \tag{2.3.4}$$

The set of all such states $\omega_{\mu,\hat{\varrho}} \in \mathcal{S}(\mathcal{C}^G)$ will be denoted by $\mathcal{S}_G$. The elements $\boldsymbol{\omega_{\mu,\hat{\varrho}}}$ of $\mathcal{S}_G$ will be called **G–states**. *Clearly $\mathcal{S}_G^{cl} \subset \mathcal{S}_G$. The functions $\hat{\varrho}$ playing the described rôle will be called here* **quantum deviation functions**. *For $\omega_{\mu,\hat{\varrho}} \in \mathcal{S}_G^{cl}$ one has $\hat{\varrho}(\nu) \equiv \nu$.* $\diamondsuit$

2.3.6. Definition (G–systems). *Let a unitary continuous representation $U(G)$ of a Lie group G be given. The model of a (quantummechanical) physical system of EQM in which the sets of its ("system determining") generators, states, and (bounded) observables coincide with the sets of the G–symmetry generators $\mathcal{G}^G$, G–classical states $\mathcal{S}_G^{cl}$, and G–observables $\mathcal{C}^G$ (resp. UG–observables $\mathcal{C}_U^G$) respectively is called the* **G–classical (***resp.* **UG–classical) quantum system**, *or just the* **G–system (***resp.* **UG–system)**, *based on the representation $U(G)$. The G–system (resp. UG–system) will be also denoted by Σ_G (resp. by Σ_{UG}). One has $\Sigma_G = \Sigma_{UG}$ for irreducible $U(G)$.* $\diamondsuit$

2.3.7. **Remark**. This (basic) definition will need, probably further elaboration. The bracketed expressions "system determining", and "bounded" has to indicate, that also other generators etc. are possibly acceptable in the theory. Similar remarks might be, probably, added to several other parts of the here presented (working) version of the theory, called here "EQM". $\heartsuit$

The definition of "G–systems" leads to a formally (and, perhaps, also intuitively) natural, and also "operationally" transparent, definition of "subsystems":

2.3.8. Definition ($\boldsymbol{G_I}$–subsystems).
Let a G–system be given by $U(G)$, and let $G_I \subset G$ be a Lie subgroup of the Lie group G. The restriction $U(G_I)$ of $U(G)$ to G_I is a continuous unitary representation of G_I. The G_I–system Σ_{G_I} (resp. UG_I–system Σ_{UG_I}) determined by this restriction is the $\boldsymbol{G_I}$**–subsystem** *(resp.* $\boldsymbol{UG_I}$**–subsystem**) *of Σ_G.* $\diamondsuit$

Let us note that the definition of states of a subsystem given in Subsection 2.1-e with a help of the "partial trace" fits into a special case of the presently introduced definition of the UG_I–subsystems: It should be chosen $G := \mathcal{U}(\mathcal{H}_{I+II})$–the unitary group of the set of all bounded operators on $\mathcal{H}_{I+II} = \mathcal{H}_I \otimes \mathcal{H}_{II}$, and as the the Lie subgroup we choose $G_I := \mathcal{U}(\mathcal{H}_I) \sim \mathcal{U}(\mathcal{H}_I) \otimes \mathbb{I}_{\mathcal{H}_{II}}$, with $U(\cdot)$ being their defining (identical) representation(s). The linear QM can be considered here as described by the subalgebra of $\mathcal{C}_U^G$ consisting of constant functions only (what is an alternative to the choice $G := \{e\}$, cf. point 3.1.1).

2.3.9. *Let us express now the Poisson bracket between the reduced function representatives of two observables $\mathfrak{f}, \mathfrak{l}$ in $\mathcal{C}^G$. This is done by a repeated use of the composite–mapping theorem, [1,*

61]. For the case $n := \dim G < \infty$, from (2.2.14)*, and* (2.2.25) *we have:*

$$\begin{aligned}\{h_{\mathfrak{f}}, h_{\mathfrak{l}}\}(\nu) \quad \equiv \quad & i\,\nu\big([\mathfrak{f}(\mathbb{F}(\nu)), \mathfrak{l}(\mathbb{F}(\nu))]\big) + i\sum_{j=1}^{n} \nu\left(\frac{\partial \mathfrak{f}(\mathbb{F}(\nu))}{\partial F_j}\right) \nu\big([X(\xi_j), \mathfrak{l}(\mathbb{F}(\nu))]\big) \\ & + i\sum_{j=1}^{n} \nu\left(\frac{\partial \mathfrak{l}(\mathbb{F}(\nu))}{\partial F_j}\right) \nu\big([\mathfrak{f}(\mathbb{F}(\nu)), X(\xi_j)]\big) \\ & + \sum_{j,k} \nu\left(\frac{\partial \mathfrak{f}(\mathbb{F}(\nu))}{\partial F_j}\right) \nu\left(\frac{\partial \mathfrak{l}(\mathbb{F}(\nu))}{\partial F_k}\right) \mathbb{F}^*\{F_j, F_k\}(\nu). \end{aligned} \tag{2.3.5}$$

One can immediately deduce from this expression also expressions for Poisson brackets of specific cases of elementary quantum and G–classical observables. ♡

We shall formulate now the solution of a quantummechanical dynamical equation of a G–system in terms of a classical equation on the group manifold G. The solution will also show us that G–systems are "self-consistent" in the sense that the G–(classical) generators generate flows leaving the sets of G–observables, G–generators, as well as the G–classical states together with their algebraic and topological structures invariant.

Let us assume $\dim G < \infty$. Let $\gamma : G \to G$ be a differentiable mapping, let $e \in G$ be the unit element, $\mathfrak{g} \equiv T_eG$. The tangent mapping $T_e\gamma : T_eG \to T_{\gamma(e)}G$ is defined by

$$T_e\gamma(\xi) := \left.\frac{d}{dt}\right|_{t=0} \gamma(\exp(t\xi)) \equiv T_{t=0}\gamma(\exp(\cdot\xi)),\ \xi \in \mathfrak{g}.$$

Let $R_g : g' \mapsto R_g g' := g'g$ $(g, g' \in G)$ be the right action of G onto itself. Let us identify the tangent space $T_F\mathfrak{g}^*$ in any point $F \in \mathfrak{g}^*$ with $\mathfrak{g}^*$ itself in the canonical way (as any tangent space to a linear space), and let its dual $T_F^*\mathfrak{g}^*$ be identified with $\mathfrak{g}^{**} = \mathfrak{g}$ (canonical identification for reflexive spaces). Then, for any $Q \in C^\infty(\mathcal{E}_\mathbb{F}, \mathbb{R})$, and any $F \in \mathcal{E}_\mathbb{F}$, we have $d_F Q \in \mathfrak{g}$. The set $\mathcal{E}_\mathbb{F}$ is $Ad^*(G)$–invariant, cf. Proposition 2.2.32.

2.3.10. Proposition. *Let $U(G)$ be as above, and $Q \in C^\infty(\mathcal{E}_\mathbb{F}, \mathbb{R})$, with complete Poisson flow φ^Q on $\mathcal{E}_\mathbb{F}$. Let* $\mathrm{Q} := \mathbb{F}^*Q \in \mathcal{G}_{cl}^G$*, i.e.* Q *is a G–classical generator, cf. Definition 2.2.26(iii). Then there is a unique infinitely differentiable solution $g_Q : \mathbb{R} \times \mathcal{E}_\mathbb{F} \to G,\ (t; F) \mapsto g_Q(t, F)$ of the differential equation on the group manifold:*

$$\frac{d}{dt} g_Q(t, F) = T_e R_{g_Q(t,F)}(d_{F_t} Q) \quad \in T_{g_Q(t,F)}G, \quad g_Q(0, F) \equiv e, \tag{2.3.6a}$$

with $F_t := \varphi_t^Q F$, for all $F \in \mathcal{E}_\mathbb{F}$. The function g_Q satisfies the cocycle identity:

$$g_Q(s, \varphi_t^Q F) g_Q(t, F) \equiv g_Q(s + t, F), \tag{2.3.6b}$$

and it determines the flow φ^Q according to the following relation:

$$\varphi_t^Q F \equiv Ad^*(g_Q(t, F))F. \tag{2.3.7}$$

The flow $\tilde{\varphi}^{\mathrm{Q}}$ generated by the Hamiltonian vector field $\mathbf{v}_{\mathrm{Q}}(\cdot)$ from (2.2.26) *is then given on $\boldsymbol{\mathcal{D}}(\mathbb{F})$ by*

$$\tilde{\varphi}^{\mathrm{Q}}_t \varrho \equiv Ad^*\big(U[g_Q(t,\mathbb{F}(\varrho))]\big)\varrho,\ \varrho \in \boldsymbol{\mathcal{D}}(\mathbb{F}), \tag{2.3.8}$$

with $g \mapsto U(g)$ being the given unitary representation of G. Hence, $\tilde{\varphi}^{\mathrm{Q}}_\cdot$ leaves all the orbits $\mathcal{O}_\varrho(G)$ invariant. ♣

Proof. The flow φ^Q leaves the $Ad^*(G)$–orbits invariant, since it is a Poisson flow and the $Ad^*(G)$–orbits are symplectic leaves of the Poisson (–Berezin) structure on $\mathfrak{g}^*$, [177, 7, 274]. Hence $F_t \in \mathcal{E}_{\mathbb{F}}$ ($F \in \mathcal{E}_{\mathbb{F}}$) for all $t \in \mathbb{R}$, and $d_{F_t}Q \in T^*_{F_t}\mathfrak{g}^* \equiv \mathfrak{g} := Lie(G)$. The vectors $T_eR_g(d_FQ) \in T_gG$ ($g \in G$) form a right–invariant vector field on G for each $F \in \mathcal{E}_{\mathbb{F}}$, and $\mathbf{v}_Q(g;t;F) := T_eR_g(d_{F_t}Q)$ ($t \in \mathbb{R}, g \in G$) are values of t–dependent vector fields (for any $F \in \mathcal{E}_{\mathbb{F}}$) on G. Their infinite differentiability follows from the properties of Q. The existence and uniqueness of the solution g_Q of (2.3.6) fulfilling (2.3.6) are then consequences of the theory of ordinary differential equations on manifolds, cf. [40].

Let $\xi \in \mathfrak{g}$. The derivative of $[Ad^*(g_Q(t,F))F](\xi) \equiv F(Ad(g_Q(t,F)^{-1})\xi)$ at $t = 0$ equals, according to (2.3.6), to $F([\xi, d_FQ])$, what can be rewritten in the form of Berezin bracket for $\xi := d_Fh,\ h \in C^\infty(\mathfrak{g}^*, \mathbb{R})$:

$$\left.\frac{d}{dt}\right|_{t=0} h\big(Ad^*(g_Q(t,F))F\big) = d_Fh\left(\left.\frac{d}{dt}\right|_{t=0} Ad^*(g_Q(t,F))F\right) = \{Q,h\}(F).$$

This, together with (2.3.6), proves (2.3.7).

The generator $\mathrm{Q} \in \mathcal{G}^G_{cl}$ generates, on the other hand, a Poisson flow $\tilde{\varphi}^Q$ on $\mathcal{S}_*$. Since $\mathrm{Q} = \mathbb{F}^*Q$, (2.3.8) is proved by (2.2.25), (2.3.6), and (2.3.7). □

2.3.11. **Interpretation.**

(i) Let us assume that a standard measuring procedure can be associated with a given mathematical quantity $\mathfrak{f} \in \mathcal{C}^G$ (or wit a quantity that can be described by an unbounded selfadjoint operator–valued function $F \mapsto Y(F)$) which leads to a numerical result λ at each individual repetition of the measuring performed on the system–object. We understand here that with each such individual measuring act there is necessarily accompanied a **registration ≡ detection** of a copy of considered system–object. This means that, contrary to often accepted definition of "measurement process" in QM, performing a statistical empirical test measuring the (average/per time) number of incoming systems in a beam (leavig a preparation apparatus), as well as of the (average/per time) number of systems approaching (entering) the apparatus, a knowledge of efficiency parameters of the apparatus, and also exact knowledge of (calculated) final state of measured objects "entered into the apparatus" (i.e. the state just before being detected by a "counter"), **all of this together is not sufficient for presence of a measuring act**. Or, in other words, the result λ of each individual measuring act should be represented by a (macroscopic) change of initial state of measuring device which is observable as a stable mark (i.e. a "trace" repeatably testable by different, namely by any "correspondingly educated", human observers with the same result of the tests with, possibly, standard statistical deviations), e.g. a "new point-

er position λ of the measuring apparatus".[76] (It might be useful to stress also here that such a measurement process is not yet satisfactorily formalized in QT.)

We assume that this λ belongs to the union of the spectra sp[ƒ(F)] of selfadjoint operators ƒ(F) (resp. spectra of generally unbounded $Y(F) \mapsto \mathfrak{f}(F)$):

$$\lambda \in \cup\{\mathrm{sp}[\mathfrak{f}(F)] : F \in \mathcal{E}_{\mathbb{F}}\} \subset \mathbb{R}.$$

(ii) We propose the following interpretation of the introduced observables $\mathfrak{f} \in \mathcal{C}^G$, or, more generally, of any ("sufficiently measurable", so that the integrals in (2.3.9) can be defined, cf. Definition 2.3.3(iii)) selfadjoint operator–valued function $Y : F(\in \mathcal{E}_{\mathbb{F}}) \mapsto Y(F) = \int_{\mathbb{R}} \lambda E_{Y(F)}(d\lambda)$, cf. also [27, 33, 31, 264, 265, 266]:

Let $\mu \in \mathcal{M}^G$ be a genuine mixture, and let $\hat{\varrho}$ be a quantum deviation function, both together defining the corresponding state $\omega_{\mu,\hat{\varrho}}$, cf. Definition 2.3.5. Let $B \subset \mathbb{R}$ be a Borel set. The probability of realization of the detected values $\lambda \in B$ at repeated measurements of the observable $Y : F \mapsto Y(F)$ in the (repeatably "identically" prepared) state $\omega_{\mu,\hat{\varrho}}$ is expressed by:

$$\mathrm{prob}(Y;\mu;\hat{\varrho})(B) \equiv \int_{\boldsymbol{\mathcal{D}}(\mathbb{F})} \hat{\varrho}(\nu)\big(E_{Y(F(\nu))}(B)\big)\mu(d\nu), \text{ with } \hat{\varrho}(\nu)(E) := Tr(\hat{\varrho}(\nu)E), \tag{2.3.9}$$

if the integral exists. ♦

Let us illustrate this general interpretation scheme on more specific examples:

2.3.12. **Examples**.

(i) Let $\mu \in \mathcal{M}^G$ be a genuine mixture describing the state $\omega_\mu \in \mathcal{S}_G^{cl}$ of a system, let $B \subset \mathbb{R}$ be a Borel set, and let $E_{Y(F)}$ be a projection (spectral) measure of the selfadjoint operator $Y(F),\ F \in \mathcal{E}_{\mathbb{F}}$. Then probability of finding in B the obtained value (i.e. the result) of a measurement of the observable: $\nu \mapsto Y(\mathbb{F}(\nu))$ in the state ω_μ is

$$\mathrm{prob}(Y \in B;\mu) \equiv \mathrm{prob}(Y;\mu)(B) = \int_{\boldsymbol{\mathcal{D}}(\mathbb{F})} \nu\big(E_{Y(\mathbb{F}(\nu))}(B)\big)\,\mu(d\nu). \tag{2.3.10a}$$

For the specific choice of the measure $\mu := \delta_\nu$, we have then

$$\mathrm{prob}(Y;\delta_\nu)(B) = \nu\big(E_{Y(\mathbb{F}(\nu))}(B)\big) \equiv Tr\big(\nu \cdot E_{Y(\mathbb{F}(\nu))}(B)\big), \tag{2.3.10b}$$

what is the usual probability distribution of the measuring results in QM of the observable described by the operator $Y(\mathbb{F}(\nu))$ performed on the system prepared in the (elementary) state $\nu \in \boldsymbol{\mathcal{D}}(\mathbb{F})$. The expectation (if it exists) of an observable $\mathfrak{f} \in \mathcal{C}^G$ in any state ω_μ is expressed by (2.3.3).

[76] According to this understanding of the content of the "process of measurement in QM", the measurement of a spin–coordinate of a $1/2$–spin particle by a Stern–Gerlach apparatus is not realized after passage of the particle across the inhomogeneous magnetic field, in spite of the fact that the wave function of the state of such a particle is splitted into two "macroscopically separated" beams: QM does not exclude a possibility of rejoining and interference of the two beams, hence they are not yet "macroscopically distinguished". The spin–component is measured only after detection of the particle described by the two–beam state vector, i.e. only after the "in which beam–question" is practically resolved (by an appearance of a "macroscopic trace" corresponding to just one of the eventualities).

(ii) Let us choose in the above formulas $Y(F) := \mathfrak{h}_\xi(F) := f_\xi(F)\mathbb{I} \equiv F(\xi)\mathbb{I}$ ($\xi \in Lie(G)$); then

$$E_{\mathfrak{h}_\xi(F)}(B) = \delta_{F(\xi)}(B)\mathbb{I} = \chi_B(F(\xi))\mathbb{I}, \tag{2.3.10c}$$

where χ_B is the characteristic function (= indicator) of the set B. Hence $\mathfrak{h}_\xi \in \mathcal{C}^G_{cl}$ is a classical observable. Let us denote $\mathbb{F}_{\boldsymbol{\xi}}(\boldsymbol{\nu}) := \mathbb{F}(\boldsymbol{\nu})(\boldsymbol{\xi}) \in \mathbb{R}$, $\xi \in \mathfrak{g}$, $\nu \in \boldsymbol{\mathcal{D}}(\mathbb{F})$. In the considered case we have

$$\mathrm{prob}(\mathfrak{h}_\xi, \mu)(B) = \int_{\boldsymbol{\mathcal{D}}(\mathbb{F})} \chi_B(\mathbb{F}_\xi(\nu))\, \mu(d\nu) = \mu\big(\mathbb{F}_\xi^{-1}[B]\big) =: \mu_\xi(B), \tag{2.3.10d}$$

where the measure $\mu_\xi \equiv \mu \circ \mathbb{F}_\xi^{-1}$ on the real line $\mathbb{R}$ was introduced. $\heartsuit$

We shall now define transformation laws τ^{Q} for observables,

$$\mathfrak{f} \mapsto \mathfrak{f}_t := \tau^{\mathrm{Q}}_t(\mathfrak{f}),\ \mathfrak{f} \in \mathcal{C}^G,$$

corresponding to the actions of the flows $\tilde{\varphi}^{\mathrm{Q}}$ on $\mathcal{S}_*$ described in the Proposition 2.3.10. We shall assume that

$$\boldsymbol{h}_{\mathfrak{f}}(\tilde{\boldsymbol{\varphi}}^{\mathbf{Q}}_{\boldsymbol{t}}(\boldsymbol{\nu})) \equiv \boldsymbol{h}_{\mathfrak{f}_t}(\boldsymbol{\nu}), \tag{2.3.11}$$

what corresponds to the transition from the Schrödinger to the Heisenberg picture in QM. This assumption is reflected in the following definitions.

2.3.13. Definitions (G–transformations).

(i) Let us consider a G–system. Let us choose some $\mathrm{Q} \in \mathcal{G}^G_{cl}$, $\mathrm{Q} = \mathbb{F}^*Q$, *with complete flow* $\tilde{\varphi}^{\mathrm{Q}}$ *on* $\mathcal{S}_*$. *Then* φ^{Q} *determined by* $\varphi^{\mathrm{Q}}_t\mathbb{F}(\nu) \equiv \mathbb{F}(\tilde{\varphi}^{\mathrm{Q}}_t\nu)$ *is the flow with Hamiltonian* Q *on* $\mathcal{E}_{\mathbb{F}}$. *Let* u_{Q} *be the solution of* (2.1.23) *(with* f *replaced by* Q*), cf. also Definition 2.3.2(v). Then, for an arbitrary G–observable* $\mathfrak{f} \in \mathcal{C}^G$, *we set:*

$$\mathfrak{f}_t(F) := \tau^{\mathrm{Q}}_t(\mathfrak{f})(F) := \mathrm{u}_{\mathrm{Q}}(t,F)^{-1}\mathfrak{f}(\varphi^{\mathrm{Q}}_t F)\mathrm{u}_{\mathrm{Q}}(t,F). \tag{2.3.12a}$$

In terms of (2.3.7) *and* (2.3.8), *we can write also* $\mathrm{u}_{\mathrm{Q}}(t,F) = U(g_{\mathrm{Q}}(t,F))$, *hence:*

$$\tau^{\mathrm{Q}}_t(\mathfrak{f})(F) \equiv U(g_{\mathrm{Q}}(t,F)^{-1})\,\mathfrak{f}(\varphi^{\mathrm{Q}}_t F)\,U(g_{\mathrm{Q}}(t,F)) \equiv Ad\big(U(g_{\mathrm{Q}}(t,F)^{-1})\big)\,\mathfrak{f}(\varphi^{\mathrm{Q}}_t F). \tag{2.3.12b}$$

We shall call $\boldsymbol{\tau}^{\mathbf{Q}}$ *the one–parameter* **G–symmetry group generated by** Q.

(ii) Let a Lie group continuous unitary representation $U(G)$ *be given. Elements of the Lie algebra* $\mathfrak{g}$ *of* G *are represented by affine functions* $h_{X(\xi)} \in \mathcal{G}^G_{cl}$, $\xi \in \mathfrak{g}$, *which are generators of one–parameter groups of symplectic isometries of our elementary phase space* $\mathcal{S}_*$. *Let a subgroup* $\sigma(G) \subset^*$*-Aut* $\mathcal{C}^G$ *of* **-automorphisms of the* C^**-algebra of observables* $\mathcal{C}^G$ *be determined by:*

$$[\sigma(g)\mathfrak{f}](F) := U(g)\,\mathfrak{f}(Ad^*(g^{-1})F)\,U(g^{-1}),\ \forall \quad \mathfrak{f} \in \mathcal{C}^G,\ g \in G,\ F \in \mathcal{E}_{\mathbb{F}}. \tag{2.3.12c}$$

The function $h_{X(\xi)}$ ($\xi \in \mathfrak{g}$) generates the flow $\tilde{\varphi}^{\xi}$ on $\mathcal{S}_*$, and for $\mathfrak{f} \in \mathcal{C}^G$ one has:

$$h_{\mathfrak{f}}(\tilde{\varphi}_t^{\xi}\nu) \equiv \nu\big((\sigma[\exp(t\xi)]\mathfrak{f})(\mathbb{F}(\nu))\big) = h_{\mathfrak{f}}(Ad^*\big(U(\exp(t\xi))\big)\nu). \tag{2.3.12d}$$

The automorphism group $\sigma(G)$ is **induced by the unitary representation** *$U(G)$. We also have the expression of an arbitrary one–parameter G–symmetry group $\tau^{\mathrm{Q}} \subset {}^*\text{-Aut}\,\mathcal{C}^G$ in terms of $\sigma(G)$, cf. Theorem 2.3.16:*

$$(\tau_t^{\mathrm{Q}}\mathfrak{f})(F) \equiv [\sigma(g_{\mathrm{Q}}(t,F)^{-1})\mathfrak{f}](F). \tag{2.3.12e}$$

The group $\sigma(G)$ is called the **G–automorphism group of** *$\mathcal{C}^G$.*

(iii) Let $\hat{\mathrm{f}}$ be a function–representative of an observable. Its evolution $\hat{\tau}_t^{\mathrm{Q}} : \hat{\mathrm{f}} \mapsto \hat{\mathrm{f}}_t$ under the G–symmetry group τ^{Q} is expressed with a help of the function $g_{\mathrm{Q}}(\cdot,\cdot)$ from (2.3.6) *as*

$$\hat{\mathrm{f}}_t(\varrho,\nu) \equiv \hat{\tau}_t^{\mathrm{Q}}(\hat{\mathrm{f}})(\varrho,\nu) := \hat{\mathrm{f}}(Ad^*\big(U(g_{\mathrm{Q}}\big(t,\mathbb{F}(\nu)\big))\big)\varrho, \tilde{\varphi}_t^{\mathrm{Q}}\nu). \tag{2.3.12f}$$

The transformation group $\hat{\boldsymbol{\tau}}_{\boldsymbol{t}}^{\mathbf{Q}}$ is the one–parameter **G–symmetry group of the function representatives generated by $\boldsymbol{Q}$.** $\diamondsuit$

2.3.14. **Remark** *(Transition probabilities).* Let us stress here that in the general case

$$Ad^*\big(U(g_{\mathrm{Q}}\big(t,\mathbb{F}(\nu)\big))\big)\varrho \not\equiv \tilde{\varphi}_t^{\mathrm{Q}}\varrho, \quad \text{for } \mathbb{F}(\varrho) \neq \mathbb{F}(\nu). \tag{2.3.13a}$$

The transformation law for observables described in Definitions 2.3.13 leads to a natural nonlinear generalization of the usual ("linear") transformation of "transition probabilities".

(linear case): In the linear case, time evolution is described in QM by a strongly–continuous one–parameter group $U(t)$ of unitary transformations, i.e. $U(t) \equiv \exp(-itX)$ for a selfadjoint Hamiltonian operator X. Expectation values of an arbitrary ("linear") observable $Y = Y^*$ in time evolved states $\varrho_t \equiv Ad^*(U(t))\varrho$ are

$$Tr(\varrho_t Y) = Tr(U(t)\varrho U(-t)Y) = Tr(\varrho U(-t)YU(t)) =: Tr(\varrho Y_t) \tag{2.3.13b}$$

where the "**Heisenberg picture**" of the time evolution $t \mapsto Y_t := U(-t)YU(t)$ (expressed in terms of observables, instead of the evolution of states) was introduced. It is now trivial to see that the expression

$$Tr(\varrho_t Y_{-t}) \equiv Tr(\varrho Y) \tag{2.3.13c}$$

remains constant in $t \in \mathbb{R}$ for any selfadjoint "observable Y".

If one inserts now into $Tr(\varrho Y)$ for the observable $Y^* = Y$ a one–dimensional projection P_y, and for the density matrix another projection P_x, then one obtains the well known "**conservation of transition probabilities**"[77]

$$Tr(P_x P_y) \equiv |\langle U(t)x|U(t)y\rangle|^2 = |\langle x|y\rangle|^2. \tag{2.3.13d}$$

[77] This interpretation of "transition probabilities", by which one of the vectors represents state preparation ("source"), and the another corresponds to a detector, connected with their invariance at symmetry transformations, is also in accordance with [120, I.3.1].

This seems to be usually interpreted as a trivial consequence of equal unitary transformation of the two vectors $x, y \in \mathcal{H}$ entering into the scalar product. Hence it is usually interpreted as an expression of the fact that "the transition amplitude **between two state vectors** $x, y \in \mathcal{H}$" does not depend on time, if both states are evolved by the same time transformation $U(t)$.

This (mis–)interpretation is repeatedly presented in connections with definitions of "symmetries" in QM, [280], and with the celebrated Wigner's theorem, which can be formulated in the following way:

(Wigner's theorem): Let $\phi : P(\mathcal{H}) \to P(\mathcal{H})$ be a bijection conserving "transition amplitudes", i.e.

$$Tr(P_x P_y) \equiv Tr(\phi(P_x)\phi(P_y)), \ \ \forall x, y \in \mathcal{H}, \tag{2.3.13e}$$

then there is either unitary or antiunitary bijection $U_\phi : \mathcal{H} \to \mathcal{H}$ such that $\phi(P_x) \equiv P_{U_\phi x}, \ \forall x \in \mathcal{H}$.

Symmetries in QM are then defined as transformations ϕ, resp. U_ϕ, satisfying conditions of the Wigner's theorem.

After reformulating the two mentioned interpretations of the "transformations of probability amplitudes" in the nonlinear case, we shall return to the problem of a choice between these two interpretations in Interpretation 2.3.15.

(nonlinear case 1): Extending the above last mentioned (mis–)interpretation mechanically to nonlinear case, one obtains **"non-conservation of transition probabilities"**:[78]

$$\begin{aligned} &Tr(\tilde{\varphi}^{\mathrm{Q}}_t(P_x)\tilde{\varphi}^{\mathrm{Q}}_t(P_y)) = \\ &Tr\big(U(g_{\mathrm{Q}}(t,\mathbb{F}(P_x)))P_x U^*(g_{\mathrm{Q}}(t,\mathbb{F}(P_x)))U(g_{\mathrm{Q}}(t,\mathbb{F}(P_y)))P_y U^*(g_{\mathrm{Q}}(t,\mathbb{F}(P_y)))\big), \end{aligned} \tag{2.3.13f}$$

what *cannot be constant in time* $t \in \mathbb{R}$ if

$$U^*(g_{\mathrm{Q}}(t,\mathbb{F}(P_x)))U(g_{\mathrm{Q}}(t,\mathbb{F}(P_y))) \not\equiv e^{i\alpha} I_{\mathcal{H}}, \ \alpha \in \mathbb{R}.$$

Hence, if we calculate the "transition probabilities" according to the algorithm taken from the linear QM in the case of nonlinear evolutions, we obtain "generically" their dependence on the parameter of transformations (on the time). This seems to be in contradiction with the usual meaning of "transformation groups" in quantum theory.

(nonlinear case 2): Let us now, however, accept the first mentioned interpretation of the "transition amplitudes", i.e. that $|\langle x|y\rangle|^2$ is the expectation value of the "observable P_y" in the "state P_x" (or vice versa). "Observables" in our generalized (nonlinear) quantum mechanics are represented by operator valued functions of "elementary states" $\varrho \in \mathcal{S}_*$, possibly via the momentum mapping $\mathbb{F}$, or by the corresponding function representatives. The transformation groups act on them in accordance with the equations (2.3.12), hence the transformations depend (generally) on points $\mathbb{F}(\varrho)$ of $Lie(G)^*$, hence on the states ϱ. Expectation of an observable $\varrho \mapsto \mathfrak{f}(\mathbb{F}(\varrho))$ in the elementary state ϱ equals $Tr(\varrho\mathfrak{f}\big(\mathbb{F}(\varrho)\big))$, cf.(2.3.3). If we transform ϱ as $\tilde{\varphi}^{\mathrm{Q}}_t(\varrho)$, and the observable $\mathfrak{f}$ is transformed simultaneously by the automorphism group transformation $\tau^{\mathrm{Q}}_{-t}\mathfrak{f}$, (2.3.12),

[78] We write here $U^*(g) \equiv U(g)^*$.

and we calculate then the expectation of the transformed observable in the transformed state, we obtain in accordance with (2.3.13e)

$$\begin{aligned} & Tr\big(\tilde{\varphi}_t^{\mathrm{Q}}\varrho\cdot(\tau_{-t}^{\mathrm{Q}}\mathfrak{f})(\mathbb{F}(\tilde{\varphi}_t^{\mathrm{Q}}\varrho))\big) \\ & = Tr\big(U(g_{\mathrm{Q}}(t,\mathbb{F}(\varrho)))\varrho U^*(g_{\mathrm{Q}}(t,\mathbb{F}(\varrho)))U(g_{\mathrm{Q}}(t,\mathbb{F}(\varrho)))\mathfrak{f}(\mathbb{F}(\varrho))U^*(g_{\mathrm{Q}}(t,\mathbb{F}(\varrho)))\big) \\ & = Tr\big(\varrho\mathfrak{f}(\mathbb{F}(\varrho))\big), \end{aligned} \tag{2.3.13g}$$

i.e. the result independent of $t \in \mathbb{R}$, as it is usually required. If the observable is, e.g. $\mathfrak{f}(\mathbb{F}(\varrho)) \equiv P_y$, i.e. it is independent of ϱ, then again it should be transformed by the same way, so that the transformed observable becomes, in general case, a function of ϱ. Hence, for $\varrho := P_x$, one has

$$Tr\big(U(g_{\mathrm{Q}}(t,\mathbb{F}(P_x)))P_xU^*(g_{\mathrm{Q}}(t,\mathbb{F}(P_x)))U(g_{\mathrm{Q}}(t,\mathbb{F}(P_x)))P_yU^*(g_{\mathrm{Q}}(t,\mathbb{F}(P_x)))\big) \equiv Tr(P_xP_y), \tag{2.3.13h}$$

and the time invariance of transition probabilities is, trivially, again obtained. ♡

We shall now return to the interpretation question of the "transition probability" $Tr(P_xP_y)$.

2.3.15. **Interpretation (Probabilities and measurements).** If we use the concept "probability" in connection with our empirical experience, it is always (perhaps) connected with a quantification of "observed phenomena", or of "occurred events". A meaning of sentences like: "The probability of the chosen *value of possible eventuality* is $\alpha > 0$" appears to us (in *empirical* sciences) unspecified without the "eventuality" being in some sense "realizable". After an experience with QM, we know that "an event" is always correlated with a change of some *macroscopically observable* (hence classical, in a general sense) parameter value. We conclude from this that probabilities ascribed to states in QM should be connected with the quantummechanical "process of measurement": They express some "weights" connected with (macroscopic) results of measurement; these weights are usually interpreted as "frequencies of occurrence" of specific results at repeated preparations of "the same microscopic state" and consecutive measurements of "the same physical observable" (let's note that this time–ordering corresponds to our, perhaps a priori, demand of causality).

All empirically interpretable (and verifiable) assertions of QM are formulated in terms of "probabilities", expressed usually by squares of moduli of "probability amplitudes". These probabilities are often called, cf. [201], the **"transition probabilities"**. Let us ask now, what "transitions", or/and transitions between what things are meant in this formulation? The mentioned probabilities are of the form $Tr(P_xP_y) \equiv |\langle x|y\rangle|^2$ for normalized vectors $|\cdot\rangle \in \mathcal{H}$ corresponding to pure states of the considered microscopic system. The standard interpretation scheme of QM (cf. [74, 201]) tells us that if a system is prepared in the state $|x\rangle$ and the measured observable Y has nondegenerate pure point spectrum (i.e. a complete orthonormal set of eigenvectors $|y_j\rangle,\ j \in J \equiv$ an index set, $Y|y_j\rangle = \lambda_j|y_j\rangle,\ \lambda_j \in \mathbb{R}, \forall j \in J,\ \lambda_j \neq \lambda_k$ for $j \neq k$), hence if it is possible to write

$$|x\rangle = \sum_{j\in J}\langle y_j|x\rangle|y_j\rangle, \quad \forall x \in \mathcal{H},$$

then only possible results of the measurement of the quantity Y are the numbers $\lambda_j,\ j \in J$, and the probability of obtaining the result λ_j in a vector state $|x\rangle$ at measuring of Y equals to

$$\mathrm{prob}(Y = \lambda_j; x) = |\langle y_j|x\rangle|^2.$$

This interpretation is the generally accepted one (according to the present author's knowledge). The denotation of this probability as "transition probability" can be understood in connection with the Dirac–von Neumann **"projection (resp. reduction) postulate"**, [74, 189], stating that after obtaining the result λ_j the measured microsystem changes abruptly its initial state $|x\rangle$ into the eigenstate of the measured quantity $|y_j\rangle$ corresponding to the obtained result λ_j. Hence, there is assumed a "transition $x \mapsto y_j$" of the microsystem.[79]

A remarkable (in the presented formulation mathematically trivial) fact is the symmetry of $\mathrm{prob}(Y = \lambda_j; x)$ with respect to interchange of the vectors x and y_j. This formal mathematical symmetry (although not being without some deep physical content) might (mis–)lead us to consider occurrence of the vectors x and y_j in the "transition probability" also as physically symmetric. We have to keep in mind, however, that the eigenvectors y_j are here in the rôle of labels of macroscopic "pointer positions", whereas the vector x represents a preparation procedure for the microsystem. This can be expressed with a help of the spectral measure E_Y of Y:

$$\mathrm{prob}(Y \in B; x) = Tr(P_x E_Y(B)), \quad B \in \mathcal{B}(\mathbb{R}),$$

where we have $E_Y(\{\lambda_j\}) \equiv P_{y_j}$, in the considered specific case. This physical asymmetry remains valid irrespective of (non-)acceptance of the "projection postulate" of Dirac and von Neumann.

To conclude, we hope that it is seen from the above considerations that in the (mathematically symmetric) expression $Tr(P_x P_y)$ for probability of a certain measurable (i.e. observable) phenomenon described in QM, the interpretation of the two vectors $x,\ y$ should be mutually different: One of the vectors represents a given (prepared) state of the micro-object, and the second represents a measured observable. This leads also to formulation of the symmetry transformation rule for these expressions generalized to our nonlinear EQM. Those symmetry transformations leave the "transition probabilities" invariant also for nonlinear generators. An a priori requirement for such an invariance is, however, of little determinative power, from the point of view of our presently defended interpretation, cf. also [35]. ♦

2.3.16. Theorem. *Any G–symmetry group τ^{Q} of a G–system (resp. UG–system) is a $\sigma(\mathcal{C}^G, \mathcal{S}_G^{cl})$–continuous one–parameter group of *-automorphisms $\tau^{\mathrm{Q}} \subset {}^*$-Aut $\mathcal{C}^G$ (resp. $\subset {}^*$-Aut $\mathcal{C}_U^G$). The relation*

$$h_{\mathfrak{f}}(\tilde{\varphi}_t^{\mathrm{Q}}\nu) = \nu(\tau_t^{\mathrm{Q}}\mathfrak{f}(\mathbb{F}(\nu))),\ \forall \mathfrak{f} \in \mathcal{C}^G,\ \forall \nu \in \boldsymbol{\mathcal{D}}(\mathbb{F}),\ \forall t \in \mathbb{R}. \tag{2.3.14}$$

is satisfied for this group of automorphisms of the C^*-algebra of G–observables $\mathcal{C}^G$. ♣

Proof. The algebraic properties of τ^{Q}, and also the τ^{Q}–invariance of $\mathcal{C}_U^G$ are consequences of (2.3.12), and of the cocycle identities (2.3.6), (2.1.24). The relation (2.3.14) is a consequence of (2.3.7), (2.3.12), (2.2.17), and of the relation (2.3.8). The $\sigma(\mathcal{C}^G, \mathcal{S}_G^{cl})$–continuity, i.e. that for all $\mu \in \mathcal{M}^G$, $\mathfrak{f} \in \mathcal{C}^G$ the functions $t \mapsto \omega_\mu(\tau_t^{\mathrm{Q}}\mathfrak{f})$ are continuous, and $\omega_\mu \circ \tau_t^{\mathrm{Q}} \in \mathcal{S}_G^{cl}$ $(\forall t \in \mathbb{R})$, follows from (2.3.12), (2.3.3), the continuity properties of $\mathfrak{f}$, g_{Q}, and U, as well as from the Lebesgue dominated convergence theorem. □

[79] This postulate, however, needn't be accepted: It cannot be usually (or even always?) verified if the measured system is really detected. As an exception might be considered the "indirect" measurement, when a correlated system is detected, what is the case of EPR–like processes. We prefer not to formulate any assumptions on the form of states of measured systems arising after measurements of a general type.

2.3.17. **Remark**. The flow $\tilde{\varphi}^{\mathrm{Q}}$ is determined by the automorphism group τ^{Q} uniquely. This association needn't be, however, injective: Different automorphism groups of $\mathcal{C}^G$ can, for a general $U(G)$, lead to the same flow $\tilde{\varphi}^{\mathrm{Q}}$ on the elementary state space $\mathcal{S}_*$. This possible ambiguity can be seen from (2.1.26), where different operator–valued functions $\nu \mapsto \mathbf{f}^0(\nu)$ with values in the commutant $\{\nu\}'$ can be chosen, cf. also [31, eqs. (2.29), (2.30)]. The whole state space–transformation groups of $\mathcal{S}(\mathcal{C}^G)$ defined as the dual mappings to the one–parameter groups τ^{Q} are, of course, different for the different τ^{Q}. We could try, e.g., to transform by them general states from $\mathcal{S}_G$. ♡

2.3.18. **Interpretation.** The theorem 2.3.16 shows, that our nonlinear dynamics can be described with a help of a *-automorphism group of our algebra of observables $\mathcal{C}^G$, resp. of $\mathcal{C}^G_U$, which is a C^*-algebra, hence it corresponds to standard *linear* descriptions of quantum systems, cf. [118, 42, 91, 120]. Since our C^*-algebra $\mathcal{C}^G$ is essentially (a weak completion of) the tensor–product algebra $\mathcal{L}(\mathcal{H}) \otimes C(\mathcal{E}_{\mathbb{F}}, \mathbb{C})$ (let us ignore here some topological aspects of definitions), it corresponds intuitively to a quantummechanical system composed of the "traditional" one, described by observables in $\mathcal{L}(\mathcal{H})$, and of a "classical subsystem" with the "generalized phase–space" $\mathcal{E}_{\mathbb{F}}$. Hence, our nonlinear quantum dynamics can be considered as a specific restricted description of dynamics (in Schrödinger picture) of a general quantum ("linear") system obtained by expressing just the evolution of "microscopic elementary states (resp. mixtures) $\in \mathcal{S}_*$" (as states on the algebra of "microscopic observables" in $\mathcal{L}(\mathcal{H})$) only, and leaving the evolution of other degrees of freedom of the composed system explicitly unnoticed. For some further comments of this point cf. Section 3.4. ♦

2.3.19. **Remark**. We shall be interested now in the possibility to represent the Lie algebra elements $\xi \in \mathfrak{g}$ by some nonlinear generators $\mathrm{h}_\xi \in \mathcal{G}^G_{cl}$, and, correspondingly, to represent the group G by continuous "nonunitary" Poisson automorphisms of $\mathcal{S}_*$. We shall formulate here one of such possibilities obtained "trivially" by a "nonlinear" Poisson morphism from the linear representation $U(G)$. This possibility was in [273] classified as "equivalent" to the linear representation. This equivalence is, of course present from the abstract mathematical point of view of theory of Poisson systems. But the quantummechanical interpretation depends on the metric structure Γ_ν on $\mathcal{S}_*$, which is not invariant with respect to such Poisson morphisms. Hence, the **physics obtained by such a "trivial delinearization" of $U(G)$**, as well as of other "G–structures" based on U(G) **might be quite different from physics coming by traditional way from the linear representation $U(G)$**. ♡

The following proposition describes an example of mechanism of the mentioned "delinearization" (cf. Remark 2.3.19) of the *G–structures* based on $U(G)$.

2.3.20. Proposition (Nonlinear G–realizations). *Let the G–system based on a unitary continuous representation U(G) be given. Let ψ be a Poisson automorphism of $\mathcal{E}_{\mathbb{F}}$ (specified, e.g. with a help of an open neighbourhood of $\mathcal{E}_{\mathbb{F}}$) leaving each symplectic leaf invariant:*

$$\psi^*\{f,h\} = \{\psi^* f, \psi^* h\} \text{ for } f,h \in C^\infty(\mathcal{E}_{\mathbb{F}}, \mathbb{R}). \tag{2.3.15}$$

Let $\mathrm{h}_\xi := \mathrm{f}^\psi_\xi := \mathbb{F}^* \circ \psi^* f_\xi,\ \xi \in \mathfrak{g}$. *Then* $\mathrm{f}^\psi_\xi \in \mathcal{G}^G_{cl}$, *and*

$$\{\mathrm{f}^\psi_\xi, \mathrm{f}^\psi_\eta\}(\nu) = -\mathrm{f}^\psi_{[\xi,\eta]}(\nu) \text{ for } \nu \in \boldsymbol{\mathcal{D}}(\mathbb{F}),\ \xi,\eta \in \mathfrak{g}, \tag{2.3.16}$$

and the association $h_{X(\xi)} \mapsto \mathrm{f}^{\psi}_{\xi}$ $(\xi \in \mathfrak{g})$ is a Poisson Lie algebra isomorphism.

Let $\Phi^{\psi}(g) := \psi^{-1} \circ Ad^*(g) \circ \psi : \mathcal{E}_{\mathbb{F}} \to \mathcal{E}_{\mathbb{F}}$; the mappings $\Phi^{\psi}(g)$ form a group of Poisson automorphisms of $\mathcal{E}_{\mathbb{F}}$ such, that its one–parameter subgroups

$$\Phi^{\psi}_{\xi} : t \mapsto \Phi^{\psi}_{\xi}(t) := \Phi^{\psi}(\exp(t\xi))$$

are the flows generated by $f^{\psi}_{\xi} := \psi^* f_{\xi}$ $(\xi \in \mathfrak{g})$. Then f^{ψ}_{ξ} are generators of their "lifts" $\tilde{\Phi}^{\psi}_{\xi}$ to the Poisson automorphism groups of $\mathcal{S}_*$ determined by the G–symmetry groups $\tau^{\xi,\psi} := \tau^{\mathrm{Q}}$ with $\mathrm{Q} := \mathrm{f}^{\psi}_{\xi}$ according to the equations (2.3.12)*, hence also*

$$h_{\mathfrak{k}}(\tilde{\Phi}^{\psi}_{\xi}(t)\nu) \equiv \nu\big((\tau^{\xi,\psi}_t(\mathfrak{k}))(\mathbb{F}(\nu))\big), \quad \forall \mathfrak{k} \in \mathcal{C}^G. \ \clubsuit$$

Proof. Recall that (cf. Definition 2.2.17)

$$h_{X(\xi)}(\nu) \equiv f_{\xi}(\mathbb{F}(\nu)) = \mathbb{F}^* f_{\xi}(\nu), \ \nu \in \boldsymbol{\mathcal{D}}(\mathbb{F}),$$

and the pull–back has trivial kernel in $C(\mathcal{E}_{\mathbb{F}}, \mathbb{R})$. Since Ran($\mathbb{F}$) consisting of $Ad^*(G)$–orbits is dense in $\mathcal{E}_{\mathbb{F}}$, a continuous function f_{η} identically vanishes on each orbit lying in $\mathcal{E}_{\mathbb{F}}$, hence vanishes on $\mathcal{E}_{\mathbb{F}}$, iff there vanishes $\psi^* f_{\eta}$. It follows that the association $h_{X(\xi)} \mapsto \mathrm{f}^{\psi}_{\xi}$ $(\forall \xi \in \mathfrak{g})$ is a bijection. It is linear in ξ, and the formulas (2.2.25), (2.3.15), and (2.2.22) show the conservation of the Poisson brackets, hence the validity of (2.3.16).

It remains to prove, that the "deformed" flows Φ^{ψ}_{ξ} are generated by f^{ψ}_{ξ}. Let $h \in C^{\infty}(\mathcal{E}_{\mathbb{F}}, \mathbb{R})$. Then

$$\frac{d}{dt}\Big|_{t=0} h\big(\Phi^{\psi}_{\xi}(t)\mathbb{F}\big) = d_{\psi F}\big((\psi^*)^{-1}h\big) \circ \mathrm{ad}^*_{\xi}(\psi F) = (\psi F)([d_{\psi F}((\psi^*)^{-1}h), \xi])$$
$$= \{f_{\xi}, (\psi^*)^{-1}h\}(\psi F) = \psi^*\{f_{\xi}, (\psi^*)^{-1}h\}(F) = \{f^{\psi}_{\xi}, h\}(F), \quad (2.3.17)$$

where we define $(-\mathrm{ad}^*_{\xi}) := (ad_{\xi})^*$, the dual mapping of the inner differentiation of the Lie algebra, $ad_{\xi} : \eta \mapsto [\xi, \eta]$. This proves the proposition. □

2.3.21. **Examples**. As a large class of examples of mappings ψ occurring in the Proposition 2.3.20, we can choose $\psi := \varphi^Q_t$ for any nonlinear $Q \in C^{\infty}(\mathcal{E}_{\mathbb{F}}, \mathbb{R})$ with complete Hamiltonian vector field (hence flow) on $\mathcal{E}_{\mathbb{F}}$, with a fixed value of $t \in \mathbb{R}$. The question of a physical interpretation of such "nonlinear deformations" of the "linear" one is left open here. ♡

Let us consider now the specific case of a physical system described (in the sense of EQM) by a C^*-algebra $\mathcal{C} := C(\mathcal{E}, \mathfrak{A})$, with $\mathcal{E}$ an Hausdorff compact, and $\mathfrak{A}$ a simple unital C^*-algebra, cf. Definition B.2.5; the continuity of $\mathfrak{f}(\in \mathcal{C}) : F(\in \mathcal{E}) \mapsto \mathfrak{f}(F)(\in \mathfrak{A})$ is here uniform in the norm of $\mathfrak{A}$. E.g., we can use $\dim \mathcal{H} < \infty$ and $U(G)$ irreducible in our previous costructions, and then we shall have $\mathfrak{A} := \mathcal{L}(\mathcal{H})$, and $\mathcal{E}(\subset \mathfrak{g}^*)$ some compact convex Ad^*–invariant set. In this case $\mathcal{C} \sim \mathfrak{A} \otimes C(\mathcal{E})$, [227], and the structure of such systems can be described now with some additional details. Let us mention first, [31, Proposition 2.6]:

2.3.22. Lemma. *The pure states $\omega \in \mathcal{S}(\mathcal{C})$(i.e. extremal points of the $\sigma(\mathcal{C}^*, \mathcal{C})$–compact $\mathcal{S}(\mathcal{C}) = \mathcal{C}^*_{+1}$) are of the form*

$$\omega(\mathfrak{f}) = \omega_{\mathfrak{A}}\big(\mathfrak{f}(F_{\omega})\big), \ \forall \mathfrak{f} \in \mathcal{C}, \quad (2.3.18)$$

where $\omega_{\mathfrak{A}} \in \mathcal{S}(\mathfrak{A})$ are pure states on $\mathfrak{A}$, and $F_{\omega} \in \mathcal{E}$ is fixed.

It could be useful to compare this assertion with Definition 2.3.5 to see what states of the C^*-algebra $\mathcal{C}$ are not contained in the set of states determined by that definition.

Let us now describe the general form of symmetry–transformations (i.e. the automorphisms of $\mathcal{C}$) of such a system, cf. [36], and [32, Remarks 3.15] for more complete (but there unproved) formulations:

2.3.23. Proposition. *Let a C^*-algebra $\mathcal{C} := C(\mathcal{E}, \mathfrak{A})$ be given as above. Then there is a canonical bijection between $\gamma \in {}^*$-Aut $\mathcal{C}$, and couples $\{\varphi_\gamma; \hat{\gamma}\}$, where φ_γ is an arbitrary homeomorphism of $\mathcal{E}$, and $\hat{\gamma}$ is an arbitrary mapping $\hat{\gamma} : \mathcal{E} \to {}^*$-Aut $\mathfrak{A},\ F \mapsto \hat{\gamma}_F$, with the functions $F \mapsto \hat{\gamma}_F(\mathrm{x})$ $(\forall \mathrm{x} \in \mathfrak{A})$ being all norm–continuous. The bijection is determined by the identity*

$$\gamma(\mathfrak{f})(F) \equiv \hat{\gamma}_F\big(\mathfrak{f}(\varphi_\gamma F)\big), \tag{2.3.19}$$

valid for all $\mathfrak{f} \in \mathcal{C}$.

Proof. Due to the simplicity of $\mathfrak{A}$, the abelian subalgebra $C(\mathcal{E})$ of $\mathcal{C}$ coincides with its center $\mathcal{Z} := \mathcal{Z}(\mathcal{C})$. The center $\mathcal{Z}$ is invariant with respect to any *-automorphism of $\mathcal{C}$, hence the restriction of γ to $C(\mathcal{E})$ is also an automorphism. The Gelfand–Najmark theory of commutative C^*-algebras (cf. [187, 103, 101], and also Example B.2.3(iii)) implies that the *–automorphisms of $C(\mathcal{E})$ are in a bijective correspondence with homeomorphisms φ of $\mathcal{E}$ onto itself.

(i) Let $\gamma \in {}^*$-Aut $\mathcal{C}$. Then the corresponding homeomorphism φ_γ is defined by:

$$(\gamma f)(F) =: f(\varphi_\gamma F), \quad \forall f \in C(\mathcal{E}) \subset \mathcal{C},\ \forall F \in \mathcal{E}.$$

Let an arbitrary $\mathrm{x} \in \mathfrak{A}$ be considered as a constant function – an element $\hat{\mathrm{x}} \in \mathcal{C} := C(\mathcal{E}, \mathfrak{A}),\ \hat{\mathrm{x}}(F) \equiv \mathrm{x} \equiv \mathrm{x}\cdot\mathbb{I}(F)$, where $\mathbb{I}(F) = 1,\ \forall F \in \mathcal{E}$. Then the value $\gamma(\hat{\mathrm{x}})(F),\ f \in \mathcal{E}$ of $\gamma(\hat{\mathrm{x}}) \in \mathcal{C}$ will be denoted by

$$\hat{\gamma}_F(\mathrm{x}) := \gamma(\hat{\mathrm{x}})(F), \quad \forall F \in \mathcal{E}, \quad \forall \mathrm{x} \in \mathfrak{A}.$$

The pointwise character of algebraic operations in $C(\mathcal{E}, \mathfrak{A})$ implies that in this way defined $\hat{\gamma} : F \mapsto \hat{\gamma}_F$ is a *–morphism of $\mathfrak{A}$ into itself.

We shall show that $\hat{\gamma}_F$ is a nonzero morphism (hence an isomorphism, due to simplicity of $\mathfrak{A}$) for any $F \in \mathcal{E}$. A general element $\mathfrak{f}$ of $\mathcal{C}$ is uniformly approximated by elements of the form

$$\mathfrak{f}' := \sum_j \mathrm{x}_j \cdot f_j,\ \mathfrak{f}'(F) \equiv \sum_j \hat{\mathrm{x}}_j(F)\cdot f_j(F), \quad \mathrm{x}_j \in \mathfrak{A},\ f_j \in C(\mathcal{E}) \subset \mathcal{C}, \tag{2.3.20}$$

hence also by the elements of the form $\gamma(\mathfrak{f}')$, since γ is a *-automorphism of $\mathcal{C}$. For the elements of the form (2.3.20) one has

$$\gamma(\mathfrak{f}')(F) \equiv \sum_j \hat{\gamma}_F(\mathrm{x}_j)\cdot f_j(\varphi_\gamma F). \tag{2.3.21}$$

For a zero morphism $\hat{\gamma}_{F_0}$ it would be $\gamma(\mathfrak{f})(F_0) = 0$ for all $\mathfrak{f} \in \mathcal{C}$, what cannot happen, since both $\mathfrak{A}$, and $C(\mathcal{E})$ are unital. It follows that $\hat{\gamma}_F \in {}^*$-Aut $\mathfrak{A},\ \forall F \in \mathcal{E}$.

The formula (2.3.21) implies then (2.3.19) due to continuity of all the $\hat{\gamma}_F$, as well as of γ:

$$\gamma(\mathfrak{f}')(F) \equiv \hat{\gamma}_F \left(\sum_j \mathrm{x}_j \cdot f_j(\varphi_\gamma F) \right).$$

The continuity of $\hat{\gamma} : F \mapsto \hat{\gamma}_F$ follows from the continuity of each function $F \mapsto \gamma(\mathfrak{f})(F)$, $\mathfrak{f} \in \mathcal{C}$.

(ii) Let us now have given any homeomorphism φ_γ of the Hausdorff compact $\mathcal{E}$ onto itself, as well as an arbitrary strongly continuous family $\hat{\gamma} : \mathcal{E} \to {}^*\text{-Aut}\,\mathcal{C},\ F \mapsto \hat{\gamma}_F$. Let us define the mappings $\varphi : \mathcal{C} \to \mathcal{C}$, and $\gamma_0 : \mathcal{C} \to \mathcal{C}$ as follows:

$$\varphi(\mathfrak{f})(F) := \mathfrak{f}(\varphi_\gamma F), \qquad \gamma_0(\mathfrak{f})(F) := \hat{\gamma}_F\big(\mathfrak{f}(F)\big), \qquad \forall \mathfrak{f} \in \mathcal{C},\ F \in \mathcal{E}. \tag{2.3.22}$$

The continuity and the morphism properties of the given $\hat{\gamma}_\cdot$, and φ_γ show that both the mappings φ, and γ_0 introduced in (2.3.22) are *–automorphisms of $\mathcal{C}$. Hence the formula (2.3.19) determines an automorphism $\gamma : \mathcal{C} \to \mathcal{C}$ as the composition of these two automorphisms: $\gamma := \gamma_0 \circ \varphi \in {}^*\text{-Aut}\,\mathcal{C}$. □

This proposition allows us to view also a degree of generality of the before introduced "G–transformations", cf. Definition 2.3.13, at least in the present simple case: Without further symmetry requirements, the general automorphism group contains much more continuous subgroups than we have introduced in Definition 2.3.13. It might be worth mentioning that, under some continuity requirements onto $\gamma \in {}^*\text{-Aut}\,\mathfrak{A}$ ($\pi_{\mathfrak{g}}$–normality, [36, 32]), the homeomorphism φ_γ can be, in specific models (mean–field, cf. also Subsection 1.1-b, and Section 3.4) uniquely determined by the set of automorphisms $\{\hat{\gamma}_F : F \in \mathcal{E}\}$.

3 Specifications and Applications

It will be shown in this chapter that the general scheme for dynamical theories developed in Chapter 2 applies to a wide scale of existing physical theories. Different specifications of the formal scheme mainly consist in choices of classes of "generators", "observables", and "states", cf. Definitions 2.3.2, 2.3.3, and 2.3.5.

A choice of the representation $U(G)$ belongs to important general tools for such a specification, i.e. a determination of a G-system, cf. Definition 2.3.6. There are, however, also other possible ways for determination of a specification; some of them are connected with models already published in literature. Let us give first a review of some of these specifications.

3.1 A Review of Considered Specifications

There is a whole array of general physical theories and/or their "caricatures", resp. "approximations" covered by the general model of our **"Extended Quantum Mechanics" (EQM)** described in Chapter 2. Let us mention here briefly some typical of them, resp. some of possible applications of EQM; most of these will be described in some details later in this chapter:

3.1.1 (Quantum Mechanics).

Traditional (linear) quantum mechanics (QM) is obtained as the specification corresponding to the G–system with trivial group $G \equiv \{e\} \equiv \{$*one–point set consisting of the unit element* $e \in G\}$, cf. Section 3.3, esp. Subsection 3.3-b. Another possibility of obtaining QM from EQM is the restriction of any G–system to the subalgebra $\mathcal{C}_q^G \subset \mathcal{C}^G$, and accepting only such generators the flows of which leave $\mathcal{C}_q^G$ invariant. ♠

3.1.2 (Nonlinear Quantum Mechanics).

Nonlinear extensions of QM "living" on the projective Hilbert space $P(\mathcal{H})$, and containing in their sets $\mathcal{G}^G$ all "relevant" generators is called here the nonlinear QM (NLQM). A specific choice of "observables" also depends on the accepted interpretation scheme; the same concerns the set of states of a chosen theory. Note here that, in the framework of this "specification", it is possible to describe also the "general theory", because, e.g. all density matrices could be expressed by unit rays in the Hilbert space $\mathfrak{H}$ of Hilbert–Schmidt operators, cf. Remark 3.2.1; such an approach seems, however, in a certain sense "unnatural", because it needs some additional restrictions. NLQM will be shortly discussed in Subsections 3.3-a, and 3.3-e. ♠

3.1.3 (Subsystems in Macroscopic Environment).

The ideas leading to the theory presented in this work are closely connected with models of infinite quantum systems with specific dynamics "of mean–field (MF) type", cf. [130, 31, 32, 33, 264, 185, 87]. We shall not go into details on this point in this paper. It was shown, however, in Theorem 2.3.16, that all the nonlinear evolutions generated by G–(classical) generators $Q \in \mathcal{G}_{cl}^G$ can be described as one–parameter groups of *-automorphisms of a C^*-algebra, namely the C^*-algebra $\mathcal{C}^G$. Let us note here that this C^*-algebraic description allows us, in this formal framework, to **distinguish elementary mixtures from genuine ones**: In the case if the domain $\mathcal{E}_{\mathbb{F}} \subset \mathfrak{g}^*$ can be identified with the whole $\mathcal{S}_*$ (what is possible in the case of the choice $G := \mathfrak{U} :=$

$\mathcal{U}(\mathcal{H})$), elementary mixtures are just the pure (i.e. extremal) states of the abelian C^*-subalgebra $\mathcal{C}^G_{cl} \subset \mathcal{C}^G$. For some further comments see Section 3.4. ♠

3.1.4 (Classical Mechanics).

Classical mechanics (CM) of a system with a symmetry group G is also contained in EQM:

Let a "kinematical" symmetry group G of a classical Hamiltonian system be given; we shall assume for simplicity that it is a connected and simply connected Lie group. Let the phase space of this system be a homogeneous space of G, the action of G being there symplectic. This phase space can be identified with a coadjoint orbit of the group G, or of its one–dimensional central extension, [148]. Natural generalizations of these systems are Poisson systems "living" on an $Ad^*(G)$–invariant part of $Lie(G)^* \equiv \mathfrak{g}^*$, cf. Section 1.4, and also Appendix A.4 (for literature on classical mechanics see also [277, 1, 7, 258, 172, 8, 179]). We can see from Section 2.3 that our scheme of EQM restricted to the algebra $\mathcal{C}^G_{cl}$ leads to description of any such "G–symmetric" CM–system as a subsystem of our (quantal) G–system. ♠

3.1.5 (Hartree–Fock Theory).

Specific "quasiclassical" and/or "selfconsistent" approximations for QM described as dynamics on manifolds of generalized coherent states, [221, 149, 199, 200], i.e. the "classical projections of QM", [27], are contained in EQM as well; these specifications include the systems obtained by the "time–dependent variational principle", [200, 154]. An important special case of these is the time–dependent Hartree–Fock approximation; the corresponding (infinite dimensional) set of generalized coherent states consists now from all Slater determinants of an N–fermion system, and the group G is the whole unitary group of one–particle Hilbert space, cf. Subsection 3.3-d. ♠

3.1.6 (Specific Time–Dependent QM).

A class of quantummechanical systems with time–dependent Hamiltonians can be found as a subtheory of EQM; it appears to be identical with the corresponding (time independent) NLQM. This class includes, as a special case, the nonlinear dynamics (for pure states) proposed by Weinberg in [273].[80] The "integrability" of such systems is determined by integrability of corresponding classical Hamiltonian systems; cf. Sections 3.5, 3.6. ♠

3.1.7 (Aspects of Quantum Measurement).

The developed theory provides a possible framework for dealing with the old fundamental question of QM – the measurement problem, resp. the problem of "collaps of wave packets". Such a possibility is here, however, except of several included remarks and notes at various places of the text, left just on the level of this unspecified hypothesis. ♠

Since the large part of this chapter will consist of formal constructions on a unique Kählerian orbit $\mathcal{O}_\varrho(\mathfrak{U})$, i.e. the projective Hilbert space $P(\mathcal{H})$ consisting of one–dimensional projections $\varrho \equiv \varrho^2$, it would be useful first to examine the manifold structure of $P(\mathcal{H})$ in some details. Our analysis is mainly based on the earlier author's works [26, 27]; the obtained structures, as well as used mathematical devices are essentially identical with independently composed papers by Cirelli et al., cf. mainly [63, 62].

[80] There is no need of any restriction by $U(G)$ in finite dimensional Hilbert spaces.

3.2 Structure of Projective Hilbert Space

Let $\mathcal{H}$ be a complex Hilbert space. Elements x of $\mathcal{H}$ will be naturally associated with the corresponding elements x^* of the topological dual space $\mathcal{H}^*$ of $\mathcal{H}$ via the Riesz lemma, i.e. $x^*(y) := (x, y)$ $(\forall y \in \mathcal{H})$; the mapping $x \mapsto x^*$ is an antilinear isometry of $\mathcal{H}$ onto $\mathcal{H}^*$. The space of Hilbert–Schmidt operators $\mathfrak{H}$ will be (linearly and isometrically) identified with the tensor product $\mathcal{H} \otimes \mathcal{H}^*$ in such a way, that the operator (in the Dirac notation, [74]) $|x\rangle\langle y| \equiv x \otimes y^* \in \mathcal{L}(\mathcal{H})$ acts as follows:

$$|x\rangle\langle y|z := |x\rangle\langle y|z\rangle := (y, z)x, \quad \forall z \in \mathcal{H}.$$

The scalar product in $\mathfrak{H}$ is then

$$\big(x \otimes y^*, z \otimes u^*\big)_2 = Tr\big(|y\rangle\langle x|\cdot|z\rangle\langle u|\big) = (x, z)(u, y) \equiv \langle x|z\rangle\langle u|y\rangle.$$

3.2.1. **Remark**. There is a natural question whether the dynamics (in general – nonlinear) of density matrices described in Chapter 2 as dynamics on the orbits $\mathcal{O}_\varrho(\mathfrak{U})$ with arbitrary $\dim(\varrho)$ can be described equivalently as a "corresponding" dynamics on the projective Hilbert space $P(\tilde{\mathfrak{H}}) \ni \varrho$ of some "another" Hilbert space $\tilde{\mathfrak{H}}$, i.e. a dynamics on the "one–dimensional" orbit $\mathcal{O}_\varrho(\tilde{\mathfrak{U}})$ with $\dim(\varrho) = 1$, $\tilde{\mathfrak{U}} := \mathcal{U}(\tilde{\mathfrak{H}}) \equiv$ (the unitary group of $\mathcal{L}(\tilde{\mathfrak{H}})$). This question can be motivated, e.g., by the fact, that any density matrix ϱ in a separable Hilbert space $\mathcal{H}$ can be considered as the "partial trace", [71, Section 10.1], of a one–dimensional projection P_x on a tensor–product space $\mathcal{H} \otimes \mathcal{K}$ interpreted usually as the Hilbert space of a composed QM–system containing the considered system (occurring in the state ϱ) as a subsystem described in $\mathcal{H}$ (this assertion is almost trivial: it can be proved by an explicit construction of P_x from ϱ); cf. also (iii) below. Let us mention here three possibilities of description of "mixed states dynamics" by a dynamics of vector–states projected to some $P(\tilde{\mathfrak{H}})$:

(i) Let us recall that the trace–class operators are $\mathfrak{T} \subset \mathfrak{H} \subset \mathcal{L}(\mathcal{H})$, where the space $\mathfrak{H}$ is the space of Hilbert–Schmidt operators endowed with a canonical Hilbert space structure. This shows that one could formulate all the theory of Chapter 2 "in principle" on the orbit of the unitary group $\mathcal{U}(\mathfrak{H})$ of $\mathfrak{H}$ consisting of one–dimensional projections of $\mathfrak{H}$, i.e. the projective Hilbert space $P(\mathfrak{H})$ of $\mathfrak{H}$. This would need, however, an additional work to distinguish what elements of $P(\mathfrak{H})$ are relevant in what physical situations and, moreover, what unitary transformations in $\mathfrak{H}$ correspond to those used in Chapter 2.

(ii) Another possibility to describe also density matrices and their (possibly nonlinear) dynamics in framework of a projective Hilbert space comes from elements of the Tomita–Takesaki theory of modular Hilbert algebras, cf. [253, 42]: Let $\tilde{\mathfrak{H}}$ be Hilbert space of a faithful weakly*–continuous representation of the considered von Neumann algebra of observables (in our case it is $\mathcal{L}(\mathcal{H})$) with a cyclic and separating vector; that part of $P(\tilde{\mathfrak{H}})$ which is the image by the canonical projection $\tilde{\mathfrak{H}} \to P(\tilde{\mathfrak{H}})$ of the natural positive cone $\mathcal{P}$ in $\tilde{\mathfrak{H}}$, cf. [42, Section 2.5.4], describes the whole $\mathcal{S}_*(\mathcal{L}(\mathcal{H}))$.

(iii) The last possibility to be mentioned here is that one considering the density matrix $\varrho \in \mathfrak{T}(\mathcal{H})$ as the partial trace of some $P_x \in P(\mathcal{H} \otimes \mathcal{K})$. Let us assume that both the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ are infinite–dimensional separable. If $\{\varphi_k^{(1)}; k \in \mathbb{Z}_+\} \subset \mathcal{H}$ is an orthonormal basis of $\mathcal{H}$ such

that the given density matrix ϱ is

$$\varrho = \sum_{j=1}^{\infty} \lambda_j |\varphi_j^{(1)}\rangle\langle\varphi_j^{(1)}|,$$

then, for any orthonormal basis $\{\psi_k^{(2)}, k \in \mathbb{Z}_+\}$ of $\mathcal{K}$, the vector $x \in \mathcal{H} \otimes \mathcal{K}$ defined by

$$x := \sum_{j \in \mathbb{Z}_+} \sqrt{\lambda_j}\varphi_j^{(1)} \otimes \psi_j^{(2)}$$

has the desired property: For any $A \in \mathcal{L}(\mathcal{H})$, with $I_{\mathcal{K}}$ the identity in $\mathcal{L}(\mathcal{K})$, one has

$$Tr(\varrho A) := Tr_{\mathcal{H}}(\varrho \cdot A) \equiv Tr_{\mathcal{H}\otimes\mathcal{K}}(P_x \cdot A \otimes I_{\mathcal{K}}).$$

This formula defines the mapping $Tr_{\mathcal{K}} : P_x \mapsto \varrho$ called the **partial trace**, cf. [71]. The mapping $Tr_{\mathcal{K}}$ can be extended by linearity to whole space $\mathfrak{T}(\mathcal{H}\otimes\mathcal{K})$ of trace–class operators on the Hilbert space of the "composed system". Let, e.g., $\mathrm{u}_f(t, \varrho)$ be a unitary cocycle describing (nonlinear) evolution of $\varrho \in \mathfrak{T}(\mathcal{H})$ according to Proposition 2.1.15. Then

$$\varrho(t) \equiv \mathrm{u}_f(t, \varrho)\varrho\mathrm{u}_f(t, \varrho)^{-1},$$

and the corresponding evolution of $x \in \mathcal{H} \otimes \mathcal{K}$ can be chosen as

$$x(t) \equiv \sum_{j=1}^{\infty} \sqrt{\lambda_j}\big(\mathrm{u}_f(t, \varrho)\varphi_j^{(1)}\big) \otimes \psi_j^{(2)}.$$

Now one has to solve the problem whether and how this evolution can be described by a unitary cocycle $\tilde{\mathrm{u}}_h(t, x)$ acting on $P(\mathcal{H} \otimes \mathcal{K})$.

We do not intend to elaborate further these remarks in this work. They were mainly mentioned here to stress importance of the special orbit of $\mathcal{U}(\mathcal{H})$: the projective Hilbert space $P(\mathcal{H})$. ♡

The projective Hilbert space $P(\mathcal{H})$ will be considered as a complex–analytic manifold, the structure of which will be presently described.

3.2.2. Notation. *The elements of $P(\mathcal{H})$ will be identified with one–dimensional projections and denoted also by boldface lowercase letters: $\mathbf{y} \equiv P_y \in P(\mathcal{H}),\ y \in \mathbf{y}$, i.e. we shall consider elements of $P(\mathcal{H})$ interchangeably as equivalence classes in $\mathcal{H}$: $\mathbf{y} := \{x \in \mathcal{H} : \exists\lambda \in \mathbb{C}, x = \lambda y\}$, and as one–dimensional projections $P_y \equiv P_{\mathbf{y}}$. In the case if $(0 \neq)y \in \mathcal{H}$ is expressed by a* formula *written in any type of letters, then we shall use the boldface expression in boldface brackets to write down the corresponding symbol for the class $\mathbf{y} \in P(\mathcal{H})$, $y \in \mathbf{y} :=$ $(\boldsymbol{formula})$.* ♡

Let us define now an atlas on the manifold $P(\mathcal{H})$:

3.2.3. Definitions (Atlas on $P(\mathcal{H})$).

(i) The **topology of** *$P(\mathcal{H})$ will be defined as the factor–topology coming from the Hilbert–space norm–topology of $\mathcal{H}$. It can be shown, [27], that this topology is equivalent to several other natural topologies induced on $P(\mathcal{H})$ by its embedding to the Banach spaces $\mathcal{L}(\mathcal{H})$, $\mathfrak{H}$, $\mathfrak{T}_s$, or also to several weak topologies coming from the duality relation $(P_x; C) \mapsto Tr(CP_x) \equiv \langle C; P_x\rangle$.*

(ii) The **charts on** $P(\mathcal{H})$ *consist of neighbourhoods*

$$\mathcal{V}_{\mathbf{y}} := \{P_x \in P(\mathcal{H}) : Tr(P_x P_y) \neq 0\} \tag{3.2.1a}$$

of the points $\mathbf{y} \in P(\mathcal{H})$*, and their (y–dependent) mappings*

$$\theta_y : \mathcal{V}_{\mathbf{y}} \to [y]^{\perp}, P_x \mapsto \theta_y(\mathbf{x}) := \|y\|^2 (y,x)^{-1}(I - P_y)x \tag{3.2.1b}$$

onto the complex orthogonal complements $[y]^{\perp}$ *(considered as complex Hilbert subspaces of* $\mathcal{H}$*) of nonzero* $y \in \mathcal{H},\ y \in \mathbf{y}$.

3.2.4. Proposition. *The mapping* θ_x *is a homeomorphism of* $\mathcal{V}_{\mathbf{x}}$ *onto* $[x]^{\perp}$ *(with the norm–topology of* $\mathcal{H}$*). The set*

$$\{(\mathcal{V}_{\mathbf{x}}; \theta_x) : 0 \neq x \in \mathcal{H}\} \tag{3.2.2a}$$

is an atlas on $P(\mathcal{H})$ *defining a complex–analytic manifold structure consistent with the topology of* $P(\mathcal{H})$. ♣

Proof. For any $\mathbf{y}_j \in \mathcal{V}_{\mathbf{x}}$, and any $y_j \in \mathbf{y}_j$ $(j = 1, 2)$, it is $\mathbf{y}_1 \neq \mathbf{y}_2$ iff $(x, y_2)y_1 \neq (x, y_1)y_2$, hence according to (3.2.1), θ_x is injective.

For any $z \in [x]^{\perp}$ and $y := z + x$, we have $\mathbf{y} \in \mathcal{V}_{\mathbf{x}}$ (since $x \neq 0$), and $\theta_x(\mathbf{y}) = z$, hence θ_x is bijective. Let $\|x\| := 1$. For $z_j \in [x]^{\perp},\ y_j := z_j + x$ $(j = 1, 2)$ the identity

$$1 - Tr(P_{y_1} P_{y_2}) = \frac{1}{(\|z_1\|^2 + 1)(\|z_2\|^2 + 1)} \left(\|z_1 - z_2\|^2 + \|z_2\|^2 \cdot \|(I - P_{z_2})(z_1 - z_2)\|^2\right) \tag{3.2.2b}$$

implies the bicontinuity of θ_x. For any $0 \neq x_j \in \mathcal{H},\ j = 1, 2$, and for $z \in \theta_{x_1}(\mathcal{V}_{\mathbf{x}_1} \cap \mathcal{V}_{\mathbf{x}_2})$, we have

$$\theta_{x_2} \circ \theta_{x_1}^{-1}(z) = \|x_2\|^2 \frac{x_1 + z}{(x_2, x_1 + z)} - x_2, \tag{3.2.2c}$$

and we can see, cf. [40, 58], that the mapping

$$\theta_{x_2} \circ \theta_{x_1}^{-1} : \theta_{x_1}(\mathcal{V}_{\mathbf{x}_1} \cap \mathcal{V}_{\mathbf{x}_2}) \to \theta_{x_2}(\mathcal{V}_{\mathbf{x}_1} \cap \mathcal{V}_{\mathbf{x}_2}) \tag{3.2.2d}$$

is a complex analytic function. □

The tangent space $T_{\mathbf{y}}P(\mathcal{H})$ of $P(\mathcal{H})$ at $\mathbf{y} \in P(\mathcal{H})$ will be identified with the linear space of classes of mutually tangent differentiable curves at $\mathbf{y}$ as in the finite dimensional case, [1, 61, 151]; this is in accordance with our results from Subsection 2.1-b, cf. Definitions 2.1.3, and Proposition 2.1.5. For any differentiable mapping θ of a neighbourhood of $\mathbf{y}$ onto a neighbourhood of $\theta(\mathbf{y})$ in another differentiable manifold, the corresponding tangent mapping $T_{\mathbf{y}}\theta$ maps the vector $\mathbf{v} \in T_{\mathbf{y}}P(\mathcal{H})$ represented by a curve $t \mapsto c_{\mathbf{v}}(t)$ $(c_{\mathbf{v}}(0) = \mathbf{y})$ onto the vector tangent at $\theta(\mathbf{y})$ represented by the curve $t \mapsto \theta(c_{\mathbf{v}}(t))$. If $\mathbf{x} \in \mathcal{V}_{\mathbf{y}}$ (hence $\mathbf{y} \in \mathcal{V}_{\mathbf{x}}$), $T_{\mathbf{y}}\theta_x$ maps $T_{\mathbf{y}}P(\mathcal{H})$ onto $[x]^{\perp}$, and the choice of $x := y \in \mathbf{y}$ (in the index of θ_x) leads to a natural

(y–dependent) identification of $T_{\mathbf{y}}P(\mathcal{H})$ with $[y]^{\perp}$ (and also with $\mathcal{V}_{\mathbf{y}}$). The vector $\mathbf{v} \in T_{\mathbf{y}}P(\mathcal{H})$ (let **y** be fixed) is mapped onto $\mathbf{v}_x := T_{\mathbf{y}}\theta_x(\mathbf{v}) \in [x]^{\perp}$,

$$\mathbf{v}_x := \frac{d}{dt}\bigg|_{t=0} \theta_x(c_{\mathbf{v}}(t)). \tag{3.2.3a}$$

One can choose, e.g.,

$$c_{\mathbf{v}}(t) := \mathbf{(exp(-itB(v))y)} = Ad^*\big(\exp(-itB(\mathbf{v}))\big)P_y, \tag{3.2.3b}$$

where $B(\mathbf{v})$ is a selfadjoint element of $\mathcal{L}(\mathcal{H})$ representing an arbitrarily chosen vector $\mathbf{v} \in T_{\mathbf{y}}P(\mathcal{H})$ in this way. With such a choice of $B(\mathbf{v})$, one has the expression

$$\mathbf{v}_x = -i\,(x,y)^{-1}\|x\|^2\left(I - \frac{P_yP_x}{Tr(P_yP_x)}\right)B(\mathbf{v})y. \tag{3.2.3c}$$

For $x = y$, this leads to

$$\mathbf{v}_y = -i\,(I - P_y)B(\mathbf{v})y. \tag{3.2.3d}$$

Specifying **v** by the choice of any $v \in [y]^{\perp}$, and by the choice

$$B(\mathbf{v}) := i\,\|y\|^{-2}(|v\rangle\langle y| - |y\rangle\langle v|), \tag{3.2.3e}$$

one obtains $\mathbf{v}_y = v$. With a chosen $y \in \mathbf{y}$, and the corresponding "identification θ_y" of $T_{\mathbf{y}}P(\mathcal{H})$ with $[y]^{\perp}$, one can identify $\mathbf{v} \equiv v \equiv \mathbf{v}_y$. For different choices of *x* in (3.2.3), on the other hand, one obtains expressions $\mathbf{v}_x$ and $\mathbf{v}_z$ of **v** in different charts θ_x and θ_z related mutually by

$$\mathbf{v}_z = \|x\|^{-2}(z,y)^{-1}\|z\|^2(x,y)\left(I - \frac{|y\rangle\langle z|}{(z,y)}\right)\mathbf{v}_x. \tag{3.2.4}$$

Let us note that $(z,y)^{-1}|y\rangle\langle z| = P_yP_z/Tr(P_yP_z)$. One can now also check validity of the following two mutually inverse relations:

$$\mathbf{v}_x = (x,y)^{-1}\|x\|^2\left(I - \frac{P_yP_x}{Tr(P_yP_x)}\right)\mathbf{v}_y, \tag{3.2.5a}$$

and

$$\mathbf{v}_y = \|x\|^{-2}(x,y)(I - P_y)\mathbf{v}_x. \tag{3.2.5b}$$

We shall consider $P(\mathcal{H})$ as a real analytic manifold endowed with (integrable, [63]) complex structure

$$J \in \mathcal{T}_1^1(P(\mathcal{H}))$$

($\mathcal{T}_s^r(M)$ denotes the vector space of all r–times contravariant and s–times covariant smooth tensor fields on a manifold M) defined as the section $\mathbf{y} \mapsto J_{\mathbf{y}} \in \mathcal{L}([y]^{\perp})$ with $(J_{\mathbf{y}}\mathbf{v})_y := i\,\mathbf{v}_y$, i.e. the complex structure is determined by the given multiplication by the imaginary unit "i" in

$\mathcal{H}$. The Kähler metrics $\Gamma \in \mathcal{T}_2^0(P(\mathcal{H}))$ on $P(\mathcal{H})$, cf. [26, 27, 63], called also the *Fubini–Study metrics*, can be expressed in the following form:

$$\Gamma_{\mathbf{y}}(\mathbf{v},\mathbf{w}) := 2\|y\|^{-2}\mathrm{Re}(\mathbf{v}_y,\mathbf{w}_y),\ \mathbf{v},\mathbf{w} \in T_{\mathbf{y}}P(\mathcal{H}). \tag{3.2.6}$$

The corresponding symplectic form $\Omega \in \mathcal{T}_2^0(P(\mathcal{H}))$ is then expressed by:

$$\Omega_{\mathbf{y}}(\mathbf{v},\mathbf{w}) := \Gamma_{\mathbf{y}}(\mathbf{v},J\mathbf{w}) = -2\|y\|^{-2}\mathrm{Im}(\mathbf{v}_y,\mathbf{w}_y). \tag{3.2.7}$$

These structures coincide on $P(\mathcal{H})$ with those coming from the tensor field Ψ, cf. (2.1.27).

3.2.5. Lemma. *The two–form* Ω *in* (3.2.7) *coincides with the restriction to the orbit* $P(\mathcal{H})$ *of the form* Ω *from* (2.1.27). ♣

Proof. The mapping β_ν defined in (2.1.7) for $\nu := \mathbf{y} \in P(\mathcal{H})$ has the form $\beta_{\mathbf{y}}(\mathrm{c}) = i[\mathrm{c}, P_y]$, where $\mathrm{c} \in T_{\mathbf{y}}\mathcal{O}_{\mathbf{y}}(\mathfrak{U}) \subset \mathfrak{T}_s$ is represented (cf. Definitions 2.1.3(iii)) by a bounded operator. If a vector $\mathbf{v} \in T_{\mathbf{y}}P(\mathcal{H})$ corresponds to the curve

$$t \mapsto \mathrm{c}_{\mathbf{v}}(t) := \boldsymbol{(\exp(-itB(\mathbf{v}))y)} = \mathrm{Ad}^*\big(\exp(-itB(\mathbf{v}))\big)P_y, \tag{3.2.8a}$$

then the corresponding operator is

$$\mathrm{c} = \dot{\mathrm{c}}_{\mathbf{v}} := \frac{d}{dt}\bigg|_{t=0} \mathrm{c}_{\mathbf{v}}(t) = i[P_y, B(\mathbf{v})]. \tag{3.2.8b}$$

By a use of Definitions 2.1.3(iv), one obtains $\beta_{\mathbf{y}}(\dot{\mathrm{c}}_{\mathbf{v}}) = q_{\mathbf{y}}(B(\mathbf{v}))$. Inserting these expressions to (2.1.27), we obtain the relation

$$\Omega_{\mathbf{y}}(\mathbf{v},\mathbf{w}) = i\,Tr\big(P_y[q_{\mathbf{y}}(B(\mathbf{v})), q_{\mathbf{y}}(B(\mathbf{w}))]\big), \tag{3.2.8c}$$

what is identical with the result of the corresponding insertions from equations (3.2.3) into (3.2.7). □

Expressed in the chart θ_x, the Kähler structure Ψ on $P(\mathcal{H})$ has the form:

$$\Gamma_{\mathbf{y}}(\mathbf{v},\mathbf{w}) - i\Omega_{\mathbf{y}}(\mathbf{v},\mathbf{w}) = 2\|y\|^{-2}Tr(P_xP_y)(\mathbf{v}_x,(I-P_y)\mathbf{w}_x). \tag{3.2.9}$$

Inserting from (3.2.3) into (3.2.6) and (3.2.7), one obtains an expression of the Kähler structure in terms of the selfadjoint operators $B(\mathbf{v}(\mathbf{x}))$ and $B(\mathbf{w}(\mathbf{x}))$ representing the vector fields $\mathbf{v}$ and $\mathbf{w}$ in any point $\mathbf{x} \in P(\mathcal{H})$, cf. also (2.1.27):

$$\begin{aligned}&\Psi_{\mathbf{y}}(\mathbf{v},\mathbf{w}) = 2\|y\|^{-2}(\mathbf{v}_y,\mathbf{w}_y) = \\ &2Tr\big(P_yB(\mathbf{v}(\mathbf{y}))B(\mathbf{w}(\mathbf{y}))\big) - 2Tr\big(P_yB(\mathbf{v}(\mathbf{y}))\big)Tr\big(P_yB(\mathbf{w}(\mathbf{y}))\big).\end{aligned} \tag{3.2.10}$$

It can be shown, [62], that the distance function d(**x**,**y**) on $P(\mathcal{H})$ corresponding to the Riemannian metrics Γ is expressed by[81]

$$\mathrm{d}(\mathbf{x},\mathbf{y}) = \sqrt{2}\arccos\sqrt{Tr(P_xP_y)}, \tag{3.2.11}$$

[81] The derivation of the distance d(**x**,**y**) is easy after accepting the (plausible looking) assumption, that any geodesic is contained in the submanifold of $P(\mathcal{H})$ homeomorphic to a real two–dimensional sphere representing the projective Hilbert space of the two–dimensional complex subspace of $\mathcal{H}$ spanned by $\{x, y\}$. The nontrivial part of the proof consists in justification of this assumption, [30].

with values in the interval $\left[0; \frac{\pi}{\sqrt{2}}\right]$.

The linearity of conventional quantummechanical time–evolutions, as well as other symmetry transformations is closely connected with the metrics (3.2.11). The corresponding mathematical formulation is in fact a rephrasing of the very well known Wigner theorem, cf. Remark 2.3.14, [280, 283]:

3.2.6. Proposition. *Let Φ be any bijection of $P(\mathcal{H})$ onto itself conserving the distance function* d *from* (3.2.11)*. Then there is a linear, or antilinear isometry u_Φ of $\mathcal{H}$ onto itself representing Φ in the sense that $\Phi(\mathbf{y}) = (\mathbf{u_\Phi} \boldsymbol{y})$ for all $\mathbf{y} \in P(\mathcal{H})$ $(0 \neq y \in \mathbf{y})$. If u_Φ is linear, then Φ conserves also the symplectic form Ω:*

$$\left(\Phi^* \Omega\right)_{\mathbf{y}} (\mathbf{v}, \mathbf{w}) := \Omega_{\Phi(\mathbf{y})} \left(\Phi_* \mathbf{v}, \Phi_* \mathbf{w}\right) = \Omega_{\mathbf{y}}(\mathbf{v}, \mathbf{w}), \tag{3.2.12}$$

i.e. Φ is an isometric symplectomorphism of $P(\mathcal{H})$. The mapping Φ changes the sign at the symplectic form Ω in the case of antilinear u_Φ: $\Phi^* \Omega \equiv -\Omega$. ♣

Proof. Conservation of d means conservation of the "transition probabilities" $Tr(P_x P_y)$, $\forall x, y \in \mathcal{H} \setminus \{0\}$; this means also conservation of the metric tensor Γ. According to the Wigner theorem there is unitary or antiunitary bijection $\mathrm{u}_\Phi : \mathcal{H} \to \mathcal{H}$, as stated in the proposition. But the symplectic form is invariant with respect to unitary transformation, as was shown in the Remark 2.2.11. The last part of the proposition is a consequence of the fact that antiunitary mappings u change the value of the scalar product in $\mathcal{H}$ to its complex conjugate: $(\mathrm{u}x, \mathrm{u}y) = (y, x)$. For more details cf. [62, 27, 63]. □

3.2.7. **Remark**. A general ("nonlinear") symplectomorphism of $P(\mathcal{H})$ does not conserve Γ (equivalently: the distance function d). This might be considered as a strong argument for linearity of QM, since, as we shall see soon in Section 3.3, the metric tensor leading to this distance function is a tool for geometric reformulation of the probability interpretation of QM. By introducing the "nonlinear observables" and their nonlinear transformations, and also the corresponding interpretation based on the "two point function representatives" of observables, cf. Definitions 2.3.3, and 2.3.4, we have overcame the difficulty with noninvariance of this "interpretational device" with respect to general symplectomorphisms. ♡

3.3 Symplectic Form of QM and NLQM; Restrictions of QM

The traditional (linear) quantum mechanics (QM) is completely described by kinematics and dynamics on $P(\mathcal{H})$, i.e. the effects connected with other parts of the "elementary quantum phase space" $\mathcal{S}_*$ containing density matrices $\varrho \neq \varrho^2$ which are described by the formalism of Chapter 2 can be reproduced by the restriction of that formalism to the "one–dimensional" orbit $P(\mathcal{H})$ only, and by "dynamics independent" manipulations with objects defined on it. This is due to linearity, since the used transformations (time evolutions, symmetries) of $\mathcal{S}_*$ are then affine mappings, and expectations also affinely depend on $\varrho \in \mathcal{S}_*$.

In the terminology of Chapter 2, QM can be obtained as the G–system on an infinite–dimensional separable Hilbert space $\mathcal{H}$ with the trivial group $G := \{e\}$. In this case, the set $\mathcal{G}^G_{cl}$ of G–classical generators consists of constants. The set $\mathcal{G}^G$ of G–symmetry generators, on the other hand, contains (densely defined) functions h_Y corresponding to all selfadjoint operators Y.

Observables can be represented by affine functions $h_\mathfrak{f}$ only, since their function representatives $\hat{h}_\mathfrak{f}$ do not depend on the G–classical variable ν in the ascription $(\varrho;\nu) \mapsto \hat{h}_\mathfrak{f}(\varrho,\nu)$. The "genuine mixtures" $\mu, \mu' \in \mathcal{M}^G$ corresponding to the same barycentres $\mathfrak{b}(\mu) = \mathfrak{b}(\mu')$ are not mutually distinguishable by measurements of the G–observables, neither they could be distinguished after a use of symmetry transformations (resp. evolutions) in the framework of this G–system. The "permitted" (possibly unbounded) generators and observables include (densely defined) affine functions $\nu \mapsto h_X(\nu) := \nu(X)$ corresponding to selfadjoint operators X.

We shall consider "nonlinear extensions" of this QM–system (i.e. of the G–system with trivial $G := \{e\}$) by allowing *evolutions of states by nonlinear generators*.[82] To be able to deal also with the questions of "integrability" of also nonlinear functions of these h_X's, see Definitions 2.2.13, it is useful to (choose and to) consider these X's as selfadjoint generators of some unitary representation $V(S)$ of a "symmetry group S" associated with the considered system (e.g. S could be the $2n+1$–dimensional Weyl–Heisenberg group, i.e. the standard one–dimensional central extension, [148, 267], of the commutative $2n$–dimensional group of translations in classical linear $2n$–dimensional phase space, see below in this section). These versions of nonlinear quantum mechanics (NLQM) are not symmetric with respect to transitions between *Schrödinger and Heisenberg pictures*: They can be used in Schrödinger picture only, since a nonlinear (i.e. nonaffine) transformation of $\mathcal{S}_*$ cannot be expressed by some "transition to adjoints", [41], as a transformation of the *algebra of linear observables*: this algebra could not stay invariant with respect to such a transformation.

Another way of "transitions to nonlinearity" in QM consists in restrictions of (linear) dynamics of QM to submanifolds of $P(\mathcal{H})$ (or also of $\mathfrak{T}_s$), e.g. to some orbits $\mathcal{O}_\varrho(S)$ of a representation $V(S)$. We can obtain in that way also usual "quasiclassical", or "self-consistent" approximations, e.g. WKB, or Hartree–Fock approximations as versions of NLQM, cf. also our Subsections 3.3-c and 3.3-e. The group S needn't be interpreted, however, as a group of transformations of a "classical background" (cf. Section 3.4) being *dynamically connected* with the system, as it is in the case of G–systems with nontrivial G and general (nonlinear) G–generators. Only affine functions $\nu \mapsto \mathrm{f}(\nu)$ (and their restrictions)[83] defined on dense sub–domains of $\mathcal{S}_*$ are used here in the rôles of the generators as well as observables. All the "traditional" quantities are "essentially contained" in the sets of corresponding quantities of *any G–system*: $\boldsymbol{\mathcal{D}}(\mathbb{F})$ is dense in $\mathcal{S}_*$, and for calculation of any bounded (hence continuous) observable $\mathfrak{f}^* = \mathfrak{f} \in \mathcal{C}_q^G$ ($\equiv \mathcal{L}(\mathcal{H})$, cf. Definition 2.3.3) one can use values $h_\mathfrak{f}(\nu)$ for $\nu \in \boldsymbol{\mathcal{D}}(\mathbb{F})$ (cf.(2.3.3), and Interpretation 2.3.11). The general observables h_X used in the rôle of generators could, however, violate the relation $\tilde{\varphi}_t^X \boldsymbol{\mathcal{D}}(\mathbb{F}) \subset \boldsymbol{\mathcal{D}}(\mathbb{F})$ for some G–systems.

We shall describe in this section the symplectic reformulation (equivalent to the usual Hilbert space formulation) of traditional (linear) QM, as well some of its restrictions to submanifolds of $P(\mathcal{H})$ leading to nonlinear dynamics (corresponding, e.g., to some "quasiclassical approximations") the general form of which was described in Chapter 2. Let us first, however, formulate briefly a general nonlinear quantum mechanics (NLQM) on the projective Hilbert space $P(\mathcal{H})$ to point out some differences between QM and NLQM.

[82] Such an extension of QM can be obtained *by restriction* of a G–system with nontrivial G in the way, that we shall admit *linear observables only*, i.e. the observables represented by nonconstant operator–valued functions on $\mathcal{E}_\mathbb{F}$ will be ignored (cf. Definitions 2.3.3).

[83] Restrictions of affine functions to submanifolds $\mathcal{O}_\varrho(S)$ considered as Hamiltonians on the phase spaces $\mathcal{O}_\varrho(S)$ lead generally, however, to nonlinear dynamics on these submanifolds.

3.3-a Generalized quantum mechanics on $P(\mathcal{H})$

We shall consider here a general (nonlinear) EQM, but we shall restrict our attention to dynamics and kinematics restricted to $P(\mathcal{H})$ only. Let us choose also a Lie group G and its unitary representation $U(G)$ such, that the space of generators $\mathcal{G}^G$ includes all the (nonlinear) generators we want to use in the theory. Let us consider, however, only elementary quantum observables $\mathcal{C}_q^G$, cf. Definition 2.3.3(ii), in the rôle of bounded observables we intend to interpret in the considered model. Hence, for nonlinear evolutions, the Heisenberg picture will not be used. We shall call the chosen system a **restricted G–system** (i.e. restricted to the "restricted quantum phase space" $P(\mathcal{H}) \subset \mathcal{S}_*$, with the restricted set of observables $\mathcal{C}_q^G$).

If $X_j = X_j^*$ are elements of the representation $dU(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G, then the typical form of the ("restricted") generators $\mathrm{Q} \in \mathcal{G}^G$ will be

$$\mathrm{Q} : \nu(\in P(\mathcal{H})) \mapsto \mathrm{Q}(\nu) \equiv Q(\nu(X_1), \nu(X_2), \dots),$$

with $Q \in C^\infty(\mathfrak{g}^*, \mathbb{R})$. The corresponding nonlinear Schrödinger equations are discussed in Sections 3.5, 3.6, and also in Subsection 3.3-e.

Let us denote $\mathcal{F}_{P(\mathcal{H})} := C^\infty(P(\mathcal{H}), \mathbb{R})$ the differentiable functions on the Banach manifold $P(\mathcal{H})$. The differential $df \in \mathcal{T}_1^0(P(\mathcal{H}))$ of $f \in \mathcal{F}_{P(\mathcal{H})}$ can be determined by the formula

$$d_{\mathbf{x}} f(\mathbf{w}(\mathbf{x})) := \left.\frac{d}{dt}\right|_{t=0} f\big((\exp(-i t \boldsymbol{B}(\mathbf{w}(\mathbf{x})))\boldsymbol{x})\big), \tag{3.3.1}$$

with $B(\mathbf{w})$ specified in (3.2.3), for any vector field $\mathbf{w} \in \mathcal{T}_0^1(P(\mathcal{H}))$. The symplectic form Ω is strongly nondegenerate [61] on $P(\mathcal{H})$ (cf. Theorem 2.1.19), hence it associates with each $f \in \mathcal{F}_{P(\mathcal{H})}$ a unique Hamiltonian vector field $\mathbf{v}_f$ on $P(\mathcal{H})$ such that

$$\Omega(\mathbf{v}_f, \mathbf{w}) = -df(\mathbf{w}), \forall \mathbf{w} \in \mathcal{T}_0^1(P(\mathcal{H})). \tag{3.3.2}$$

The (local) flow $\tilde{\varphi}^f$ of $\mathbf{v}_f$ leaves Ω invariant, hence for the Lie derivative $\pounds_{\mathbf{v}_f}$ we have:

$$\pounds_{\mathbf{v}_f} \Omega = 0.$$

The Poisson bracket $\{f, h\} := \Omega(\mathbf{v}_f, \mathbf{v}_h) \in \mathcal{F}_{P(\mathcal{H})}$ determines the differential equation (equivalent to the Schrödinger equation for affine f) for the Hamiltonian flow $\tilde{\varphi}^f$. Also the following formula (well known from CM, [7, 1]) is valid here:

$$dh(\mathbf{v}_f) = \{f, h\} \quad (\forall h \in \mathcal{F}_{P(\mathcal{H})}).$$

We shall formulate now a necessary and sufficient condition under which a function $f \in \mathcal{F}_{P(\mathcal{H})}$ is affine, i.e. is expressed by a linear operator:

3.3.1. Proposition. *Let $f \in P(\mathcal{H})$, and let $\mathbf{v}_f$ is the corresponding Hamiltonian vector field on $P(\mathcal{H})$. Let Γ be the canonical (Kählerian) metrics on $P(\mathcal{H})$. Then $\pounds_{\mathbf{v}_f}\Gamma \equiv 0$ iff there is a bounded selfadjoint operator $\mathrm{a} = \mathrm{a}^* \in \mathcal{L}(\mathcal{H})$ such that:*

$$f(\mathbf{x}) \equiv h_{\mathrm{a}}(\mathbf{x}) := Tr(P_x \mathrm{a}), \quad 0 \neq x \in \mathbf{x}. \tag{3.3.3a}$$

In the case of $f = h_{\mathrm{a}}$, $\mathbf{v}_f$ is complete, and one has

$$\tilde{\varphi}_t^f(\mathbf{x}) = (\mathbf{exp}(-\boldsymbol{it}\mathbf{a})\boldsymbol{x}),\ t \in \mathbb{R},\ \mathbf{x} \in P(\mathcal{H}),\ 0 \neq x \in \mathbf{x}. \tag{3.3.3b}$$

Hence the flows of those Hamiltonian vectors fields $\mathbf{v}_f$ which conserve the metrics Γ on $P(\mathcal{H})$ correspond to norm continuous one–parameter unitary groups on $\mathcal{H}$. ♣

A proof is contained in [63, Propositions 3.4, and 3.5], resp. in [27].

Let us introduce also the **Riemann bracket** $[[\cdot,\cdot]]$ in accordance with [63]:

$$[[f,h]] := \Gamma(\mathbf{v}_f, \mathbf{v}_h). \tag{3.3.4}$$

An immediate consequence of (3.2.10) and of the Proposition 3.3.1 is the following lemma, cf. also [63]:

3.3.2. Lemma. *Let $h_{\mathrm{a}} \in \mathcal{F}_{P(\mathcal{H})}$ be defined for any $\mathrm{a} \in \mathcal{L}(\mathcal{H})$ by (3.3.3). Then for any selfadjoint $\mathrm{a}, \mathrm{b} \in \mathcal{L}(\mathcal{H})$, the following formula holds:*

$$2h_{\mathrm{a}} * h_{\mathrm{b}} := 2h_{\mathrm{a}\circ\mathrm{b}} = [[h_{\mathrm{a}}, h_{\mathrm{b}}]] - i\{h_{\mathrm{a}}, h_{\mathrm{b}}\} + h_{\mathrm{a}}h_{\mathrm{b}}. \tag{3.3.5}$$

The mappings $\mathrm{a} \mapsto h_{\mathrm{a}}(\mathbf{x})$ $(\mathbf{x} \in P(\mathcal{H}))$ are continuous in the weak operator topology. ♣

Due to (3.3.5), we can calculate $Tr(P_x \mathrm{a}^n)$ $(\forall n \in \mathbb{N})$ in terms of the function h_{a}, what allows us to express the probability interpretation of QM in differential geometrical terms on $P(\mathcal{H})$: The formula (3.3.5) leads us (via the functional calculus) to a rule for calculation of $h_{f(\mathrm{a})}$ for an arbitrary real bounded Borel function f defined on the spectrum σ(a) of $\mathrm{a} = \mathrm{a}^* \in \mathcal{L}(\mathcal{H})_s$. Then the number $h_{f(\mathrm{a})}(\mathbf{x}) \equiv Tr(P_x f(\mathrm{a}))$ can be interpreted as the expectation value of the "observable" (represented by the operator) $f(\mathrm{a})$ obtained by averaging of repeated measurements of $f(\mathrm{a})$ in the (repeatedly prepared) pure quantum state $\mathbf{x} \in \mathcal{S}_*$. The probability of finding measured values of a selfadjoint ain an interval $J \subset \mathbb{R}$ is then expressed by taking for f the characteristic function χ_J of that interval:

$$\mathrm{prob}(\mathbf{x}, \mathrm{a} \in J) = Tr(P_x \chi_J(\mathrm{a})). \tag{3.3.6}$$

Calculating expectations of arbitrary selfadjoint a in $\mathcal{L}(\mathcal{H})$ for any (elementary) mixture $\varrho \in \mathcal{S}$ in the standard way from the expectations in pure states (by corresponding convex combinations), we obtain the result $Tr(\varrho \mathrm{a})$, in accordance with QM.

Let us stress also here that each ϱ such that $\varrho^2 \neq \varrho \in \mathcal{S}_*$ can be decomposed in uncountably many different ways into (not necessarily orthogonal) convex combinations

$$\varrho = \sum_j \lambda_j P_{x(j)}, \quad \lambda_j \geq 0,\ \sum_j \lambda_j = 1,$$

of one–dimensional projections $P_{x(j)}, x(j) \in \mathcal{H}$ (representing pure states).[84] Different decompositions $(P_{x(j)}; \lambda_j; j \in J)$, and $(P_{x'(j')}; \lambda'_{j'}; j' \in J')$ of a given ϱ can be represented, in

[84] This is an essential difference of QM from CM.

another language, by probability measures $\mu_\varrho,\ \mu'_\varrho$ on the state space $\mathcal{S}_*$ with the same **barycentre $\mathfrak{b}(\mu_\varrho) = \mathfrak{b}(\mu'_\varrho) = \varrho$**, the measures being concentrated on at most countable sets of points (i.e. on the sets $\{P_{x(j)} : j \in J\}$):

$$\mu_\varrho(\{P_{x(j)}\}) = \lambda_j,\ \forall j \in J,$$

hence the states ω_μ ($\mu = \mu_\varrho, \mu'_\varrho, \dots$), all representing the same ϱ, give the following expressions for expectation values of $\mathrm{a} \in \mathcal{L}(\mathcal{H})_s$:

$$\begin{aligned}\omega_\mu(a) &:= \int_{\mathcal{S}_*} \nu(a)\mu(d\nu) \equiv \int_{\mathcal{S}_*} Tr(a\nu)\mu(d\nu)\\ &= \int_{P(\mathcal{H})} Tr(a\nu)\mu(d\nu) = \int_{P(\mathcal{H})} h_{\mathrm{a}}(P_x)\mu(dP_x)\\ &= \sum_{j\in J} h_{\mathrm{a}}(P_{x(j)})\mu(\{P_{x(j)}\}) =: Tr(\mathfrak{b}(\mu)\mathrm{a}) = Tr(\varrho\mathrm{a}).\end{aligned} \tag{3.3.7}$$

In "orthodox" linear QM the states corresponding to measures on $\mathcal{S}_*$ with the same resultant are indistinguishable. This is one of the important differences of QM from NLQM (also in the framework of our restricted model of EQM).

3.3.3. **Note**. Let Q be a nonlinear generator of time evolution in our theory. Then, according to Proposition 3.3.1, its flow $\tilde{\varphi}^{\mathrm{Q}}$ does not conserve the canonical metrics Γ, hence it does not conserve the distance function $\mathrm{d} : P(\mathcal{H}) \times P(\mathcal{H}) \to \mathbb{R}_+$. From the expression (3.2.11) of the distance function $\mathrm{d}(P_x, P_y)$ we see that, in turn, it does not conserve the "transition probabilities" $Tr(P_xP_y) = |\langle x|y\rangle|^2$ between the states $\mathbf{x}, \mathbf{y} \in P(\mathcal{H})$. This shows, however, that different measures $\mu \neq \mu'$ with the same barycentres $\mathfrak{b}(\mu) = \mathfrak{b}(\mu')$ can have different barycentres after some time $t \neq 0 : \mathfrak{b}(\mu \circ \tilde{\varphi}^{\mathrm{Q}}_{-t}) \neq \mathfrak{b}(\mu' \circ \tilde{\varphi}^{\mathrm{Q}}_{-t})$, and validity of some of the equalities in (3.3.7) will depend on time (cf. also Subsection 2.1-e). This might lead to prediction of superluminal communication (for a specific, but rather conventional, interpretation of the process of measurement in QM), as is pointed out in the Interpretation 2.1.24. ♡

3.3.4. **Interpretation.** In the traditional interpretation of QM, the expectation value of the numerical results of measurement of an "observable f" (i.e. a scalar–valued function f of quantum states **y**, in this case an "affine" one, resp. Kähler function in the terminology of [63]) in an arbitrary (pure) state $\mathbf{y} \in P(\mathcal{H})$ equals to its value f(**y**), i.e. for f:=h_X, the expectation is $h_X(\mathbf{y}) = Tr(P_yX) = \langle y|X|y\rangle$, if $\|y\| = 1$. The calculations of these expectations are closely connected, in the orthodox QM, with eigenstates of the operators X (assume now, that X has pure point spectrum). In terms of the presented "geometric formulation", the eigenvectors $x(k) \equiv |x(k)\rangle \in \mathcal{H}$,

$$X|x(k)\rangle = \kappa_k|x(k)\rangle,\ k \in K, \quad \sum_{k\in K} P_{x(k)} = I_{\mathcal{H}},$$

resp. the one–dimensional eigenprojections $P_{x(k)} \equiv \mathbf{x}(k) \in P(\mathcal{H})$, are exactly the "stationary points" of the generators h_X, cf.3.3.5.

Stationarity of the points $\mathbf{x}(k)$ is rather a "dynamical property". The observable probabilities can be expressed with a help of the projection measure E_X of X : $E_X(J) := \chi_J(X)$, as above, see (3.3.6).

Let us denote the **eigenprojections of X** corresponding to single eigenvalues κ_k by E_k,

$$E_k := E_X(\{\kappa_k\}) := \sum_{j\in K:\kappa_j=\kappa_k} P_{x(j)}.$$

Then the probability of obtaining the result κ_k, if X is measured on the system prepared in the state $\mathbf{y} \in P(\mathcal{H})$, is

$$\mathrm{prob}(\mathbf{y}; X = \kappa_k) = Tr(E_k P_y).$$

The values of these probabilities, for $\dim E_k = 1$, i.e. $E_k = P_{x(k)}$, are the above discussed (cf. Remark 2.3.14) **"transition probabilities"**, and the values of the function h_X in these points $P_{x(k)}$ are just the measured eigenvalues, $h_X(\mathbf{x}(k)) = \kappa_k$. Hence the expectation value of X with pure point spectrum in an arbitrary $\mathbf{y} \in P(\mathcal{H})$ is

$$\begin{aligned}\langle X\rangle_{\mathbf{y}} :=& h_X(\mathbf{y}) = \sum_{k\in K} Tr(P_{x(k)}P_y)h_X(\mathbf{x}(k)) \equiv Tr\left(\sum_{k\in K} h_X\big(\mathbf{x}(k)\big)P_{x(k)}P_y\right)\\ =& Tr\left(\sum_{k\in K}\kappa_k P_{x(k)}P_y\right) \equiv Tr(XP_y).\end{aligned} \tag{3.3.8}$$

The first sum is often interpreted in the sense of classical probability, [95], by considering occurrences of different κ_k (better: of different orthogonal eigenstates) as independent "events", and the function $\mathbf{x}(k) \mapsto Tr(P_y P_{x(k)})$ is a measure on the space of these "events" determined by the state $\mathbf{y} \in P(\mathcal{H})$, and consisting of the "transition probabilities". If the concept of the "transition probabilities" (which is coming from an interpretation of quantum measurement) were taken seriously also for NLQM, and the rôle accepted for the stationary points $\{\mathbf{x}(k) \in P(\mathcal{H}) : d_{\mathbf{x}(k)}f = 0\} =: \boldsymbol{S(f)}$ of a "nonlinear observable" f (cf. [273]) were formulated as above, in the case of linear observables,[85] with keeping unchanged the above formula (3.3.8) for calculation of expectations, i.e. if we postulated something like

$$\langle f\rangle_{\mathbf{y}} := \sum_{\mathbf{x}\in S(f)} Tr(P_x P_y)f(\mathbf{x}),\ \forall \mathbf{y} \in P(\mathcal{H}),$$

then we would come to a contradiction: The "nonlinear" function f would be affine:

$$f \equiv h_Y, \quad Y := \sum_{\mathbf{x}\in S(f)} f(\mathbf{x})P_x.$$

This consideration indicates that a "traditional–like" interpretation of observables expressed as numerical functions on $P(\mathcal{H})$ (our "reduced function representations", cf. Definitions 2.3.4) cannot be used in NLQM. ♦

[85] This means that we would work with such an "observable" as with a random variable in the sense of Kolmogorov formulation of probability theory, by which the stationary states form the the whole space of "elementary events".

3.3.5. Lemma. *Let $X = X^*$ be any selfadjoint operator on $\mathcal{H}$ with corresponding (densely defined) function h_X on $P(\mathcal{H})$. Let $\mathbf{x} \mapsto d_{\mathbf{x}}h_X$ be its generalized differential (cf. Definition 2.2.9) defined on the domain $\mathcal{D}(h_X)$, cf. also Lemma 2.2.7, and Proposition 2.2.8. Then the points $\mathbf{x}(k)$ lying in the domain of dh_X in which the differential vanishes, satisfy the relation*

$$d_{\mathbf{x}(k)}h_X = 0 \Leftrightarrow X|x(k)\rangle = \kappa_k|x(k)\rangle, \quad (x(k) \in \mathbf{x}(k) \in P(\mathcal{H})),$$

i.e. they are exactly the one–dimensional eigenspaces of X. ♣

Proof. The differential $d_{\mathbf{x}}h_X$ can be represented, according to considerations in Subsection 2.2-b, on its domain by the bounded operator

$$d_{\mathbf{x}}h_X = \mathrm{q}_{\mathbf{x}}(X) \equiv P_x X(I_{\mathcal{H}} - P_x) + (I_{\mathcal{H}} - P_x)XP_x = P_x X + XP_x - 2P_x X P_x,$$

and its vanishing implies commutativity of P_x with X, i.e. invariance of the one–dimensional subspace **x** with respect to the action of X. For proof of the converse, the arguments go in the reversed order. □

We stop here with general considerations, and we shall turn now to more specific cases.

3.3-b The Weyl–Heisenberg group and CCR

The $2n+1$–dimensional **Weyl–Heisenberg group** G_{WH} (it is also called the *Heisenberg group*) can be chosen in our theory either in the rôle of the group G defining a G–system, cf. Definition 2.3.6, or in the rôle of the above mentioned Lie group S determining domains for generalized fields (cf. Definition 2.2.13). We shall investigate here the action of the standard irreducible Schrödinger representation $U(G_{WH})$ of G_{WH} on $\mathcal{H}$ in some details, as well as the quantum kinematics and dynamics constructed with a help of it. As an expression of the corresponding Lie algebra relations between generators we obtain the usual definitions of **canonical commutation relations** (CCR).

Let us recall that the $2n+1$–dimensional group G_{WH} can be defined as the group of square $(n+2)\times(n+2)$–matrices, [148, 287]:

$$g(q,p,s) := \begin{pmatrix} 1 & -q & s \\ 0 & I_n & p^T \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.3.9a}$$

where $q := \{q_1, q_2, \dots q_n\} \in \mathbb{R}^n$, $p := \{p_1, p_2, \dots p_n\} \in \mathbb{R}^n$, $s \in \mathbb{R}$, I_n is the unit $n\times n$–matrix, p^T is the transposed row p (i.e. the column vector), and 0's have an appropriate meaning of zero submatrices according to their place in the matrix. The group multiplication is represented by the matrix multiplication:

$$g(q,p,s)g(q',p',s') = g(q+q', p+p', s+s'-q\cdot p'), \tag{3.3.9b}$$

with $q\cdot p' := \sum_j q_j p'_j$. Let's note that p^T (or p) can be considered as an element of the dual $(\mathbb{R}^n)^*$, hence its value on $q \in \mathbb{R}^n$ is $\langle p^T; q\rangle := q\cdot p := \sum_j q_j p_j$.

The group G_{WH} is a central extension, [148, 267], of the commutative group $\mathbb{R}^{2n} \ni (q;p) \equiv x$ (with respect to the addition $x + x'$) by the additive group $\mathbb{R}$, corresponding to the **multiplier** (in additive notation) [267, Chap. X] $\tilde{m}(x,x') \equiv -p'\cdot q$.

3.3.6. **Note** *(Multipliers and quantization).* The commutative group $\mathbb{R}^{2n}$ is naturally identified with a classical phase space, or with the group of its translations. As any commutative group, it has only one–dimensional linear (unitary) irreducible representations. It has, however, many (mutually inequivalent) infinite–dimensional **projective representations**, i.e. "unitary representations up to a phase factor". Namely multipliers $m(x,y),\ x,y\in\mathbb{R}^{2n}$, i.e. real–valued functions on the direct product of two copies of the group, $\mathbb{R}^{2n}\times\mathbb{R}^{2n}$, satisfying

$$m(x+y,z)+m(x,y)\equiv m(x,y+z)+m(y,z),\ m(x,0)\equiv m(0,x)\equiv 0, \tag{3.3.10}$$

are the (logarithms/i of the) phase factors of the (noncommutative) projective representations. In a more general setting, let G be a Lie group, and say $V(G)$ be its continuous projective representation with a multiplier m: Let $g_1\cdot g_2\in G$ denotes the multiplication in G (e.g. addition in $\mathbb{R}^{2n}$). Let m be a multiplier of G, i.e. $m:\ G\times G\to\mathbb{R}$ satisfying the relations in (3.3.10), with, e.g., $m(g_1\cdot g_2,g_3)\mapsto m(x+y,z)$, etc. The projective $\boldsymbol{m}$**–representation** $V(G)$ is characterized by unitarity of the all $V(g)$'s, and by the relation:

$$V(g_1\cdot g_2)\equiv\exp(i\cdot m(g_1,g_2))V(g_1)V(g_2). \tag{3.3.11a}$$

One can make from these "unitary up to factors" representations $V(G)$ genuine unitary representations of larger noncommutative groups G_m constructed from the original group G (e.g. from our $G:=\mathbb{R}^{2n}$) with a help of the corresponding multipliers m. These **central extensions** $\boldsymbol{G_m}$ of a Lie group G are constructed as follows:

Let $(g;\lambda)\in G\times S^1$, with $S^1:=\{\lambda\in\mathbb{C}:|\lambda|=1\}$. Then the central extension G_m of the group G by the commutative group S^1 (resp. by $\mathbb{R}$, if the "corresponding logarithms" are taken) corresponding to the multiplier m consists of the couples $(g;\lambda)$, and the group multiplication is defined by

$$(g_1;\lambda_1)\cdot(g_2;\lambda_2):=\Big(g_1\cdot g_2;\exp\big(i\cdot m(g_1,g_2)\big)\lambda_1\lambda_2\Big). \tag{3.3.11b}$$

This simple procedure makes from a (say, commutative) group G another (noncommutative) group G_m, provided m is not **exact**; exactness of m means the existence of a real function $a:G\to\mathbb{R}$ such, that

$$m(g_1,g_2)\equiv a(g_1\cdot g_2)-a(g_1)-a(g_2). \tag{3.3.11c}$$

If the difference of two multipliers m_1-m_2 (what is always again a multiplier) is exact, m_1 and m_2 are (mutually) similar, or **cohomologous**.

Let us take now the m–representation $V(G)$. It can be "translated" into a unitary representation $V(G_m)$ of G_m as follows:

$$V\big((g;\lambda)\big):=\lambda^{-1}V(g). \tag{3.3.11d}$$

There is a certain "both-sided" correspondence between projective "m–representations" of G, and a class of unitary representations of G_m. For details cf. [267, Theorem 10.16]. As we shall see in a while, traditional "quantization" of classical flat phase spaces corresponds to specific choice of a multiplier of $G:=\mathbb{R}^{2n}$, determined by the experimental value of the Planck constant $\hbar$. ♡

To any **similar multiplier** $\tilde{m}'$ related to $\tilde{m}$ by a real–valued function $a : (G \ni)x \mapsto a(x),\ a(0) = 0$:

$$\tilde{m}'(x, x') \equiv \tilde{m}(x, x') + a(x + x') - a(x) - a(x') \tag{3.3.12a}$$

corresponds the central extension isomorphic to G_{WH}. The choice $a(q; p) := \frac{1}{2} p \cdot q$ gives the following group–multiplication in G_{WH} (corresponding to a reparametrization of the abstract group G_{WH})

$$\tilde{g}(q, p, s)\tilde{g}(q', p', s') = \tilde{g}(q + q', p + p', s + s' + \frac{1}{2}(q' \cdot p - p' \cdot q)), \tag{3.3.12b}$$

and the corresponding matrix representation is

$$\tilde{g}(q, p, s) := \begin{pmatrix} 1 & -q & s - p \cdot q/2 \\ 0 & I_n & p^T \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.3.12c}$$

what corresponds to the form usually used in QM, as will be clear soon.

The Lie algebra of G_{WH} can be described as the matrix algebra consisting of derivatives of matrices $g(q, p, s)$ with respect to the parameters. Let the basis $\{\xi_j, \xi_0; j = 1, 2, \ldots 2n\}$ in the Lie algebra $Lie(G_{WH})$ be chosen such, that an arbitrary element $\xi \in Lie(G_{WH})$ is of the form

$$\xi(\alpha, \gamma, \beta) \equiv \sum_{j=1}^{n} \big(\alpha_j \xi_{n+j} + \gamma_j \xi_j\big) + \beta \xi_0 \equiv \begin{pmatrix} 0 & -\alpha & \beta \\ 0 & 0_n & \gamma^T \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.3.13a}$$

where $\alpha_j, \gamma_j, \beta \in \mathbb{R},\ j = 1, 2, \ldots n$. We shall use this parametrization here.

The commutation relations on $Lie(G_{WH})$ are expressed by this basis as

$$[\xi_{j+n}, \xi_{k+n}] = 0, \quad [\xi_j, \xi_k] = 0, \tag{3.3.13b}$$

$$[\xi_j, \xi_{k+n}] = \delta_{jk} \xi_0, \quad j, k = 1, 2, \ldots n. \tag{3.3.13c}$$

Those elements F of the dual $Lie(G_{WH})^*$ for which $F(\xi_0) \neq 0$ can be parametrized in the basis dual to the chosen one in $Lie(G_{WH})$ by parameters $q_0, p_0 \in \mathbb{R}^n,\ s_0 \in \mathbb{R} \setminus \{0\}$ in such a way, that they can be conveniently described by the matrix

$$F(q_0, p_0, s_0) \equiv \begin{pmatrix} 0 & 0 & 0 \\ s_0 p_0^T & 0_n & 0 \\ s_0 & s_0 q_0 & 0 \end{pmatrix}. \tag{3.3.13d}$$

The value of the linear functional F with $F(\xi_0) \neq 0$ on the element ξ is then

$$F(\xi) := \langle F; \xi \rangle \equiv Tr\big[F(q_0, p_0, s_0)\xi(\alpha, \gamma, \beta)\big] = (q_0 \cdot \gamma - p_0 \cdot \alpha + \beta) s_0. \tag{3.3.13e}$$

For F's with $F(\xi_0) = 0$, we have

$$F(\xi) \equiv \sum_{j=1}^{n} \big(\alpha_j F(\xi_{n+j}) + \gamma_j F(\xi_j)\big) + \beta F(\xi_0) = \sum_{j=1}^{n} \big(\alpha_j F(\xi_{n+j}) + \gamma_j F(\xi_j)\big) \tag{3.3.13f}$$

The coadjoint action on elements with $F(\xi_0) \neq 0$ is expressed then by

$$Ad^*\big(g(q,p,s)\big)\begin{pmatrix} 0 & 0 & 0 \\ s_0 p_0^T & 0_n & 0 \\ s_0 & s_0 q_0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ s_0\big(p_0+p\big)^T & 0_n & 0 \\ s_0 & s_0\big(q_0+q\big) & 0 \end{pmatrix}. \tag{3.3.13g}$$

It is easy to see that the points $F \in Lie(G_{WH})^*$ with $F(\xi_0) = 0$ are all stable with respect to $Ad^*(G_{WH})$–action. Hence, the $Ad^*(G_{WH})$–orbits are either single points F with $F(\xi_0) = 0$ covering a $2n$–dimensional hyperplane, or the whole hyperplanes with fixed $s_0 \neq 0$. Let us note that the action of $Ad^*\big(g(q,p,s)\big)$ does not depend on the parameter $s \in \mathbb{R}$, hence its above expression is independent also on the considered reparametrization of G_{WH}.

All irreducible unitary representations π_λ of G_{WH} (which are more than one–dimensional) can be parametrized by a real parameter $\lambda \neq 0$, and they are realized in $\mathcal{H}$:=$L^2(\mathbb{R}^n)$ as follows (in the parametrization of (3.3.9), [287]):

$$[\pi_\lambda(g(q,p,s))\psi](q') = e^{i\lambda(s+p\cdot q')}\psi(q'-q),\ \forall\, q,q',p \in \mathbb{R}^n, s \in \mathbb{R}, \psi \in L^2(\mathbb{R}^n). \tag{3.3.14}$$

The generators of π_λ corresponding to the chosen parameters are

$$-i\,\lambda P_j := \ \tfrac{\partial}{\partial q_j}\Big|_0 \pi_\lambda = \ -\frac{\partial}{\partial q'_j} \ = -i\,X(\xi_{j+n}), \tag{3.3.15a}$$

$$i\,\lambda Q_j := \ \tfrac{\partial}{\partial p_j}\Big|_0 \pi_\lambda = \ i\,\lambda q'_j\cdot \text{ (i.e. multiplication by the variable } q'_j) \\ = -i\,X(\xi_j),\ j = 1,\dots n, \tag{3.3.15b}$$

$$i\,(\lambda)^2 X_0 := \ \tfrac{\partial}{\partial s}\Big|_0 \pi_\lambda = \ i\,\lambda I \ = -i\,X(\xi_0), \tag{3.3.15c}$$

where the labels "0" at the derivatives denote differentiations in the unit element of G_{WH}.

The Schrödinger representation of CCR can be considered as that one given by the generators of π_λ with $\lambda := \frac{1}{\hbar}$. We shall need, however, the "corresponding" group representation of G_{WH} expressing the Weyl form of CCR. If we use the parametrization of G_{WH} from (3.3.11), we obtain the rewriting of the representation π_λ from (3.3.14) in the form W_λ (the Weyl form):

$$[W_\lambda(q,p,s)\psi](q') = e^{i\lambda(s+p\cdot q'-\frac{1}{2}p\cdot q)}\psi(q'-q),\ \forall q,q',p \in \mathbb{R}^n, s \in \mathbb{R}, \psi \in L^2(\mathbb{R}^n). \tag{3.3.16}$$

3.3.7. Notation. *Let us denote the* **projective representation** *of the commutative group $\mathbb{R}^{2n}$ usually referred to as "the Weyl form of (the representation of) CCR", by $W_\lambda(x) := W_\lambda(q,p,0)$ with $\{q;p\} =: x \in \mathbb{R}^{2n}$. Note that the projective representation W_λ of $\mathbb{R}^{2n}$ differs from the "corresponding" unitary representation of G_{WH} just by a "phase factor":*

$$W_\lambda(q,p,s) \equiv W_\lambda(x)\cdot e^{i\lambda s},$$

hence the $Ad^*(\cdot)$–action of both representations on $\mathfrak{T}(\mathcal{H})$ is identical – it depends on elements of the factorgroup $G_{WH}/\mathbb{R} = \mathbb{R}^{2n}$ only.

Let $X_j := Q_j,\ X_{j+n} := P_j,\ j = 1,2,\dots n,\ X_0 := \frac{1}{\lambda}I$, cf.(3.3.15)*, be selfadjoint generators of W_λ. Let $S^T = -S = S^{-1}$ be the $2n \times 2n$–symplectic matrix with elements*

$$S_{j\ j+n} = -S_{j+n\ j} := 1,\ \ j = 1,2,\dots n;\ \ S_{jk} = 0 \text{ otherwise}.$$

For selfadjoint operators X, Y on $\mathcal{H}$, let $[X, Y] \equiv XY - YX$ denote the commutator on its domain, and $[X, Y] = i\,Z$, for selfadjoint X, Y, Z on $\mathcal{H}$ will mean equality of operators restricted to their common domain. $\diamondsuit$

The operators $X_j, j = 0, 1, 2, \ldots 2n$ satisfy the (Heisenberg form of) **the canonical commutation relations** (CCR) on a common dense domain:

$$[X_j, X_k] = i\,S_{jk}X_0, \;\; j, k \neq 0, \tag{3.3.17a}$$

$$[X_j, X_0] = 0, \;\; j = 1, 2, \ldots 2n. \tag{3.3.17b}$$

The **Weyl form of CCR** is

$$W_\lambda(x + x') = \exp\left(\frac{i\lambda}{2} x \cdot S \cdot x'\right) W_\lambda(x) W_\lambda(x'), \tag{3.3.18}$$

and the unitary operators W_λ are expressed by

$$W_\lambda(x) \equiv \exp\left(i\,\lambda X \cdot S \cdot x\right) := \exp\left(i\,\lambda \sum_{j,k=1}^{2n} X_j S_{jk} x_k\right). \tag{3.3.19}$$

The following useful relation is then valid:

$$W_\lambda(x)^{-1} X_j W_\lambda(x) \equiv X_j + x_j I, \quad \forall j \in \{1, 2, \ldots 2n\},\; x_j \in \mathbb{R}. \tag{3.3.20}$$

3.3.8. Notation. *Let us write $W(x) \equiv W_{1/\hbar}(x)$. Let us define $\mathcal{V}_\nu := \{Ad^*(W(x))\nu : x \in \mathbb{R}^{2n}\} = \mathcal{O}_\nu(G)$ with $G := G_{WH}$, and $\mathcal{V} := \{\mathcal{V}_\nu : \nu \in \boldsymbol{\mathcal{D}}_r(\mathbb{F})\}$, with $\mathbb{F}$ given by $W_{1/\hbar}(G_{WH})$, cf. Definition 2.2.17 . Let $X(x) := X \cdot S \cdot x$ be a selfadjoint generator of the (projective) representation $W(\mathbb{R}^{2n})$, i.e. the generator of the one–parameter unitary group $t \mapsto \exp(-itX(x))$. The densely defined Hamiltonian function generating the corresponding flow on the Poisson manifold $\mathfrak{T}_s$ is $h_{X(x)}$. Let us denote also $x \cdot \nu := W(x)\nu W(x)^* \equiv Ad^*(W(x))\nu$.* $\diamondsuit$

3.3.9. Proposition. *(i) With the notation from 3.3.8, the (densely defined) function $h_{X(x)}$ has $\boldsymbol{\mathcal{D}}_r(\mathbb{F})$–generalized differential, which is $\mathcal{V}$–integrable.*

(ii) The orbits $\mathcal{V}_\nu$ are embedded submanifolds of $\mathfrak{T}_s$, each diffeomorphic to the "flat phase space" $\mathbb{R}^{2n}$.

(iii) The restrictions of the symplectic forms Ω_ν introduced on $\mathcal{O}_\nu(\mathfrak{U})$ to the orbits $\mathcal{V}_\nu$ are non-degenerate, and the restrictions of the Momentum mapping $\mathbb{F}$ to these orbits are symplectomorphisms onto the coadjoint orbit of G_{WH} "corresponding" to the choice of $s_0 = -\lambda := -\frac{1}{\hbar}$, cf.(3.3.21). $\clubsuit$

Proof. (i) The proof of integrability trivially follows from Lemma 2.2.16 and Proposition 2.2.14 , since the integral curves of the Hamiltonian vector field corresponding to $dh_{X(x)}$ leave all $\mathcal{V}_\nu$, $(\nu \in \boldsymbol{\mathcal{D}}_r(\mathbb{F}))$ invariant.

(ii) The second assertion follows from the Lemma 2.2.16, and from its proof. A more intuitive argument is seen from the Momentum mapping $\mathbb{F}$ restricted to any $\mathcal{V}_\nu$ with a help of (3.3.20): According In view of (2.2.17), the j–th component of $\mathbb{F}(x\cdot\nu)$ can be expressed as:

$$F_{x\cdot\nu}(\xi_j) = Tr\big(\nu W(x)^* X(\xi_j) W(x)\big) = \begin{cases} F_\nu(\xi_j) - \lambda q_j, & \text{for } j = 1, 2, \dots n, \\ F_\nu(\xi_j) + \lambda p_{j-n}, & \text{for } j = n+1, \dots 2n. \end{cases} \tag{3.3.21}$$

what proves bijection of $\mathcal{V}_\nu$ onto $\mathbb{R}^{2n}$.

(iii) The (densely defined) vector fields $\mathbf{v}_j := \mathbf{v}_{\xi_j}$ corresponding to the basis $\{\xi_j,\ j = 0, 1, \dots 2n\}$ of G_{WH} form a basis of $T_\varrho(\mathcal{V}_\nu)$ for any $\varrho \in \mathcal{V}_\nu$ for all $\mathcal{V}_\nu$. These vector fields are proportional to the Hamiltonian vector fields corresponding to h_{X_j} for the selfadjoint generators X_j of the representation $W_{1/\hbar}(G_{WH})$. The vector field $\mathbf{v}_0$ is identical zero. According to (2.1.30), (2.2.7), and (2.2.4), one has

$$\Omega_\varrho(\mathbf{v}_\xi, \mathbf{v}_\eta) = i\, Tr\big(\varrho[X(\xi), X(\eta)]\big), \tag{3.3.22}$$

resp. from (2.2.25) one has

$$\mathbb{F}^*\{h_\xi, h_\eta\} = \{\mathbb{F}^* h_\xi, \mathbb{F}^* h_\eta\}.$$

The Kirillov–Kostant symplectic form on an $Ad^*(G_{WH})$ orbits through F has the form (2.2.23):

$$\Omega^K_F(\mathbf{v}_\xi, \mathbf{v}_\eta) = -F([\xi, \eta]).$$

From the CCR (3.3.17) one has

$$\begin{aligned} \Omega_\varrho(\mathbf{v}_j, \mathbf{v}_{j+n}) = -\Omega_\varrho(\mathbf{v}_{j+n}, \mathbf{v}_j) &= i\, Tr\big(\varrho[X(\xi_j), X(\xi_{j+n})]\big) && (3.3.23)\\ &= -i\,\lambda^2 Tr\big(\varrho[Q_j, P_j]\big) && (3.3.24)\\ &= -Tr(\varrho X(\xi_0)) = -s_0 = \lambda, && (3.3.25) \end{aligned}$$

and for the remaining indices $j, k:\ \Omega_\varrho(\mathbf{v}_j, \mathbf{v}_k) = 0$. For the Kirillov–Kostant form we have

$$\Omega^K_F(\mathbf{v}_j, \mathbf{v}_{j+n}) = -F([\xi_j, \xi_{j+n}]) = -F(\xi_0), \tag{3.3.26}$$

what corresponds to (3.3.23) in accordance with the equation (2.2.25d); this proves the symplectomorphism property of $\mathbb{F}$. The commutation relations (3.3.17) show nondegeneracy of the restricted form $\Omega^{\mathcal{V}}_\varrho := \iota^*_{\mathcal{V}}\Omega_\varrho$ for all relevant ϱ.

It remains to prove that $\mathbb{F}$ maps all $\mathcal{V}_\nu$ onto a unique orbit. The basis $\{\xi_j,\ j = 0, 1, \dots 2n\} \subset Lie(G_{WH})$ determines global coordinates on the dual $Lie(G_{WH})^*$, $F(\xi_j)$ being the coordinates of $F \in Lie(G_{WH})^*$ in the dual basis. It is clear from (3.3.21) that on any $\mathcal{V}_\nu$ there is a point ϱ_0 such that $\mathbb{F}(\varrho_0)(\xi_j) = 0,\ j = 1, 2, \dots 2n$. The coordinates of other points on those orbits are then $\mathbb{F}(x\cdot\varrho_0)(\xi_j) = (-1)^{\left[\frac{n+1+j}{n+1}\right]}\lambda x_j,\ j = 1, 2, \dots 2n$, and the remaining coordinate $\mathbb{F}(x\cdot\varrho_0)(\xi_0) \equiv Tr(\nu X(\xi_0)) = -\lambda = -\hbar^{-1}$, (3.3.15), hence it is constant on the orbit and of equal value on all orbits, i.e. on all the images $\mathbb{F}(\mathcal{V}_\nu) \subset Lie(G_{WH})^*$. This proves the last statement. □

3.3.10. **Remark**. These considerations showed that the choice of a specific value of Planck constant in QM corresponds mathematically to the choice of the coadjoint orbit of G_{WH} labelled by $s_0 = -\lambda = -\frac{1}{\hbar}$ determining a unitary representation of this group and, in this way, also determining Heisenberg uncertainty relations and many physical effects connected with them. Since unitary equivalent representations lead, as a rule, to indistinguishable physics, validity of the mathematical theorem about unitary inequivalence of representations (3.3.14), or (3.3.16), for different real values of λ, "can be seen" also from the known physical measurability of the Planck constant $\hbar$. ♡

3.3-c Restricted flows with linear generators on $\mathcal{O}_\varrho(G_{WH})$

Let X be a selfadjoint operator on a Hilbert space $\mathcal{H}$, and let $U(G)$ be a continuous unitary representation of a connected Lie group G. Assume that the orbit of $Ad^*(U(G))$ through $\varrho \in \boldsymbol{\mathcal{D}}_r(\mathbb{F})$, cf. Definition 2.2.17, belongs to the domain of X, $\mathcal{O}_\varrho(G) \subset \mathcal{D}_{rd}(\delta_X)$. *Let us assume further in this subsection that the function* h_X *is constant on the submanifolds* $Ad^*\big(U(G_{\mathbb{F}(\nu)})\big)\nu$ *for all* $\nu \in \mathcal{O}_\varrho(G)$, i.e. that it is a ϱG–classical generator, cf. Definition 2.2.26(iv). This would be trivially the case, if the momentum mapping $\mathbb{F}$ is injective on the orbit $\mathcal{O}_\varrho(G)$, i.e. if the orbit $\mathcal{O}_\varrho(G)$ is diffeomorphic to the coadjoint orbit $Ad^*(G)\mathbb{F}(\varrho)$. For further examples of ϱG–classical generators cf.2.2.27. We can now define the corresponding classical Hamiltonian h^ϱ_X

$$h^\varrho_X : Ad^*(G)\mathbb{F}(\varrho) \to \mathbb{R},\ h^\varrho_X(F_\nu) := \mathrm{h}_X(\nu) \equiv Tr(\nu X), \forall \nu \in \mathcal{O}_\varrho(G) \tag{3.3.27}$$

what is an infinitely differentiable function on the coadjoint orbit through F_ϱ. The function h^ϱ_X will be also called **the (classical) Hamiltonian induced by** $(X; U(G))$ on the orbit $Ad^*(G)\mathbb{F}(\varrho)$. The restriction of h_X:

$$\mathrm{h}^\varrho_X : \mathcal{O}_\varrho(G) \to \mathbb{R},\ \nu \mapsto \mathrm{h}^\varrho_X(\nu) := Tr(\nu X) \tag{3.3.28}$$

generates the restricted flow of X, to the orbit $\mathcal{O}_\varrho(G)$.

Let us choose now $G := G_{WH}$, and $X := H$, with

$$H := \frac{1}{2}\sum_{j=1}^{n} \frac{1}{m_j} P_j^2 + V(Q_1, Q_2, \dots Q_n), \tag{3.3.29}$$

(cf.3.3.7) with some "convenient" real function $V : \mathbb{R}^n \to \mathbb{R}$. The "correct quantum evolution" given by $z(\in \mathcal{H}) \mapsto z_t := \exp(-itH)z$ (we set here $\lambda = \hbar = 1$) leads to the **Ehrenfest's relations** for expectations $\langle X_j\rangle_t := \langle z_t|X_j|z_t\rangle,\ (j = 1, 2, \dots, n)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Q_j\rangle_t = \frac{1}{m_j}\langle P_j\rangle_t, \tag{3.3.30a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle P_j\rangle_t = -\langle \partial_j V(Q_1, \dots, Q_n)\rangle_t. \tag{3.3.30b}$$

These relations *are not differential equations* for the functions $t \mapsto \langle X_j\rangle_t$, if the potential energy V is not at most quadratic polynomial in Q's, cf. the text following Eq. (3.3.32). In the case of

quadratic $H := A$ from (3.3.32), the Hamiltonian evolutions given by the Hamiltonians (3.3.31) lead to the results identical with those of QM (hence satisfying also (3.3.30) with $x_j(t) \equiv \langle X_j \rangle_t$).

Let us express h^ϱ_H corresponding to H from (3.3.29) according to (3.3.27). Let $\varrho := \varrho_0$ (cf. the text on page 117 following (3.3.26)) with $Tr(\varrho X_j) = 0,\ \forall j = 1, 2, \dots 2n$, with the notation of Subsection 3.3-b. We write (q, p) instead of $\mathbb{F}(\nu)$ with components $\mp\lambda x_j$, cf. (3.3.21):

$$h^\varrho_H(q,p) \equiv \frac{1}{2}\sum_{j=1}^{n} \frac{1}{m_j} p_j^2 + V_\varrho(q_1, q_2, \dots q_n), \tag{3.3.31}$$

with

$$V_\varrho(q) := Tr\big(\varrho V(Q+q)\big) + \sum_j \frac{1}{2m_j} Tr\big(\varrho P_j^2\big).$$

The last sum in this expression is a constant term on the orbit $\mathcal{O}_\varrho(G)$, hence the flow generated by h^ϱ_H on $\mathcal{O}_\varrho(G)$ is independent of this constant. This flow (the restricted "linear" flow of H) is projected by the momentum mapping $\mathbb{F}$ onto the flow on the coadjoint orbit of G_{WH} with $s_0 := s_0(\mathbb{F}(\varrho)) = -\frac{1}{\hbar}$ generated by the Hamiltonian $h^\varrho_H(q,p)$ via the standard symplectic form $\mathrm{d}p \wedge \mathrm{d}q$.

3.3.11. **Example**. Let us take, e.g. $\varrho := P_z$ with $0 \neq z \in L^2(\mathbb{R}^n) : \langle z|X_j|z\rangle = 0, \forall j$. Let $\tilde{z}(q) := z(-q)$. Then

$$V_\varrho(q) = \text{const.} + V_z(q), \quad \text{with } V_z(q) := |\tilde{z}|^2 * V(q),$$

the symbol $a * b(q) := \int a(q-q')b(q')\mathrm{d}^n q'$ denoting convolution of two complex–valued functions $a,\ b$ on $\mathbb{R}^n$. Let, e.g., $n = 3$, and $V(q) := \frac{\alpha}{|q|}$ be the Coulomb potential. Let the above $z \in L^2(\mathbb{R}^3)$ be rotationally (i.e. $O(3)$) symmetric normalized function with support "concentrated" near $q = 0$. Then $q \mapsto V_z(q)$ is again, for large values of $|q|$, approximately (resp. exactly, for compact support of z) of the Coulomb form. ♡

We see that the ϱG_{WH}–restrictions of the flow $\tilde{\varphi}^H$ are identical to the flows of classical Hamiltonian mechanics on $\mathbb{R}^{2n}$ with the Hamiltonian function h^ϱ_H from (3.3.31) differing from the usually considered "classical limit" of the quantum flow $\tilde{\varphi}^H$ by the "ϱ–smearing" of the potential V only.

A specific interesting choice of V in (3.3.29) is a quadratic function, describing, e.g. harmonic oscillators. This case can be generalized to any quadratic operator $X := A$:

$$A := \frac{1}{2}\sum_{j,k=1}^{2n} a_{jk} X_j X_k, \tag{3.3.32}$$

with real constants $a_{jk} \equiv a_{kj}$, and with $X_j,\ j = 1, 2, \dots 2n$ defined in Notation 3.3.7. This case is specific in that the operator A is essentially selfadjoint on a common domain of all X_j's, and it generates, together with the X_j's, a unitary representation of a $2n+2$–dimensional Lie group containing G_{WH} as a subgroup. This follows from the following considerations.

It is clear that the operators $\{A; X_j,\ j = 0, 1, \dots 2n\}$ form a basis of a Lie algebra of (unbounded) operators in $\mathcal{H}$=$L^2(\mathbb{R}^n)$. It is also easily seen that an arbitrary number of quadratic

symmetric operators of the form (3.3.32) together with all X_j's can be included into a *finite dimensional* Lie algebra $X(\mathfrak{g})$ of operators (with respect to the operator commutation $i\,[A,B]$) containing operators at most quadratic in X_j's. Less trivial is the assertion, that these Lie algebras of operators are composed of essentially selfadjoint operators on the domain $D^\omega(G_{WH})$ of essential selfadjointness of all X_j's, and *that they are integrable into continuous unitary representations of some Lie groups*. The maximal Lie algebra obtained in this way is (isomorphic to) the algebra called $\mathfrak{st}(n,\mathbb{R})$ (see also [148, §15.3, and §18.4]). Let us formulate and prove the mentioned facts for the algebra $\mathfrak{st}(n,\mathbb{R})$:

3.3.12. Proposition. *Let $X(\mathfrak{st}(n,\mathbb{R}))$ denote the above mentioned "maximal" Lie algebra of "at most quadratic" symmetric operators acting on the Hilbert space $\mathcal{H}$ of representation $W_\lambda(G_{WH})$. Let $\tilde{St}(n,\mathbb{R})$ be the corresponding connected, simply connected Lie group with the Lie algebra $\mathfrak{st}(n,\mathbb{R})$. Then the representation $W_\lambda(G_{WH})$ has a unique extension to the continuous unitary representation $\tilde{W}_\lambda(\tilde{St}(n,\mathbb{R}))$ in $\mathcal{H}$ such that all the closures of the operators from $X(\mathfrak{st}(n,\mathbb{R}))$ are exactly all of its selfadjoint generators.* ♣

Proof. The analytic domain $D^\omega(G_{WH})$ is a common invariant domain for all operators from $X(\mathfrak{st}(n,\mathbb{R}))$. According to Nelson's theorem (cf. [13, Theorem 11.5.2]) it suffices to prove essential selfadjointness of the operator Δ, what is sum of squares of a basis of $X(\mathfrak{st}(n,\mathbb{R}))$. We chose here the basis consisting of the generators X_j of $W_\lambda(G_{WH})$, and of all their symmetrized products $\frac{1}{2}(X_jX_k+X_kX_j)$. Then

$$\begin{aligned}\Delta &:= \sum_{j=1}^{2n} X_j^2 + \frac{1}{4}\sum_{j,k=1}^{2n}(X_jX_k+X_kX_j)^2\\ &= \frac{3}{2}nI + \sum_{j=1}^{n}(P_j^2+Q_j^2)(I+\sum_{k=1}^{n}(P_k^2+Q_k^2)),\end{aligned}$$

where we used notation from 3.3.7, and the CCR (3.3.17). From the known properties of the Hamiltonians $P_j^2+Q_j^2$ of independent linear oscillators, we conclude (with a help of, e.g. [218, Theorem VIII.33] on operators on tensor products of Hilbert spaces) that Δ is essentially selfadjoint. The Nelson's theorem states now integrability of $X(\mathfrak{st}(n,\mathbb{R}))$ onto a unitary representation. Selfadjointness and uniqueness now easily follow. □

Hence, also any Lie subalgebra of $X(\mathfrak{st}(n,\mathbb{R}))$ integrates onto a continuous group representation. Let us denote by AG_{WH} the simply connected Lie group represented by this unitary representation with the basis of generators $\{A;\ X_j,\ j=0,1,2,\dots 2n\}$, with A from (3.3.32). The $2n+2$–dimensional group AG_{WH} contains G_{WH} as its normal subgroup.

In the "quadratic case" (3.3.32) the expression (3.3.31) has the form

$$h_A^\varrho(x) \equiv \frac{1}{2}\sum_{j,k=1}^{2n} a_{jk}x_jx_k + const., \tag{3.3.33}$$

with the $const.$ depending on the choice of orbit only (we always assume $\varrho := \varrho_0$, according to the definition of ϱ_0 in the notes on page 117). This is valid regardless the orbit $\mathcal{O}_\varrho(AG_{WH})$ is $2n$–, or $2n+1$–dimensional. Hence for the (at most) quadratic Hamiltonian A, the projected

quantal evolutions $\tilde{\varphi}_t^A(\nu)$ to the orbits and the "corresponding" classical evolution "coincide" in the sense that the $2n$ coordinates $Tr(\tilde{\varphi}_t^A(\nu)X_j),\ j = 1,\dots 2n$ (of the possible total $2n+1$) satisfy classical equations with the Hamiltonian h_A^o from (3.3.33) corresponding to the canonical symplectic form $\mathrm{d}p \wedge \mathrm{d}q$.

3.3.13. **Remark**. Let us stay on $P(\mathcal{H})$, and let $\varrho_0 = P_z$. Let A be quadratic as in (3.3.32). Then a general assertion tells us that the dimension of the AG_{WH}–orbit through P_z is $2n$ iff it contains an eigenstate of A, [27]. In that case, each point $W(x)P_zW(-x)$ of the orbit is (i.e. represents) an eigenstate of some selfadjoint operator of the form $\sum_j c_jX_j + A$. Other orbits are $2n+1$–dimensional. This assertion is a consequence of the fact that the dimension of any connected finite dimensional manifold is constant in all of its points and equals to the dimension of the tangent spaces which are in turn generated by vector fields corresponding to the flows $\tilde{\varphi}_t^X$ ($X = A, X_1, \dots, X_{2n}$). These $2n+1$ vectors in any point of the orbit are mutually linearly independent except in a stationary point P_z where $\tilde{\varphi}_t^X(P_z) \equiv P_z$, for some $X := \sum_j c_jX_j + A$, i.e. $z \in \mathcal{H}$ ($z \neq 0$) is an eigenvector of X (the linear independence of the $2n$ vectors determined by the X_j's is a consequence of CCR). ♡

3.3-d Time dependent Hartree–Fock theory

We shall consider here the "approximation to QM" which is very well known in nonrelativistic quantummechanical many–particle theory as the *Hartree–Fock theory*. It consists, expressed briefly in our terminology, in the restriction of a given (linear) QM problem to a manifold (a G-orbit) of quantum states Ψ, and, in its stationary setting, in looking for the points Ψ_0 of the manifold that minimize the expectation value $\langle\Psi|H|\Psi\rangle$ of a given Hamiltonian H. In the Hartree–Fock theory of systems consisting of N interacting fermions in an external potential the manifold consists of all "Slater determinants" for the considered N fermions. The point Ψ_0 then satisfies the Hartree–Fock equation (3.3.51), what is a condition for the zero value of the derivative $\mathrm{q}_{P_\Psi}(D_{P_\Psi}h_H)$ of the corresponding restricted generator. It is assumed that in many interesting cases stationary point(s) Ψ_0 (resp., more correctly: P_{Ψ_0}) of the orbit approximate(s) the ground state (states) of the unrestricted system. This theory is expressible in terms of "one–particle states", due to a natural bijection (see e.g., also [221]) between the set of all Slater determinants and an orbit in the one–particle state space $\mathcal{S}_*(\mathcal{L}(\mathcal{H}))$ of unitary group $\mathfrak{U}$:= all unitary operators on the one–particle Hilbert space $\mathcal{H}$.

Let us consider a system of N identical fermions described in the Hilbert space $\mathcal{H}_N := \otimes^N\mathcal{H}$, where $\mathcal{H} := \mathcal{H}_1$ is "the one–particle Hilbert space". The vectors in $\mathcal{H}_N$ are expressed by linear combinations of "product–vectors" $\Phi := \phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_N,\ \phi_k \in \mathcal{H}$, and the scalar product linear in the second factor is determined by

$$\langle\Phi'|\Phi\rangle := \prod_{k=1}^{N}(\phi'_k, \phi_k).$$

Let $\psi_j \in \mathcal{H}, j \in \mathbb{N}$ be an orthonormal basis in $\mathcal{H}$. Then an orthonormal basis in $\mathcal{H}_N$ consists of vectors labelled by ordered N–tuples $(j) := (j_1; j_2; \dots j_N) \in \mathbb{N}^N$:

$$\Psi_{(j)} \equiv |(j_1; j_2; \dots j_N)\rangle := \psi_{j_1} \otimes \psi_{j_2} \otimes \cdots \otimes \psi_{j_N},\ \forall j_1, j_2, \dots j_N \in \mathbb{N}. \tag{3.3.34}$$

Let $\Sigma(N)$ be the permutation group (="symmetric group") of N elements, and, for any $\sigma \in \Sigma(N)$, let its action on ordered N–tuples of integers be denoted by

$$\sigma(1;2;\dots N) \equiv (\sigma(1);\sigma(2);\dots \sigma(N)).$$

Then a unitary representation $\sigma \mapsto p_\sigma$ of $\Sigma(N)$ in $\mathcal{H}_N$ is determined by

$$p_\sigma \Psi_{(j)} \equiv \Psi_{\sigma(j)} := |(j_{\sigma(1)}; j_{\sigma(2)}; \dots j_{\sigma(N)})\rangle.$$

Let the **Fermionic subspace** $\mathcal{H}_N^F$ of $\mathcal{H}_N$ consists of all vectors $\Psi \in \mathcal{H}_N$ satisfying

$$p_\sigma \Psi = \varepsilon_\sigma \Psi, \quad \varepsilon_\sigma \varepsilon_{\sigma'} \equiv \varepsilon_{\sigma\sigma'} \in \{-1;1\}, \forall \sigma, \sigma' \in \Sigma(N),$$

with $\varepsilon_\sigma := -1$ for σ corresponding to a mere interchange of two elements. The orthogonal projection P_N^F onto $\mathcal{H}_N^F$ is the maximal of all projections P satisfying: $p_\sigma P \equiv \varepsilon_\sigma P$. It can be expressed:

$$P_N^F = \frac{1}{N!} \sum_\sigma \varepsilon_\sigma p_\sigma.$$

An orthonormal basis in $\mathcal{H}_N^F$ is given by *"Slater determinants"* $\Psi_{\{j\}}$ labelled by the ordered N–tuples $(j) := (j_1; j_2; \dots j_N)$ with $j_1 < j_2 < \dots < j_N$, and defined by

$$\Psi_{\{j\}} := \frac{1}{\sqrt{N!}} \sum_\sigma \varepsilon_\sigma \Psi_{\sigma(j)}. \tag{3.3.35}$$

Let $\mathfrak{U}$ be the unitary group of $\mathcal{L}(\mathcal{H})$, and let its unitary representation in $\mathcal{H}_N$ be given by its action on the product vectors

$$\mathrm{u} \mapsto \mathrm{u}^{\otimes N}, \quad \mathrm{u}^{\otimes N}\Phi \equiv \mathrm{u}^{\otimes N}(\phi_1 \otimes \phi_2 \otimes \dots \otimes \phi_N) \tag{3.3.36}$$

$$:= \mathrm{u}\phi_1 \otimes \mathrm{u}\phi_2 \otimes \dots \otimes \mathrm{u}\phi_N, \ \forall \mathrm{u} \in \mathfrak{U}. \tag{3.3.37}$$

One orbit of this representation in $\mathcal{H}_N^F$ consists of all the Slater determinants, and its canonical projection to the projective Hilbert space $P(\mathcal{H}_N)$ is homeomorphic, cf. also (3.3.40), to the coadjoint orbit of $\mathfrak{U}$ in $\mathcal{T}_{1+}(\mathcal{H}) =: \mathcal{S}_*$ (cf. page 42) consisting of all the "N–dimensional" density matrices $\varrho_{\{j\}}$ with maximally degenerate spectrum, cf. also [221]. These density matrices are obtained as "partial traces" from Slater determinants by restriction to "one particle observables", i.e. for all $\mathrm{a} \in \mathcal{L}(\mathcal{H})$ one has

$$Tr(\varrho_{\{j\}} \cdot \mathrm{a}) := \langle \Psi_{\{j\}} | \mathrm{a} \otimes \mathbb{I}_{\mathcal{H}}^{\otimes(N-1)} | \Psi_{\{j\}} \rangle, \tag{3.3.38}$$

and the resulting density matrix has an explicit expression of the form

$$\varrho_{\{j\}} = \frac{1}{N} \sum_{k=1}^N |\psi_{j_k}\rangle\langle\psi_{j_k}| =: \frac{1}{N} p_{\{j\}}, \tag{3.3.39}$$

where the projector $p_{\{j\}}$ onto the subspace of $\mathcal{H}$ spanned by the N orthonormal vectors $\{\psi_k : k \in (j) = (j_1; j_2; \dots j_N)\}$ was introduced. Conversely, as can be proved by elementary techniques, any N–dimensional subspace of $\mathcal{H}$ determines a Slater determinant (uniquely

up to a numerical factor), namely the space spanned by N one–particle orthonormal vectors ψ_j determines the Slater determinant constructed by the same vectors, cf.(3.3.35), (3.3.34), and different orthonormal bases of that subspace give (up to a phase factor) the same Slater determinant. Hence the mapping

$$\mathbb{G}: \quad \varrho := \varrho_{\{j\}} (\in Ad^*(\mathfrak{U})\varrho_{\{j'\}}) \mapsto \mathbb{G}(\varrho) := P_{\Psi_{\{j\}}},\ (\Psi_{\{j\}} \in \{\mathrm{u}^{\otimes N}\Psi_{\{j'\}} : \mathrm{u} \in \mathfrak{U}\}) \tag{3.3.40}$$

is a bijection (here $\Psi_{\{j'\}}$ is an arbitrary Slater determinant).

Reduced one particle density matrices corresponding to arbitrary states $\Phi \in \mathcal{H}_N^F$ cannot be expressed in such a simple way.

Let $H := H_N$ be a selfadjoint Hamiltonian on $\mathcal{H}_N$, and let us assume that it is permutation symmetric, i.e. that

$$p_\sigma H_N \equiv H_N p_\sigma, \quad \forall \sigma \in \Sigma(N).$$

The corresponding generator, as a function on (a dense subset of) $P(\mathcal{H}_N)$, is $h_H^N(P_\Phi) := Tr(P_\Phi \cdot H_N)$. Its restriction to the $\mathfrak{U}$–orbit of Slater determinants is

$$\tilde{h}_H^{Sl}(P_{\Psi_{\{j\}}}) := \langle \Psi_{\{j\}} | H_N | \Psi_{\{j\}} \rangle = \sum_\sigma \varepsilon_\sigma \langle p_\sigma \Psi_{(j)} | H_N | \Psi_{(j)} \rangle = N! \langle \Psi_{(j)} | P_N^F H_N | \Psi_{(j)} \rangle. \tag{3.3.41}$$

With a help of the bijection $\mathbb{G}$ from (3.3.40), we can write the restricted function $\tilde{h}_H^{Sl}$ as a function on the orbit $Ad^*(\mathfrak{U})\varrho_{\{j\}} \subset \mathcal{S}_*$, i.e. as a generator on one–particle states. We shall write the corresponding "one–particle energy" as $h_H^{Sl} := N^{-1}\tilde{h}_H^{Sl}$:

$$N \cdot h_H^{Sl}(\varrho_{\{j\}}) := \langle \Psi_{\{j\}} | H_N | \Psi_{\{j\}} \rangle = \langle \mathbb{G}(\varrho_{\{j\}}) | H_N | \mathbb{G}(\varrho_{\{j\}}) \rangle. \tag{3.3.42}$$

This relation can be made more explicit for specific choices of H_N.

Let us take for H_N a nonrelativistic spin–independent Hamiltonian of N point particles with symmetric pair potential interaction, i.e.

$$H_N := \sum_{j=1}^N \mathrm{h}_{0j} + \frac{1}{2}\sum_{j \neq k} \mathrm{v}_{jk}, \tag{3.3.43}$$

where the indices $j, k = 1, \ldots N$ specify "one–, resp. two–particle factors" $\mathcal{H}$ in the tensor product $\mathcal{H}_N$ on which the corresponding operators act; the h_{0j}'s are copies of the same one particle Hamiltonian h_0 (kinetic energy plus external fields) "acting at the j–th factor" in the tensor product $\mathcal{H}_N$, and $\mathrm{v}_{jk} \equiv \mathrm{v}_{kj}$ are also copies of a two–particle operator $\mathrm{v} \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$ "acting on the j–th and k–th factor" in $\mathcal{H}_N$. Let us introduce the linear **exchange operator** $p_\leftrightarrow$ on the two–particle spaces (commuting with v) by

$$p_\leftrightarrow(\phi \otimes \psi) := \psi \otimes \phi,\ \forall \phi, \psi \in \mathcal{H}.$$

We can calculate now (3.3.41) with $H := H_N$ from (3.3.43):

$$\begin{aligned} N\cdot h_H^{Sl}(\varrho_{\{j\}}) &:= \langle\Psi_{\{j\}}|H_N|\Psi_{\{j\}}\rangle = N!\langle\Psi_{(j)}|P_N^F H_N|\Psi_{(j)}\rangle \\ &\equiv \sum_\sigma \varepsilon_\sigma\langle\psi_{\sigma(j_1)}\otimes\cdots\otimes\psi_{\sigma(j_N)}|\left[\sum_{k=1}^N \mathrm{h}_{0k}+\frac{1}{2}\sum_{k\neq l}\mathrm{v}_{kl}\right]|\psi_{j_1}\otimes\cdots\otimes\psi_{j_N}\rangle \\ &= \sum_{k=1}^N(\psi_{j_k},\mathrm{h}_0\psi_{j_k})+\frac{1}{2}\sum_{k\neq l}(\psi_{j_k}\otimes\psi_{j_l}-\psi_{j_l}\otimes\psi_{j_k}|\mathrm{v}_{kl}|\psi_{j_k}\otimes\psi_{j_l}) \\ &= N\cdot Tr(\varrho_{\{j\}}\mathrm{h}_0)+\frac{1}{2}\sum_{k\neq l}(\psi_{j_k}\otimes\psi_{j_l}|\mathrm{v}\cdot(\mathbb{I}_{\mathcal{H}}\otimes\mathbb{I}_{\mathcal{H}}-p_{\leftrightarrow})|\psi_{j_k}\otimes\psi_{j_l}) \\ &= N\cdot Tr(\varrho_{\{j\}}\mathrm{h}_0)+\frac{N^2}{2}Tr(\varrho_{\{j\}}\otimes\varrho_{\{j\}}\cdot\mathrm{v}\cdot(\mathbb{I}_{\mathcal{H}}\otimes\mathbb{I}_{\mathcal{H}}-p_{\leftrightarrow})). \end{aligned} \tag{3.3.44}$$

This function h_H^{Sl} can be used as a generator of the (quantum nonlinear) motion on the $\mathfrak{U}$–orbit of N–dimensional projections (which, divided by N, are density matrices of the domain of $\mathbb{G}$, (3.3.40)) in the one–particle state space $\mathcal{S}_*$. The resulting dynamical equation describes the **time dependent Hartree–Fock theory**, cf. [221, 222, 154], and the equations describing its stationary points are just the **Hartree–Fock equations**, cf. [126, 175, 12]. Let us show how it looks in our formalism.

We shall need an expression for the derivative $D_\varrho h_H^{Sl}$ to be able to write a dynamical equation for ϱ, e.g. the "Schrödinger equation" (2.1.23), resp. (2.1.26).[86] The differential will be calculated according to (2.1.11), and with a help of (2.1.14). We shall need, however, derivatives along the curves $t\mapsto\exp(-itb)\varrho\exp(itb)$, corresponding to tangent vectors $i[\varrho,b]$, cf. Notes 2.1.4:

$$D_\varrho h_H^{Sl}(i[\varrho,b]) = i\,Tr(\mathrm{h}_0[\varrho,b])+i\frac{N}{2}Tr\left([\varrho,b]\otimes\varrho\cdot(\mathbb{I}_{\mathcal{H}}^{\otimes 2}-p_{\leftrightarrow})\mathrm{v}+\varrho\otimes[\varrho,b]\cdot(\mathbb{I}_{\mathcal{H}}^{\otimes 2}-p_{\leftrightarrow})\mathrm{v}\right). \tag{3.3.45}$$

This equation can be rewritten by inserting the unit operators $\mathbb{I}_{\mathcal{H}}$ expressed with a help of convenient complete systems $\{\varphi_k\}\subset\mathcal{H}$ of orthonormal vectors into several places in between of the multiplied operators in the above formula. E.g., from the trace in $\mathcal{L}(\mathcal{H})\otimes\mathcal{L}(\mathcal{H})$ of the product $(A\otimes\varrho)\cdot B$ ($B\in\mathcal{L}(\mathcal{H})\otimes\mathcal{L}(\mathcal{H}),\forall A\in\mathcal{L}(\mathcal{H})$) one can find an operator $D\in\mathcal{L}(\mathcal{H})$ defined by

$$Tr(A\cdot D) := Tr(A\otimes\varrho\cdot B)$$

as follows:

$$\begin{aligned} Tr(A\cdot D) &:= \sum_{j,k}\sum_{l,m}(\varphi_j\otimes\varphi_k|A\otimes\varrho|\varphi_l\otimes\varphi_m)(\varphi_l\otimes\varphi_m|B|\varphi_j\otimes\varphi_k) \\ &= \sum_{j,l}(\varphi_j|A|\varphi_l)\underbrace{\sum_{k,m}(\varphi_k|\varrho|\varphi_m)(\varphi_l\otimes\varphi_m|B|\varphi_j\otimes\varphi_k)}_{(\varphi_l|D|\varphi_j)}. \end{aligned} \tag{3.3.46}$$

[86] Our considerations will be a little "heuristic" from mathematical point of view in this Subsection: we shall not consider here the domain questions, hence we shall not be able to write equations as (2.2.10) with precisely defined generalized differentials.

Since

$$(\varphi_m\otimes\varphi_l|(\mathbb{I}_{\mathcal{H}}^{\otimes 2}-p_{\leftrightarrow})\mathrm{v}|\varphi_k\otimes\varphi_j)=(\varphi_m\otimes\varphi_l|\mathrm{v}|\varphi_k\otimes\varphi_j)-(\varphi_m\otimes\varphi_l|\mathrm{v}|\varphi_j\otimes\varphi_k),$$

we can write for the operator representation of $D_\varrho h_H^{Sl}\in T_\varrho^*\mathfrak{T}_s\sim\mathcal{L}(\mathcal{H})_s$:

$$\begin{aligned}(\varphi|D_\varrho h_H^{Sl}|\psi)=(\varphi|\mathrm{h}_0|\psi)+\frac{N}{2}\sum_{k,m}(\varphi_k|\varrho|\varphi_m)\big[(\varphi\otimes\varphi_m|\mathrm{v}|\psi\otimes\varphi_k)-\\(\varphi\otimes\varphi_m|\mathrm{v}|\varphi_k\otimes\psi)+(\varphi_m\otimes\varphi|\mathrm{v}|\varphi_k\otimes\psi)-(\varphi_m\otimes\varphi|\mathrm{v}|\psi\otimes\varphi_k)\big].\end{aligned}\tag{3.3.47}$$

Let us consider now that, on the chosen orbit $\mathcal{O}_\varrho(\mathfrak{U})$, one has

$$\varrho_t\equiv\varrho_{\{j\}}^t:=\mathrm{u}_H(t,\varrho)\varrho_{\{j\}}\mathrm{u}_H(t,\varrho)^*:=N^{-1}\sum_{k=1}^N|\psi_k^t\rangle\langle\psi_k^t|.$$

Let us denote by $\boldsymbol{E_{\{j\}}^t:=N\varrho_{\{j\}}^t}$ the N–dimensional projection corresponding to the Slater vector–determinant $\Psi_{\{j\}}^t$, according to the mapping $\mathbb{G}$ from (3.3.40). For each such $\varrho=\varrho_{\{j\}}^t$ occurring in (3.3.47), let us choose in the rôle of the complete orthonormal system $\{\varphi_k\}$ such a complete orthonormal system $\{\psi_j^t:j=1,2,\dots\}$ that contains the $\varrho_{\{j\}}^t$–defining one–particle vectors $\{\psi_j^t:j=1,\dots,N\}$ numbered as the initial segment. Then we have for the obtained orthonormal bases $\{\{\psi_j^t:j\in\mathbb{N}\}:\ t\in\mathbb{R}\}$:

$$(\varphi_k|\varrho_t|\varphi_m)\equiv(\psi_k^t|\varrho_t|\psi_m^t)=\begin{cases}\frac{1}{N}\delta_{k,m} & \text{for } k,m\le N,\\ 0 & \text{for } \max(k;m)>N;\end{cases}$$

these should be inserted into the formulas like (3.3.47). Matrix elements of the Schrödinger equation (2.1.23) for arbitrary $\varphi,\psi\in\mathcal{H}$ are then of the form

$$\begin{aligned}i\frac{d}{dt}(\varphi|\mathrm{u}_H(t,\varrho)|\psi)=(\varphi|D_{\varrho_t}h_H^{Sl}\cdot\mathrm{u}_H(t,\varrho)|\psi)=(\varphi|\mathrm{h}_0\cdot\mathrm{u}_H(t,\varrho)|\psi)+\\\frac{1}{2}\sum_{k=1}^N\sum_j\big[(\varphi\otimes\psi_k^t|\mathrm{v}|\psi_j^t\otimes\psi_k^t)-(\varphi\otimes\psi_k^t|\mathrm{v}|\psi_k^t\otimes\psi_j^t)+\\(\psi_k^t\otimes\varphi|\mathrm{v}|\psi_k^t\otimes\psi_j^t)-(\psi_k^t\otimes\varphi|\mathrm{v}|\psi_j^t\otimes\psi_k^t)\big](\psi_j^t|\mathrm{u}_H(t,\varrho)|\psi).\end{aligned}\tag{3.3.48}$$

Let us rewrite this equation in "configuration representation", if $\mathcal{H}:=L^2(\mathbb{R}^n)$, and operators A are (formally) expressed by they "kernels":

$$A(x,y):=\sum_{j,k}\psi_j(x)(\psi_j|A|\psi_k)\overline{\psi_k(y)}.$$

Let us, moreover, consider that $\mathrm{v}_{12}=\mathrm{v}_{21}$, hence for the multiplication operator v (in this representation) one has: $\mathrm{v}(x,y)\equiv\mathrm{v}(y,x)$. The projections $E_{\{j\}}^t$ have now the kernels

$$E_{\{j\}}^t(x,y):=\sum_{j=1}^N\psi_j^t(x)\overline{\psi_j^t(y)},\quad\|\psi_j^t\|\equiv 1.$$

We obtain then, with $\psi_t^\varrho := \mathrm{u}_H(t,\varrho)\psi$, the usual **time–dependent Hartree–Fock equation**:

$$i\frac{d}{dt}\psi_t^\varrho(x) = \left[\mathrm{h}_0 + \int \mathrm{d}y\, E_{\{j\}}^t(y,y)\mathrm{v}(y,x)\right]\psi_t^\varrho(x) - \int \mathrm{d}y\, \mathrm{v}(x,y)E_{\{j\}}^t(x,y)\psi_t^\varrho(y). \tag{3.3.49}$$

We can insert into (3.3.49) $\psi_t^\varrho := \psi_j^t,\ j = 1,2,\dots N$, to obtain coupled nonlinear equations for ψ_j^t's. Evolution of the whole density matrices on $\mathcal{O}_\varrho(\mathfrak{U})$ is expressed by

$$i\frac{d}{dt}\varrho_t = [D_{\varrho_t}h_H^{Sl}, \varrho_t]. \tag{3.3.50}$$

Its stationary solutions $\varrho_t \equiv \varrho = N^{-1}E_{\{j\}}$ commute with $D_\varrho h_H^{Sl}$, hence the selfadjoint operator $D_\varrho h_H^{Sl}$ leaves the subspace $E_{\{j\}}\mathcal{H}$ of the Hilbert space $\mathcal{H}$ invariant: $D_\varrho h_H^{Sl}E_{\{j\}}\mathcal{H} \subset E_{\{j\}}\mathcal{H}$. This means that its restriction to the subspace $E_{\{j\}}\mathcal{H}$ can be diagonalized, and the basis $\{\psi_j : j \in \mathbb{N}\}$ can be chosen (in the point ϱ: the bases are point–dependent, according to their definition) such that the vectors $\psi_1, \psi_2, \dots, \psi_N$ are eigenvectors of $D_\varrho h_H^{Sl}$. Hence we have from (3.3.49) the corresponding eigenvalue equation, what is the (time independent) **Hartree–Fock equation**, cf. [175, §10]:

$$\left[\mathrm{h}_0 + \int \mathrm{d}y E_{\{j\}}(y,y)\mathrm{v}(y,x)\right]\psi_k(x) - \int \mathrm{d}y E_{\{j\}}(x,y)\mathrm{v}(x,y)\psi_k(y) = \epsilon_k\psi_k(x). \tag{3.3.51}$$

We have shown how the time dependent, as well as the stationary Hartree–Fock theory is described in the framework of our formalism.

3.3-e Nonlinear Schrödinger equation and mixed states

Let us give here another example of description of "a system" in the framework of NLQM. We shall show here that a traditional "nonlinear Schrödinger equation" [47, 11] can be included in the scheme of EQM. We shall partly proceed, in the following example (taken from [34]), in a heuristic way, by "plausible" formal manipulations; the necessary mathematical comments will be omitted here. This example will be also used to show, in a nontrivial concrete case, that the barycentre of a genuine mixture $\mu \in \mathcal{M}(\mathcal{S}_*)$ evolves under nonlinear evolution differently from the evolution of the elementary mixture ϱ being its initial barycentre, (2.1.32): $Tr(\varrho\mathrm{a}) := \int_{\mathcal{S}_*} Tr(\nu\mathrm{a})\mu(\mathrm{d}\nu),\ \forall \mathrm{a} \in \mathcal{L}(\mathcal{H})_s$.

Let us first recall that, for a given generator $Q \in C^\infty(\mathcal{S}_*, \mathbb{R})$, the Schrödinger equation (resp. the Liouville–von Neumann equation) for the flow φ^Q corresponding to the Poisson structure on $\mathcal{S}_*$ (cf. Subsection 2.1-c) can be written on $\mathcal{S}_*$ in the form (cf. (2.1.23) and (2.2.9))

$$i\,\frac{d}{dt}\varrho(t) = [D_{\varrho(t)}Q, \varrho(t)], \tag{3.3.52}$$

where

$$\varrho(t) \equiv \varphi_t^Q(\varrho) := \mathrm{u}_Q(t,\varrho)\varrho\mathrm{u}_Q(t,\varrho)^*, \quad \varrho(0) := \varrho, \tag{3.3.53}$$

and $\mathrm{u}_Q(t,\varrho)$ satisfies the equation (one can use alternatively, in the rôle of the "Hamiltonian operator" in the following equation, an operator of the form $D_{\varrho(t)}Q + \mathbf{f}^0(\varrho(t))$, with $\mathbf{f}^0(\varrho) \in \{\varrho\}'$, cf. Remark 2.1.18)

$$i\,\frac{d}{dt}\mathrm{u}_Q(t,\varrho) = D_{\varrho(t)}Q\cdot\mathrm{u}_Q(t,\varrho), \quad \mathrm{u}_Q(0,\varrho)\equiv 0. \tag{3.3.54}$$

The equation (3.3.52) can be rewritten for wave functions $\psi\in\mathcal{H}$, $\psi(t) := \mathrm{u}_Q(t,P_\psi)\psi \in \mathcal{H}$ (we set $D_\psi := D_{P_\psi}$):

$$i\,\frac{d}{dt}\psi(t) = D_{\psi(t)}Q\cdot\psi(t). \tag{3.3.55}$$

Let us take now $\mathcal{H}$=$L^2(\mathbb{R}^n)$ with $\langle\psi|\varphi\rangle := \int\overline{\psi}(x)\varphi(x)\,d^nx$. Let us write density matrices ϱ "in the x–representation" with a help of their operator kernels $\varrho(x,y)$:

$$[\varrho\psi](x) \equiv \int \varrho(x,y)\psi(y)\,d^ny, \quad \psi\in\mathcal{H}. \tag{3.3.56}$$

Projection operators P_ψ have their kernels $P_\psi(x,y) \equiv \|\psi\|^{-2}\psi(x)\overline{\psi}(y)$. Let the Hamiltonian function Q will be taken as the (unbounded) functional

$$Q(P_\psi) := Tr(P_\psi\cdot H_0) + \frac{\varepsilon}{\alpha+1}\int P_\psi(x,x)^{\alpha+1}d^nx, \tag{3.3.57}$$

with H_0 some selfadjoint (linear) operator on $L^2(\mathbb{R}^n)$, and $\alpha>0$. Let $t\mapsto P_{\psi(t)}, \psi(0) := \psi$ be any differentiable curve through $P_\psi\in P(\mathcal{H})$, and let $\dot{P}_\psi \in T_{P_\psi}P(\mathcal{H})$ be its tangent vector expressed by an operator according to equations (2.1.14). Then the (unbounded, nonlinear) Hamiltonian $D_\psi Q$ can be expressed by:

$$Tr(D_\psi Q\cdot\dot{P}_\psi) := \left.\frac{d}{dt}\right|_{t=0} Q(P_{\psi(t)}), \tag{3.3.58}$$

what leads to the corresponding form of "nonlinear Schrödinger wave–equation" for $\psi_t := \psi(t)$:

$$i\left[\frac{d}{dt}\psi_t\right](x) = [H_0\psi_t](x) + \varepsilon|\psi_t(x)|^{2\alpha}\psi_t(x), \quad \|\psi_t\|\equiv 1. \tag{3.3.59}$$

One possible extension of this nonlinear dynamics from $P(\mathcal{H})$ to the whole space $\mathcal{S}_*$ is obtained by "the substitution $\varrho\mapsto P_\psi$", i.e. by the choice of the Hamiltonian

$$Q(\varrho) := Tr(\varrho\cdot H_0) + \frac{\varepsilon}{\alpha+1}\int \varrho(x,x)^{\alpha+1}d^nx, \tag{3.3.60}$$

and the corresponding dynamics is then described by (3.3.52) with[87]

$$D_\varrho Q(\nu) \equiv Tr(D_\varrho Q\cdot\nu) \equiv Tr(\nu\cdot H_0) + \varepsilon\int \varrho(x,x)^\alpha\nu(x,x)\,d^nx. \tag{3.3.61}$$

[87] The notation $D_\varrho Q$ represents here the linear functional according to the standard definition of the Fréchet derivative, as well as its operator representative, cf. (2.1.14), and Definitions 2.2.13.

We shall compare the evolutions of the mixed states described by the same initial barycentre $\equiv$ density matrix ϱ

$$\varrho := \sum_j \lambda_j P_{\psi_j}, \quad \sum_j \lambda_j = 1,\ \lambda_j \geq 0. \tag{3.3.62}$$

for the two distinguished interpretations. The evolution of the elementary mixture ϱ is described by (3.3.52), while the evolution of the "corresponding" genuine mixture μ,

$$\mu := \sum_j \lambda_j \delta_{\psi_j}, \tag{3.3.63}$$

(where $\delta_\psi \equiv \delta_{P_\psi}$ is the Dirac measure concentrated on $P_\psi \in P(\mathcal{H}) \subset \mathcal{S}_*$) is described by

$$(t;\mu) \mapsto \mu_t \equiv \mu \circ \varphi^Q_{-t}. \tag{3.3.64}$$

This corresponds to such an evolution of the measure μ representing a state, when each of the vectors ψ_j entering into (3.3.63) evolves according to the equation (3.3.55).

Let us illustrate, by explicit calculation, the difference between time evolutions of the same initial density matrix considered in its two different interpretations.

Let us take the system with its above determined "extended" dynamics, and let us fix a non-trivial (i.e. $\lambda_j < 1,\ \forall j$) mixture ϱ of several vector states P_{ψ_j}, as in (3.3.62). Let us calculate the difference between the derivatives with respect to the time in $t = 0$ of the two evolutions: (i) of the *barycentre of the time evolved genuine mixture* $\sum_j \lambda_j \varphi^Q_t(P_{\psi_j})$, and (ii) of the *elementary mixture evolution* $\varphi^Q_t(\varrho)$. We shall calculate the right hand side of (3.3.52) for the two cases and take their difference. Let us write the kernel "in x–representation" of ϱ as the convex combination of the vector–state kernels:

$$\varrho(x,y) \equiv \sum_j \lambda_j \|\psi_j\|^{-2} \psi_j(x)\overline{\psi}_j(y). \tag{3.3.65}$$

The (symbolic) "kernel" of the Hamiltonian $D_\varrho Q$ can be written:

$$D_\varrho Q(x,y) = H_0(x,y) + \varepsilon\delta(x-y)\varrho(x,x)^\alpha.$$

Here, $\delta(\cdot)$ is the Dirac distribution on $\mathbb{R}^n$. We have to express the difference $\Delta_t^{\{\varrho\}}(x,y)$ between the kernels (in x–representation) of the operators

$$\sum_j \lambda_j [D_{\psi_j(t)}Q, P_{\psi_j(t)}], \text{ and } [D_{\varrho(t)}Q, \varrho(t)],$$

what expresses the difference between time derivatives of "the same density matrix" $\varrho = \sum \lambda_j P_{\psi_j}$ in the two interpretations. The linear operator H_0 does not contribute into this difference. The kernels of commutators entering into the calculation are (for all $\nu \in \mathcal{S}_*$) of the form

$$[D_\nu Q, \nu](x,y) = [H_0, \nu](x,y) + \varepsilon\nu(x,y)(\nu(x,x)^\alpha - \nu(y,y)^\alpha).$$

We can (and we shall) take all $\|\psi_j\| \equiv 1$. Let us denote

$$\chi_j^{\{\varrho\}}(x) := |\psi_j(x)|^{2\alpha} - \left(\sum_k \lambda_k |\psi_k(x)|^2\right)^{\alpha}.$$

Then the wanted difference at $t = 0$ is

$$\Delta_{\{\varrho\}}(x,y) := \Delta_0^{\{\varrho\}}(x,y) = \varepsilon \sum_j \lambda_j \psi_j(x) \overline{\psi}_j(y) (\chi_j^{\{\varrho\}}(x) - \chi_j^{\{\varrho\}}(y)). \tag{3.3.66}$$

By proving that the operator $\Delta_{\{\varrho\}}$ is not identical zero for all $\{\varrho\}$, we can prove nontrivial difference of the two time evolutions explicitly. This can be proved easily for $\lambda_1 := 1 - \lambda_2$, and for ψ_1, ψ_2 chosen to be specific three–valued (i.e. with two mutually distinct nonzero values) functions concentrated on disjoint compact subsets of $\mathbb{R}^n$: Each $\psi_j,\ (j = 1, 2)$ has its nonzero constant values on domains with different nonzero Lebesgue ($\mathrm{d}^n x$) measure.

Analogical examples could be constructed for, e.g. unbounded functions Q on dense domains of $\mathcal{S}_*$ expressed by the formula

$$Q(\varrho) := Tr(\varrho \cdot H_0) + \varepsilon \int_{\mathbb{R}^n} \mathcal{K}(\varrho(x,x)) \mathrm{d}^n x, \tag{3.3.67}$$

where $\mathcal{K} \in C^\infty(\mathbb{R}, \mathbb{R})$ can be chosen (in this abstract approach) rather arbitrarily. Such possibilities were mentioned (in a framework of Schrödinger equations (3.3.55) for wave functions) also in [11]; they include, e.g. the equations proposed in [18], and also WKB–equations.

Differentiation of (3.3.67) gives a formula for the corresponding "Hamiltonian" $D_\varrho Q$ (cf. Footnote 87):

$$D_\varrho Q(\nu) \equiv Tr(D_\varrho Q \cdot \nu) = Tr(H_0 \cdot \nu) + \varepsilon \int \mathcal{K}'(\varrho(x,x)) \cdot \nu(x,x) \mathrm{d}^n x,$$

or in terms of formal "operator kernels" (with $\mathcal{K}'(s) := \frac{\mathrm{d}\mathcal{K}(s)}{\mathrm{d}s},\ s \in \mathbb{R}$):

$$D_\varrho Q(x,y) = H_0(x,y) + \varepsilon \cdot \delta(x-y) \cdot \mathcal{K}'(\varrho(x,x)),$$

We did not consider any domain questions here: that would need more time and space. It seems, however, that the above formally given operators $D_\varrho Q$ could be correctly defined on (a dense subset of) $\mathcal{H}$, at least as symmetric operators.

3.3.14. **Remark** *(Koopmanism).* We have restricted, in the above considerations, our attention to the "Schrödinger picture", hence the algebra of observables $\mathcal{A}$ was not investigated: It could be chosen $\mathcal{A} = \mathcal{L}(\mathcal{H})$, as in linear QM. Its completion to an algebra of operator–valued functions could give a "linear extension" of the system. Let us note, however, that such extensions might be considered as a version of "noncommutative Koopmanism", cf. a Koopman formalism in CM (i.e. the "commutative" one), e.g. in [218, Chap.X.14], resp. [91, 152]. This can be expressed schematically as follows.

In Hamiltonian classical mechanics, the system is described by a $2n$–dimensional symplectic manifold $(M; \Omega)$, where the symplectic form Ω provides for ascription to Hamiltonian functions

(i.e. generators) $h \in C^\infty(M, \mathbb{R})$ the symplectic flows φ_t^h on M. Since the n–th exterior power of Ω, i.e. the Liouville volume $\wedge^n\Omega$ (corresponding to a measure μ_Ω on M) is conserved by φ^h, it is possible to introduce the Hilbert space $L^2(M, \mu_\Omega)$, where the flow φ^h (let us assume, that it is complete) is described by the continuous unitary group $U^h(t)$ defined by:

$$f_t(m) := [U^h(t)f](m) := f(\varphi_t^h(m)), \quad \text{for all } f \text{ in classes } f \in L^2(M, \mu_\Omega).$$

The selfadjoint (cf. [1, Proposition 2.6.14]) generator L_h of $U^h(t) \equiv \exp(-itL_h)$ is called the Liouville operator of the CM system. It is in fact the differential operator given by the Poisson bracket (up to domain questions concerning possible choice of f):

$$L_h f \equiv i\cdot\{h, f\}, \quad f \in D(L_h) \subset L^2(M, \mu_\Omega).$$

In this way, the nonlinear finite–dimensional Hamilton's equations are transformed formally into a linear Schrödinger–like "Liouville equation"

$$i\cdot\frac{d}{dt}f = L_h f$$

on infinite–dimensional Hilbert space. This Hilbert space description of CM is called the **Koopman formalism**. Let us note that here, in the "commutative case" of CM, the transformation $U^h(t)f$ of elements $f \in L^2(M, \mu_\Omega)$ of this "extended phase space" is uniquely given by the transformation φ_t^h of the phase space M, i.e. of the space of arguments of scalar–valued functions (the real values of $f(m)$ should stay real also for $f_t(m)$, due to their physical interpretation).

The situation in EQM can be considered in analogy with the preceding Koopman transition in CM, cf. also [35]: The (nonlinear) transformations $\tilde{\varphi}_t^Q$ of $\mathcal{S}_*$ are extended to one–parameter *-automorphism group τ^{Q} of a C^*-algebra $\mathcal{C}_{bs}$ (cf. Definition 2.3.13 and Theorem 2.3.16), or some of its subalgebras, what is a standard picture of *linear quantum theories*. The difference from the "commutative case" is, that $\mathcal{C}_{bs}$ is generated by functions on the "quantum phase space" $\mathcal{S}_*$ *with values in noncommutative C^*-algebra $\mathcal{L}(\mathcal{H})$*. Hence, to obtain the automorphism group τ^{Q} corresponding to nontrivial φ_t^Q, we have to introduce in a consistent way *also transformations of values* of these functions. These are, however, nonunique, and the nonuniqueness is pointed out, e.g., in Remark 2.1.18. ♡

3.4 "Macroscopic" Reinterpretation of EQM

It might be interesting from technical, as well as from physically intuitive point of view to show a simple way how our *nonlinear quantum–mechanical dynamical systems* considered in this paper (in the framework of EQM) can be considered as subsystems of infinite physical systems described in a framework of traditional (linear) quantum theory. A Hilbert space description of such a "large" system would necessitate, however, also usage of a nonseparable Hilbert space, e.g. the space of universal representation of a certain algebra $\mathcal{A}$ having uncountably many mutually inequivalent faithful representations (each one corresponding to a specific value of macroscopic variables), cf., e.g. [227, 196, 238]. It is not very comfortable to have all these representations simultaneously as subrepresentations in one nonseparable Hilbert space. A way to describe this situation in a more transparent way can be found in the framework of formalisms used in quantum field theory (QFT), or in theories of systems "with infinite number of degrees

of freedom", cf. [118, 119, 91, 42, 43, 120]. The main mathematical tool of these theories are C^*-algebras and their automorphism groups, cf. also [228, 196]. These theories are usually used to describe "thermodynamic systems" considered as infinitely large in the sense of intuitive notion of spatial extension, and also containing an infinite number of particles. Such an *infinite approximation to finite, but large systems* is conceptually acceptable and technically useful: It allows clear mathematical description of **macroscopic subsystems** of physical systems consisting of very large number of microscopic constituents – so large that any detailed practical (e.g. numerical) description and measurement of their states, taken even in any nontrivial a priori restricted precision for individual subsystems, is hopeless; their macroscopic subsystems consist, on the other side, of manageable sets of classically described parameters. Mathematically clear description of states and dynamics of such sets of classical parameters of quantal systems is up to now possible, however, only in the framework of "infinitely extended" systems. Its possibilities include, e.g., a description of phase transitions, cf. [42, 238].

We shall sketch briefly in this section a possibility, how to introduce a C^*-algebra $\mathcal{C}$ describing a "large" QM–system, "containing" in a certain sense the traditional observable algebra $\mathcal{L}(\mathcal{H})$ of a finite quantum system, as well as a commutative subalgebra of continuous complex–valued functions $C(M)$. This subalgebra $C(M)$ is interpreted as the C^*-algebra of a classical subsystem in such a way that $\mathcal{C}$ *is determined by these two subalgebras*, together with an infinite index set Π containing labels of the "elementary" (mutually equal) finite subsystems. This algebra $\mathcal{C}$ can be chosen so that it "contains" the C^*-algebra $\mathcal{C}^G$ (cf. Definition 2.3.3) describing any of the infinite number of equal "microscopic" subsystems composing the large system, as well as its extension by a classical system (= a "mean–field"); the later can describe collective influence of all the other subsystems onto the specified one, [130, 31, 33, 263, 264, 265]. In such systems, the dynamics can be determined by a sequence of local Hamiltonians. If a Lie group G is given so that it determines (via its unitary continuous representation) selfadjoint generators entering into the expressions of the local Hamiltonians of the (arbitrarily large but) finite subsystems, the spectrum space M of the classical subalgebra is the range $\mathcal{E}_{\mathbb{F}}$=$\mathbb{F}(\boldsymbol{\mathcal{D}}(\mathbb{F}))$ in the dual $\mathfrak{g}^*$ of the Lie algebra of G, cf. Definition 2.2.17; we can even have $\mathcal{E}_{\mathbb{F}} = \mathcal{S}_*$, for $G := \mathfrak{U} := \mathcal{U}(\mathcal{H})$. This approach can be considered either as a "phenomenological" introduction of a formal classical system to "complete" a given nonlinear quantum system to a linear one, or as a dynamical theory of a large system with a long range interaction. The dynamics is then a *-automorphism subgroup of the *-automorphism group of $\mathcal{C}$.

Since the C^*-algebra $\mathcal{C}^G$ is "essentially" (i.e. up to its completions in weaker–than–norm topologies) the tensor product $\mathcal{L}(\mathcal{H}) \otimes \mathcal{C}^G_{cl}$, it corresponds (in the sense of usual QM constructions – again "essentially") to a compound system consisting of a "standard (with finite number of degrees of freedom) QM–system" described by the algebra of observables $\mathcal{L}(\mathcal{H})$, and of a "classical subsystem" described by the commutative C^*-algebra $\mathcal{C}^G_{cl}$ which is isomorphic to the space of all complex valued continuous functions on the quantum phase space of elementary mixtures $\mathcal{S}_*$, $C(\mathcal{S}_*, \mathbb{C})$. Hence, the "nonlinear" EQM can be embedded as a subsystem theory to a linear quantum theory, cf. also Theorem 2.3.16. This linear theory can be considered in turn as a subtheory of a (nonrelativistic) quantum field theory (QFT), i.e. a theory of an infinite number of "standard" QM systems, [118, 91, 119, 42, 31]; this can be done not only kinematically, i.e. by construction of the sets of observables, but also by postulating a "microscopic" evolution in local subalgebras (given by local – linear – Hamiltonians *of mean–field type*, [130], depending on size of the local subsystems) and taking the thermodynamic limit, [31, 32]. As

a result of such a limiting procedure, it is obtained, besides the **quasilocal algebra** $\mathcal{A}$ of observables of arbitrarily large, but finite subsystems, also the **algebra of classical quantities** $\mathcal{C}_{cl}^G$ (belonging to the "algebra of observables at infinity", [223, 131, 42, 238]), without which the time evolution cannot be defined as a (semi–) group of transformations of an algebra of observables, [185, 31]. A "simplest" and a "most natural" quasilocal algebra of an infinite system in nonrelativistic QFT is $\mathcal{A} := \otimes_{p\in\Pi}\mathcal{L}(\mathcal{H}_p)$, as it is introduced below. If the "standard" QM system under consideration (the extension of which is the "considered" system described by EQM) is described in finite–dimensional Hilbert space, then we have, as the algebra of observables of the corresponding infinite system, directly the tensor product C^*-algebra $\boldsymbol{\mathcal{C} = \mathcal{A} \otimes \mathcal{C}_{cl}^G}$. [In the case of infinitedimensional Hilbert spaces $\mathcal{H}$, the algebra $\mathcal{C}$ contains $\mathcal{A} \otimes \mathcal{C}_{cl}^G$ as a (possibly proper) subalgebra, [27], (the fact, that $\mathcal{C} \neq \mathcal{A} \otimes \mathcal{C}_{cl}^G$ in this case is a consequence of weak, but not norm, continuity of corresponding unitary groups, resp. of unboundedness of generators, cf. also [42, 228] for some mathematical refinements).] In these cases of infinite systems, the elements of the classical subalgebra $\mathcal{C}_{cl}^G$ are naturally interpreted as **(global) intensive quantities** of the infinite system; hence, they correspond to **macroscopic variables** of this large quantal system.

Let us introduce an example of such macroscopic algebraic elements. The description is realized on, e.g., infinite tensor product $\mathcal{H}_\Pi := \otimes_{p\in\Pi}\mathcal{H}_p$ (with $\Pi :=$ an infinite index set labelling the "constituent microsystems"), [190], of equal copies of the Hilbert space $\mathcal{H} \equiv \mathcal{H}_p$, and the quasilocal algebra $\mathcal{A}$ is generated (via algebraic operations and norm limits) by the subalgebras (isomorphic to)

$$\mathcal{A}_\Lambda := \otimes_{p\in\Lambda}\mathcal{L}(\mathcal{H}_p), \quad \Lambda \subset \Pi,\ |\Lambda| < \infty, \tag{3.4.1a}$$

where $|\Lambda| :=$ *the number of elements in* $\Lambda \subset \Pi$, with the natural inclusions

$$\Lambda \subset \Lambda' \Rightarrow \mathcal{A}_\Lambda \subset \mathcal{A}_{\Lambda'}, \tag{3.4.1b}$$

acting on $\mathcal{H}_\Pi \equiv \mathcal{H}_\Lambda \otimes \mathcal{H}_{\Pi\setminus\Lambda}$ (the tensor product of Hilbert spaces is, for finite number of factors, associative, [190]) in the obvious way: Let us define an "identification" of the Hilbert spaces $\mathcal{H}_p,\ p \in \Pi$ with $\mathcal{H}$ by defining unitary mappings $\mathrm{u}_p,\ p \in \Pi$ of $\mathcal{H}_p$ onto $\mathcal{H}$. $\mathcal{A}_\Lambda$ is generated by elements $\mathrm{u}_p\mathrm{a}_p\mathrm{u}_p^{-1} \in \mathcal{L}(\mathcal{H})$, with a_p's "acting on the p-th factor" $\mathcal{H}_p,\ \forall p \in \Lambda$, i.e. for a vector $\Phi \in \mathcal{H}_\Pi$, one has

$$\Phi := \otimes_{p\in\Pi}\phi_p, \quad \mathrm{a}_q\Phi := (\mathrm{a}_q\phi_q) \otimes \left(\otimes_{p\in\Pi\setminus\{q\}}\phi_p\right). \tag{3.4.1c}$$

Now we choose any $X \equiv \mathrm{u}_pX_p\mathrm{u}_p^{-1} \in \mathcal{L}(\mathcal{H})_s$,[88] corresponding to "an observable of individual subsystems", and define

$$X_\Lambda := \frac{1}{|\Lambda|}\sum_{p\in\Lambda} X_p \in \mathcal{A}_\Lambda, \tag{3.4.1d}$$

Let a Lie group G be unitarily and continuously represented in $\mathcal{H}$ by $U(G)$, and let $U_p(G) := \mathrm{u}_p^{-1}U(G)\mathrm{u}_p$ be the corresponding action on each $\mathcal{H}_p$. Let the $X = X^*$ above be an arbitrary generator of $U(G)$. The set of "intensive observables" in $\mathcal{C}_{cl}^G$ is generated by *limits*

[88] We can choose also unbounded X's here, but in that case nontrivial domain questions should be considered, [27].

X_Π *in some topology*[89] of X_Λ, for $\Lambda \nearrow \Pi$, for all selfadjoint generators $X := X_\xi,\ \xi \in \mathfrak{g}$ of $U(G)$. These limits do not belong to $\mathcal{A}$; their introduction to the algebra of observables is, however, necessary for the "standard–type" description of dynamics with long–range interactions, [185], e.g. for MF–type evolutions, [31]. The quasilocal algebra $\mathcal{A}$ is canonically included into its second topological dual $\mathcal{A}^{**}$ which is in turn a W^*-algebra in a canonical way [227, 196]; the limits X_Π are then associated, [227, Definition 2.7.13], with a certain W^*-subalgebras of $\mathcal{A}^{**}$. On the algebra of functions $\mathcal{C}^G_{cl}$, Poisson brackets can be naturally defined. Then the mentioned mean–field type dynamics defined with a help of local Hamiltonians on $\mathcal{A}_\Lambda$'s does not leave, in the thermodynamic limit, the C^*-algebra $\mathcal{A}$ invariant (invariant with respect to such evolutions is, however, the "classical subalgebra" $\mathcal{C}^G_{cl}$).[90] The dynamics of the classical algebra $\mathcal{C}^G_{cl}$ is Hamiltonian (with respect to the mentioned Poisson structure), and *the dynamics of any local subsystem (described by $\mathcal{L}(\mathcal{H}_p)$) "essentially coincides" with some of the nonlinear dynamics described by Theorem 2.3.16, and by Definitions 2.3.13*: The considered dynamics of the infinite system is constructed as follows: Let us consider a function $Q \in C^\infty(\mathfrak{g}^*, \mathbb{R})$ as a Hamiltonian for the dynamics (with respect to the natural Poisson structure on $\mathfrak{g}^*$) described by the Poisson diffeomorphisms $\varphi^Q_t : \mathfrak{g}^* \to \mathfrak{g}^*$. The local Hamiltonians of the infinite quantum system are (consider, in the following formulas, Q as a polynomial in $F_j := F(\xi_j),\ F \in \mathfrak{g}^*$, for simplicity, otherwise cf. [86])

$$H_\Lambda := |\Lambda| \cdot Q(X_{\xi_1 \Lambda}, \ldots, X_{\xi_n \Lambda}), \tag{3.4.2a}$$

with $\{\xi_j, j = 1, \ldots, n\}$ a basis in $\mathfrak{g}$, and the "ordering" of operators is such, that all H_Λ's are selfadjoint.[91] Then the limiting dynamics

$$\tau^Q_t(\mathrm{x}) = (?) - \lim_{\Lambda \nearrow \Pi} \exp(-itH_\Lambda)\mathrm{x}\exp(itH_\Lambda), \quad \mathrm{x} \in \mathcal{A} \tag{3.4.2b}$$

can be defined, but only (for nonlinear Q) as a dynamics of the "extended" algebra containing also the classical (macroscopic) quantities, cf. [31, 33, 86]. If in an initial state $\omega = \omega_0 \in \mathcal{S}(\mathcal{A})$ the values $s - \lim_{\Lambda \nearrow \Pi} \pi_\omega(X_{\xi\Lambda})$ exist, then in its time evolved states $\omega_t := \omega \circ \tau^Q_t$ (if canonically extended to the states on $\mathcal{A}^{**}$) we have

$$\omega_t(X_{\xi\Pi}) \equiv Ad^*(g_Q(t, F_0))F_0(\xi), \qquad F_0(\xi) := \omega_0(X_{\xi\Pi}), \forall \xi \in \mathfrak{g}, \tag{3.4.2c}$$

where the cocycle $g_Q(t, F)$ is as in Proposition 2.3.10.

The subdynamics of the system "living" on any one of Hilbert spaces $\mathcal{H}_p,\ p \in \Pi$, say on $\mathcal{H}$ (if the index p is skipped), is given then by a unitary cocycle $\{U(g_Q(t, F)) : t \in \mathbb{R}, F \in \mathfrak{g}^*\}$[92], cf. Proposition 2.3.10:

$$\psi(t) \equiv U(g_Q(t, F))\psi(0),\ \psi(0) \in \mathcal{H}, \tag{3.4.3a}$$

[89] The topology, in which this limit exists is rather special: It cannot be norm–topology of $\mathcal{A}$ for X such, that their spectrum contains at least two points, [31]. The set $\{X_\Lambda : \Lambda \subset \Pi, |\Lambda| < \infty\}$ considered as a net in the von Neumann algebra $\mathcal{A}^{**}$ (:= the second topological dual of $\mathcal{A}$) has more than one cluster points, [31]. Hence, the topology on $\mathcal{A}^{**}$ has to be chosen weaker than its w^*–topology, [31].

[90] The rôle of the group G is here in choice of the topology mentioned in the preceding footnote, as well as in determination of dynamics: It is very useful especially in the presence of unbounded generators (i.e. local Hamiltonians).

[91] The ordering and symmetry of the operators is not here very important, since in the limit $\Lambda \nearrow \Pi$ the elements $X_{\xi_j \Pi}$ commute with all observables: They belong to (a subspace of) the centre of $\mathcal{A}^{**}$.

[92] These cocycles are nonunique, cf. Remark 2.1.18.

if the initial condition $\psi(0)$ is chosen such, that for the initial value of "the macroscopic field $F = F_0$", $F_0(\xi) := \omega_0(X_{\xi\Pi})$ at $t = 0$, it is fulfilled

$$F(\xi) := \langle F;\xi\rangle \equiv (\psi(0), X_\xi\psi(0)), \quad \forall\xi \in \mathfrak{g}. \tag{3.4.3b}$$

We can see that for

$$F_t := Ad^*(g_Q(t,F_0))F_0 \equiv \varphi_t^Q(F_0), \quad t \in \mathbb{R}, \tag{3.4.3c}$$

the following relation is valid:

$$F_t(\xi) \equiv (\psi(t), X_\xi\psi(t)), \text{ for } F_0(\xi) \equiv (\psi(0), X_\xi\psi(0)). \tag{3.4.3d}$$

After insertion of (3.4.3d) for F_t into the time dependent Schrödinger equation for $\psi(t)$ obtained from (3.4.3a) by differentiation, we obtain a nonlinear Schrödinger equation of EQM, describing now the evolution of a "small" subsystem of an infinite (linear) system of traditional QT, cf. (3.5.1). Hence, the subdynamics of an infinite quantum system with an automorphic (hence "linear") time evolution appears as nonlinear evolution in a NLQM.

We shall return to the relations (3.4.3) in Section 3.5, where we shall rewrite nonlinear QM–equations as a couple of equations: one nonlinear classical Hamilton's equation, and a linear time–dependent Schrödinger one.

3.5 Solution of Some Nonlinear Schrödinger Equations

Let $\dim G = n < \infty$, for a Lie group G. Let an arbitrary $U(G)$–system be given, with $\{\xi_j : j = 1,2,\ldots,n\}$ a basis of $\mathfrak{g} := Lie(G)$, and $F_j := F(\xi_j)$, $j = 1,2,\ldots,n$, $\forall F \in \mathfrak{g}^*$. Let $Q \in C^\infty(\mathcal{E}_\mathbb{F},\mathbb{R})$ be chosen. The selfadjoint generators of unitary one–parameter subgroups $U(\exp(t\xi_j))$ in the Hilbert space $\mathcal{H}$ are $X_j \equiv X(\xi_j)$. Let us consider the function Q as a function of n real variables $\{F_j\}$, i.e. $Q(F) \equiv Q(F_1,F_2,\ldots,F_n)$ is expressed by vector coordinates of the linear space $\mathfrak{g}^*$. Let there exists complete classical flow φ_t^Q on $\mathcal{E}_\mathbb{F}$, and let $F(t) \equiv \varphi_t^Q(F(0))$, with $F(0) := \mathbb{F}(P_{x_0}) \equiv \mathbb{F}(\mathrm{x}_0)$, where $x_0 \in \mathcal{D}^\omega(G)$.

We intend to look for continuously differentiable curves $t \mapsto x_t \in \mathcal{D}^\omega(G) \subset \mathcal{H}$, $x_{t=0} := x_0$ satisfying the following nonlinear Schrödinger equation:

$$i\frac{d}{dt}|x_t\rangle = \sum_{j=1}^n \frac{\partial}{\partial F_j} Q(\langle x_t|X_1|x_t\rangle, \langle x_t|X_2|x_t\rangle, \ldots, \langle x_t|X_n|x_t\rangle)\cdot X_j|x_t\rangle, \tag{3.5.1}$$

where the quantities $\langle x_t|X_j|x_t\rangle$ are inserted for the components F_j of $F \in \mathfrak{g}^*$ into

$$\frac{\partial Q(F)}{\partial F_j}, \quad F \in \mathfrak{g}^*.$$

It depends on the choice of the group G, and of its representation $U(G)$, and also on the choice of realization of the Hilbert space what a specific form this abstract differential equation will attain: It can be partial differential equation, and possibly also an integro–differential equation, and for nonlinear (in variables F_j) function Q it is always nonlinear. We shall show, however, that in all

these cases the equation (3.5.1) can be equivalently rewritten (for solutions lying in $\mathcal{D}^\omega(G)$) in a more transparent form of two connected problems:

(i) The problem of finding solutions of CM–problem for Hamilton's equations on the (generalized) classical phase space $\mathfrak{g}^*$ with its canonical Poisson structure, and with the Hamiltonian Q, leading to the Poisson flow φ_t^Q on $\mathfrak{g}^*$.

(ii) Then, after insertion into the expression of $d_F Q$ in the equation (3.5.1) for the argument F the appropriate solution (specified by the initial conditions) $F(t) := \varphi_t^Q(F(0))$, solving the obtained time dependent linear Schrödinger equation (resp. equivalently: solving (2.3.6) to find the cocycle $g_{\mathrm{Q}}(t, F)$).

Let us formulate and prove this result:

3.5.1. Theorem. *Let the conditions imposed above on the objects entering into the equation (3.5.1) are fulfilled. Then, for any $x_0 \in \mathcal{D}^\omega(G)$, there is a solution $\{x_t : t \in \mathbb{R}\}$ of (3.5.1) lying in $\mathcal{D}^\omega(G)$. It can be obtained as a solution of the time dependent linear equation*

$$i\,\frac{\mathrm{d}|x_t\rangle}{\mathrm{d}t} = \sum_{j=1}^{n} \frac{\partial Q(F(t))}{\partial F_j} X_j |x_t\rangle, \tag{3.5.2}$$

where $F(t)$ is the solution of the classical Hamilton's equations corresponding to the symplectic flow φ^Q on that $Ad^*(G)$–orbit which contains the initial classical state $F(0) := \mathbb{F}(P_{x_0})$. If $g_{\mathrm{Q}}(t, F(0))$ is the solution of the equations (2.3.6), then a solution $|x_t\rangle$ can be expressed by the relation:

$$|x_t\rangle \equiv U\big(g_{\mathrm{Q}}(t, F(0))\big)|x_0\rangle. \tag{3.5.3}$$

Each (global) solution of (3.5.1) satisfies also (3.5.2), with $F_t := F(t)$, $\langle x_t|X(\xi)|x_t\rangle \equiv F_t(\xi)$ satisfying the classical equations: $F_t \equiv \varphi_t^Q((F(0))$. ♣

Proof. $\mathcal{D}^\omega(G)$ is $U(G)$–invariant, $x_0 \in \mathcal{D}^\omega(G)$, hence also $U\big(g_{\mathrm{Q}}(t, F(0))\big) \in \mathcal{D}^\omega(G)$ for all $t \in \mathbb{R}$. The function $|x_t\rangle$ from (3.5.3) leads to the identity

$$\langle x_t|X(\xi)|x_t\rangle \equiv \varphi_t^Q F_\xi(P_{x_0}), \tag{3.5.4}$$

what is a consequence of (2.3.8), (2.3.7), and Definitions 2.2.17. Hence we have $F_j(t) \equiv \langle x_t|X_j|x_t\rangle$. Differentiation of (3.5.3) with a help of (2.3.5), (2.3.6), and of the group–representation property of U gives:

$$\begin{aligned} \frac{d}{dt} U\big(g_{\mathrm{Q}}(t, F(0))\big)|x_0\rangle &= \frac{d}{ds}\Big|_{s=0} U\big(g_{\mathrm{Q}}(s, F(t))\big) U\big(g_{\mathrm{Q}}(t, F(0))\big)|x_0\rangle \\ &= -i\,X(d_{F(t)}Q) U\big(g_{\mathrm{Q}}(t, F(0))\big)|x_0\rangle, \end{aligned} \tag{3.5.5}$$

what is the relation (3.5.2) with $|x_t\rangle$ from (3.5.3). Insertion of (3.5.4) into (3.5.2) gives (3.5.1), what proves that the function $|x_t\rangle$ from (3.5.3) solves the equation (3.5.1).

Let $|x_t\rangle$ be some global solution of (3.5.2) with $\|x_0\| = 1$, and fulfilling $F_j(0) \equiv \langle x_0|X_j|x_0\rangle$. Then it satisfies (3.5.4), what follows from the differentiation of $\langle x_t|X_j|x_t\rangle$ with a help of (3.5.2), (2.2.14), and (2.2.17). Consequently, this $|x_t\rangle$ satisfies also (3.5.1).

Conversely, let $|x_t\rangle$ be some global solution of (3.5.1). Again by differentiation of $F_j(t) := \langle x_t|X_j|x_t\rangle$, one obtains, as above, the identity (cf. also Notation 3.2.2)

$$\frac{d}{dt}F_k(t) \equiv \sum_{j=1}^{n} \frac{\partial Q(F(t))}{\partial F_j}\{F_j, F_k\}(\mathbb{F}(\mathbf{x}_t)),$$

with $F(t) \equiv \mathbb{F}(\mathbf{x}_t)$. This means that each global solution of (3.5.1) fulfills also the equations (3.5.2) and (3.5.4). □

We can see that all the norm–differentiable solutions of (3.5.1) conserve their norms: $\langle x_t|X_j|x_t\rangle \equiv \langle x_0|X_j|x_0\rangle$, since the generator on the right side of (3.5.2) is selfadjoint for all $t \in \mathbb{R}$ (because it belongs to the generators of $U(G)$). It follows also, that for $\|x_0\| = 1$, one has $F_j(t) := Tr(P_{x_t}X_j) \equiv \langle x_t|X_j|x_t\rangle$.

3.6 On an Alternative Formulation of NLQM

It might be fair and also useful to look onto another, a rather popular formulation of general NLQM (on the set of pure states $P(\mathcal{H})$) published by Weinberg in [273]. His proposal contained some ambiguities, and also it had some physically unacceptable consequences discussed already in literature, cf. e.g. [106, 272].[93] Its mathematical framework can be, however, consistently presented if it is restricted to $P(\mathcal{H})$. In that case, it is in fact equivalent to our formulation of NLQM on $P(\mathcal{H})$.

Weinberg mostly worked with finite dimensional Hilbert spaces, and he used formalism depending on components in a chosen basis of Hilbert space. We shall try to reformulate the Weinberg's theory [273] in a coordinatefree way, but simultaneously preserving, as far as possible, the main ideas[94] of the original formulation.

Let the *nonlinear observables (and generators)* be differentiable functions $a, b, \ldots$, of two variables $x \in \mathcal{H}$, and $y^* \in \mathcal{H}^*$ from the Hilbert space $\mathcal{H}$ and its dual: $(x; y^*) \mapsto a(x, y^*) \in \mathbb{C}$. It is assumed that the functions $a, \ldots$, are homogeneous of the first degree in each of the variables, i.e.

$$a(\lambda x, y^*) \equiv a(x, \lambda y^*) \equiv \lambda a(x, y^*),\ \forall \lambda \in \mathbb{C} \setminus \{0\}. \tag{3.6.1}$$

Another requirement is the "reality condition": $a(x, x^*) \in \mathbb{R}, \forall x \in \mathcal{H}$. A specific "observable" is $n(x, y^*) := y^*(x) \equiv (y, x)$; the observables $a, b, \ldots$, corresponding to traditional "observables" of QM determined by selfadjoint operators $A, B, \ldots$, are of the form $a(x, y^*) \equiv (y, Ax), \ldots$, where $\lambda(y^*) \equiv (\overline{\lambda}y)*$ (since the bijective mapping $x \mapsto x^*$ of $\mathcal{H}$ onto $\mathcal{H}^*$ according to the Riesz lemma is antilinear). In Ref. [273], only values $a(x, x^*), \ldots$, of observables $a, b, \ldots$, in "diagonal" points $(x; x^*) \in \mathcal{H} \times \mathcal{H}^*$ corresponding to a specific vector

[93] In the presented formulation of EQM, some of these "unacceptable consequences" remain valid, as it was discussed, e.g. , in Subsection 2.1-e. We have overcome here, as the present author believes, at least the difficulties connected with the inappropriate work with mixed states and subsystems (resp. composed systems; these we do not try to introduce here as a general concept) in the Weinberg's papers. We also proposed a consistent interpretation scheme, in which possible ambiguities are well understood.

[94] We mean here mainly the *formalism determining – mathematical ideas*, as they were understood by the present author.

state $x \in \mathcal{H}$ are used, except of the instants when $a, b. \ldots$, are differentiated according to only one of the variables x, y^* (in points with $y = x$). The Fréchet differentials are

$$\begin{aligned} D_x a &:= \left.\frac{\partial a(x, y^*)}{\partial x}\right|_{y=x} \in \mathcal{H}^*, \\ D_x^* a &:= \left.\frac{\partial a(y, x^*)}{\partial x^*}\right|_{y=x} \in \mathcal{L}(\mathcal{H}^*, \mathbb{C}) = \mathcal{H}. \end{aligned} \tag{3.6.2}$$

Then we can write the nonlinear Schrödinger equation, [273, (b):Eq.(2.12)], with a generator $h \equiv h(x, y^*)$ in the form:

$$i\,\frac{\partial x(t)}{\partial t} = D^*_{x(t)} h. \tag{3.6.3}$$

As concerns the interpretation, let us only mention that the expectation value of an observable *a* in the state described by a vector $x \neq 0$ is expressed by the number

$$\mathrm{a}(\mathbf{x}) := \frac{a(x, x^*)}{n(x, x^*)}. \tag{3.6.4}$$

This is in accordance with our interpretation from (2.3.4), and (2.3.9): The function a in (3.6.4) depends on elements $\mathbf{x} \in P(\mathcal{H})$ only; it can be identified with one of our observables and/or generators restricted to $P(\mathcal{H})$. In the case if finite–dimensional $\mathcal{H}$, any a in (3.6.4) can be written as a function $\tilde{a}(\mathrm{f}_1, \mathrm{f}_2, \ldots, \mathrm{f}_n)$ of a finite number of quantities $\mathrm{f}_j(\mathbf{x})$ given by an equation:

$$\mathrm{f}_j(\mathbf{x}) := \mathrm{f}_j(x, x^*), \text{ with } \mathrm{f}_j(x, y^*) := \frac{(y, \mathrm{X}_j x)}{(y, x)}, \tag{3.6.5}$$

with $\mathrm{X}_j \in \mathcal{L}(\mathcal{H})_s$. In the finite dimensional case, we can insert into the nonlinear Schrödinger equation (3.6.3) the function

$$h(x, y^*) := n(x, y^*) Q(\mathbf{f}(x, y^*)), \quad \mathbf{f} := (\mathrm{f}_1; \mathrm{f}_2; \ldots; \mathrm{f}_n),$$

where we write Q instead of $\tilde{h}$ from the text above (3.6.5). Let also $\mathbb{F}(\mathbf{x}) :=\mathbf{f}(x, x^*)$, with components $F_j(\mathbf{x})$. An easy computation then gives:

$$\begin{aligned} D^*_{x(t)} h \equiv & \sum_{j=1}^{n} \frac{\partial Q\big(\mathbb{F}(\mathbf{x}(t))\big)}{\partial F_j} X_j |x(t)\rangle \\ & + \left(Q\big(\mathbb{F}(\mathbf{x}(t))\big) - \sum_{j=1}^{n} \frac{\partial Q\big(\mathbb{F}(\mathbf{x}(t))\big)}{\partial F_j} F_j(\mathbf{x}(t)) \right) |x(t)\rangle, \end{aligned} \tag{3.6.6}$$

what expresses the right hand side of the nonlinear Schrödinger equation written in the form (3.6.3) in accordance with Ref. [273]. The notation in (3.6.6) literally corresponds to that introduced in Section 2.3, because the selfadjoint operators $\{X_j : j = 1, \ldots, n\}$ in finite dimensional Hilbert space $\mathcal{H}$ generate a Lie algebra of operators $dU(\mathfrak{g})$ of a (finite–dimensional)

simply connected Lie group G with representation $U(G)$ in $\mathcal{H}$ generated by (integration of) $dU(\mathfrak{g}) \ni X_j$, and $\mathbb{F}$ is then the corresponding momentum mapping, cf. also [32, Sec.IV]. Direct inspection shows that if $t \mapsto \mathbf{x}(t) := P_{x(t)}$ corresponds to a solution of (3.6.3), then the function $t \mapsto \mathbb{F}(\mathbf{x}(t))$ is solution of Hamilton's equations on $\mathfrak{g}^*$ with Poisson brackets

$$\{F_j, F_k\}(\mathbb{F}(\mathbf{x})) := i\,Tr(P_x[X_j, X_k]),$$

with the Hamiltonian function $\mathbf{F} \mapsto Q(\mathbf{F})$, in correspondence with the canonical Poisson structures on $P(\mathcal{H})$ and on $\mathfrak{g}^* := Lie(G)^*$. Let us denote

$$\alpha(\mathbf{x}(t)) := Q\big(\mathbb{F}(\mathbf{x}(t))\big) - \sum_{j=1}^{n} \frac{\partial Q\big(\mathbb{F}(\mathbf{x}(t))\big)}{\partial F_j}\, F_j(\mathbf{x}(t)).$$

Since this is a real numerical function of time (for a given solution $x(t),\ t \in \mathbb{R}$), any solution $|x(t)\rangle$ of (3.6.3) can be transformed into a solution $|x_t\rangle$ of the corresponding equation of the form (3.5.1) by a gauge–transformation, namely by multiplication of the vectors $|x(t)\rangle$ by a phase factor $\exp(i\,\beta(t, x_0))$:

$$|x_t\rangle \equiv \exp(i\,\beta(t, x_0))|x(t)\rangle,$$

where the phase $\beta(t, x_0)$ is a solution of the equation

$$\frac{\mathrm{d}\beta}{\mathrm{d}t} = \alpha(\mathbf{x}(t)), \tag{3.6.7}$$

corresponding to the initial condition $x(0) := x_0$. The two solutions, $|x_t\rangle$ of (3.5.1), and $|x(t)\rangle$ of (3.6.3), corresponding to the same initial condition $x(0) = x_0$ are mutually physically indistinguishable. A comparison of the "Weinberg type" nonlinear Schrödinger equations with that of geometric formulation of QM (essentially identical with the ours one) was presented in [11].

A Selected Topics of Differential Geometry

We shall give in this appendix a brief review of some basic definitions, illustrative examples, and some facts (theorems) concerning the elements of differential geometry and some of related topics.

A.1 Introduction to topology

The general concept of topology is basic for mathematical description of continuity, stability, connectedness, compactness, etc. This concept is useful for clear understanding of several issues of this paper.

Let, for a given set $\mathcal{X}$ the collection of all its subsets (i.e. **the power set of $\boldsymbol{\mathcal{X}}$**) be denoted by $\mathcal{P}(\mathcal{X})$.

A.1.1. Definitions (Topology).

(i) A **topology** *on the set $\mathcal{X}$ is a collection $\mathcal{T} \subset \mathcal{P}(\mathcal{X})$ of subsets $\mathcal{U}, \mathcal{V}, \cdots \subset \mathcal{X}$ satisfying:*

t1. Union of an arbitrary set of members of $\mathcal{T}$ also belongs to $\mathcal{T}$.

t2. Intersection of an arbitrary finite set of members of $\mathcal{T}$ is a member of $\mathcal{T}$.

t3. The empty set $\emptyset$, as well as the whole $\mathcal{X}$, are members of $\mathcal{T}$.

The elements $\mathcal{U} \in \mathcal{T}$ of the given topology $\mathcal{T}$ are the **open sets** *(in this specific topology!). The complements $\mathcal{X} \setminus \mathcal{U}$ are called the* **closed sets***. Topologies $\{\mathcal{T}_\gamma\}$ are naturally ordered by inclusion: $\mathcal{T}_1 \prec \mathcal{T}_2$ iff $\mathcal{T}_1 \subset \mathcal{T}_2$, and, in this case, $\mathcal{T}_2$* **is stronger ($\equiv$ finer) than** *$\mathcal{T}_1$ (also: $\mathcal{T}_1$* **is weaker ($\equiv$ coarser) than** *$\mathcal{T}_2$). The set of all possible topologies on $\mathcal{X}$ is a directed set; it is, moreover, a complete lattice (i.e. each subset has supremum and infimum). The strongest of all topologies is the* **discrete topology** *for which each subset of $\mathcal{X}$ is both open and closed (=:* **clopen sets***). The weakest topology is the trivial one: only open (and closed) subsets of $\mathcal{X}$ are the empty set $\emptyset$, and the whole space $\mathcal{X}$. For any subset $\mathcal{M} \subset \mathcal{X}$, there is unique minimal (with respect to the set inclusion) closed subset $\overline{\mathcal{M}}$ of $\mathcal{X}$ containing $\mathcal{M}$, called the* **closure of $\boldsymbol{\mathcal{M}}$: $\boldsymbol{\overline{\mathcal{M}}}$***; as well as there is a unique maximal open subset of $\mathcal{X}$ contained in $\mathcal{M}$, called the* **interior of $\boldsymbol{\mathcal{M}}$***, denoted by $\boldsymbol{\mathcal{M}^\circ}$. If the closure of $\mathcal{M}$ is the whole space $\mathcal{X}$, then $\mathcal{M}$* **is dense in** *$\mathcal{X}$. Given an arbitrary subsystem $S \subset \mathcal{P}(\mathcal{X})$, there is a minimal topology on $\mathcal{X}$ containing S; it is the* **topology generated by** *S. The couple $(\mathcal{X};\mathcal{T})$ is a* **topological space***, or also the* **topological space $\boldsymbol{\mathcal{X}}$***. If cardinality of a dense subset of $\mathcal{X}$ is at most countable, then the* **topological space $\boldsymbol{\mathcal{X}}$ is separable***. Any subset $\mathcal{M}$ of $\mathcal{X}$, such that $x \in \mathcal{M}^\circ$, is a* **neighbourhood of $\boldsymbol{x \in \mathcal{X}}$***.*

(ii) Any subset $\mathcal{Y} \subset \mathcal{X}$ of the topological space $(\mathcal{X};\mathcal{T})$ is endowed with the **relative (**or **induced) topology** *$\mathcal{T}_\mathcal{Y} := \{\mathcal{Y} \cap \mathcal{V} : \mathcal{V} \in \mathcal{T}\}$. With this topology, the subset $\mathcal{Y}$ is a* **topological subspace** *of $\mathcal{X}$.*

(iii) A topological space is **disconnected** *iff it is union of two nonempty disjoint open (equivalently: closed) subsets. In the opposite case it is* **connected***. The union of all connected topological subspaces each of which contains the point ϱ is the* **connected component** *of the point $\varrho \in \mathcal{X}$.*

(iv) A topological space $(\mathcal{X};\mathcal{T})$ is **compact** *iff* for any *collection $\{\mathcal{V}_j : j \in J\} \subset \mathcal{T}$ (with J an arbitrary index set) covering $\mathcal{X}$: $\cup_{j\in J}\mathcal{V}_j = \mathcal{X}$, there exists a* **finite subcovering***, i.e. there is a*

finite subset $K \subset J$ such, that $\cup_{j\in K}\mathcal{V}_j = \mathcal{X}$. A subset $\mathcal{Y} \subset \mathcal{X}$ of any topological space $(\mathcal{X};\mathcal{T})$ is compact, if it is compact in the relative topology.

(v) Topologies used usually in analysis are **Hausdorff**, *i.e. for any two distinct points ϱ, σ of the considered topological space there are disjoint open sets $\mathcal{V}_\varrho, \mathcal{V}_\sigma$ each containing one of the chosen points. This is one of the types of possible topologies which* **separate points** *of topological spaces, cf., e.g. [148]. In Hausdorff spaces, each one–point set is closed, and any compact subset is also closed. A Hausdorff space $\mathcal{X}$ is* **locally compact** *iff each point $x \in \mathcal{X}$ has a compact neighbourhood.*

(vi) Let a topological space $\mathcal{X}$ be decomposed into a collection of its (mutually disjoint nonempty) subsets: $\mathcal{X} = \cup_j N_j : j \in J$, the decomposition being denoted by N. Let us form the **factor–space** *$\mathcal{X}/N$ (resp. also the* **quotient–space***) the points of which are the subsets N_j (it is essentially equivalent to the index set J – as a set). Let p_N be the natural projection of $\mathcal{X}$ onto $\mathcal{X}/N$, $x \in N_j \Leftrightarrow p_N(x) = N_j$. The natural topology on $\mathcal{X}/N$,* **the factor–topology** *(resp.* **quotient–topology***), is the strongest topology for which p_N is continuous, cf. the Definitions A.1.2.*

(vii) Let $\mathcal{X}$, $\mathcal{Y}$ be topological spaces, $\mathcal{X} \times \mathcal{Y}$ be their Cartesian product, i.e. the set of ordered couples $(x;y), x \in \mathcal{X}, y \in \mathcal{Y}$. The **product topology** *is generated on this space by Cartesian products of all the couples of their open subsets $\mathcal{U} \times \mathcal{V}$,$\mathcal{U} \in \mathcal{T}_\mathcal{X}, \mathcal{V} \in \mathcal{T}_\mathcal{Y}$. This concept is uniquely extended to products of any finite numbers of topological spaces (by associativity of the Cartesian product).* $\diamondsuit$

The perhaps most important "topological" concept is that of continuity.

A.1.2. Definitions (Continuity).

(i) Let f *be a mapping (i.e. a function) from a topological space $(\mathcal{X}, \mathcal{T}_\mathcal{X})$, into $(\mathcal{Y}, \mathcal{T}_\mathcal{Y})$,* $\mathrm{f} : \mathcal{X} \mapsto \mathcal{Y}$. *Then* f *is* **continuous** *iff* $\mathrm{f}^{-1}(\mathcal{U}) \in \mathcal{T}_\mathcal{X}, \forall \mathcal{U} \in \mathcal{T}_\mathcal{Y}$.

(ii) The mapping $\mathrm{f} : \mathcal{X} \to \mathcal{Y}$ *is continuous in the point $x \in \mathcal{X}$ iff for any open neighbourhood $\mathcal{U}$ of* $\mathrm{f}(x) \in \mathcal{Y}$, $\mathrm{f}(x) \in \mathcal{U}$, *there is an open neighbourhood $\mathcal{V}$ of x, $\mathcal{V} \ni x$ such, that its image under* f *is contained in $\mathcal{U}$:* $\mathrm{f}(\mathcal{V}) \subset \mathcal{U}$.

(iii) Any continuous bijection f of a topological space $\mathcal{X}$ onto another topological space $\mathcal{Y}$ such, that its inverse f^{-1} is also continuous is a **homeomorphism** *of the spaces $\mathcal{X}$ and $\mathcal{Y}$. Spaces mutually homeomorphic are indistinguishable from the topological point of view - they are* **topologically isomorphic**.

(iv) Any given set of functions $\{f_j : \mathcal{X} \to \mathcal{Y}_j, j \in J\}$, where $\mathcal{Y}_j$ are arbitrary topological spaces, determines a unique topology on $\mathcal{X}$ such, that it is the weakest topology for which all the functions $\{f_j, j \in J\}$ are continuous. This topology on $\mathcal{X}$ is the **topology determined by the functions** *$\{f_j, j \in J\}$.* $\diamondsuit$

A.1.3. **Examples** *(Various topologies).* We shall introduce here some examples of topologies.

(i) The topology on a metric space ($\mathcal{X}$,d) generated by the **open balls** $B_{\varepsilon,x} := \{y \in \mathcal{X} : d(x,y) < \varepsilon\}$ $(x \in \mathcal{X}, \varepsilon > 0)$ is the **metric topology**. For $\mathcal{X} := \mathbb{R}$, the metric topology given by the **distance function** $d(x,y) \equiv |x - y|$ is the "usual topology". The metric topology is always Hausdorff. The "usual" topology on $\mathbb{R}^n$ is the product topology of n copies of the spaces $\mathbb{R}$

with their "usual topologies". The complex line $\mathbb{C}$ is considered as topologically equivalent (i.e. homeomorphic) to $\mathbb{R}^2$.

(ii) Very different kind of topology on $\mathbb{R}$ is the following one: Let the open sets on $\mathbb{R}$ be $\{x \in \mathbb{R} : x < a\},\ \forall a \in \mathbb{R}$. The obtained topology is not Hausdorff. Observe that there are no nonempty mutually disjoint open subsets now. This implies that the only continuous real-valued functions (where the image-space $\mathbb{R}$ is endowed with the "usual topology") on $\mathbb{R}$ endowed with this topology are constants.

(iii) Consider the identity mapping $id_{\mathbb{R}} : x \mapsto x, \forall x \in \mathbb{R}$; it is discontinuous if the image-topology is finer than the "domain-topology". E.g., the identity mapping on an arbitrary set $\mathcal{X}$ from discrete to arbitrary topology is continuous, and the inverse mapping is also continuous only in the case, if both copies of the mapped set are endowed by the same (now discrete) topology.

(iv) Let $\mathcal{B}(I)$ be any set of real–valued functions (i.e. the image–space $\mathbb{R}$ is endowed with the "usual" topology) on the unit interval $I := \{r \in \mathbb{R} : 0 \leq r \leq 1\}$. It generates the weakest topology on I such, that all functions $\mathrm{f} \in \mathcal{B}(I)$ are continuous. Consider, e.g. the cases, where $\mathcal{B}(I)$ contains also some characteristic functions of subintervals of I: such a topology makes the interval I disconnected. $\heartsuit$

We are often working with linear spaces endowed with some topologies. Finite–dimensional spaces $\mathbb{R}^{3N}$ of particle configurations, as well as infinite–dimensional spaces of functions with values in linear spaces (with pointwise additions), are linear spaces in a natural way . To be useful in dealing with linear mappings, topologies introduced on such spaces should be in a "correspondence" with the existing linear structures on them.

A.1.4. Definitions (Topological linear spaces).

(i) Let $\mathcal{L}$ be a linear space over $\mathbb{K} \in \{\mathbb{R}; \mathbb{C}\}$, where $\mathbb{K}$ is considered with its canonical (:="usual") topology. Let a topology $\mathcal{T}$ on $\mathcal{L}$ be given. Let us consider the multiplication of elements $x \in \mathcal{L}$ by scalars $\lambda \in \mathbb{K}$ as mapping from the topological product–space $\mathbb{K} \times \mathcal{L}$ into $\mathcal{L} : (\lambda; x) \mapsto \lambda x \in \mathcal{L}$, and the addition: $(x; y)(\in \mathcal{L} \times \mathcal{L}) \mapsto x + y(\in \mathcal{L})$, also with the product–topology of $\mathcal{L} \times \mathcal{L}$. Then the topological space $(\mathcal{L}; \mathcal{T})$ is a **topological linear space (=t.l.s.)** *iff the addition and multiplication by scalars are (everywhere) continuous functions. This allows us to define any topology of a topological linear space on $\mathcal{L}$ by giving just all the open sets containing an arbitrarily chosen point (e.g. x= 0).*

(ii) Most often used in applications are such t.l.s. which are Hausdorff, and their topology is determined by **seminorms***: T.l.s. $\mathcal{L}$ is* **locally convex space (= l.c.s.)** *iff its topology is determined by a set $\{p_j : j \in J\}$ of mappings (=seminorms) $p_j : \mathcal{L} \to \mathbb{R}_+,\ x \mapsto p_j(x) \geq 0$ such that*

$$p_j(\lambda x) \equiv |\lambda| p_j(x),\ p_j(x + y) \leq p_j(x) + p_j(y),\ \forall x, y \in \mathcal{L},\ \forall j \in J.$$

It is supposed (to be the topology Hausdorff) that the set of seminorms is "sufficient", resp. that it **separates points***:*

$$\forall x \in \mathcal{L}, (x \neq 0)\, \exists j \in J : p_j(x) > 0.$$

The topology is the weakest one for which all the seminorms are continuous. On finite–dimensional linear spaces there is just one such a **l.c. (locally convex)–topology**.

(iii) If the topology of l.c.s. $\mathcal{L}$ is determined by just one seminorm p_α, it is necessarily a **norm** *(i.e. $p_\alpha(x) = 0 \Rightarrow x = 0$). A norm topology is naturally metric topology with the distance function $d(x,y) := p_\alpha(x-y)$. If the space is complete as the metric space, $\mathcal{L}$ is called a* **Banach space**, *simply* **B–space**. *The norm of $x \in \mathcal{L},\ p_\alpha(x)$ will be usually denoted by $\|x\|_\alpha$, where the index α can distinguish different norms on $\mathcal{L}$.*

(iv) Let $\mathcal{L}$ be a B–space, its norm being denoted $\|\cdot\|$. A linear mapping $\varrho :\ x(\in \mathcal{L}) \mapsto \varrho(x) \equiv \langle \varrho; x\rangle \in \mathbb{K}$ is a **linear functional** *on $\mathcal{L}$. On general (infinite–dimensional) B–spaces, there are also discontinuous linear functionals. The set of all* **continuous linear functionals on** *$\mathcal{L}$ is denoted by $\mathcal{L}^*$, and it is called the* **topological dual (space) of** *$\mathcal{L}$. In $\mathcal{L}^*$, there is a canonical norm–topology determined by that of $\mathcal{L}$:*

$$\|\varrho\| \equiv \sup\left\{\frac{|\varrho(x)|}{\|x\|} : 0 \neq x \in \mathcal{L}\right\}, \quad \varrho \in \mathcal{L}^*.$$

With this norm, $\mathcal{L}^*$ is a B–space. Its dual space $\mathcal{L}^{**}$ contains, as a canonically isometrically embedded subspace, the original B–space $\mathcal{L}$: $x \in \mathcal{L}$ is interpreted as the mapping $\varrho \mapsto \varrho(x) \equiv \langle \varrho; x\rangle$, i.e. an element of $\mathcal{L}^{**}$.

(v) Let $\mathcal{M}$ be a linear set of linear functionals on a linear space $\mathcal{L}$. Assume, that $\mathcal{M}$ separates points of $\mathcal{L}$, i.e. $\varrho(x) = 0, \forall \varrho \in \mathcal{M} \Rightarrow x = 0$. The topology on $\mathcal{L}$ determined by all $\varrho \in \mathcal{M}$ is called the **$\mathcal{M}$–weak topology on** *$\mathcal{L}$, or the $\sigma(\mathcal{L},\mathcal{M})$–topology. If we consider $\mathcal{L}$ as linear functionals on $\mathcal{M}$, and if $\mathcal{L}$ separates points of $\mathcal{M}$, then we have also the $\sigma(\mathcal{M},\mathcal{L})$–topology on $\mathcal{M}$. If $\mathcal{L}$ is a B–space, the $\sigma(\mathcal{L},\mathcal{L}^*)$–topology is the* **weak topology on** *$\mathcal{L}$. The $\sigma(\mathcal{L}^*,\mathcal{L})$–topology on the dual space $\mathcal{L}^*$ is called the w^*–***topology on*** *$\mathcal{L}^*$.The closed unit ball $B_1 := \{\varrho \in \mathcal{L}^* : \|\varrho\| \leq 1\}$ of the dual to a B–space $\mathcal{L}$ is compact in the w^*–topology (=Banach-Alaoglu theorem).* $\diamondsuit$

A.2 Elements of differentiation on Banach spaces

The differential calculus of mappings $f : \mathfrak{T} \to \mathfrak{R}$ between two Banach spaces $\mathfrak{T}$, and $\mathfrak{R}$ is largely similar to calculus in finite dimensional spaces. A formal difference appears because of coordinate free notation, what is useful also in the case, when the B-spaces $\mathfrak{T}, \mathfrak{R}$ are finite-dimensional. We shall need mainly the case of an infinite dimensional $\mathfrak{T}$ (e.g. $\mathfrak{T} = \mathfrak{T}_s$) and of the one dimensional $\mathfrak{R} = \mathbb{R}$. Let us define the **Fréchet differential** $D_\nu f$ at the point $\nu \in \mathfrak{T}$ of an $\mathfrak{R}$-valued function $f : \mathfrak{T} \to \mathfrak{R}$:

A.2.1. Definitions.

(i) Let $\mathfrak{T}, \mathfrak{R}$ be Banach spaces with (arbitrary) norms $\|\cdot\|$ (equally denoted for both spaces) leading to their Banach-space topologies. Let $U \subset \mathfrak{T}$ be an open subset containing ν. The Fréchet differential, *resp. the* **Fréchet derivative**, *of f at the point $\nu \in \mathfrak{T}$ is the unique (if it exists) continuous linear mapping $D_\nu f : \mathfrak{T} \to \mathfrak{R},\ \eta \mapsto D_\nu f(\eta)$ $(\forall \eta \in \mathfrak{T})$ satisfying*

$$\lim_{\eta \to 0} \|\eta\|^{-1} \|f(\nu+\eta) - f(\nu) - D_\nu f(\eta)\| = 0. \tag{A.2.1a}$$

If the derivative $D_\nu f$ exists, the function f is **differentiable at the point** *ν. If* **the derivative** *$Df : \nu \mapsto D_\nu f \in \mathcal{L}(\mathfrak{T}, \mathfrak{R})$ exists in U, f is* **differentiable on** ***U****; if, in that case, $U = \mathfrak{T}$, then f is called* Fréchet differentiable function, *or just: f is* ***F*–differentiable**.

(ii) Let, by the above notation, the derivative of $t \mapsto f(\nu + t\eta)$*:*

$$D_{t=0} f(\nu + \cdot\eta)(1) \equiv \left.\frac{\mathrm{d}f(\nu + t\eta)}{\mathrm{d}t}\right|_{t=0} =: Df(\nu, \eta), \quad \forall \eta \in \mathfrak{T}, \tag{A.2.1b}$$

exists for all $\nu \in U$*. Then* f *is* **G–differentiable**, $Df(\nu, \cdot)$ *is* **Gateaux derivative** *of* f *at* $\nu \in U$*, and its value* $Df(\nu, \eta)$ *is the* **derivative of f at ν in the direction η**.

F–differentiability implies G–differentiability, and then it is $Df(\nu, \eta) \equiv D_\nu f(\eta)$*. Conversely, if* f *is G–differentiable in* $U = U^\circ \subset \mathfrak{T}$*, if the G–derivative* $\eta \mapsto Df(\nu, \eta)$ *is bounded linear*[95] *for all* $\nu \in U$*, and if the function* $\nu\ (\in U) \mapsto Df(\nu, \cdot)\ (\in \mathcal{L}(\mathfrak{T}, \mathfrak{R}))$ *is continuous, then* $Df(\nu, \eta) \equiv D_\nu f(\eta)$*, cf. [234, Lemmas 1.13–1.15].* $\diamondsuit$

In formulation of this definition, we have included also important assertions on uniqueness, and on the relation of the two concepts, [58].

It is seen that the derivative $f \mapsto D.f$ is a linear operation (functions f with values in a linear space $\mathcal{R}$ form naturally a linear space).

A.2.2. **Notes**.

(i) In finite–dimensional case, i.e. for $\dim \mathfrak{T} < \infty$, and also $\dim \mathfrak{R} < \infty$, $D_\nu f$ (expressed in some bases of $\mathfrak{T}$, and of $\mathfrak{R}$) is just the Jacobi matrix of f at ν. For $\mathfrak{R} := \mathbb{R}$, the function $\nu \mapsto D_\nu f$ is the ordinary first differential of f (understood as a linear functional on $\mathfrak{T}$: the "differentials of coordinates $d\nu_j,\ j = 1, 2, \ldots \dim \mathfrak{T}$" are coordinates of vectors in $\mathfrak{T}$); in the case $\mathfrak{T} := \mathbb{R}$, $D_\nu f(1) \in \mathfrak{R}$ is just the derivative of the ($\mathfrak{R}$ – valued) function f according to the parameter $\nu \in \mathbb{R}$ (here $1 \in \mathbb{R} = \mathfrak{T}$ is the "number 1" considered as a vector from $\mathfrak{T}$).

(ii) If $\mathfrak{T}$ is a function space, the derivative $D_\nu f$ is the **functional derivative**, cf. [85, 61]. If f is expressed in a form of integral over the space M of arguments of the functions $\eta \in \mathfrak{T},\ \eta : x(\in M) \mapsto \eta(x)(\in \mathbb{R})$, then $D_\nu f$ is usually expressed as an integral kernel:

$$D_\nu f_{(x)} \equiv \frac{\delta f(\nu)}{\delta \nu(x)}, \quad \nu \in \mathfrak{T},$$

and $D_\nu f(\eta)$ is the integral containing in its integrand the function η (and its derivatives with respect to its arguments, denoted together by a vector–symbol $\tilde{\eta}$) linearly, e.g.

$$D_\nu f(\eta) = \int_M D_\nu f_{(x)} \cdot \tilde{\eta}(x) \mu(dx).$$

To be more specific, f can be here, e.g., an "action integral" of the classical field theory, [192, 160], $f(\nu) := \int_M L(x, \tilde{\nu}(x)) \mathrm{d}^4 x$, the functions $\nu : x \mapsto \nu(x) \in \mathbb{R}^K$ are finite collections ($K < \infty$) of classical fields on the Minkowski space M, and the function L is a *Lagrangian density*, i.e. it is a numerical differentiable function of a finite number of $rK + 4$ real variables, x and $\tilde{\nu}(x) \in \mathbb{R}^{rK}$, attaining values $x \in M$, resp. equal to values of components of $\nu(x)$, and of (a finite number of) their derivatives taken simultaneously in the same point $x \in M$ (locality): $\tilde{\nu} := \{\nu, \partial_0^{\alpha_0} \partial_1^{\alpha_1} \ldots \partial_3^{\alpha_3} \nu : 1 \leq \sum \alpha_j \leq r\}$. Then the derivative of $f :\ D_\nu f(\eta)$, is expressed by an integral over M of the integrand $D_\nu L_{(x)} \cdot \tilde{\eta}(x)$, what can be considered as an application of

[95] Let us stress, that the linearity of G–derivative is a nontrivial requirement in general B–spaces.

the chain rule for composed mappings, (A.2.2): $\nu \mapsto L(\cdot, \tilde{\nu}(\cdot)) \in C(M, \mathbb{R})$ might be considered as a mapping between the Banach spaces $\mathfrak{T} := C^r(M, \mathbb{R}^K) \ni \nu$, and $\mathfrak{B} := C(M, \mathbb{R})$ (endowed with some "appropriate" norms, if, e.g., the domain of integration in M is bounded), with its derivative $D_\nu L_{(\cdot)}$ $(=$ *a "multiplication operator"* $\in \mathcal{L}(\mathfrak{T}, \mathfrak{B}))$: $\eta \mapsto D_\nu L_{(\cdot)} \cdot \tilde{\eta}(\cdot)$, and the integral is the next (linear) mapping in the chain.

(iii) The derivative (A.2.1a) of $f : \mathfrak{T} \to \mathbb{R}$ in any point ν belongs to $\mathcal{L}(\mathfrak{T}, \mathbb{R}) = \mathfrak{T}^*_{\mathbb{R}}$. Hence, for $\mathfrak{T} := \mathcal{L}(\mathcal{H})^*_s \supset \mathcal{S}$ the derivative would be in the double dual $\mathcal{L}(\mathcal{H})^{**}_s$, what is strictly larger than $\mathcal{L}(\mathcal{H})_s = \mathfrak{T}^*_s$, whereas the space $\mathfrak{T}_s$ is the ($\mathbb{R}$-linear envelope of the) normal state space, i.e. the "density matrices space", which is, in turn, the space of *all symmetric linear functionals*, i.e. the state space of the C^*-algebra $\mathfrak{C}$ of compact operators on $\mathcal{H}$.

(iv) It might be useful to stress here a (rather trivial) fact, that the derivative of a linear function $f \in \mathcal{L}(\mathfrak{T}, \mathfrak{R})$ equals, in any point $\nu \in \mathfrak{T}$, to the element $f \equiv D_\nu f \in \mathcal{L}(\mathfrak{T}, \mathfrak{R})$ itself. $\heartsuit$

An important formula can be proved for differentiation of composed mappings [58, Ch.1.§2.2]. Let $\mathfrak{T}, \mathfrak{R}, \mathfrak{L}$ be three B-spaces, and let $f : \mathfrak{T} \to \mathfrak{R}$, $g : \mathfrak{R} \to \mathfrak{L}$ be differentiable mappings. Then the composed mapping $h := g \circ f : \mathfrak{T} \to \mathfrak{L}$ is differentiable, and

$$D_\nu h \equiv D_{f(\nu)} g \circ D_\nu f. \tag{A.2.2a}$$

Since $D_\nu f$ is a linear mapping from $\mathfrak{T}$ into $\mathfrak{R}$, and $D_{f(\nu)} g$ is a linear mapping from $\mathfrak{R}$ into $\mathfrak{L}$, we have for all $\eta \in \mathfrak{T}$:

$$D_\nu h(\eta) = D_{f(\nu)} g\big(D_\nu f(\eta)\big) \equiv \big(D_{f(\nu)} g \circ D_\nu f\big)(\eta). \tag{A.2.2b}$$

Specifications of these concepts lead to infinite dimensional analogs of **partial derivatives**, cf. [58, Chap.1,§5.2].

A.2.3. Definitions.

(i) The **second derivative** $D^2_\nu f(\cdot, \cdot)$ *of the differentiable function* $f : \mathfrak{T} \to \mathfrak{R}$*, i.e. the first derivative of the function* $D.f : \mathfrak{T} \to \mathcal{L}(\mathfrak{T}, \mathfrak{R})$*,* $\eta \mapsto D_\eta f$*, at a point* $\nu \in \mathfrak{T}$ *belongs to a subspace* $\mathcal{L}^{(2)}_s(\mathfrak{T}, \mathfrak{R})$ *of the space* $\mathcal{L}(\mathfrak{T}, \mathcal{L}(\mathfrak{T}, \mathfrak{R}))$*, what is canonically isomorphic to the space* $\mathcal{L}(\mathfrak{T} \times \mathfrak{T}, \mathfrak{R})$ *of bilinear continuous functionals on* $\mathfrak{T}$*. The subspace* $\mathcal{L}^{(2)}_s(\mathfrak{T}, \mathfrak{R}) := \mathcal{L}_s(\mathfrak{T} \times \mathfrak{T}, \mathfrak{R})$ *is the space of* symmetric *bilinear functionals:* $D^2_\nu f(\varrho, \omega) \equiv D^2_\nu f(\omega, \varrho)$.

(ii) Similarly as above, the $\boldsymbol{n}-$**th derivative** $D^{(n)}_\nu f(\underbrace{\cdot, \cdot, \ldots, \cdot}_{n-times})$ *is a symmetric continuous* $n-$*linear functional on* $\mathfrak{T}$*, an element of the canonically defined Banach space* $\mathcal{L}^{(n)}_s(\mathfrak{T}, \mathfrak{R}) := \mathcal{L}_s(\times^n \mathfrak{T}, \mathfrak{R})$.

(iii) The space of $k-$*times continuously differentiable functions* f *on the B-space* $\mathfrak{T}$ *with values in* $\mathfrak{R}$ *will be denoted by* $C^k(\mathfrak{T}, \mathfrak{R})$*. The space of all infinitely differentiable functions on* $\mathfrak{T}$ *will be denoted by* $C^\infty(\mathfrak{T}, \mathfrak{R})$ *(*$\equiv \mathcal{F}(\mathfrak{T})$*, if* $\mathfrak{R} := \mathbb{R}$*).* $\diamondsuit$

Also the notion of the Taylor expansion can be introduced similarly as in finite–dimensional case, [58, 234, 61]. It is clear from the point (iv) in Notes A.2.2 that the second derivative of any linear function (with respect to to the same argument) equals to zero.

To deal with differential equations, resp. dynamical systems in different conditions, it is useful to generalize the differential calculus to more general spaces M replacing the linear space $\mathfrak{T}$. Such convenient spaces M are for us topological spaces endowed with the structures called "manifold structures", and these M's are called "manifolds".

A.3 Basic structures on manifolds

We shall start with the concept of differentiable manifold as a basis of further geometrical constructions, cf. [219, 40]. Intuitively, a manifold is a set "piecewise similar" to a t.l.s.

A.3.1. Definitions.

(i) A **chart** *on a topological space M is a triple $c := (U; \varphi; \mathcal{L})$, where $U^\circ = U \subset M$, φ is a homeomorphism of U onto an open subset of a Banach space $\mathcal{L}$. We shall often take $\mathcal{L} := \mathbb{R}^n$ for some natural number $n < \infty$; in this case, the existence of such a chart means possibility of introducing n continuous (local) coordinates on the open subset U of M. U is the* **domain** *of c, resp. (also) of φ: We shall call the "chart c" alternatively also "the chart φ".*

(ii) A **topological (Banach) manifold** *M (simply:* a manifold*) is a Hausdorff topological space M every point of which has an open neighbourhood homeomorphic to some open subset of a Banach space $\mathcal{L}$. This means that M can be covered by domains of charts defined on it.*

(iii) A **C^m-atlas** *on a manifold M is a collection of charts $\{c_j := (U_j; \varphi_j; \mathcal{L}_j) : j \in J\}$ such, that the open subsets $\{U_j : j \in J\}$ (J is an index set) cover M: $\bigcup_{j\in J} U_j = M$, satisfying simultaneously the condition that for the set of homeomorphisms φ_j the mappings $\varphi_j \circ \varphi_k^{-1} : \varphi_k(U_j \cap U_k) \to \varphi_j(U_j \cap U_k)$ are, for all $j, k \in J$, C^m-diffeomorphisms, i.e. the mappings together with their inverses are m-times continuously differentiable in all local coordinates. Two C^m-atlases are* **equivalent** *if their union is again a C^m-atlas. All the equivalent atlases compose the* **maximal atlas**. *If all the B-spaces $\mathcal{L}$ of the charts of the atlases are finite dimensional $\mathbb{R}$-spaces, and an atlas is $\{c_j := (U_j; \varphi_j; \mathbb{R}^{n(j)}) : j \in J\}$, the numbers $n(j)$ occurring in the specifications of charts are* **local dimensions** *of M. For a connected M it follows that $n(j) \equiv n$ in which case n is the* **dimension of M**, *$n = \dim(M)$. In the case of a manifold M with the image-spaces $\mathcal{L}$ being infinite–dimensional B-spaces, M is a* **manifold of infinite dimension**. *The manifold M endowed with a C^m-atlas (equivalently: with an equivalence class of C^m-atlases) is called a C^m-*manifold*: The atlas(-es) defines a* **structure of (C^m-)differentiable manifold** *on M. Equivalent atlases determine* **equivalent manifold structures** *on M.* $\diamondsuit$

It is a theorem, [135], that on any finite dimensional C^m-manifold with $m \geq 1$ there is also a C^∞-atlas in the equivalence class defining the manifold structure. Hence, on differentiable manifolds of finite dimension we can always introduce local coordinates the transformations of which on the intersections of their domains are all infinitely differentiable. In the following, any manifold will be a C^∞-manifold. Let us note also that on a given (topological) manifold it might be possible to introduce many nonequivalent differentiable structures; e.g., on the sphere S^n, for $n \leq 6$, it can be introduced exactly one differentiable structure, but for $n \geq 7$ there are several dozens of nonequivalent differentiable structures, cf. [148].

Let us introduce now some examples.

A.3.2. **Examples**.

(i) Let $M = \mathbb{R}^n$ considered with the (unique) locally convex topology of $\mathbb{R}^n$. Let an atlas consisting of a unique chart with domain M and φ being the identity map be given. This atlas defines a C^∞-manifold structure on M.

(ii) Let $M := S^n \subset \mathbb{R}^{n+1}$ be the n-dimensional unit sphere. We can construct charts of an atlas on M by stereographic projections onto hyperplains $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ orthogonal to coordinate axes: If M is described by the equation $\sum_{k=1}^{n+1} x_k^2 = 1$, then, for the j-th projection φ_j, the point with coordinates $\{x_k : k = 1, \dots n+1\}$ is mapped into the point $\{y_l := 2x_l/(1-x_j), l \neq j\}$, for all the points in $\{x \in S^n : x_j \neq 1\} =: U_j$ composing the domain of φ_j. As a simplest case of these manifolds, the circle S^1 needs at least two charts to compose an atlas.

(iii) The torus $T^n = (S^1)^n$ is an example for multiply–connected (cf. below) manifold. Its charts are constructed, e.g., as Cartesian products of the charts of circles.

(iv) Let a set N be homeomorphic to the subset of $\mathbb{R}^2$ consisting of several mutually different straight lines intersecting in some points, with the induced topology. Then N cannot be endowed with a structure of manifold, since any point of intersection has not a neighbourhood homeomorphic to $\mathbb{R}$ (or to $\mathbb{R}^n$, for any $n \geq 0$). $\heartsuit$

The real line $\mathbb{R}$ will be always (if not mentioned contrary) considered with its usual topology generated by open intervals. Similarly, the complex plane $\mathbb{C}$ is considered with the usual product topology of $\mathbb{R}^2$. The manifold structures of these spaces are given as in Example A.3.2(i). We shall define now important subsets of a manifold, that are endowed with canonically induced manifold structures.

A.3.3. Definition.

(i) A subset $N \subset M$ is a **submanifold of M**, *$dim(M) = n$, if every point $x \in N$ is in the domain U of such a chart $(U;\varphi)$, that for all $x \in U \cap N$ one has $\varphi(x) = \{x^1, x^2, \dots x^k, a^1, a^2, \dots a^{n-k}\}$, where $\{a^1, \dots a^{n-k}\}$ is a constant in $\mathbb{R}^{n-k}$. The obvious manifold structure on N determined by these charts is the* **induced manifold structure** *from the manifold M. Dimension of the manifold N is $\dim N = k$.* $\diamondsuit$

The usual model of a submanifold N in $M := \mathbb{R}^n$ is realized as the a "surface" in $\mathbb{R}^n$, i.e. as the inverse image $f^{-1}(\{a\}) =: N$ of a point $a \in \mathbb{R}^{n-k}$ by a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^{n-k}$ (i.e. $n-k$ real differentiable functions of n real variables) with its Jacobi matrix of constant maximal rank on N; this means, that N (with $dim(N) = k$) consists of roots $x \in \mathbb{R}^n$ of the equation

$$f(x) - a = 0 \ (\in \mathbb{R}^{n-k}). \tag{A.3.1}$$

Hypersurfaces of the dimension $n-1$ are determined by real-valued functions f on M with nonvanishing differential df at points x satisfying (A.3.1) .

Let M, N be two manifolds, and let a function $f : M \to N, x \mapsto f(x)$ be given.

A.3.4. Definition. *A function (resp.* mapping*) $f : M \to N, x \mapsto f(x)$ is* **differentiable** *in $x \in M$ iff there are charts $(U;\varphi), (V;\psi)$ on M, N, respectively, with $x \in U, f(x) \in V$ such that the function $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is differentiable in $\varphi(x)$. That* ***f*** **is differentiable**

means differentiability in each point $x \in M$. If $f : M \to N$ is a bijection and both f and f^{-1} are differentiable, then f is a **diffeomorphism** *of the manifolds M and N.*

Let $I \subset \mathbb{R}$ be an open interval containing 0. A differentiable function $c : I \to M$ is a **differentiable curve on M**. $\diamondsuit$

These concepts do not depend on a specific choice of charts in an equivalence class of atlases. We shall mean in the following by "differentiability" the infinite differentiability, if not stated otherwise. Differentiable mappings $f : M \to \mathbb{R}$ compose the space $\mathcal{F}(M)$ of infinite differentiable real-valued functions on M. The real linear space $\mathcal{F}(M)$ is also **an associative algebra** with respect to pointwise multiplication: $(fh)(x) \equiv f(x)h(x)$.

These concepts allow us to introduce an intrinsic definition of tangent space to M at a point $x \in M$. This has an advantage with respect to intuitive notions of tangent spaces as a certain "plains" in some higher dimensional linear space containing our manifold M as a submanifold: Such intuitive notions needn't be invariant with respect to diffeomorphisms, since after a diffeomorphic deformation of M the "tangent plain" might become "tangent" in more than one points of M, or even intersect M if this is not embedded in an "appropriate way". Our definition is, however, physically intuitive, since it directly defines tangent vectors as invariantly specified "instantaneous velocities" of motions along curves lying on the manifold.

A.3.5. Definition. *Let $c_j, j \in J$ be differentiable curves on a manifold M through a point $x \in M : c_j(0) = x, \forall j \in J$. Let $(U; \varphi; \mathcal{E})$ be a chart on M at $x \in U$. Then the derivatives*

$$\mathbf{v}_j^\varphi := \left.\frac{\mathrm{d}\varphi(c_j(t))}{\mathrm{d}t}\right|_{t=0} := D_{t=0}(\varphi \circ c_j)(1) \in \mathcal{E}$$

exist. If they are equal for different $j \in J$, as vectors $\mathbf{v}_j^\varphi$ in the B-space $\mathcal{E}$, this mutual equality is independent of a chosen chart φ. We shall call c_j and c_k, with $\mathbf{v}_j^\varphi = \mathbf{v}_k^\varphi$, **equivalent curves at $x \in M$**. *Hence, the differentiable curves at $x \in M$ are distributed into* **equivalence classes $[c]_x$** *of curves c at $x \in M$.* $\diamondsuit$

Let φ be a chart on M as above in Definition A.3.5, and let $\mathcal{E}$ be considered as a manifold with the atlas consisting of single chart given by the identity mapping $id_\mathcal{E}$ on $\mathcal{E}$. Then the equivalence classes $[d]_\eta$ of all curves $d_j : I_j \to \mathcal{E}, d_j(0) \equiv \eta \in \mathcal{E}$ through η are in canonical bijection with vectors in $\mathcal{E}$ given by $[d]_\eta \leftrightarrow D_0 d_j(1) \in \mathcal{E}, \quad d_j \in [d]_\eta$. Any curve d_j through $\eta := \varphi(x) \in \mathcal{E}$ gives a curve $t \mapsto c_j(t) := \varphi^{-1}(d_j(t))$ through $x \in M$. This helps us to see that there is a bijection between the above defined equivalence classes $[c]_x$ of curves on M, and vectors in $\mathcal{E}$. Now it is possible to introduce linear operations into the set $\{[c]_x : c$ is a differentiable curve on M through $x\}$ of equivalence classes of the curves, by extending the above bijection to a linear mapping. It is important that *the linear structure on the set of classes $[c]_x$ does not depend on a chosen chart.* This leads us to important

A.3.6. Definitions.

(i) Let M be a differentiable manifold, $x \in M$. The above introduced linear space of equivalence classes $[c]_x$ of differentiable curves through x is called the **tangent space to M at x**, *and will be denoted by $T_x M \equiv T_x(M)$. An element $\mathbf{v}_x^c := [c]_x \in T_x M$ is a* **tangent vector** *at x to M. If $U \subset M$ is an open subset (considered as a submanifold of M) containing x, we shall identify the tangent spaces $T_x U \equiv T_x M$, since $T_x M$ is determined by "the local structure" of M.*

(ii) Let $f : M \to N$ be a differentiable mapping (cf. Definition A.3.4) of manifolds. Let $c' \in [c]_x \in T_xM$. Then the equivalence class of the curves $t \mapsto f(c'(t)) \in N$ through $f(x)$ is independent of a representative $c' \in [c]_x$, hence the mapping f induces a well defined mapping T_xf of classes $[c]_x$ into classes $[f \circ c]_{f(x)} \in T_{f(x)}N$:

$$T_xf \equiv T_x(f) : T_xM \to T_xN, \quad \mathbf{v}^c_x := [c]_x \mapsto T_xf(\mathbf{v}^c_x) := [f \circ c]_{f(x)}. \tag{A.3.2}$$

The mapping T_xf is called the **tangent of f at x**.

(iii) Let a manifold M with an atlas $\{(U_j; \varphi_j; \mathcal{E}_j) : j \in J\}$ be given. Let TM be the manifold determined as the set

$$\{[c]_x \in T_xM : x \in M\}$$

of all tangent vectors in all points of the manifold M, endowed by the atlas consisting of charts

$$\big(\cup\{T_xM : x \in U_j\}; \Phi_j; \mathcal{E}_j \times \mathcal{E}_j\big),$$

where the mapping Φ_j is defined:

$$\Phi_j([c]_x) := \big(\varphi_j(x); T_x\varphi_j([c]_x)\big) \in \varphi(U_j) \times \mathcal{E}_j \subset \mathcal{E}_j \times \mathcal{E}_j.$$

In the last relation, the image of the tangent of φ_j on the vector $[c]_x$ at $x \in U_j \subset M$ equals to the derivative of the curve $t \mapsto \varphi_j \circ c(t) \in \mathcal{E}_j$ in point $t = 0$ taken at "the vector" $1 \in T_0\mathbb{R} \cong \mathbb{R}$, cf. Definition A.2.1, and $T_{\varphi_j(x)}\mathcal{E}_j$ is identified with $\mathcal{E}_j$. Moreover, let the **projection π_M** *be defined on the manifold TM by*

$$\pi_M : TM \to M; \quad [c]_x \mapsto x. \tag{A.3.3}$$

The differentiable manifold TM endowed with the projection (A.3.3) is **the tangent bundle of M**. *The projection π_M is the* **tangent bundle projection** *of M.*

(iv) The tangent bundle is an example of a **vector bundle (P, π_M, E)** *, i.e. of a manifold P with a differentiable mapping $\pi_M : P \to M$ onto another manifold M with a given open covering $\mathcal{U}_M := \{U_j : j \in J\}$ by domains U_j of its charts, and a topological vector (let it be Banach) space E (considered with its natural manifold structure) such that $\pi_M^{-1}(\{x\}) =: E_x \subset P$ is homeomorphic to E, the homeomorphism being the restriction of a diffeomorphism of $\pi_M^{-1}(U_j)$ onto $U_j \times E$, and the homeomorphisms corresponding to $j \neq k$ and to points $x \in U_j \cap U_k$ induce a group of linear transformations on E in a natural way, [61, 151, 1, 40], called the* **structural group** *of the bundle. Such homeomorphisms $E_x \leftrightarrow E$ allow us to introduce a natural linear structure on all E_x, $x \in M$, by transferring it from that on E.*

(v) Let $T^*_x(M) := (T_xM)^*$ be the topological dual of T_xM. This space is called the **cotangent space to M at x**. *Let us take p copies of $T_x(M)$, and q copies of $T^*_x(M)$, and let us form the tensor product spaces*

$$T^p_{qx}M := \otimes^p_{j=1}T_x(M) \bigotimes \otimes^q_{k=1}T^*_x(M),\ x \in M. \tag{A.3.4}$$

Let us denote $T^p_qM \equiv T^p_q(M)$ the set theoretic union of these linear sets. With a use of the manifold structure on M, they can be "sewed together", i.e. there can be introduced a manifold

structure on the set $T^p_q(M)$ in an obvious analogy with that of TM. The resulting manifold will be denoted by the same symbol $T^p_q(M)$; it will be called the (vector) **bundle of tensors of type $\binom{p}{q}$**, *or of the tensors* **contravariant of order p**, *and* **covariant of order q**. *The manifold $T^0_1(M) =: T^*(M)$ is the* **cotangent bundle** *of the manifold M, and $T^1_0(M) = TM$.*

(vi) Let a bundle (P, π_M, E) be given, and let $\mathbf{v} : M \to P, x \mapsto \mathbf{v}(x)$ be a differentiable mapping such, that

$$\pi_M(\mathbf{v}(x)) \equiv x. \tag{A.3.5}$$

Such mappings are called **sections of the (vector) bundle**. *A section of the tangent bundle $TM := (TM, \pi_M, \mathcal{E})$ is a* **vector field on M**. *Sections of the tensor bundle $T^p_q(M)$ are* **tensor fields of type $\binom{p}{q}$**. *The tensor fields of the type $\binom{p}{q}$ form an infinite dimensional vector space $\mathcal{T}^p_q(M)$. The space of vector fields is $\mathcal{T}^1_0(M)$, the space $\mathcal{T}^0_0(M)$ is identified with $\mathcal{F}(M)$. The direct sum $\mathcal{T}(M) := \oplus_{p\geq 0, q\geq 0}\mathcal{T}^p_q(M)$ is the* **algebra of tensor fields** *on M, the algebraic operation being the pointwise tensor product.*

(vii) Let $f : M \to N$ be as in (ii). The **tangent mapping of f** *is the mapping $Tf : TM \to TN$ defined by (cf. eq. (A.3.2))*

$$Tf : \mathbf{v}_x\ (\in T_xM) \mapsto T_xf(\mathbf{v}_x) \equiv T_xf \cdot \mathbf{v}_x\ (\in T_{f(x)}N). \tag{A.3.6}$$

The tangent mapping is also denoted by $f_* := Tf$. If f is a diffeomorphism, then we denote by f_* also the unique natural extension *of this mapping to the whole algebra of tensor fields, $f_* : \mathcal{T}(M) \to \mathcal{T}(N)$, determined by its "commuting with contraction", and conserving the type $\binom{p}{q}$, [151, Chap.I, Propositions 2.12 and 3.2].* $\diamondsuit$

Any vector field $\mathbf{v}$ on M uniquely determines a differentiation $\pounds_{\mathbf{v}}$ (i.e. a linear mapping satisfying the Leibniz rule for its action on products) of the associative algebra $\mathcal{F}(M)$. Let $\mathbf{v}(x)$ corresponds to the class $[c^{\mathbf{v}}]_x$ of curves through $x \in M$, and let $c^{\mathbf{v}}$ be in this class. Then $\pounds_{\mathbf{v}}$ is defined by the formula

$$\pounds_{\mathbf{v}} f(x) := \mathbf{v}_x(f) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(c^{\mathbf{v}}(t)). \tag{A.3.7}$$

Let us stress that this definition depends on vectors $\mathbf{v}_x \in T_xM$ only, independently of their possible inclusions as values of some vector fields: The mapping $\mathbf{v}(\in T_xM) \mapsto \pounds_{\mathbf{v}}$ is well defined for any fixed $x \in M$. On finite dimensional manifolds, any differentiation on the algebra $\mathcal{F}(M)$ is given by a vector field $\mathbf{v}$ according to (A.3.7); cf. [61] for comments on infinite dimensional cases. Hence, each vector field $\mathbf{v}$ determines a differential operator $\pounds_{\mathbf{v}}$, and the mapping $\mathbf{v}_x(\in T_xM) \mapsto \pounds_{\mathbf{v}}$ is a linear injection into the set of differential operators on the "algebra of germs of functions $\mathcal{F}(M)$ in the point $x \in M$"; this injection is also onto (i.e. surjective) for $\dim M < \infty$. We shall often identify $\pounds_{\mathbf{v}}$ with $\mathbf{v} \in TM$. The derivation $\pounds_{\mathbf{v}}$ can be naturally (under the requirement of "commutativity with contractions", [151, 1], and of satisfaction of the Leibniz rule) uniquely extended to a derivation on all spaces $\mathcal{T}^p_q(M)$. It acts on the vector fields as

$$\pounds_{\mathbf{v}}\mathbf{w} = [\pounds_{\mathbf{v}}, \pounds_{\mathbf{w}}] := \pounds_{\mathbf{v}}\pounds_{\mathbf{w}} - \pounds_{\mathbf{w}}\pounds_{\mathbf{v}} \equiv \pounds_{[\mathbf{v},\mathbf{w}]}, \tag{A.3.8}$$

and, for given vector fields $\mathbf{v}$ and $\mathbf{w}$, it represents a vector field, [40], denoted by $[\mathbf{v}, \mathbf{w}]$.

A.3.7. Definition. *The above determined mapping* $\pounds_{\mathbf{v}} : \mathcal{T}(M) \to \mathcal{T}(M)$ *(leaving each* $\mathcal{T}^p_q(M)$ *invariant) is the* **Lie derivative of tensor fields with respect to** $\mathbf{v} \in \boldsymbol{\mathcal{T}^1_0(M)}$. *The result of its action on a vector field* $\mathbf{w} : \pounds_{\mathbf{v}}(\mathbf{w}) = [\mathbf{v}, \mathbf{w}]$ *is* **the commutator (or Lie bracket) of the vector fields v and w**. *This Lie bracket satisfies the* **Jacobi identity***;*

$$[\pounds_{\mathbf{u}}, [\pounds_{\mathbf{v}}, \pounds_{\mathbf{w}}]] + [\pounds_{\mathbf{w}}, [\pounds_{\mathbf{u}}, \pounds_{\mathbf{v}}]] + [\pounds_{\mathbf{v}}, [\pounds_{\mathbf{w}}, \pounds_{\mathbf{u}}]] \equiv 0,$$

what is a consequence of the definition. $\diamondsuit$

Let us note that the mapping

$$d_x f : T_x M \to \mathbb{R}, \mathbf{v} \mapsto d_x f(\mathbf{v}) := \pounds_{\mathbf{v}}(f) \equiv \mathbf{v}(f), \forall \mathbf{v} \in T_x M, \tag{A.3.9}$$

is a bounded linear functional on $T_x M$ (this is a consequence of definition of Fre'chet differentiability of $f \in \mathcal{F}(M)$; $d_x f$ equals to $T_x f$, if $T_{f(x)}\mathbb{R} \equiv \mathbb{R}$ is the canonical identification): $d_x f \in T^*_x M$. Each element of $T^*_x M$ has the form $d_x f$ of **differential of the function** $\boldsymbol{f}$ for some $f \in \mathcal{F}(M)$. Hence, each tensor in $T^p_{qx}(M)$ can be expressed as a linear combination of tensor products of the form $\otimes^p_{j=1}\mathbf{v}_j \bigotimes \otimes^q_{k=1} d_x f_k, \mathbf{v}_j \in T_x M, f_k \in \mathcal{F}(M)$.

Any vector field $\mathbf{v} \in \mathcal{T}^1_0(M) =: \mathcal{X}(M)$ determines a **differential equation** on the manifold M, written symbolically for an **initial condition** $x(0) = x$:

$$\dot{x}(t) = \mathbf{v}(x(t)), \quad x(0) := x \in M. \tag{A.3.10}$$

Its solutions are **integral curves of the vector field v**, i.e. curves $t(\in I_x = I^\circ_x \subset \mathbb{R}) \mapsto x(t)$ through x such that for any $t_0 \in I_x$, the curve $\{t \mapsto x(t+t_0)\} \in$ "*the class of curves determined by* $\mathbf{v}(x(t_0))$". The open interval I_x can be (and is supposed to be) chosen maximal. Let us define the set $\boldsymbol{\mathcal{D}_{\mathbf{v}}} := \{(t; x) : t \in I_x, x \in M\}$, called the **domain of the (local) flow of** $\mathbf{v} \in \boldsymbol{\mathcal{X}(M)}$. There is defined on it the mapping

$$\varphi^{\mathbf{v}}_\cdot : (t, x)(\in \boldsymbol{\mathcal{D}_{\mathbf{v}}}) \mapsto \varphi^{\mathbf{v}}_t(x) := x(t), x(0) = x, \tag{A.3.11}$$

where $x(t)$ is the solution of (A.3.10); the mapping $\varphi^{\mathbf{v}}_\cdot$ is called the **(local) flow of v**. The locality means, that there might be for some $x \in M : I_x \neq \mathbb{R}$. If for all the intervals one has: $I_x \equiv \mathbb{R}, \forall x \in M$, the vector field **v** as well as its flow are called **complete**. On an arbitrary compact manifold M, any vector field is complete. Any (local) flow satisfies *on its domain* the **group property**:

$$\varphi^{\mathbf{v}}_{t_1+t_2} = \varphi^{\mathbf{v}}_{t_1} \circ \varphi^{\mathbf{v}}_{t_2}. \tag{A.3.12}$$

Vector fields are typically used to determine flows on manifolds as solutions of the corresponding differential equations. There are, on the other hand, other kinds of (covariant) tensor fields typically used for integration on manifolds. We shall not review here the integration theory on (finite dimensional) manifolds leading to the general Stokes theorem generalizing the particular Stokes, Gauss', Green's, and Stokes' theorems connecting some integrals on *manifolds* N *with boundary*[96] ∂N with corresponding integrals on the boundary ∂N. The formal expression

[96] A **manifold with boundary** has, besides the usual manifold charts, also charts φ_α whose ranges are intersections of open subsets of linear spaces $\mathcal{E}_j$ with their closed "halfspaces", [40, §11.1]. The boundary of the manifold consists of its points lying in inverse images of the boundaries of the halfspaces with respect to the chart-mappings φ_α, cf. also [1, p. 137].

of the general Stokes' theorem is the following **Stokes formula**:

$$\int_N d\omega = \int_{\partial N} \omega. \tag{A.3.13}$$

If $N \subset M, \dim N = n$, and M is a manifold ($\dim M > \dim N :=$ the dimension of the submanifold N°), inducing on N its structure of a *submanifold with boundary*, the objects entering into the Stokes formula are tensor fields $\omega \in \mathcal{T}^0_{n-1}(M)$, $d\omega \in \mathcal{T}^0_n$ of special kind called *differential forms*. Another usage of differential forms is in formulation of some partial differential equations on manifolds with a help of *exterior differential systems*, [61]. We need such tensor fields, in the present work, in connection with Hamilton's formulation of mechanics on "nondegenerate" phase spaces (i.e. on *symplectic manifolds*), and also in some modified situations (e.g. on *Poisson manifolds*).

Let us consider the elements of $T^0_{px}(M)$ as p–linear forms on $T_x(M)$, e.g. $d_x f_1 \otimes d_x f_2 \otimes \cdots \otimes d_x f_p \in T^0_{px}(M)$ is determined by specification of the mapping

$$(\mathbf{v}_1; \mathbf{v}_2; \ldots; \mathbf{v}_p) \left(\in \times^p T_x(M)\right) \mapsto \prod_{j=1}^{p} \mathbf{v}_j(f_j); \tag{A.3.14}$$

the space of bounded p–linear forms $\mathcal{L}_p(T_x M, \mathbb{R})$ can be identified with $T^0_{px}(M)$ by the linear extension of this correspondence. Let us introduce the **alternation mapping** $\mathbb{A}$ of this space into itself. Let for $\sigma \in \Sigma(p) :=$ the permutation group of p elements, and let $\epsilon_\sigma = \pm 1$ be the "parity" of σ, i.e. the nontrivial one–dimensional representation of $\Sigma(p)$. Let now $\mathbb{A}$ be the linear mapping determined by

$$\mathbb{A}\mathbf{t}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p) := \frac{1}{p!} \sum_{\sigma \in \Sigma(p)} \epsilon_\sigma \mathbf{t}(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \ldots, \mathbf{v}_{\sigma(p)}), \forall \mathbf{t} \in \mathcal{L}_p(T_x M, \mathbb{R}). \tag{A.3.15}$$

One can see that this mapping is idempotent: $\mathbb{A} \circ \mathbb{A} = \mathbb{A}$. Let us define now the subspace $\mathbf{\Lambda}^{\boldsymbol{p}}_{\boldsymbol{x}}$(M) of $T^0_{px}(M)$ by

$$\mathbf{\Lambda}^{\boldsymbol{p}}_{\boldsymbol{x}}(M) := \mathbb{A} T^0_{px}(M). \tag{A.3.16}$$

Let us denote $\mathbf{\Lambda}^{\boldsymbol{p}}(M)$ the space of tensor fields $\omega : x \mapsto \omega_x$ on M with values $\omega_x \in \mathbf{\Lambda}^{\boldsymbol{p}}_{\boldsymbol{x}}(M)$, for any integer $0 \leq p < \dim M + 1 \leq \infty$. Such ω are called ***p*–forms on M**. We identify 0–forms with differentiable functions, i.e. $\mathbf{\Lambda}^{\mathbf{0}}(M) := \mathcal{F}(M)$. A useful associative algebraic structure on the space $\mathbf{\Lambda}(M) := \oplus_{p=0}^{\dim M} \mathbf{\Lambda}^{\boldsymbol{p}}(M)$ can be introduced: The **wedge–product** $\wedge : \mathbf{\Lambda}^{\boldsymbol{p}}(M) \times \mathbf{\Lambda}^{\boldsymbol{q}}(M) \to \mathbf{\Lambda}^{\boldsymbol{p+q}}(M), (\omega_1; \omega_2) \mapsto \omega_1 \wedge \omega_2$, where $\mathbf{\Lambda}^{\boldsymbol{p}}(M) := \{0\}$, if $p > \dim M$. For arbitrary $f_j \in \mathcal{F}(M)$ we define the wedge–product of their differentials (for the consistency of various definitions of $\wedge$ cf. [7]):

$$df_1 \wedge df_2 \wedge \cdots \wedge df_p := p! \cdot \mathbb{A}(df_1 \otimes df_2 \otimes \cdots \otimes df_p),\ p = 2, 3, \ldots, \dim M, \tag{A.3.17}$$

where the alternation mapping $\mathbb{A}$ acts pointwise on M. More general formula for an arbitrary wedge–product of a p_1–form ω_1, and a p_2–form ω_2 reads:

$$\omega_1 \wedge \omega_2 = \frac{(p_1 + p_2)!}{p_1! p_2!} \mathbb{A}(\omega_1 \otimes \omega_2). \tag{A.3.18a}$$

Then we have

$$\omega_1 \wedge \omega_2 = (-1)^{p_1 p_2} \omega_2 \wedge \omega_1, \tag{A.3.18b}$$

$$f \wedge \omega = \omega \wedge f \equiv f \cdot \omega, \ \forall f \in \mathbf{\Lambda^0}(M), \omega \in \mathbf{\Lambda^p}(M) \ (p = 0, \ldots, \dim M). \tag{A.3.18c}$$

A.3.8. Definition. *The linear space* $\mathbf{\Lambda}(M) := \oplus_{p=0}^{\dim M} \mathbf{\Lambda^p}(M)$ *endowed with the above introduced wedge–product* $\wedge$ *is called the* **algebra of exterior differential forms** *on* M*. Its elements lying in the subspace* $\mathbf{\Lambda^p}(M)$ *are called* $\boldsymbol{p}$**–forms on** $\boldsymbol{M}$*; specifically, the elements of* $\mathbf{\Lambda^1}(M) = \mathcal{X}^*(M)$ *are* **one–forms***, and the elements of* $\mathbf{\Lambda^0}(M) = \mathcal{F}(M)$ *are* **zero–forms**. $\diamondsuit$

Let us introduce now some operations on the algebra $\mathbf{\Lambda}(M)$, i.e. some linear mappings of $\mathbf{\Lambda}(M)$ into itself. Let us first note that the **Lie derivative** $\pounds_{\mathbf{v}}$, as it was extended to the whole tensor algebra $\mathcal{T}(M)$, leaves its linear subspace $\mathbf{\Lambda}(M)$ invariant, and the Leibniz rule with respect to the wedge–product is fulfilled:

$$\pounds_{\mathbf{v}}(\omega_1 \wedge \omega_2) = (\pounds_{\mathbf{v}} \omega_1) \wedge \omega_2 + \omega_1 \wedge (\pounds_{\mathbf{v}} \omega_2). \tag{A.3.19}$$

Another important linear mapping $\mathrm{d} : \mathbf{\Lambda}(M) \to \mathbf{\Lambda}(M)$ called the **exterior differential** is uniquely determined by the below listed properties, [1, Theorem 2.4.5]:

A.3.9. Theorem. *The following properties determine a unique linear mapping* d *on* $\mathbf{\Lambda}(M)$ *(called the exterior differential on* M*):*

(i) $\mathrm{d}\mathbf{\Lambda^p}(M) \subset \mathbf{\Lambda^{p+1}}(M)$*;*

(ii) $\mathrm{d}(\omega_1 \wedge \omega_2) = (\mathrm{d}\omega_1) \wedge \omega_2 + (-1)^{p_1} \omega_1 \wedge (\mathrm{d}\omega_2), \ \forall \omega_j \in \mathbf{\Lambda^{p_j}}(M)$*;*

(iii) $\mathrm{d} \circ \mathrm{d} \equiv 0$*;*

(iv) For any $f \in \mathcal{F}(M), \mathrm{d}f \in \mathbf{\Lambda^1}(M) = \mathcal{X}^*(M)$*:* $\mathrm{d}f(\mathbf{v}) \equiv \mathbf{v}(f) := \pounds_{\mathbf{v}}(f), \ \forall \mathbf{v} \in \mathcal{X}(M)$*. This means, that the exterior differential of a function* f *coincides with the differential df introduced above, in (A.3.9).* $\clubsuit$

Explicit expression of the differential $\mathrm{d}\omega$ of an $\omega \in \mathbf{\Lambda^p}(M)$ given by

$$\omega := \sum_{j_1 < j_2 < \cdots < j_p} h_{j_1 j_2 \ldots j_p} \mathrm{d}f_{j_1} \wedge \mathrm{d}f_{j_2} \wedge \cdots \wedge \mathrm{d}f_{j_p}, \tag{A.3.20a}$$

with $h_{j_1 j_2 \ldots j_p}, f_j \in \mathcal{F}(M)$, is easily obtained by linearity and by the (modified) "Leibniz rule", as well as by the property $\mathrm{d} \circ \mathrm{d} \equiv 0$:

$$\mathrm{d}\omega = \sum_{j_1 < j_2 < \cdots < j_p} \mathrm{d}h_{j_1 j_2 \ldots j_p} \wedge \mathrm{d}f_{j_1} \wedge \mathrm{d}f_{j_2} \wedge \cdots \wedge \mathrm{d}f_{j_p}. \tag{A.3.20b}$$

A.3.10. Definition. *Let a vector field* $\mathbf{v}$ *on* M *be given,* $\mathbf{v} \in \mathcal{X}(M)$*. Then the linear mapping* $\boldsymbol{i}_{\mathbf{v}} : \mathbf{\Lambda}(M) \to \mathbf{\Lambda}(M), \mathbf{\Lambda^p}(M) \to \mathbf{\Lambda^{p-1}}(M)$*, determined by*

$$(\boldsymbol{i}_{\mathbf{v}} \omega)(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{p-1}) := \omega(\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{p-1}), \quad \boldsymbol{i}_{\mathbf{v}} f := 0 \ (\forall f \in \mathbf{\Lambda^0}(M))$$

is the **inner product of v and** $\boldsymbol{\omega}$**.** $\diamondsuit$

One of the main statements of this section will be a list of mutual relations between introduced operations on exterior differential forms. Before quoting it, let us introduce still one transformation which allows us to "transfer" differential forms (and other tensor fields) from a manifold to another one.

A.3.11. Definition. *Let $\beta : N \to M$ be a differentiable mapping of a (differentiable) manifold N into a manifold M, and let $T\beta : TN \to TM$ be its tangent mapping. For any p–form on M : $\omega \in \mathbf{\Lambda}^{\boldsymbol{p}}(M)$, let us define a p–form $\beta^*\omega \in \mathbf{\Lambda}^{\boldsymbol{p}}(N)$ on the manifold N by the formula:*

$$(\beta^*\omega)_y(\mathbf{w}_1, \ldots, \mathbf{w}_p) \equiv \omega_{\beta(y)}(T_y\beta\cdot\mathbf{w}_1, \ldots, T_y\beta\cdot\mathbf{w}_p),\ \forall y \in N, \mathbf{w}_j \in T_yN.$$

The mapping $\beta^* : \mathbf{\Lambda}(M) \to \mathbf{\Lambda}(N)$ is called the **pull–back by $\boldsymbol{\beta}$**. *Let us note, that in the particular case $p = 0$ we have for $f \in \mathcal{F}(M)$: $\beta^* f(y) \equiv f \circ \beta(y)$.* $\diamondsuit$

We can now present a basic tool of the "machinery" for such a differential computation on manifolds which does not need introducing any coordinates on them; we shall collect also some earlier recognized relations, cf. [40, 151, 1, 61].

A.3.12. Theorem. *For above defined operations on (infinitely) differentiable manifolds represented by the symbols $\beta^*, \pounds., \mathrm{d}, \boldsymbol{i}.$, as well as by the* **commutator** *(if it is defined) of any operations τ_j:*

$$[\tau_1, \tau_2] := \tau_1 \circ \tau_2 - \tau_2 \circ \tau_1,$$

with **v**, **w** *any differentiable vector fields on a manifold, the following identities are valid:*

(i) $[\pounds_{\mathbf{v}}, \pounds_{\mathbf{w}}] = \pounds_{[\mathbf{v},\mathbf{w}]}$;

(ii) $[\pounds_{\mathbf{v}}, \mathrm{d}] = 0$;

(iii) $[\pounds_{\mathbf{v}}, \boldsymbol{i}_{\mathbf{w}}] = \boldsymbol{i}_{[\mathbf{v},\mathbf{w}]}$;

(iv) $[\beta^*, \mathrm{d}] = 0$,

where, for $\beta \in C^\infty(N, M)$, d *acts interchangeably on $\mathbf{\Lambda}(M)$, and on $\mathbf{\Lambda}(N)$;*

(v) $\mathrm{d} \circ \mathrm{d} = 0$;

(vi) $\mathrm{d} \circ \boldsymbol{i}_{\mathbf{v}} + \boldsymbol{i}_{\mathbf{v}} \circ \mathrm{d} = \pounds_{\mathbf{v}}$;

(vii) $\boldsymbol{i}_{\mathbf{v}} \circ \boldsymbol{i}_{\mathbf{w}} + \boldsymbol{i}_{\mathbf{w}} \circ \boldsymbol{i}_{\mathbf{v}} = 0$;

If $\beta : N \to M$ is a diffeomorphism, and, for any $\mathbf{v} \in \mathcal{X}(M)$, we define $\beta^*\mathbf{v} \in \mathcal{X}(N)$ by the identity

$$(\mathrm{d}g)_y(\beta^*\mathbf{v}) := \big((\beta^{-1})^*\mathrm{d}g\big)_{\beta(y)}(\mathbf{v}),\ \forall g \in \mathcal{F}(N),\ \forall y \in N,$$

then the following two items, (viii), and (ix), also express identities:

(viii) $\beta^* \circ \pounds_{\mathbf{v}} = \pounds_{\beta^*\mathbf{v}} \circ \beta^*$;

(ix) $\beta^* \boldsymbol{i}_{\mathbf{v}} = \boldsymbol{i}_{\boldsymbol{\beta^*\mathbf{v}}} \beta^*$.

Moreover, the following elementary properties are identically valid (with $f \in \mathbf{\Lambda}^{\mathbf{0}}(M) = \mathcal{F}(M)$, and $f\cdot$ means poinwise multiplication, e.g. $f\cdot\alpha \equiv f \wedge \alpha,\ \alpha \in \mathbf{\Lambda}(M)$):

(x) $\pounds_{\mathbf{v}+\mathbf{w}} = \pounds_{\mathbf{v}} + \pounds_{\mathbf{w}}$*;*

(xi) $\pounds_{f\cdot\mathbf{v}} = \mathrm{d}f \wedge \boldsymbol{i}_{\mathbf{v}} + f\cdot\pounds_{\mathbf{v}}$*;*

(xii) $\boldsymbol{i}_{\mathbf{v}+\mathbf{w}} = \boldsymbol{i}_{\mathbf{v}} + \boldsymbol{i}_{\mathbf{w}}$*;*

(xiii) $\boldsymbol{i}_{f\cdot\mathbf{v}} = f\cdot\boldsymbol{i}_{\mathbf{v}}$.

Let us give also the following useful formula for coordinate–free calculation of the exterior differential:

(xiv)

$$\begin{aligned}\mathrm{d}\omega(\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_p) = &\sum_{j=0}^{p}(-1)^j \pounds_{\mathbf{v}_j}\big(\omega(\mathbf{v}_0,\mathbf{v}_1,\ldots,\hat{\mathbf{v}}_j,\ldots,\mathbf{v}_p)\big)\\ &+\sum_{0\le j<k\le p}(-1)^{j+k}\omega([\mathbf{v}_j,\mathbf{v}_k],\mathbf{v}_0,\ldots,\hat{\mathbf{v}}_j,\ldots,\hat{\mathbf{v}}_k,\ldots\mathbf{v}_p),\end{aligned} \tag{A.3.21}$$

for all $\omega \in \boldsymbol{\Lambda^p}(M)$*, where* $\hat{\mathbf{v}}_j$ *means* skipping of *the vector field* $\mathbf{v}_j$ *in the arguments, so that it is replaced by* $\mathbf{v}_{j-1}$*, and other arguments are also shifted by keeping their original order unchanged.* ♣

We shall introduce here also the following standard terminology:

A.3.13. Definition. *Let* $\omega \in \boldsymbol{\Lambda^p}(M)$ *be such a* p*–form, that its differential vanishes:* $\mathrm{d}\omega = 0$*, hence it equals to the zero element of* $\boldsymbol{\Lambda^{p+1}}(M)$*. In this situation,* ω *is a* **closed** $\boldsymbol{p}$**–form**. *Clearly, if* $\omega = \mathrm{d}\alpha$ *for some* $p-1$*–form* α*, then* $\mathrm{d}\omega = 0$*; for such a closed form we say, that* ω *is an* **exact** $\boldsymbol{p}$**–form**. *Let us assume now, that* $\dim M < \infty$*. Since exact* p*–forms form a linear subspace in the subspace of all closed* p*–forms, one can form the factorspace of the later* p*–forms according to its subspace consisting of the former ones. The resulting linear space is denoted by* $H^p(\boldsymbol{\Lambda}(M)) \equiv H^p(M)$*, and it is called the* $\boldsymbol{p}$**–th cohomology group of** $\boldsymbol{M}$*, where the group operation is the vector addition, cf. [148, 61].* ♢

The mentioned cohomology groups are important algebraic–topological characterizations of manifolds, but we leave it here without giving any further comments and results, cf. [246, 219, 148, 80, 61].

If there is given a bilinear continuous form Ψ on a vector space $\mathcal{E}$, it determines a linear mapping $\Psi^\flat$ from $\mathcal{E}$ into its topological dual $\mathcal{E}^*$ by

$$\Psi^\flat : \mathcal{E} \to \mathcal{E}^*,\ x \mapsto \Psi^\flat_x,\ \text{with } \langle \Psi^\flat_x; y\rangle := \Psi(x,y),\ \forall x,y \in \mathcal{E}. \tag{A.3.22}$$

The mapping $\Psi^\flat$ is injective iff

$$x \neq 0 \Rightarrow \Psi^\flat_x \neq 0. \tag{A.3.23}$$

In the case of finite dimensional $\mathcal{E}$, this condition means that $\Psi^\flat$ is a linear isomorphism (hence also bicontinuous in the natural l.c. topologies). Otherwise, $\Psi^\flat$ needn't be even a bijection: it might injectively map the space $\mathcal{E}$ onto a proper subspace of $\mathcal{E}^*$. It is useful to distinguish several cases, [1, 178, 61]:

A.3.14. Definition (Nondegenerate 2-tensors).

(i) Let the above introduced mapping $\Psi^\flat$ fulfills the condition (A.3.23). Then the we say that the bilinear form $\mathbf{\Psi}$ **is weakly nondegenerate**. *If $\Psi^\flat$ is bijective (hence, $\mathcal{E}$ is mapped also onto $\mathcal{E}^*$), then Ψ is called* **strongly nondegenerate**.

(ii) Let now $\Psi \in \mathcal{T}_2^0(M)$ be a two-covariant tensor field on a manifold M, $\Psi : x(\in M) \mapsto \Psi_x(\in T_{2x}^0 M)$. Let us assume, that Ψ is either symmetric (i.e. $\Psi_x(\mathbf{v},\mathbf{w}) \equiv \Psi_x(\mathbf{w},\mathbf{v})$, $\forall x \in M, \mathbf{v},\mathbf{w} \in \mathcal{X}(M)$), or antisymmetric (i.e. $\Psi_x(\mathbf{v},\mathbf{w}) \equiv -\Psi_x(\mathbf{w},\mathbf{v})$, $\forall x \in M, \mathbf{v},\mathbf{w} \in \mathcal{X}(M)$). Then Ψ is **weakly (resp. strongly) nondegenerate**, *if all Ψ_x, $\forall x \in M$, are weakly (resp. strongly) nondegenerate.*

(iii) Let $\Gamma \in \mathcal{T}_2^0(M)$ be symmetric. If it is weakly (strongly) nondegenerate, then it is called **weak (strong) pseudo–Riemannian metric on M**. *If Γ is, moreover, positive definite (i.e. $\Gamma_x(\mathbf{v},\mathbf{v}) > 0, \forall \mathbf{v} \neq 0, \mathbf{v} \in T_x M, \forall x \in M$), then it is called a* **weak (resp. strong) Riemannian metric**.

(iii) Let $\Omega \in \mathbf{\Lambda^2}(M)$, and assume, moreover, that it is closed: $\mathrm{d}\Omega \equiv 0$. If the two–form Ω is weakly (strongly) nondegenerate, it is called **weak (strong) symplectic form on M**. $\diamondsuit$

The Riemannian metrics are the basic objects of Riemannian geometry, [129, 61, 1], providing a mathematical formalism for the relativistic theory of gravitation (i.e. **"general relativity"**), [88, 195, 226], and it is useful also for a description of classical "continuous media" (i.e. the phase spaces are infinite–dimensional), e.g. [7, Appendix 2], [178, 110]. The symplectic forms are basic for (finite–, or infinite–dimensional) classical Hamiltonian mechanics (CM), cf., e.g. [1, 59, 178]. In our extension of quantum mechanics (EQM), symplectic forms on manifolds of density matrices generate dynamics and symmetries with a help of scalar–valued functions ("Hamiltonians"), and simultaneously canonically defined Riemannian metrics on that manifolds of density matrices are tools for determination of specifically quantum probability interpretation of the theory.

A.4 Elementary concepts of Lie groups

We shall restrict our present brief exposition mainly to finite dimensional Lie groups; for infinite dimensional Lie groups see, e.g. [39, 155]. Let us start, however, with some basic definitions and relations, [207, 19, 267], concerning general groups.

A.4.1. Definitions (Abstract and topological groups).

(i) A **group** *G is a set with a distinguished element $e \in G$ called the* **unit element** *of G, and with two mappings: (a) a bijection of G onto itself, $g(\in G) \mapsto g^{-1}(\equiv$***the inverse of** g*); and (b) the* **group multiplication (**equiv.: **product)**,

$$(g_1; g_2)\ (\in G \times G)\ \mapsto g_1 \cdot g_2 \equiv g_1 g_2\ (\in G),$$

which **is associative** *and such, that*

$$e \cdot g = g,\ g^{-1} \cdot g = e,\ \forall g \in G.$$

Then it is also $ge \equiv g$, $gg^{-1} \equiv e$. If $g \cdot h = h \cdot g$ $(\forall g, h \in G)$, then G is **abelian (**equiv.: **commutative) group**.

(ii) A subset $H \subset G$ such, that it is invariant with respect to taking inverse and also with respect to group multiplication of its elements: $h_1 \cdot h_2 \in H,\ \forall h_j \in H,\ j = 1, 2$, is called a **subgroup of $\boldsymbol{G}$**. *Any subgroup H of G is a group with the induced operations from G.*

(iii) If the group G is a topological space, and the inverse operation and group multiplication are in this topology continuous (by which $G \times G$ is endowed by the product topology), then G is **topological group**. *If H is a subgroup of G* and a closed subspace, *it is a* **topological subgroup of $\boldsymbol{G}$**.

(iv) Let G be a (topological) group, and H its (topological) subgroup. If $gHg^{-1} := \{ghg^{-1} : h \in H\} = H,\ \forall g \in G$, then H is **normal (**equiv.: **invariant) subgroup of $\boldsymbol{G}$**. *Since any subgroup contains the unit element e of G, the subsets $g \cdot H \subset G, g \in G$ (called* **left cosets of $\boldsymbol{G}$)** *cover whole G, and any two of them are either equal, or disjoint: They define an equivalence relation on G. The factor spaces G/H corresponding to this decomposition of G to left cosets are important in the theory of actions of G on some arbitrary spaces. Similarly, another equivalence relation on G determined by the* **right cosets** *$\{Hg : g \in G\}$ of G; for normal subgroups H (and only for them) these two decompositions of G coincide. If H is a normal subgroup, the space G/H is again a (topological) group with the group multiplication*

$$(g^{-1} \cdot H) \cdot (g' \cdot H) := (g^{-1} \cdot g') \cdot H,\ \forall g, g' \in G.$$

In this case, the factor space G/H is called the **factor group of $\boldsymbol{G}$ by $\boldsymbol{H}$**.

(v) Let G, G' be two (topological) groups and $\phi : G \to G'$ be such a (continuous) mapping, that

$$\phi(g_1 \cdot g_2) \equiv (\phi g_1) \cdot (\phi g_2),\ \phi e := e';$$

the mapping ϕ is a **group homomorphism** *of G into G', with e' = the identity of G'. If ϕ is bijective (i.e. injective and onto) (resp. homeomorphism), it is called* **isomorphism** *of (topological) groups G, and G'. An isomorphism of G onto itself is an* **automorphism of $\boldsymbol{G}$**. *The set of all automorphisms of G forms, with respect to the group multiplication given by the compositions of mappings, a group $\boldsymbol{Aut(G)}$, called the* **automorphisms group of $\boldsymbol{G}$**. *Let any fixed $g \in G$ be given. Then the mapping*

$$g' (\in G) \mapsto g \cdot g' \cdot g^{-1},$$

defines an **inner automorphism** *of G, and all of them form the* **group of inner automorphisms $\boldsymbol{In(G)}$**. *The group $In(G)$ is a normal subgroup of $Aut(G)$, and the factor group $\boldsymbol{Aut(G)/In(G)}$ is called, [19], the* **group of external automorphisms** *of the group G.* $\diamondsuit$

The groups defined above are certain abstract sets endowed with their "inner" operations. We find usually in applications groups as some sets of transformations of some other sets of well defined (i.e. formalized) elements, e.g. some reversible motions of physical systems. Having defined a group, on the other hand, we could find some transformations of a set which act as a homomorphic image of the given group; e.g. a group of some mechanical motions can act on electromagnetic field in some electronic device. To enforce intuition about transformations of an arbitrary (in general infinite) set, we can imagine them as some "permutations" of elements of that set: The "number of elements" remains the same (transformation is invertible and onto), but at least some of elements are "replaced to places occupied before by some other replaced elements".

A.4.2. Definitions (Actions of groups).

(i) The set of (all such "permutations", i.e. of) all transformations of a set X form a group $\mathbb{G}(X)$; it will be called **the transformations group of $\boldsymbol{X}$**. *If the set X is endowed by a structure (e.g. topology, algebra, metrics, ...), the subgroup of all transformations of X consisting of the transformations preserving this structure (e.g. homeomorphism, algebraic automorphisms, isometries, ...) will be denoted by $Aut(X)$ (with a corresponding specification), and called the* **automorphism group of $\boldsymbol{X}$**. *If there is no structure specified on X (i.e. the only structure is the set-structure), then we shall use $\mathbb{G}(X)$ and $Aut(X)$ interchangeably.*

(ii) Let G be a group, and let X be a set. Let $T : g \mapsto T_g \in Aut(X),\ g \in G$, be a homomorphism of G into $Aut(X)$. The mapping T is called an **action (**or **realization) of $\boldsymbol{G}$ on $\boldsymbol{X}$**, *and the space (resp. set) X endowed with such an action is called a* **$\boldsymbol{G}$ –space**. *For any fixed $x \in X$, the set of elements $\{y \in X : \exists g \in G,\ T_g x = y\}$ is called the* **orbit of $\boldsymbol{T}$ (**equiv.: **of $\boldsymbol{G}$) through $\boldsymbol{x} \in \boldsymbol{X}$**. *The belonging to orbits is an equivalence relation. If the whole space X coincides with an orbit, it is a* **homogeneous (**equiv.: **transitive) $\boldsymbol{G}$ –space**. *We shall usually use notation $g \cdot x := T_g(x) := T_g x,\ \forall x \in X, g \in G$. Each orbit of any G–space is a transitive G–space. If G is a topological group and X a topological space, the mapping $g \mapsto T_g$ is assumed to be continuous in a certain topology on $Aut(X)$; usually it is assumed continuity on the topological product space: $G \times X \to X, (g; x) \mapsto T_g x$ is jointly continuous, cf. e.g. [207, §24]. If X is a linear space, and $T_G \subset Aut(X) = \mathcal{L}(X)$, then* **$\boldsymbol{T_G}$ is a representation of $\boldsymbol{G}$**.

(iii) Let X be a transitive G–space, and let $x \in X$ be fixed. Clearly, $G \cdot x = X$. It is $e \cdot x = x$, and the set of all $h \in G$ such, that $h \cdot x = x$ forms a (closed) subgroup $H \equiv G_x$ of G. The group G_x is the **stability subgroup (of $\boldsymbol{G}$) at $\boldsymbol{x}$**. *It is called also the* **stationary subgroup of $\boldsymbol{x}$**. *Since the left coset $g \cdot H$ consists of all the elements transforming x into $g \cdot x$, the homogeneous space X is isomorphic to the factor space G/H.* $\diamondsuit$

A.4.3. Definition. *Let G be any group. Take the space $X := G$, and define the* **left translation $\boldsymbol{g \mapsto L_g}$** *as an action of G on itself by $L_g(g') := g \cdot g'$. Then G is a transitive G–space. Similarly, another action of G on itself is defined by the* **right translations** *$R_g,\ R_g(g') := g' \cdot g$, by taking the group homomorphism $G \to \mathbb{G}(G) : g \mapsto R_{g^{-1}}$. These two actions mutually commute: $L_g R_h \equiv R_h L_g$. The mapping $g \mapsto A(g) := L_g \circ R_{g_{-1}}$ is also an action of G on itself, $A(g) \in Aut(G)$.* $\diamondsuit$

Let us turn our attention to Lie groups now.

A.4.4. Definition (Lie groups). *Let G be a manifold with such a group structure, that the group mapping $(g_1; g_2) \mapsto g_1 \cdot g_2^{-1}$ is differentiable (equiv.: continuous, equiv.: smooth, if $\dim G < \infty$) as a manifold–mapping of $G \times G \to G$. The group G endowed with such a manifold structure is a* **Lie group**.
Equivalently: The Lie group is a topological group with a C^r–manifold structure consistent with the group topology (for $\dim G < \infty$ one need not specify r). $\diamondsuit$

A.4.5. **Note**. Let us note, that the mentioned equivalence (i.e. sufficiency of mere topological manifold structure, and continuity of the group operations for smoothness of these) is the content of positive solution of the fifth Hilbert problem by Gleason [108], and Montgomery with Zippin [184], for $\dim G < \infty$. A partial solution is given in the book [207], according to which the original papers are cited here. $\heartsuit$

A.4.6. **Examples**. The following groups are simple examples of Lie groups:

(i) Abelian connected Lie groups are $\mathbb{R}^k \times \mathbb{T}^n$, where $\mathbb{T}$ is one dimensional torus (circle), and multiplication is componentwise addition ($\mod (2\pi)$ on the torus with a marked element).

(ii) The group $GL(n,\mathbb{R})$ of all real invertible $n \times n$ matrices with the matrix multiplication as the group operation, and the topology given by continuity of all matrix elements. Also all closed continuous subgroups of this group are Lie groups, e.g. $O(n), O(p,q), Sp(2p,\mathbb{R})$. Such groups G of matrices $g \in GL(n,\mathbb{R})$ can be obtained by specification of a matrix A, and by requiring, [148, p. 78]: $g^T A g = A,\ \forall g \in G$.

(iii) As an example of infinite-dimensional Lie group, [39, Chap.III.3.10, Proposition 37], let us take an infinite dimensional Hilbert space $\mathcal{H}$, and let $\mathfrak{U}$ be the group of all unitary operators on it. Then $\mathfrak{U}$ is a Lie group, if taken in the norm–topology of $\mathcal{L}(\mathcal{H})$, as a submanifold of $\mathcal{L}(\mathcal{H})$, what can be taken, in turn, as a manifold with the single chart with the identity mapping onto itself, as a B-space $\mathcal{L}(\mathcal{H})$. $\heartsuit$

Let us consider a Lie group G with unit element e, and let $\xi, \eta \in T_eG$ be arbitrary tangent vectors at e to G. We shall construct, to each ξ, a vector field $\mathbf{w}_\xi$ on the manifold G by a help of left translations, cf. Definition A.4.3, with a help of their tangent mappings, Definition A.3.6:

$$\mathbf{w}_\xi(g) := T_e L_g(\xi),\ g \in G,\ \mathbf{w}_\xi(e) := \xi, \quad \forall \xi \in T_eG. \tag{A.4.1}$$

These vector fields are **left invariant**, i.e. for any $g \in G$:

$$L_{g*}\mathbf{w}_\xi = \mathbf{w}_\xi,\ \text{i.e. } T_h L_g(\mathbf{w}_\xi(h)) \equiv \mathbf{w}_\xi(g{\cdot}h),\ \forall h \in G, \tag{A.4.2}$$

what is an immediate consequence of the definition (A.4.1). The mapping $\xi \mapsto \mathbf{w}_\xi (\xi \in T_eG)$ is linear. Conversely, all left–invariant vector fields on G are of this form. These vector fields are complete. Let us form a commutator, cf. Definition A.3.7, of two left–invariant vector fields, $[\mathbf{w}_\xi, \mathbf{w}_\eta] \in \mathcal{X}(G)$. It can be shown, that the commutator is again left invariant, hence

$$[\mathbf{w}_\xi, \mathbf{w}_\eta] =: \mathbf{w}_{[\xi,\eta]},\ [\xi,\eta] \equiv \mathbf{w}_{[\xi,\eta]}(e). \tag{A.4.3}$$

This shows, that the subspace of $\mathcal{X}(G)$ consisting of all left–invariant vector fields on G is also an algebra with respect to commutations. The mapping $\mathbf{w}_\xi \mapsto \mathbf{w}_\xi(e) \equiv \xi$ is a linear isomorphism of the space of left invariant vector fields onto T_eG; they are isomorphic also as algebras with the "commutation" $[\cdot,\cdot]$.

A.4.7. Definition. *A linear space X is a* **Lie algebra**, *if it is endowed by a* **Lie bracket**, *i.e. by a bilinear mapping $[\cdot,\cdot] : X \times X \to X, (\xi;\eta) \mapsto [\xi,\eta] \in X$, such that it is antisymmetric: $[\xi,\eta] \equiv -[\eta,\xi]$, and the* **Jacobi identity** *is fulfilled:*

$$[\xi,[\eta,\zeta]] + [\zeta,[\xi,\eta]] + [\eta,[\zeta,\xi]] = 0,\ \forall \xi,\eta,\zeta \in X. \tag{A.4.4}$$

The Lie bracket $[\xi,\eta]$ is called also the **commutator** *of the elements ξ and η. A mapping ϕ between two Lie algebras is a* **Lie algebra morphism**, *if it is linear, and conserves the Lie brackets: $\phi([\xi,\eta]) \equiv [\phi(\xi),\phi(\eta)]$. If ϕ is a bijection, it is a* **Lie algebra isomorphism**. $\diamondsuit$

We shall next consider the Lie algebras determined by given Lie groups.

A.4.8. Definitions.

(i) Since the commutator of vector fields satisfies the Jacobi identity, cf. Definition A.3.7, the tangent space T_eG is naturally endowed by the Lie algebra structure induced by that of vector fields $\mathbf{w}_\xi$. This linear space with the Lie algebra structure is the **Lie algebra of the Lie group G***; it will be denoted alternatively by $Lie(G) \equiv \mathfrak{g}$. It is considered also as topological space with the topology of T_eG. It is also a B-space, in this natural way, cf. [39, Chap.III]. The* **topological dual of $\mathfrak{g}$** *will be denoted $\mathfrak{g}^* = Lie(G)^*$.*

(ii) Let the integral curve through e of the left–invariant field $\mathbf{w}_\xi$ be denoted by $t(\in \mathbb{R}) \mapsto \exp(t\xi)(\in G)$. This curves form **one–parameter subgroups $\mathbb{R} \to G$ of G***:*

$$t_1 + t_2 \mapsto \exp((t_1 + t_2)\xi) \equiv \exp(t_1\xi)\cdot\exp(t_2\xi),\ t_j \in \mathbb{R}, \xi \in \mathfrak{g}.$$

The mapping $\xi(\in \mathfrak{g}) \mapsto \exp(\xi)(\in G)$ is called the **exponential mapping***; it is a local homeomorphism of neighbourhoods of $0 \in \mathfrak{g}$ and $e \in G$, hence its (local) inverse provides a chart of G around e.* ♢

Let us define now a representation of any Lie group G on its Lie algebra $\mathfrak{g}$. The action $A : g \mapsto A(g) := L_g \circ R_{g^{-1}}$ of G on itself is differentiable, it leaves the unit element e invariant, and its tangent at e, $T_eA(g)$, is a linear automorphism of the Lie algebra (identified with T_eG). It is an element of the wanted representation.

A.4.9. Proposition. *The linear automorphisms $Ad(g) := T_eA(g) : \mathfrak{g} \to \mathfrak{g},\ g \in G$, form a representation of G in linear endomorphisms of $\mathfrak{g}$:*

$$Ad(g_1\cdot g_2) \equiv Ad(g_1) \circ Ad(g_2),$$

(this is a consequence of the chain rule for the tangent mappings). They are also Lie algebra automorphisms:

$$Ad(g)([\xi, \eta]) \equiv [Ad(g)\xi, Ad(g)\eta].$$

The tangent of $Ad(\cdot)$ in the unit element is a linear mapping denoted by ad, $\mathrm{ad} : \xi \mapsto \mathrm{ad}_\xi$ of $\mathfrak{g}$ into $\mathcal{L}(\mathfrak{g},\mathfrak{g})$ such that the identity

$$T_eAd(\xi)\cdot\eta =: \mathrm{ad}_\xi(\eta) \equiv [\xi, \eta]$$

is satisfied. ♣

A.4.10. Definitions.

(i) The representation $g \mapsto Ad(g)$ is called the **adjoint representation of G***.*

(ii) Let $F, F' \in \mathfrak{g}^*$ be elements of the dual space of the Lie algebra $\mathfrak{g}$; their values on the elements $\xi \in \mathfrak{g}$ are denoted by $\langle F; \xi\rangle \equiv F(\xi)$, etc. Then the mappings $F \mapsto Ad^*(g)F,\ g \in G$, of $\mathfrak{g}^*$ into itself determined by

$$\langle Ad^*(g)F; \xi\rangle := \langle F; Ad(g^{-1})\xi\rangle, \quad \xi \in \mathfrak{g},\ g \in G,$$

form also a (linear) representation of G called the **coadjoint representation** *of the Lie group G.* ♢

Let the tangent spaces $T_F\mathfrak{g}^*,\ F \in \mathfrak{g}^*$ are all identified with $\mathfrak{g}^*$ in the canonical way (as in any linear space). Their duals $T_F^*\mathfrak{g}^*$ are then canonically identified with the second dual $\mathfrak{g}^{**}$ of the Lie algebra, and also $\mathfrak{g}$ is canonically included into $\mathfrak{g}^{**}$ as a $\sigma(\mathfrak{g}^{**}, \mathfrak{g}^*)$–dense subset, but in the norm topology it is identical with a norm-closed subspace of the (canonically defined) B-space $\mathfrak{g}^{**}$. Since the commutator $(\xi; \eta) \mapsto [\xi, \eta]$ is continuous in norm (from the continuity of Fréchet derivatives), it is also continuous in $\sigma(\mathfrak{g}^{**}, \mathfrak{g}^*)$ topology, if $\mathfrak{g}$ is considered as a $\sigma(\mathfrak{g}^{**}, \mathfrak{g}^*)$–dense subspace of $\mathfrak{g}^{**}$. Hence, the double dual $\mathfrak{g}^{**}$ is canonically endowed with a Lie bracket – it is also a Lie algebra. This is, clearly, trivial for $\dim G < \infty$, in what case $\mathfrak{g} = \mathfrak{g}^{**}$, by the canonical identification.

A.4.11. Definition. *Let $\mathcal{F}(\mathfrak{g}^*)$ be the space of infinitely differentiable functions on $\mathfrak{g}^*$. Then differentials $d_F f \in T_F^*\mathfrak{g}^*, f \in \mathcal{F}(\mathfrak{g}^*)$, can be (canonically) considered as elements of the Lie algebra $\mathfrak{g}^{**}$, according to the above written arguments. Let*

$$[d_F f, d_F h] \in \mathfrak{g}^{**}(\supseteq \mathfrak{g}),\ f, h \in \mathcal{F}(\mathfrak{g}^*)$$

be the corresponding commutator. Let us define the bilinear mapping

$$\{\cdot,\cdot\} : \mathcal{F}(\mathfrak{g}^*) \times \mathcal{F}(\mathfrak{g}^*) \to \mathcal{F}(\mathfrak{g}^*), \text{with } \{f, h\}(F) \equiv -\langle F; [d_F f, d_F h]\rangle,\ \forall f, h \in \mathcal{F}(\mathfrak{g}^*), \tag{A.4.5}$$

where the evaluations at $F \in \mathfrak{g}^*$ of linear functionals $\gamma \in \mathfrak{g}^{**}$ are denoted by $\gamma : F \mapsto \langle F; \gamma\rangle$. The mapping (A.4.5) is called the **Poisson bracket**, *defining the* **canonical Poisson structure on $\mathfrak{g}^*$**. $\diamondsuit$

A.4.12. Lemma. *Let G be a Lie group, and let the canonical Poisson structure $\{\cdot,\cdot\}$ on $Lie(G)^*$ be given. Let us accept the above mentioned identifications of $T_F^*Lie(G)^*$ with the second dual of $Lie(G)$. Then, for any $f \in \mathcal{F}(Lie(G)^*)$, and for an arbitrary $F \in Lie(G)^*$, the restriction to $Lie(G) \subset Lie(G)^{**}$ of the linear map*

$$\mathrm{d}_F h(\in Lie(G)^{**}) \mapsto -\langle F; [\mathrm{d}_F f, \mathrm{d}_F h]\rangle,\ h \in \mathcal{F}(Lie(G)^*), \tag{A.4.6}$$

to the Lie algebra $Lie(G)$, identified with the set of (differentials of) the functions

$$h_\xi(F) \equiv \langle F; \xi\rangle,\ \xi \in Lie(G),$$

is norm–continuous, cf. Definition A.4.8. Hence, as an element of $Lie(G)^*$, which in turn is identified with $T_F Lie(G)^*$, the map (A.4.6) can be considered as a tangent vector to $Lie(G)^*$ at the point F. With f fixed, these tangent vectors (for $F \in Lie(G)^*$) form a smooth vector field $\mathbf{v}_f$ on $Lie(G)^*$. $\clubsuit$

Proof. The Poisson bracket $\{f, g\}(F)$ is a norm–continuous bilinear form of the variables $\mathrm{d}_F f$,$\mathrm{d}_F h \in \mathfrak{g}^{**}$, hence (with the above mentioned identification) the linear functionals: $\xi \mapsto \langle F; [\mathrm{d}_F f, \xi]\rangle$, are norm continuous on $\mathfrak{g}$, representing some vectors $\mathbf{v}_f(F) \in T_F\mathfrak{g}^*$. Let $\tilde{\eta} \in \mathfrak{g}^{**}$ be an arbitrary element. Then the (bounded linear) function $h_{\tilde{\eta}} :\ F(\in \mathfrak{g}^*) \mapsto \langle F; \tilde{\eta}\rangle$ is smooth, $h_{\tilde{\eta}} \in \mathcal{F}(\mathfrak{g}^*)$, and its differential $\mathrm{d}_F h_{\tilde{\eta}}$ (in any point F) is identified with $\tilde{\eta}$ itself. Hence, the mapping $f(\in \mathcal{F}(\mathfrak{g}^*)) \mapsto \mathrm{d}_F f \in \mathfrak{g}^{**}$ is onto. Since the functions f, h are smooth (in the sense of the underlying norm–topology), all the functions $F \mapsto \mathrm{d}_F h(\mathbf{v}_f(F)),\ h \in \mathcal{F}(\mathfrak{g}^*)$ are also smooth. This, due the Leibniz property of derivatives, implies smoothness of $\mathbf{v}_f$. $\square$

Now we can define some of the key structures for the present paper.

A.4.13. Definitions.

(i) Let $f \in \mathcal{F}(\mathfrak{g}^*)$ be given. The vector field $\mathbf{v}_f$ on $\mathfrak{g}^*$ determined (according to Lemma A.4.12) by the canonical Poisson structure:

$$\mathrm{d}_F h(\mathbf{v}_f(F)) \equiv \{f, h\}(F),\ \forall h \in \mathcal{F}(\mathfrak{g}^*)$$

is called the **Hamiltonian vector field** *corresponding to the* **Hamiltonian function f**. *Let us denote by φ^f the local flow of $\mathbf{v}_f$, called the* **Hamiltonian flow of f**. *Let φ^ξ be the Hamiltonian flow of the linear function h_ξ. Then we have*

$$\varphi_t^\xi(F) \equiv Ad^*(\exp(t\xi))F.$$

The **stability subgroup** *G_F of the coadjoint action at $F \in \mathfrak{g}^*$ is a Lie subgroup of G generated by those $\xi \in \mathfrak{g}$, for which*

$$\langle F; [\xi, \eta]\rangle = 0,\ \forall \eta \in \mathfrak{g},$$

cf. Lemma 2.2.20. The Lie algebra generated by these elements is **stability Lie algebra of F** *with respect to the $Ad^*(G)$–representation.*

(ii) The sets $Ad^*(G)F := \{Ad^*(g)(F) : g \in G\}$ are **coadjoint orbits $\mathcal{O}_F(G)$ of G**. *They are identical with the symplectic leaves (cf. Section 1.4, Definition 1.4.1) of this Poisson structure. They are conserved by all the Hamiltonian flows: $\varphi_t^f F \in Ad^*(G)F := \mathcal{O}_F(G),\ \forall t \in \mathbb{R}$. In this sense, all the vectors $\mathbf{v}_f(F'),\ F' \in \mathcal{O}_F(G), f \in \mathcal{F}(\mathfrak{g}^*)$, are* **tangent vectors to the leaf $\mathcal{O}_F(G)$**. *(These "tangent vectors" needn't form a closed tangent space to a coadjoint orbit $Ad^*(G)F$ for a general $F \in \mathfrak{g}^*$, cf. Proposition 2.1.5. For $\dim G < \infty$, all the $Ad^*(G)F$ are smooth submanifolds on $\mathfrak{g}^*$, hence the ("tangent vectors") $\equiv$ (tangent vectors) now.)*

(iii) Let us define, on each $\mathcal{O}_F(G)$, a two form $F \mapsto \Omega_F$ by defining it for all the tangent vectors to $\mathcal{O}_F(G)$ by

$$\Omega_F(\mathbf{v}_f(F), \mathbf{v}_h(F)) := \{f, h\}(F),\ \forall f, h \in \mathcal{F}(\mathfrak{g}^*).$$

This is a well defined (i.e. it depends only on the vectors $\mathbf{v}_f(F), \ldots$, and not on the various functions $f, \ldots$ giving the same vectors), closed (from the Jacobi identity for the commutator in $\mathfrak{g}$), weakly nondegenerate two–form on $\mathcal{O}_F(G)$called the **canonical symplectic form on the coadjoint orbit $\mathcal{O}_F(G)$**. *Endowed with this form, $\mathcal{O}_F(G)$ is a (weakly) symplectic manifold, called the* **symplectic leaf of $\mathfrak{g}^*$**. $\diamondsuit$

It is clear, that the Hamiltonian flows φ^f of the canonical Poisson structure on $\mathfrak{g}^*$ are identical on each orbit $\mathcal{O}_F(G)$ with the symplectic flows corresponding to the Hamiltonian functions which are equal on $\mathcal{O}_F(G)$ to the restrictions of f's to that orbit.

B On Bounded Operators and C^*-algebras

Conventional nonrelativistic QM is (or can be) formulated with a help of the algebra $\mathcal{L}(\mathcal{H})$ of all bounded operators on a separable Hilbert space $\mathcal{H}$, [74, 230, 194, 189, 201, 181]. This is essentially true also for the conventional (but mathematically largely heuristic) quantum field theory

(QFT), [236, 142]. That such a formulation is not satisfactory for systems with infinite number of degrees of freedom became clear at least since the Haag's paper on nonexistence of the **"interaction representation"** in cases of nontrivially interacting fields, [121]. The problems of description of "infinite systems" (i.e. quantum fields, as well as infinite–particle "thermodynamic" systems) were connected with the mathematical phenomenon of appearance of "inequivalent representations", either of CCR, or CAR, or in some other way defined sets of observables. This phenomenon was formalized in the framework of QFT by Araki, Haag and Kastler in terms of C^*-algebras. It offered possibilities to describe in mathematically well defined terms also such physical phenomena as **phase transitions**, [91, 42, 238, 228], or, more generally, **collective phenomena** in "large systems", including "macroscopic (classical) variables" of large quantal systems.

We shall give here a brief description of several basic concepts of the theory of C^*-algebras important for understanding of description of "the quantum world", including our nonlinear extensions of QM: These last mentioned applications to finite systems with nonlinear "quantum rules of behaviour", can also be included into the (linear) C^*-algebraic formalism; in that connection, C^*-algebras composed of operator–valued functions on a Hamiltonian (better: Poisson) phase space consisting of, e.g., density matrices of the traditional QM, were introduced, cf. Definition 2.3.3; these density matrices are here, perhaps rather paradoxically, in a rôle of (in the presented proposal of interpretation of EQM) **classical macroscopic parameters** — they can be considered in this place as classical fields describing a "macroscopic background" of the considered microsystem, cf.3.4.

B.1 Bounded operators on Hilbert space

A **linear operator $\boldsymbol{A}$** on an infinite–dimensional Hilbert space is a linear mapping $A : D(A) \to \mathcal{H}$ of a linear subset $\boldsymbol{D(A) \subset \mathcal{H}}$ called the **domain of $\boldsymbol{A}$**, into $\mathcal{H}$. If possible, we shall assume that $D(A)$ is dense in $\mathcal{H}$. For bounded operators A it is always either $D(A) = \mathcal{H}$, or the domain is a closed subspace of $\mathcal{H}$. We shall assume that, for bounded A's, if not explicitly stated a contrary, the domain is $D(A) = \mathcal{H}$.

Bounded linear operators $A : \mathcal{H} \to \mathcal{H}$ on a complex Hilbert space $\mathcal{H}$ form a specific **Banach algebra with involution**, i.e. they are endowed with a natural norm $\|A\| := \sup\{\|A\psi\| : \psi \in \mathcal{H}, \|\psi\| \leq 1\}$ (with the Hilbert–space scalar product $(\varphi, \psi) = \overline{(\psi, \varphi)}$, and $\|\psi\| := \sqrt{(\psi, \psi)}$); their product $AB := A \circ B$, and the (adjoint–linear, i.e. antilinear, involution, i.e. the) *–operation $\langle^*\rangle : A \mapsto A^*, (A^*\psi, \varphi) \equiv (\psi, A\varphi)$, satisfying also (besides the associative linear algebra and the Banach space properties):

$$\begin{aligned} &(A^*)^* \equiv A,\ (AB)^* = B^*A^*, \\ &\|AB\| \leq \|A\| \cdot \|B\|,\ \|A^*\| \equiv \|A\|,\ \|A^*A\| \equiv \|A\|^2, \end{aligned} \tag{B.1.1}$$

and the B-space of all such operators is denoted by $\boldsymbol{\mathcal{L}(\mathcal{H})}$. The elements $A = A^*$ are **selfadjoint**. The operator $I_{\mathcal{H}} \equiv I \equiv \mathbb{I}$, for which $IA = AI = A$ $(\forall A \in \mathcal{L}(\mathcal{H}))$, is the **identity (or unit element) of $\boldsymbol{\mathcal{L}(\mathcal{H})}$**. If, for a given A, there is an $A' \in \mathcal{L}(\mathcal{H})$ such, that $A'A = AA' = I_{\mathcal{H}}$, it is called the **inverse of $\boldsymbol{A}$**, denoted $A' =: A^{-1}$, and A is called an **invertible operator**; clearly, $(A')^{-1} = A$. The subset of all invertible elements of $\mathcal{L}(\mathcal{H})$ will be denoted $\boldsymbol{GL(\mathcal{H})}$. The operators $U \in GL(\mathcal{H}) : U^* = U^{-1}$ are called **unitary**, and compose a subset of $\mathcal{L}(\mathcal{H})$ – the

infinite–dimensional Lie group, [39, Chap.III] denoted by $\mathfrak{U}$ (:= the **unitary group of $\mathcal{H}$**). For any given $A \in \mathcal{L}(\mathcal{H})$, the set of complex numbers $\rho(A) := \{\lambda \in \mathbb{C} : (\lambda\mathbb{I} - A) \in GL(\mathcal{H})\}$ is called the **resolvent set of A**; it is an open subset of $\mathbb{C}$. Its complement $\sigma(A) \equiv sp(A)$ is called the **spectrum of A: $\sigma(A) := \mathbb{C} \setminus \rho(A)$**. The spectrum contains also all the **eigenvalues of A**, i.e. the numbers $\lambda_j \in \mathbb{C}$, for which there are some (nonzero) vectors $\varphi_j \in \mathcal{H}$ such that

$$A\varphi_j = \lambda_j\varphi_j, j \in J \ (:= \text{an index set}). \tag{B.1.2}$$

Dimension of the subspace of $\mathcal{H}$ spanned by all the vectors $\varphi_j \in \mathcal{H}$ satisfying (B.1.2) for the same complex value of λ_j is called the **degeneracy of λ_j**, it will be denoted deg(λ_j). Let $A = A^*$. Then $\sigma(A) \subset \mathbb{R}$. The set of eigenvalues is denoted by $\sigma_{pp}(A)$. The closure of the set of all the eigenvalues of A: $\overline{\sigma_{pp}(A)} =: \sigma_p(A) \subset \sigma(A)$ is called the **pure–point spectrum**.

If the vectors $\varphi_j, j \in J$, (B.1.2), form a basis in $\mathcal{H}$, the spectrum of the operator A reduces to the pure–point spectrum: $\sigma(A) = \sigma_p(A)$. Otherwise, A has also some **continuous spectrum**. As subsets of $\sigma(A)$, these two parts of spectra needn't be disjoint. The spectrum of any bounded operator A is compact, enclosed in the closed disc centered in $0 \in \mathbb{C}$ of radius $\|A\|$.

The selfadjoint operators form the subspace (a real B-space) $\mathcal{L}(\mathcal{H})_s$; they have spectra lying on the real line: $\sigma(A) \subset \mathbb{R}$. The selfadjoint operators A with positive (i.e. nonnegative) spectra are called **positive operators**, what is denoted by $A \geq 0$, or also $A > 0$.

The (positive) operators $P \in \mathcal{L}(\mathcal{H})$ such, that $P = P^2 = P^*$ are called **(orthogonal) projections, or projectors**. There is a natural bijection between projectors and closed subspaces: $\mathcal{H}_P := P\mathcal{H}(\subset \mathcal{H}) \leftrightarrow P$. The projection onto the one–dimensional subspace spanned by a nonzero vector $\psi \in \mathcal{H}$ is denoted by P_ψ. Projectors P_1, P_2 are **mutually orthogonal** iff $P_1P_2 = 0$. The projector onto the subspace of $\mathcal{H}$ spanned by all eigenvectors of a selfadjoint operator A corresponding to the same eigenvalue λ is its **eigenprojector $E_A(\{\lambda\})$**. The dimension of the **eigenspace $\mathcal{H}_\lambda$**:= $E_A(\{\lambda\})\mathcal{H}$ is deg(λ).

Important objects for analysis of structure and of representations of a Banach algebra $\mathcal{A}$ are its **left (resp. right, resp. two–sided) ideals**, i.e. such linear subsets $\mathcal{J} \subset \mathcal{A}$, $\{0\} \neq \mathcal{J} \neq \mathcal{A}$, that multiplication of their elements *by an arbitrary element B of $\mathcal{A}$* from left (resp. right, resp. any) side gives again elements from $\mathcal{J}$, i.e. $\forall B \in \mathcal{A} : B\mathcal{J} \subset \mathcal{J}$, resp. $\mathcal{J}B \subset \mathcal{J}$, resp. $B\mathcal{J} \cup \mathcal{J}B \subset \mathcal{J}$. Two–sided ideals are called just **ideals**. It follows that an (also one–sided) ideal $\mathcal{J} \subset \mathcal{A}$ is also a subalgebra of $\mathcal{A}$. For $\mathcal{A} := \mathcal{L}(\mathcal{H})$, and $\mathcal{H}$ separable, there is only norm–closed ideal $\mathfrak{C}$ in $\mathcal{L}(\mathcal{H})$, [187, §22] consisting of all **compact operators**, i.e. such linear operators on $\mathcal{H}$, that map any norm–bounded subset of $\mathcal{H}$ into a norm–compact subset of $\mathcal{H}$. There are other important ideals in $\mathcal{L}(\mathcal{H})$, which are subsets of $\mathfrak{C}$, e.g. the set of all **Hilbert–Schmidt operators**: $\mathfrak{H}$, and its subset $\mathfrak{T}$ of all **trace–class operators** in $\mathcal{L}(\mathcal{H})$; these ideals are characterized below. All these sets are (as are all twosided ideals) **symmetric**, i.e. they are invariant with respect to the involution $\langle^*\rangle$. Hence, they are generated by their selfadjoint elements: each their element A can be decomposed into the complex–linear combination of its two selfadjoint elements:

$$A = \frac{A + A^*}{2} + i\frac{A - A^*}{2i}.$$

The ideal $\mathfrak{T}$ contains exactly those selfadjoint A which have pure point spectra, and the set of all their eigenvalues is absolutely summable (by respecting the degeneracy):

$$A^* = A \in \mathfrak{T} \Leftrightarrow A \textit{ has pure point spectrum and } \sum_{\lambda \in \sigma_{pp}(A)} \deg(\lambda)|\lambda| =: \|A\|_1 < \infty.$$

Then we can define the finite number (for $A = A^*$)

$$Tr(A) := \sum_{\lambda \in \sigma_{pp}(A)} \deg(\lambda)\lambda,$$

called the **trace of A**. Its value does not depend on unitary transformations: $Tr(A) = Tr(UAU^*), \forall U \in \mathfrak{U}$. The trace $A \mapsto Tr(A)$ can be uniquely extended to the whole complex space $\mathfrak{T}$ by linearity. Then it is defined for all products (since $\mathfrak{T}$ is an ideal) $BA : B \in \mathcal{L}(\mathcal{H}),\ A \in \mathfrak{T}$, and we have

$$Tr(AB) \equiv Tr(BA), \forall B \in \mathcal{L}(\mathcal{H}), A \in \mathfrak{T}.$$

It is also valid:

$$\|AB\|_1 \leq \|A\|_1\|B\|,\ \|BA\|_1 \leq \|A\|_1\|B\|, \quad \forall A \in \mathfrak{T}, B \in \mathcal{L}(\mathcal{H}). \tag{B.1.3}$$

The Hilbert–Schmidt ideal $\mathfrak{H}$ is defined:

$$A \in \mathfrak{H} \Leftrightarrow A \in \mathcal{L}(\mathcal{H})\ \&\ A^*A \in \mathfrak{T}.$$

Then also $AB \in \mathfrak{T}$ for all $A, B \in \mathfrak{H}$; the algebra $\mathfrak{H}$ of operators in $\mathcal{H}$ can be made a Hilbert space in a canonical way, by defining the scalar product by

$$(A, B)_2 := Tr(A^*B),\ \forall A, B \in \mathfrak{H}.$$

The set $\mathfrak{H}$ is closed with respect to the **Hilbert–Schmidt norm $\|A\|_2 := \sqrt{Tr(A^*A)}$**. Also $\mathfrak{T}$ is closed with respect to the **trace norm $\|A\|_1 := Tr|A|$**, where the operator $|A|$, the **absolute value of A** can be defined by a "functional calculus" as $|A| := \sqrt{A^*A}$. The elements of the subset $\mathfrak{T}_{+1} \subset \mathfrak{T}$ of positive trace–class operators with unit norm:

$$\varrho \in \mathfrak{T}_{+1} \Leftrightarrow \varrho \in \mathfrak{T}\ \&\ \varrho > 0\ \&\ \|\varrho\|_1 = 1, \tag{B.1.4}$$

are called in physics the **density matrices**.

The bilinear form

$$\langle B; A\rangle := Tr(BA),\ A \in \mathfrak{T},\ B \in \mathcal{L}(\mathcal{H}),$$

provides a **duality** between the B-spaces $\mathfrak{T}$, and $\mathcal{L}(\mathcal{H})$, and similarly between the B-space of compact operators $\mathfrak{C}$, and $\mathfrak{T}$, in the sense that the operators B from the second of a couple of spaces represent (all) continuous linear functionals l_B on the first of spaces, by the evaluations

$$A \mapsto l_B(A) := Tr(BA) \equiv \langle B; A\rangle.$$

In this sense, the following assertions are valid for the topological duals:

$$\mathfrak{C}^* = \mathfrak{T}, \quad \mathfrak{T}^* = \mathcal{L}(\mathcal{H}) = \mathfrak{C}^{**}. \tag{B.1.5}$$

Since the mathematical objects $A, B, \cdots \in \mathcal{L}(\mathcal{H})$ are not only elements of a Banach algebra, but they also realize linear transformations of $\mathcal{H}$, they are endowed by other natural l.c. topologies. Let us introduce the **weak operator topology $\mathcal{T}_w$** by the set of seminorms $\{p^w_\psi : \psi \in \mathcal{H}\}$:

$$p^w_\psi : A \mapsto p^w_\psi(A) := |(\psi, A\psi)|, \quad \psi \in \mathcal{H}, A \in \mathcal{L}(\mathcal{H}). \tag{B.1.6}$$

The **strong operator topology $\mathcal{T}_s$** on $\mathcal{L}(\mathcal{H})$ is determined by the seminorms $\{p^s_\psi : \psi \in \mathcal{H}\}$:

$$p^s_\psi : A \mapsto p^s_\psi(A) := \|A\psi\|, \quad \psi \in \mathcal{H}, A \in \mathcal{L}(\mathcal{H}). \tag{B.1.7}$$

There are important also other topologies on $\mathcal{L}(\mathcal{H})$, namely the **σ–strong (equiv.: ultrastrong) topology $\mathcal{T}_{us}$**, the **σ–weak (equiv.: ultraweak) topology $\mathcal{T}_{uw}$**, and also the **strong* topology $\mathcal{T}_{s*}$**, and the **σ–strong* (equiv.: ultrastrong*) topology $\mathcal{T}_{us*}$**, [187, 91, 42, 254]. These topologies, including also the **norm topology $\mathcal{T}_n$**, are ordered in the following hierarchy (with respect to the ordering $\prec$ introduced in Definitions A.1.1):

$$\begin{array}{ccccccc} \mathcal{T}_w & \prec & \mathcal{T}_s & \prec & \mathcal{T}_{s*} & & \\ \curlywedge & & \curlywedge & & \curlywedge & & \\ \mathcal{T}_{uw} & \prec & \mathcal{T}_{us} & \prec & \mathcal{T}_{us*} & \prec & \mathcal{T}_n \end{array} \tag{B.1.8}$$

We did not considered now the topological dual of $\mathcal{L}(\mathcal{H})$ with respect to the norm–topology: $\mathcal{L}(\mathcal{H})^*$ contains also "nonnormal states" on $\mathcal{L}(\mathcal{H})$, cf. Definition B.3.1. In QM, mainly linear functionals from $\mathfrak{T} \subset \mathcal{L}(\mathcal{H})^*$ are used in the rôle of (normal) quantum states described by density matrices. The space $\mathfrak{T}$ can be considered as the dual of $\mathcal{L}(\mathcal{H})$, if the last space is endowed with the $\sigma(\mathcal{L}(\mathcal{H}), \mathfrak{T})$–topology, which is identical with the σ–weak topology $\mathcal{T}_{uw}$ determined by all density matrices ϱ by seminorms $p^{uw}_\varrho : A(\in \mathcal{L}(\mathcal{H})) \mapsto p^{uw}_\varrho(A) := |Tr(\varrho A)|$. The "nonnormal" states from $\mathcal{L}(\mathcal{H})^* \setminus \mathfrak{T}$ include, e.g. **dispersionless states for observables with purely continuous spectra**, e.g. the "eigenstates" for position coordinates, [27].

Theoretical physics is mainly interested in selfadjoint operators, resp. in unitary operators (these all belong to the "equally nice" **normal operators** A characterized by $AA^* = A^*A$). Normal operators can have their spectrum also in $\mathbb{C} \setminus \mathbb{R}$. Selfadjoint operators A (not only bounded) are generators of one–parameter groups of unitaries: $t \mapsto \exp(itA) \in \mathfrak{U}$, and also are representatives of "observables" in QM; the most clear understanding of their interpretation is expressed, perhaps, via the **spectral theorem**. This theorem shows, that any selfadjoint operator can be, roughly speaking, expressed as a real linear combination (resp. integral) of mutually orthogonal "eigenprojections", which are multiplied by the corresponding "eigenvalues".

The key concept in this connection is a projection–valued measure (PM, or PVM). Let us introduce simultaneously its generalizations, i.e. positive operator valued measures (POV, or POVM).

B.1.1. Definitions (Projection measures, and POV measures).

(i) Let $(X; \mathcal{T})$ be a topological space, and $\mathcal{B}(X) \subset \mathcal{P}(X)$ be the set of all subsets obtained from the open and closed *subsets of X by countable unions and/or intersections. Elements $\Lambda \in \mathcal{B}(X)$ are called* **Borel sets of X**. *The class of Borel sets is topology–dependent; if, on some set Y, the topology is standard (e.g., on $\mathbb{R}^n$), then the specification of $\mathcal{B}(Y)$ is not usually given. A function f from a topological space $(X_1; \mathcal{T}_1)$ into a topological space $(X_2; \mathcal{T}_2)$ is called a* **Borel function** *iff for any $V \in \mathcal{B}(X_2)$ the inverse image $f^{-1}[V] \in \mathcal{B}(X_1)$; the set of all such uniformly bounded Borel functions will be denoted by $\mathcal{B}_b(X_1, X_2)$.*

(ii) Let a mapping $E : \mathcal{B}(X) \to \mathcal{L}(\mathcal{H})$ be such that each $E(\Lambda)$ is an orthogonal projection, and
(a) $E(X) = \mathbb{I},\ E(\emptyset) = 0$;
(b) for any at most countable collection $\Lambda_j, j \in J, |J| \leq \aleph_0$ of mutually disjoint Borel sets $\Lambda_j \in \mathcal{B}(X)$, $\Lambda_j \cap \Lambda_k = \emptyset\ (\forall j \neq k)$, one has

$$E(\cup_{j\in J}\Lambda_j) = \sum_{j\in J} E(\Lambda_j),$$

where the sum converges in the strong operator topology of $\mathcal{L}(\mathcal{H})$.

The mapping E is a **projection (valued) measure (PM,** equiv. **PVM)** *(on X with values in $\mathcal{L}(\mathcal{H})$).*

(iii) Let $S \subset \mathcal{L}(\mathcal{H})$ be any set of bounded operators. The **commutant S' of S** *is the set $S' := \{B \in \mathcal{L}(\mathcal{H}) : AB - BA = 0, \forall A \in S\} \subset \mathcal{L}(\mathcal{H})$. Clearly, $S \subset S'' := (S')'$, and S'' is called the* **bicommutant of S**.

(iv) Let a mapping $F : \mathcal{B}(X) \to \mathcal{L}(\mathcal{H})$ be such that each $F(\Lambda)$ is a positive operator, and
(a) $F(X) = \mathbb{I},\ F(\emptyset) = 0$;
(b) for any at most countable collection $\Lambda_j, j \in J, |J| \leq \aleph_0$ of mutually disjoint Borel sets $\Lambda_j \in \mathcal{B}(X)$ one has

$$F(\cup_{j\in J}\Lambda_j) = \sum_{j\in J} F(\Lambda_j),$$

where the sum converges in the strong operator topology of $\mathcal{L}(\mathcal{H})$.

The mapping F is a **normalized positive operator valued (POV) measure** *(on X with values in $\mathcal{L}(\mathcal{H})$), called also* **an observable on X,** resp. also **a POVM,***[71, Sec.3.1].* $\diamondsuit$

The POV measures are generalizations of PM; they are useful in description of "nonideal measurements" in QM, cf. [71]. Let's note, that now the operators $F(\Lambda)$ for different $\Lambda \in \mathcal{B}(X)$ needn't mutually commute. There is an important construction giving also a criterion for distinction of PM from POVM, cf. [71]:

B.1.2. Proposition. *Let X be a compact Hausdorff space, and let $F : \mathcal{B}(X) \to \mathcal{L}(\mathcal{H})$ be a normalized POV measure (i.e. an observable). Let $C(X)$ be the space of complex–valued continuous functions on X. Then the strongly convergent integral*

$$F(f) := \int_X f(x)F(\mathrm{d}x),\ f \in C(X), \tag{B.1.9a}$$

defines a bijection between observables $F(\cdot)$ and linear maps $F : C(X) \to \mathcal{L}(\mathcal{H})$ such that $f \geq 0 \Rightarrow F(f) \geq 0, F(\mathbb{I}) = I_{\mathcal{H}}$. The POV measure F is PM iff the map $F : C(X) \to \mathcal{L}(\mathcal{H})$ is a *–homomorphism of the algebra $C(X)$ into the algebra $\mathcal{L}(\mathcal{H})$. $\clubsuit$

B.1.3. Theorem (Spectral theorem). *Let $A \in \mathcal{L}(\mathcal{H})$ be a normal operator. Then there is a unique PM, E_A, on the spectrum $\sigma(A)$ such that:*
(i) $E_A(\Lambda) \in \{A\}'',\ \forall \Lambda \in \mathcal{B}(\sigma(A))$;[97]

[97] $\{A\}''$ is the bicommutant of the one–point set $\{A\} \subset \mathcal{L}(\mathcal{H})$.

(ii) For any $f \in \mathcal{B}_b(\sigma(A), \mathbb{C})$, there is an operator $E_A(f) \equiv f(A)$ given by the strongly convergent integral (being the strong-operator limit of any sequence of expressions $E_A(f_n)$ for "simple functions" $f_n(\lambda) := \sum_j c_j^{(n)} \chi_{\Lambda_j^{(n)}}(\lambda)$ approximating f by pointwise limits)

$$E_A(f) \equiv f(A) = \int_{\sigma(A)} f(\lambda) E_A(\mathrm{d}\lambda); \tag{B.1.9b}$$

(iii) The mapping $E_A : \mathcal{B}_b(\sigma(A), \mathbb{C}) \to \mathcal{L}(\mathcal{H}),\ f \mapsto E_A(f) = f(A)$, is a unique continuous *–homomorphism (called **the functional calculus***) of commutative algebras (in $\mathcal{B}_b(\sigma(A), \mathbb{C})$, the multiplication and addition are defined pointwise, and involution is the complex conjugation) determined by $E_A(\mathrm{id}_{\mathbb{C}}) = A$. Continuity is here understood so that $\|f(A)\| \leq \|f\|$, where the norm of f is the supremum norm.* ♣

Important features of the algebras of operators, also from a physical point of view, are their representations as homomorphic images in some $\mathcal{L}(\mathcal{K})$, where $\mathcal{K}$ is a complex Hilbert space. There are two kinds of nonzero representations of the algebra $\mathcal{L}(\mathcal{H})$, if $\mathcal{H}$ is separable [187]: There are orthogonal multiples of the identical representation, and the representations setting the ideal $\mathfrak{C}$ into zero, in which case the (simple) factoralgebra $\mathcal{L}(\mathcal{H})/\mathfrak{C}$ (= the *Calkin algebra*) is isomorphically represented. Representations of the first mentioned kind are "trivial", and that of the second kind are "physically irrelevant" (with respect to the standard nonrelativistic QM), since it might be difficult to interpret states, in which all finite–dimensional projections in the "given algebra of observables" $\mathcal{L}(\mathcal{H})$ are mapped to zero (probabilities of values of all quantities with pure point spectra with finite degeneracies would be zero!); cf., however, Note B.4.1. More "colourful" picture of "physically interesting" representations of algebras of observables arise for some closed symmetric subalgebras of $\mathcal{L}(\mathcal{H})$, and, more generally, for general C^*-algebras (these might not be faithfully represented on separable Hilbert spaces).

B.2 Elementary properties of C^*-algebras and W^*-algebras

We shall reformulate now algebraic properties of $\mathcal{L}(\mathcal{H})$ to be able to obtain a more general framework for quantum theories (QT). All (mathematical) fields of scalars will be the complex numbers $\mathbb{C}$, and in the natural restriction also the field of reals $\mathbb{R}$.

B.2.1. Definitions (C^*-algebras and W^*-algebras).

(i) A **Banach algebra** *$\mathfrak{A}$ is a B-space endowed with an associative and distributive multiplication (i.e. the* **algebraic product***, resp. the* **product***, converting the linear space $\mathfrak{A}$ into* **an algebra***): $(x; y)(\in \mathfrak{A} \times \mathfrak{A}) \mapsto x \cdot y \equiv xy (\in \mathfrak{A})$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, $(x + \lambda y) \cdot z = x \cdot z + \lambda y \cdot z$, $x \cdot (y + \lambda z) = x \cdot y + \lambda x \cdot z$; the multiplication is connected with the norm in $\mathfrak{A}$ by the requirement: $\|xy\| \leq \|x\| \|y\|; \forall x, y, z \in \mathfrak{A}, \lambda \in \mathbb{C}$. If $xy = yx, \forall x, y \in \mathfrak{A}$, the algebra $\mathfrak{A}$ is called* **abelian***, resp.* **commutative**.

(ii) If there is an element $e \in \mathfrak{A}$ such that $e \cdot x = x \cdot e = x, \forall x \in \mathfrak{A}$, the Banach algebra $\mathfrak{A}$ is a **unital algebra** *and the element $\boldsymbol{e}$* **is the unit element** *(or* **unit***) of $\mathfrak{A}$. If a unit exists in $\mathfrak{A}$, it is unique. If there is, for an element $x \in \mathfrak{A}$, an element (denoted by) $x^{-1} \in \mathfrak{A}$ such that $x \cdot x^{-1} = x^{-1} \cdot x = e$, the element $\boldsymbol{x}$* **is invertible***, and $\boldsymbol{x^{-1}}$* **is the inverse of** *$\boldsymbol{x}$. If x is invertible, the inverse element x^{-1} for x is unique; then also x^{-1} is invertible, and $(x^{-1})^{-1} = x$. The set of all invertible elements in $\mathfrak{A}$ is its* **general linear group** *$\boldsymbol{G(\mathfrak{A})}$, denoted also by $\boldsymbol{\mathfrak{A}^{-1}}$.*

(iii) An algebra $\mathfrak{A}$ is **symmetric**, *if there is defined in it an* **(antilinear) involution**, *i.e. a mapping $\langle^*\rangle : \mathfrak{A} \to \mathfrak{A}, x \mapsto x^* : x^{**} := (x^*)^* \equiv x,\ (x+\lambda y)^* \equiv x^* + \overline{\lambda} y^*$; and, moreover, this involution is connected with the product by: $(xy)^* \equiv y^* x^*$. We shall call the linear combination, product and involution* **the algebraic operations**.

(iv) If, for a symmetric Banach algebra $\mathfrak{A}$, it is satisfied the **C^*–property**: *$\|x^* \cdot x\| \equiv \|x\|^2$, then* **$\mathfrak{A}$ is a C^*–algebra**. *If $\mathfrak{A}$ has also the unit element, then it is a* **unital C^*–algebra**. *We shall usually assume, that $\mathfrak{A}$ is unital, and the converse will be pointed out. A Banach subspace $\mathfrak{B}$ of a C^*-algebra $\mathfrak{A}$, which is invariant with respect to all the algebraic operations applied to its elements is a* **C^*–subalgebra of $\mathfrak{A}$**. *If $\mathfrak{B} (\neq \mathfrak{A})$ is, moreover, invariant with respect to the multiplication by all elements of $\mathfrak{A}$, it is a* **closed (two–sided) ideal**; *clearly, such a C^*-subalgebra $\mathfrak{B}$ does not contain the unit element of $\mathfrak{A}$. An element x of a C^*-algebra is:* **selfadjoint** *iff $x^* = x$;* **normal** *iff $x^* x = x x^*$;* **projection** *iff $x = x^* = x^2$;* **partial isometry** *iff $x x^*$ is a projection;* **unitary** *(in a unital algebra) iff $x x^* = x^* x = e$.*

(v) If, for a C^*-algebra $\mathfrak{A}$, as a B-space, there is another B-space (denoted by) $\mathfrak{A}_*$ such that $\mathfrak{A}$ is (isomorphic to) its topological dual: $(\mathfrak{A}_*)^* = \mathfrak{A}$, the C^*-algebra $\mathfrak{A}$ is called a **W^*-algebra**, *and the Banach space $\mathfrak{A}_*$* is its **predual**. *Any C^*-algebra has at most one predual, up to isomorphisms. The W^*-algebras (originally: their specific operator realizations) are called also* **von Neumann algebras**. *Any W^*-algebra is a unital C^*-algebra. Any W^*-algebra is generated by its projections (via $\sigma(\mathfrak{A}, \mathfrak{A}_*)$–closure of their linear combinations). [A general C^*-algebra needn't have any nontrivial projection.]*

(vi) Let $\mathfrak{A}$ be a C^*-algebra, and let $\mathfrak{A}^{**} := (\mathfrak{A}^*)^*$ be its second topological dual. The C^*-algebra $\mathfrak{A}$ is canonically embedded into $\mathfrak{A}^{**}$ as a $\sigma(\mathfrak{A}^{**}, \mathfrak{A}^*)$–weakly dense linear subspace, cf. Definition A.1.4 and, in this topology, all the algebraic operations (i.e. the linear combination, addition, multiplication – with one of the multiplicands fixed, and the involution) are continuous. Hence, the algebraic structure of $\mathfrak{A}$ can be unambiguously extended to the whole $\mathfrak{A}^{**}$, endowing this by a (canonical) C^*-algebraic structure. The obtained W^*-algebra $\mathfrak{A}^{**}$ is denoted also $\mathfrak{A}''$, and it is called **the universal enveloping W^*–algebra of the C^*–algebra $\mathfrak{A}$.**[98]

(vii) The **centre $\mathcal{Z}(\mathfrak{A})$** *of a C^*-algebra $\mathfrak{A}$ is the commutative C^*-subalgebra of $\mathfrak{A}$ consisting of all elements of $\mathfrak{A}$ commuting with any element of $\mathfrak{A}$: $\mathcal{Z}(\mathfrak{A}) := \{z \in \mathfrak{A} : z \cdot x - x \cdot z = 0, \forall x \in \mathfrak{A}\}$. A von Neumann algebra $\mathfrak{B}$ with trivial centre: $\mathcal{Z}(\mathfrak{B}) = \{\lambda \cdot e : \lambda \in \mathbb{C}\}$, is called a* **factor**. $\diamondsuit$

B.2.2. **Note** *(Quotient C^*-algebra).* The factor–space (resp. the quotient space) $\mathfrak{A}/\mathfrak{B}$ of a C^*-algebra $\mathfrak{A}$ over its closed ideal $\mathfrak{B}$ is canonically endowed with the structure of a C^*-algebra. Let the canonical projection be $\beta : \mathfrak{A} \to \mathfrak{A}/\mathfrak{B}, x \mapsto \beta_x := \{y \in \mathfrak{A} : y = x - z, z \in \mathfrak{B}\}$, and $\beta_x \cdot \beta_y := \beta_{xy}, \beta_x^* := \beta_{x^*}, \|\beta_x\| := \inf\{\|x - z\| : z \in \mathfrak{B}\}$. Then all the "$C^*$–properties" for $\mathfrak{A}/\mathfrak{B} := \{\beta_x : x \in \mathfrak{A}\}$ are valid, cf. [77, 1.8.2]. $\heartsuit$

Let us give a list of examples of C^*-algebras:

B.2.3. **Examples** *(Some C^*-algebras and W^*-algebras).*

(i) Since the dual of the trace class operator–space $\mathfrak{T}$ is $\mathfrak{T}^* = \mathcal{L}(\mathcal{H})$, the algebra of all bounded operators on $\mathcal{H}$ is a von Neumann algebra.

[98] The notation $\mathfrak{A}''$ originated in the realization of this von Neumann algebra as the ultrastrong (hence weak) closure of a specific faithful representation, [91, 196], called the **universal representation $\pi_u(\mathfrak{A})$ of $\mathfrak{A}$**, hence as the bicommutant $\pi_u(\mathfrak{A})''$, cf. Example B.2.3(ii).

(ii) $\mathcal{L}(\mathcal{H})$, and all its closed symmetric subalgebras are C^*-algebras. Any such C^*-subalgebra $\mathcal{B}$, that is also closed in operator weak (resp. ultraweak, resp. strong, resp. ultrastrong) topology is also a W^*-algebra; the closure of any C^*-subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ (if $I_{\mathcal{H}} \in \mathcal{A}$) in any of these mentioned topologies equals to its double commutant $\mathcal{A}''$, [187, 227, 254], what is a form of the well known **von Neumann bicommutant theorem**. In separable Hilbert space, the only nontrivial C^*-subalgebra of $\mathcal{L}(\mathcal{H})$ which is also an ideal of $\mathcal{L}(\mathcal{H})$ is the algebra of all compact operators $\mathfrak{C}$.

(iii) Let M be a compact Hausdorff space, and let $C(M)$ be the set of all complex valued continuous functions on M. Let pointwise linear combinations, multiplication, and conjugation be defined on $C(M)$ by:

$$f,h \in C(M), \lambda \in \mathbb{C}: \quad \begin{array}{l}(f+\lambda h)(m) := f(m) + \lambda h(m), \\ (f\cdot h)(m) := f(m)h(m),\ f^*(m) := \overline{f(m)}, \quad \forall m \in M,\end{array}$$

and let the norm be the supremum norm: $\|f\| := \sup\{|f(m)| : m \in M\}$. Then $C(M)$, endowed with these structures, is a commutative C^*-algebra. Each unital commutative C^*-algebra is isomorphic to one of this form (the **Gelfand–Najmark theorem**).

(iv) The factoralgebra $\mathcal{L}(\mathcal{H})/\mathfrak{C}$ of the algebra of all bounded operators by the C^*-subalgebra of its compact operators $\mathfrak{C}$ is a unital C^*-algebra, called (according to [196, 6.1.2]) the **Calkin algebra**. It belongs to the class of **antiliminary C^*–algebras**, playing an important rôle in descriptions of infinite quantum systems. ♡

An important characterization of elements of a C^*-algebra $\mathfrak{A}$ is (as it was in $\mathcal{L}(\mathcal{H})$ for operators) their **spectrum**. Since the definitions and properties are identical in this general case with those in the case of bounded operators in $\mathcal{L}(\mathcal{H})$, we shall proceed briefly:

B.2.4. Definition (Spectrum). *Let $x \in \mathfrak{A} :=$ a unital C^*-algebra. The set $\rho(x) := \{\lambda \in \mathbb{C} : (\lambda e - x) \in \mathfrak{A}^{-1}\} \subset \mathbb{C}$ is the* **resolvent set of x**. *Its complement $\sigma(x) \equiv \mathrm{sp}(x) := \mathbb{C} \setminus \rho(x)$ is the* **spectrum of x**. *The spectrum of any element is closed in $\mathbb{C}$. The number $\|x\|_\sigma := \sup\{|\lambda| : \lambda \in \sigma(x)\}$ is called the* **spectral radius of x**. *Always is $\|x\|_\sigma \leq \|x\|$, and $\|x\|_\sigma = \|x\|$ if x is normal.* ♢

An important property of spectrum of any C^*-algebraic element x is its independence on a choice of unital C^*-subalgebras $\mathfrak{B} \subset \mathfrak{A}$ containing x, with respect to which is $\sigma(x)$ calculated (instead of $\mathfrak{A}$). Hence, the spectrum of x can be calculated with respect to the minimal C^*-subalgebra $\mathfrak{A}_x \subset \mathfrak{A}$ containing x, i.e. with respect to the subalgebra generated by the elements x, x^*, e. For a normal element x, the C^*-algebra $\mathfrak{A}_x$ is commutative, and it is isomorphic to $C(\sigma(x))$. The algebraic elements in $\mathfrak{A}$ corresponding (according to this isomorphism) to some $f \in C(\sigma(x))$ are denoted by $f(x)$. The association $f\big(\in C(\sigma(x))\big) \mapsto f(x)$ is the inverse of the **Gelfand transform**. It is a *–isomorphism of C^*-algebras, cf. Definition B.2.5, hence for, e.g., $f(\lambda) \equiv \lambda^n$ one has $f(x) = x^n, n \in \mathbb{Z}_+, x^0 := e$. This mapping of $\mathfrak{A} \times C(\mathbb{C})$ (restricted to normal elements of $\mathfrak{A}$) into $\mathfrak{A}$ is called **continuous funtional calculus** on $\mathfrak{A}$. If $\mathfrak{A}$ is a W^*-algebra, then also complex valued bounded Borel functions $f \in \mathcal{B}_b(\mathbb{C})$ have their homomorphic images $f(x)$ in $\mathfrak{A}$, for normal elements $x \in \mathfrak{A}$. The *–homomorphism determined by an arbitrary

normal element x of a W^*-algebra $\mathfrak{A}$:

$$f(\in \mathcal{B}_b(\mathbb{C})) \mapsto f(x) \in \mathfrak{A}$$

is a unique continuous (i.e. $\|f(x)\| \leq \|f\| = \|f\|_\infty$) extension of the continuous functional calculus. These extended mappings are called the **Borel functional calculus** on a W^*-algebra.

For arbitrary elements (i.e. not necessarily normal) x of $\mathfrak{A}$, we have the **analytic functional calculus**: If f is holomorphic (i.e. analytic) function on an open domain in $\mathbb{C}$ containing the spectrum of x, and c is a "Jordan" curve (i.e. continuous, closed, nonselfintersecting, of finite length, being a homeomorphic image of a circle S^1) lying in this domain and "surrounding" the spectrum $\sigma(x)$, then we can define a Banach space valued integral

$$f(x) := \frac{1}{2\pi i} \oint_c \frac{f(\lambda)\mathrm{d}\lambda}{\lambda e - x} \in \mathfrak{A}, \tag{B.2.1}$$

what can be defined by a norm–convergent sequence of Riemann sums in $\mathfrak{A}$. Restrictions of above mentioned "functional calculi" to analytic functions give the values expressed by (B.2.1).

If we define **positive elements** x of a C^*-algebra $\mathfrak{A}$ as such selfadjoint elements of $\mathfrak{A}$ that can be expressed as $x = y^*y$ for some $y \in \mathfrak{A}$, we can see that these, and only these elements correspond to positive functions in the mentioned functional calculi. The positive elements form a cone $\mathfrak{A}_+$ in $\mathfrak{A}$, i.e. any linear combination of elements in $\mathfrak{A}_+$ *with nonnegative coefficients* also belongs to $\mathfrak{A}_+$. The isomorphism of commutative C^*-algebras with spaces of continuous functions mentioned in Example B.2.3(iii) exactly corresponds to the mentioned functional calculi, but extended also to such commutative C^*-algebras, that need–not be generated by a single normal element. The compact M, corresponding to a unital commutative C^*-algebra $\mathfrak{A}$ which is *–isomorphic to $C(M)$, is called the **spectrum of the abelian C^*-algebra** $\mathfrak{A}$. If a function $f_x \in C(M)$ represents the element x via the Gelfand transform, the spectrum $\sigma(x)$ is identical with the range of f_x: $\sigma(x) = \{f_x(m) : m \in M\}$, cf. also Examples B.3.5.

Elements of algebras usually appear in physical theories represented in forms of linear operators acting on Hilbert spaces. This is naturally connected, as we shall also see in the Subsection B.3, with the physical interpretation of elements of Hilbert spaces as physical states in which the "observables" represented by the elements of algebra are measured (resp. calculated). We shall now turn to an introduction to the representation theory.

B.2.5. Definitions (Representations).

(i) A mapping $\pi : \mathfrak{A} \to \mathfrak{B}$ between two C^*-algebras $\mathfrak{A}$, and $\mathfrak{B}$, is a ***–morphism**, *iff it satisfies the properties:*

(l) π *is linear:* $\pi(x + \lambda y) = \pi(x) + \lambda\pi(y)$,

(ll) $\pi(x \cdot y) = \pi(x) \cdot \pi(y)$,

(lll) $\pi(x^*) = \pi(x)^*$,

for all $x, y \in \mathfrak{A}, \lambda \in \mathbb{C}$. *The set* $\mathbf{Ker}(\boldsymbol{\pi})$ *consisting of such elements x of $\mathfrak{A}$, that $\pi(x) = 0$, is called the* **kernel of π**. *If* $\mathrm{Ker}(\pi) = 0$ *for all morphisms π of $\mathfrak{A}$, then $\mathfrak{A}$ is called a* **simple C^*-algebra**.

(ii) If $\mathrm{Ker}(\pi) = \{0\}$ *then π is a bijection. If* $\mathrm{Ker}(\pi) = \{0\}$, *and also* $\pi(\mathfrak{A}) = \mathfrak{B}$, *the morphism is called a* ***–isomorphism** *(briefly: an isomorphism) of $\mathfrak{A}$ onto $\mathfrak{B}$, and these two C^*-algebras*

are mutually **isomorphic**. *Any morphism of C^*-algebras is continuous:*

$$\|\pi(x)\| \leq \|x\|,$$

and any isomorphism is isometric:

$$\mathrm{Ker}(\pi) = 0 \Rightarrow \|\pi(x)\| \equiv \|x\|.$$

$\mathrm{Ker}(\pi)$ *is always a closed (twosided) ideal in the C^*-algebra $\mathfrak{A}$.*

(iii) Let π be a *–morphism of a C^*-algebra $\mathfrak{A}$ into $\mathcal{L}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$. It is called a **representation of $\mathfrak{A}$ in $\mathcal{H}$**. *It will be denoted also $(\pi;\mathcal{H})$. If there is a nontrivial (i.e. $\neq 0$, and $\neq I_{\mathcal{H}}$) projection P in the commutant $\pi(\mathfrak{A})' \in \mathcal{L}(\mathcal{H})$, then the mapping*

$$\pi_P : \mathfrak{A} \to \mathcal{L}(P\mathcal{H}),\ x \mapsto P\pi(x),$$

is a **subrepresentation of $\boldsymbol{\pi}$**. *We shall assume, that any considered representation is* **nondegenerate**, *i.e. that it has* **no zero subrepresentations**, *i.e. that there is no nonzero projection $P \in \mathcal{L}(\mathcal{H})$ such, that $P\pi(\mathfrak{A}) = \{0\}$. If a representation of a C^*-algebra have no nontrivial subrepresentations, it is called an* **irreducible representation**.

(iv) Let us assume that, in the Hilbert space $\mathcal{H}_\pi$ of a representation $\pi(\mathfrak{A})$, there is a vector, say $\psi_\pi \in \mathcal{H}_\pi$ such, that the set

$$\pi(\mathfrak{A})\psi_\pi := \{\pi(x)\psi_\pi : x \in \mathfrak{A}\}$$

is dense in the Hilbert space $\mathcal{H}_\pi$. Then the representation π is called a **cyclic representation of $\mathfrak{A}$**, *and the vector ψ_π is a* **cyclic vector of the representation** π. *Each representation of a C^*-algebra can be decomposed into an orthogonal sum of cyclic subrepresentations, i.e. there is a system of mutually orthogonal projections $P_j \in \pi(\mathfrak{A})',\ j \in J$, such that $\sum_{j\in J} P_j = I_{\mathcal{H}_\pi}$ (strong convergence), and each subrepresentation $x \mapsto P_j\pi(x) \in \mathcal{L}(P_j\mathcal{H})$ is cyclic. In an irreducible representation space $\mathcal{H}_\pi$, any nonzero vector $\psi \in \mathcal{H}_\pi$ is cyclic, and $\pi(\mathfrak{A})\psi = \mathcal{H}_\pi$.*

(v) Let $(\pi_1;\mathcal{H}_1), (\pi_2;\mathcal{H}_2)$ be two representations of a C^*-algebra $\mathfrak{A}$. If there is a linear (unitary) isometry $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\pi_2(x) \equiv U\pi_1(x)U^{-1}$, the representations **$\boldsymbol{\pi_1}$ and $\boldsymbol{\pi_2}$ are unitarily (or spatially) equivalent**. *We shall denote this fact by $\pi_1 \simeq \pi_2$.* $\diamondsuit$

As it was mentioned above, any *–representation of a C^*-algebra is continuous in the norm topology. On each W^*-algebra $\mathfrak{B}$, moreover, another natural topology, namely the $\sigma(\mathfrak{B},\mathfrak{B}_*)$–topology, or briefly the w^*–topology is given. The same is true for the W^*-algebra $\mathcal{L}(\mathcal{H}_\pi)$ for any representation π. Hence the question on the w^*–continuity of π arises. As we shall show, this property is also relevant for any representation of a C^*-algebra $\mathfrak{A}$ canonically extended to $\mathfrak{A}''$, cf. Proposition B.2.7.

B.2.6. Definition (W^*–representations). *If a representation π of a W^*-algebra $\mathfrak{B}$ in the Hilbert space $\mathcal{H}_\pi$ is $\sigma(\mathfrak{B},\mathfrak{B}_*) - \sigma(\mathcal{L}(\mathcal{H}_\pi),\mathfrak{T}(\mathcal{H}_\pi))$ continuous, it is called a W^*–*__representation__.

The image $\pi(\mathfrak{B}) \subset \mathcal{L}(\mathcal{H}_\pi)$ of any W^*–representation such that $\pi(e) = I_{\mathcal{H}_\pi}$ (nondegeneracy) is again a W^*-subalgebra of $\mathcal{L}(\mathcal{H}_\pi)$. Let us mention also that an isomorphism of two W^*-algebras is always w^*–continuous, [227].

B.2.7. Proposition. *Let $\mathfrak{A}$ be a C^*-algebra, and $(\pi;\mathcal{H}_\pi)$ its arbitrary nondegenerate representation. Let us consider $\mathfrak{A}\subset\mathfrak{A}''$, in the canonical way. Then there is unique extension of π to a W^*–representation $(\pi'';\mathcal{H}_\pi)$ of the univeral enveloping algebra $\mathfrak{A}''$, [227, Proposition 2.21.13]. The image of this extension equals to the bicommutant $\pi(\mathfrak{A})''$, if $\mathfrak{A}$ is unital.* ♣

It follows from this assertion that to any representation π of a C^*-algebra $\mathfrak{A}$ there is a central projection $z(\pi)\in\mathcal{Z}(\mathfrak{A}'')$ such, that its orthogonal complement $e''-z(\pi)$ (with e'' the unit of $\mathfrak{A}''$) supports the kernel of $\pi'':\mathrm{Ker}(\pi'')=(e''-z(\pi))\mathfrak{A}''$, called alternatively the **central support**, [227, 1.21.14], resp. **central cover** of π, [196, 3.8.1]. The representations π_1,π_2 with the same central projection $z(\pi_1)=z(\pi_2)$ are called **(quasi–) equivalent**, denoting this by $\pi_1\sim\pi_2$. Unitary equivalence implies equivalence, but equivalent representations are just, roughly speaking, decomposable into various multiples of the same unitary equivalent subrepresentations, [77, 5.3.1]. If the central supports are orthogonal: $z(\pi_1)\cdot z(\pi_2)=0$, the two representations are called **disjoint**, we shall denote this by $\pi_1\mho\pi_2$.[99]

B.2.8. **Interpretation (Macro–distinguishability).** Disjoint representations are interpreted in physics as **macroscopically (or classically) distinguishable representations**: Since the "physically most relevant" seems the w^*–topology, it also seems natural to consider also (some of) elements of the enveloping algebra $\mathfrak{A}''$ of "the C^*-algebra of observables $\mathfrak{A}$" which do not belong to $\mathfrak{A}$, as representing some observable quantities of the system. The macroscopic (resp. classical) quantities of the considered quantum system are then found in the centre $\mathcal{Z}(\mathfrak{A}'')$. Any two mutually orthogonal projections of the centre then can represent macroscopically distinguishable values of some observable quantity. Hence, $\pi_1\mho\pi_2$ can be interpreted as macroscopic distinguishability. It would be, perhaps, more intuitive after a discussion of disjointness of states, [131, 28, 238], cf. also Interpretation B.3.7. ♦

In general theory of C^*-algebras, and also in physical applications, cyclic representations arise from given "states".

B.3 States and representations

We shall introduce here the mathematical definition of states on a C^*-algebra, as well as some connetions with representations, and we shall give some hints to their physical interpretations.

B.3.1. Definition (States).

(i) Let $\mathfrak{A}$ be a C^*-algebra, and $\mathfrak{A}^*$ its topological dual. A continuous linear functional $\varrho\in\mathfrak{A}^*$ is **symmetric (or real)** *if $\varrho(x^*)\equiv\overline{\varrho(x)}$. It is, moreover,* **positive**, *if $\varrho(x^*x)\geq 0,\forall x\in\mathfrak{A}$. The set $\mathfrak{A}^*_+$ consists of all positive elements of $\mathfrak{A}^*$. The elements of $\mathcal{S}(\mathfrak{A}):=\mathfrak{A}^*_{+1}:=\{\varrho\in\mathfrak{A}^*_+:\|\varrho\|=1\}$ are* **positive normalized functionals** *on $\mathfrak{A}$. They are called* **states on the C^*-algebra $\mathfrak{A}$**. *The set $\mathcal{S}(\mathfrak{A})\subset\mathfrak{A}^*$ is convex, i.e. $\varrho_j\in\mathcal{S}(\mathfrak{A})$ $(j=1,2)$, $0<\lambda<1\Rightarrow\lambda\varrho_1+(1-\lambda)\varrho_2\in\mathcal{S}(\mathfrak{A})$.*

(ii) An element of $\omega\in\mathcal{S}(\mathfrak{A})$ is called a **pure state on $\mathfrak{A}$**, *if it is not an internal point of any line segment lying in $S(\mathfrak{A})$, i.e. if for some states $\omega_j, j=1,2$, one has $\omega=\frac{1}{2}(\omega_1+\omega_2)$, then necessarily $\omega_1=\omega_2=\omega$. Such elements ω of a convex set $\mathcal{S}$ are called* **extremal points of $\mathcal{S}$**. *The set of all pure states on $\mathfrak{A}$ will be denoted by $\mathcal{ES}(\mathfrak{A})$. In the state space $\mathcal{S}=\mathcal{S}(\mathfrak{A})$, the nonextremal elements are called* **mixed states, or mixtures**.

[99] The standard symbol for disjoitness, [77, 42], was not found among the LaTeX symbols.

(iii) Let $\mathfrak{A}$ be a W^*-algebra. Then it is canonically $\mathfrak{A}_* \subset \mathfrak{A}^*$. The subset $\mathcal{S}_*(\mathfrak{A}) := \mathfrak{A}_* \cap \mathcal{S}(\mathfrak{A})$ of states consists of all **normal states on** $\mathfrak{A}$. $\diamondsuit$

The convex set $\mathcal{S} := \mathcal{S}(\mathfrak{A})$ is contained in the closed unit ball of $\mathfrak{A}^*$, what is compact in the $\sigma(\mathfrak{A}^*, \mathfrak{A})$–topology, according to the Banach–Alaoglu theorem [218]. $\mathcal{S}$ is also compact iff $\mathfrak{A}$ is unital. In other cases, it is usually considered [196] the **quasi–state space** $\mathcal{Q}$ defined by $\mathcal{Q} := \{\varrho \in \mathfrak{A}_+^* : \|\varrho\| \leq 1\}$. The quasi–state space $\mathcal{Q}$ is the convex hull of the state space $\mathcal{S}$ and the zero functional. It is always compact in the w^*–topology. We shall use the following theorem, [187, 41]:

B.3.2. Theorem (Krein–Milman). *Every compact convex set $\mathcal{S}$ in a Hausdorff l.c.s. is closure of the set of all finite convex combinations of the extremal points of $\mathcal{S}$.* $\clubsuit$

We see that there is enough pure states on any C^*-algebra so that finite convex combinations of them can approximate any state in the w^*–topology.

B.3.3. **Interpretation (States in physics).** If the selfadjoint elements of a C^*-algebra $\mathfrak{A}$ are considered as **observables of a physical system**, then the states on $\mathfrak{A}$ are interpreted as follows: The state $\omega \in \mathcal{S}(\mathfrak{A})$ represents a **physical situation** (whatever it means, we do not go to analyze its meaning here) of the described system and, for any given $x = x^* \in \mathfrak{A}$, the real number $\omega(x)$ equals to the expectation value (i.e. the arithmetical average – in "finite approximations") of results of repeated measurements of the observable x in the (repeatedly prepared) "situation" described by ω .

Hence, the above mentioned possibility of approximation of any state by convex combinations of pure ones means an approximation by convex combinations of (potential) measurement results; this shows in what sense the w^*–topology on the state space $\mathfrak{A}_s^*$:= (the set of symmetric elements of $\mathfrak{A}^*$) is "more physical", than the topology of norm, cf. [118]. $\blacklozenge$

The following proposition can be proved by the well known Gelfand–Najmark–Segal (GNS) construction of a canonical representation π_ω corresponding to any given state ω , with a use of algebraic concepts, [187, 77, 227, 91, 42]:

B.3.4. Proposition. *A cyclic representation $(\pi_\omega; \mathcal{H}_\omega; \psi_\omega)$ in $\mathcal{H}_\omega$ with the cyclic vector $\psi_\omega \in \mathcal{H}_\omega : \overline{\pi_\omega(\mathfrak{A})\psi_\omega} = \mathcal{H}_\omega$, $\|\psi_\omega\| = 1$, such that*

$$\omega(x) = (\psi_\omega, \pi_\omega(x)\,\psi_\omega), \quad \forall x \in \mathfrak{A},$$

corresponds to any state $\omega \in \mathcal{S}(\mathfrak{A})$ on a C^*-algebra $\mathfrak{A}$. Such a representation is unique, up to unitary equivalence. Any cyclic representation of $\mathfrak{A}$ can be obtained in this way. The cyclic representation π_ω is irreducible iff ω is a pure state. In that case, $\pi_\omega(\mathfrak{A})\psi_\omega = \mathcal{H}_\omega$ (without taking the closure in $\mathcal{H}_\omega$). $\clubsuit$

The canonical representations satisfying the conditions of Proposition B.3.4 are also called the **GNS representations**. Two **states** $\omega_1, \omega_2 \in \mathcal{S}(\mathfrak{A})$ **are mutually disjoint**, cf. page 172, $\boldsymbol{\omega_1 \mho \omega_2}$, iff $\pi_{\omega_1} \mho \pi_{\omega_2}$; such a two states can be considered as macroscopically different, [91, 131, 42, 238], and any two macroscopically different states are mutually disjoint, cf. Interpretations B.2.8, and B.3.7.

B.3.5. **Examples** *(Abelian C^*-algebras and W^*-algebras).*

(i) Any unital abelian C^*-algebra $\mathfrak{A}$ is isomorphic to the space of all continuous complex valued functions $C(M)$ on a Hausdorff compact M. The set M can be constructed as the set of all irreducible representations π_χ (which are all one–dimensional), and their kernels are **maximal ideals** of $\mathfrak{A}$. The corresponding pure states $\chi \in \mathcal{S}(\mathfrak{A})$ are **characters on** $\mathfrak{A}$, i.e. they satisfy also: $\chi(x\cdot y) \equiv \chi(x)\chi(y)$. Since the three sets: the set of irreducible representations, the set of maximal ideals, and the set of pure states are in bijective correspondence, they can be endowed, together with the **"spectrum space"** M, with the induced topology from the w^*–topology of $\mathfrak{A}^*$. Let us denote by χ_m the pure state on $C(M)$ corresponding to the maximal ideal $m \in M$, i.e. to the irreducible representation with the kernel m. Let $f \in C(M)$ be any element of the commutative C^*-algebra. Then

$$\chi_m(f) = f(m), \quad \forall m \in M,$$

hence the pure states χ_m correspond to the Dirac measures δ_m on M. Arbitrary states ω are then realized as probability "regular Borel measures" μ_ω on M, symbolically

$$\omega(f) = \int_M f(m)\mu_\omega(\mathrm{d}m).$$

The correspondence between states on $C(M)$ and probability measures on M is a bijection, according to Riesz-Markov theorem, [218, Theorems IV.14, and IV.18]. Hence, a decomposition of an arbitrary state on an abelian C^*-algebra into a convex combination (here: integral) of pure states, so called extremal decomposition, is unique.

(ii) Let us assume, that the abelian C^*-algebra $C(M)$ is a W^*-algebra. We know, that it is generated by its projections. But a projection in $C(M)$ is just a characteristic function χ_B of a subset $B \subset M$, which is also continuous: Hence the set $B = \chi_B^{-1}(\{1\})$ should be a clopen subset in M. It can be shown, [101, 254], that the topology of M is now generated by its clopen sets. The chracteristic functions of one–point sets corresponding to *normal pure states* χ_m can be considered also as elements of the W^*-algebra: $\delta_{m,m'} =: f_m(m'), f_m \in C(M)$, because these m's are just the isolated points of M. ♡

Let $\omega \in \mathcal{S}(\mathfrak{A})$ be a mixed state on a unital C^*-algebra $\mathfrak{A}$. Then it can be **decomposed** into a (generally "continuous") convex combination of other states. Looking for such convex decompositions of a given $\omega \in \mathcal{S}(\mathfrak{A})$, we are interested in such probability measures μ_ω on the compact $\mathcal{S} := \mathcal{S}(\mathfrak{A})$, that for all affine continuous functions $f \in C(\mathcal{S}) : f(\lambda\omega_1 + (1-\lambda)\omega_1) \equiv \lambda f(\omega_1) + (1-\lambda)f(\omega_2)$, we have

$$f(\omega) = \int_{\mathcal{S}} f(\nu)\mu_\omega(\mathrm{d}\nu).$$

The state ω is a **barycentre of the measure μ_ω**. Specific examples $\hat{x}$ of the affine functions are given by arbitrary elements $x \in \mathfrak{A} :\ \hat{x}(\omega) := \omega(x), \forall\omega \in \mathcal{S}$. The measures μ_ω can be "concentrated" on various subsets of $\mathcal{S}$. We have seen that if one assumes that μ_ω is concentrated on pure states $\mathcal{ES}$, then μ_ω is uniquely determined in the commutative case. For general C^*-algebras, this uniqueness is absent (it is in a sense equivalent to commutativity of the C^*-algebra). Hence, we have to choose some of the measures with the barycentre ω to obtain some (barycentric)

decomposition. These decompositions might be chosen in different ways. We cannot give here details; let us see at least some of techniques for construction of such decompositions.

If $\omega = \sum_j \lambda_j \omega_j$ is a (discrete, convex) decomposition of a state ω to states $\omega_j, \lambda_j \neq 0$, then the states ω_j are *majorized by* ω . Important might be the following lemma stating conditions for $f \in \mathfrak{A}_+^*$ to be **majorized by ω**, i.e. stating conditions when there is a number λ for which $(\lambda\omega - f) \in \mathfrak{A}_+^*$.

B.3.6. Lemma. *Let, with the introduced notation, $\omega \in \mathcal{S}$, and $(\pi_\omega; \mathcal{H}_\omega; \psi_\omega)$ is the corresponding cyclic representation. Then there is a bijection between the set of all $f \in \mathfrak{A}_+^*$ majorized by ω , and the set of all positive elements: $B \geq 0, B \in \pi_\omega(\mathfrak{A})'$, of the commutant of $\pi_\omega(\mathfrak{A})$. The correspondence $f \mapsto B_f \in \pi_\omega(\mathfrak{A})' \cap \mathcal{L}(\mathcal{H}_\omega)_+$ is by the relation*

$$f(x) = (\psi_\omega, \pi_\omega(x) \cdot B_f \psi_\omega), \quad \forall x \in \mathfrak{A}$$

determined uniquely. ♣

Most simple and useful decompositions are such that are derived from some abelian W^*-subalgebras $\mathfrak{B}$ of the von Neumann algebra $\pi_\omega(\mathfrak{A})'$. Let us mention here just a very simple example when the algebra $\mathfrak{B}$ is generated by a ("discrete") projection measure E_d defined on $\mathbb{Z}_+$: $E_d : j\ (\in \mathbb{Z}_+) \mapsto E_d(j) \in \pi_\omega(\mathfrak{A})', \sum_{j=0}^{\infty} E_d(j) = \mathbb{I}$. Then we define the states $\omega_j \in \mathcal{S}$, for those values of the indices j for which $(\psi_\omega, E_d(j)\psi_\omega) =: \lambda_j \neq 0$, by the relation

$$\omega_j(x) := \lambda_j^{-1}(\psi_\omega, \pi_\omega(x) E_d(j) \psi_\omega)\ (\forall x \in \mathfrak{A}).$$

It is trivially clear that now we can write a ("orthogonal") decomposition of the ω by:

$$\omega(x) \equiv \sum_j \lambda_j \omega_j(x).$$

If the W^*-algebra $\mathfrak{B}$ is contained in the centre $\mathcal{Z}(\pi_\omega(\mathfrak{A})') := \pi_\omega(\mathfrak{A})'' \cap \pi_\omega(\mathfrak{A})'$, the decomposition is called a **(sub–) central decomposition**.

B.3.7. **Interpretation.** By the extended representation $\pi_\omega'' : \mathfrak{A}'' \to \mathcal{L}(\mathcal{H}_\omega)$, cf. Proposition B.2.7, the centre $\mathcal{Z}(\mathfrak{A}'')$ is mapped onto the centre $\mathcal{Z}(\pi_\omega(\mathfrak{A})'')$. Hence, the subcentral decompositions might be interpreted physically as decompositions according to values of macroscopic observables. ♦

General theories of decompositions can be found in [227, 42, 196].

B.4 Symmetries and automorphisms

Symmetries appear in quantum theory (QT) either in a form of transformations of "states" (the *Schrödinger picture*), or as transformations of "observables" (the *Heisenberg picture*). Although, in the "standard" QM, these two forms of symmetry transformations are usually considered as equivalent, for more general formulations of QT it needn't be so. Some relations between these two descriptions of symmetry operations in QT are described in [42, Chap. 3.2]. We shall restrict here our attention mainly to the formulation in the "Heisenberg form", what is the most usual form of the description of time evolution in quantum theories of large systems.

Theory of symmetries, resp. automorphism groups of C^*-algebras is a rather extensive field, cf. [224, 227, 42, 196, 228]. There are known many related and mutually connected fields like the ergodic theory, decomposition theory, various kinds of "spectra" connected with analysis of structure of algebras and of their automorphism groups α_G etc. which we shall not consider in this paper. We shall present here just some notes for a first orientation in approaches to formulations and investigation of several problems concerning symmetries of physical systems described by C^*-algebraic theories, and such which are connected with techniques used in this work.

Let $\mathfrak{A}$ be a C^*-algebra. The *-automorphisms of $\mathfrak{A}$ (i.e. the *–isomorphisms of $\mathfrak{A}$ onto itself) form a group $Aut(\mathfrak{A})$ with respect to composition as the group multiplication. Each $\alpha \in Aut(\mathfrak{A})$ is a continuous linear transformation of the B–space $\mathfrak{A}$, hence $Aut(\mathfrak{A}) \in \mathcal{L}(\mathfrak{A})$ (=the space of bounded linear mappings of $\mathfrak{A}$ into itself), where $\mathcal{L}(\mathfrak{A})$ is again, canonically, a B–space. With the induced topology, $Aut(\mathfrak{A})$ is a topological group; it is also a closed subset of $\mathcal{L}(\mathfrak{A})$, [227, Proposition 4.1.13]. There are also several other useful "natural" (weaker–than–norm) topologies introduced on $Aut(\mathfrak{A})$, namely the **strong topology** given by the seminorms

$$\mathfrak{p}_x(\alpha) := \|\alpha(x)\|, \quad \forall \alpha \in Aut(\mathfrak{A}), \quad x \in \mathfrak{A},$$

in which $Aut(\mathfrak{A})$ is also a topological group, [228], and also some of the $\boldsymbol{\sigma(Aut(\mathfrak{A}), \mathcal{F}^*)}$**–weak topologies**, where $\mathcal{F}^*$ is a "conveniently chosen" subset of linear functionals on the B–space $\mathcal{L}(\mathfrak{A})$, to make $Aut(\mathfrak{A})$ a Hausdorff space, cf. also Definition A.1.4(v). The subset $\mathcal{F}^*$ is often given by the requirement of continuity of the mappings

$$\alpha(\in Aut(\mathfrak{A})) \mapsto \omega(\alpha(x)), \quad \forall x \in \mathfrak{A},\ \omega \in \mathcal{F}',$$

where we have different useful possibilities, [42, Definition 2.5.17], for a choice of the set $\mathcal{F}' \subset \mathfrak{A}^*$. If $\mathfrak{A}$ is a W^*-algebra, then its automorphisms are continuous mappings of $\mathfrak{A}$ onto itself not only in the norm–topology, but also in the $\sigma(\mathfrak{A},\mathfrak{A}_*)$–topology determined by its normal states, cf. Definition B.3.1. These states are "often" chosen in the rôle of the set $\mathcal{F}'$ above, in the case of a W^*-algebras $\mathfrak{A}$. The automorphism α of $\mathfrak{A}$ is called **inner** if there is a unitary element $u_\alpha \in \mathfrak{A}$ such that $\alpha(x) \equiv u_\alpha x u_\alpha^*$. For $\mathfrak{A} := \mathcal{L}(\mathcal{H})$ is each automorphism inner , [227, Corollary 2.9.32].

Any $\alpha \in Aut(\mathfrak{A})$ determines a unique affine isometry $\alpha^* : \mathcal{S}(\mathfrak{A}) \to \mathcal{S}(\mathfrak{A})$ of the state space of $\mathfrak{A}$ by the transposing:

$$\alpha^*(\omega)(x) \equiv \omega(\alpha(x)), \quad \omega \in \mathcal{S}(\mathfrak{A}),\ x \in \mathfrak{A}.$$

If $\mathfrak{A}$ is a W^*-algebra, the transposed map leaves its normal states invariant:

$$\alpha^* : \mathcal{S}_*(\mathfrak{A}) \to \mathcal{S}_*(\mathfrak{A}). \tag{B.4.1}$$

This is the transition to the "Schrödinger picture". The converse needn't be so immediate:

If it is given an affine mapping $\alpha^* : \mathcal{S}(\mathfrak{A}) \to \mathcal{S}(\mathfrak{A})$, its transpose α^{**} determines a linear map of the double dual $\mathfrak{A}^{**}$ into itself, that leaves its (in a canonical way defined) subset $\mathfrak{A}$ invariant only in specific cases: some "sufficient continuity" conditions should be satisfied, cf. [42, Theorem 3.2.11]; only then one can consider the restriction α of α^{**} to the subspace $\mathfrak{A}$ of $\mathfrak{A}^{**}$ and ask, whether is $\alpha \in Aut(\mathfrak{A})$, hence, whether there exists the corresponding "Heisenberg picture". In the case of a W^*-algebra $\mathfrak{A}$, if the condition (B.4.1) is fulfilled, there is a unique

Jordan[100] automorphism α of $\mathfrak{A}$ obtained by the above mentioned "transposing" of α^* and by the subsequent restriction.

A physical meaning is usually given to an automorphism α according to its belonging to some subgroup of $Aut(\mathfrak{A})$ which is a homomorphic image of a topological (usually Lie) group G: $\alpha \in \alpha_G$, where

$$\alpha_{g_1 g_2} = \alpha_{g_1} \cdot \alpha_{g_2}, \quad \forall g_j \in G,\ j = 1, 2.$$

The homomorphisms $g(\in G) \mapsto \alpha_g$, i.e. **representations of G** represent groups of "physical motions", or transformations. If the group is $G = \mathbb{R}$, we have a one parameter transformations group $\alpha_{\mathbb{R}}$; such groups describe also time evolutions of the physical systems with $\mathfrak{A}$ as the "algebra of observables".

There are traditional reasons in QM (e.g. the spectra of generators of U_G^π represent measurable values) for interest in such representations $\{\pi, \mathcal{H}_\pi\}$ of the C^*-algebra $\mathfrak{A}$ with a given **symmetry α_G**, in which the automorphisms $\alpha_g,\ g \in G$, are expressed by a unitary strongly continuous representation $U_G^\pi \in \mathcal{U}(\mathcal{H}_\pi)$ of G "in the usual way", i.e.

$$\pi\big(\alpha_g(x)\big) \equiv U_g^\pi \pi(x) U_{g^{-1}}^\pi, \quad \forall g \in G. \tag{B.4.2}$$

Such representations $\pi(\mathfrak{A})$ are called **covariant representations**. A simple important case of a covariant representation is obtained (we omit here specification of necessary continuity conditions imposed to α_G), if there is an **α_G^*–invariant state ω** given, [237, 224]; the corresponding cyclic representation $(\pi_\omega; \mathcal{H}_\omega; \psi_\omega)$ ensures existence of a unique (continuous) unitary representation U_G^ω satisfying (B.4.2) (with $\pi_\omega \mapsto \pi,\ U^\omega \mapsto U^\pi$), and such that the cyclic vector corresponding to the state ω is U_G^ω–invariant:

$$U_g^\omega \psi_\omega = \psi_\omega, \quad \forall g \in G.$$

In more general situations (e.g. of cyclic representations with noninvariant cyclic vectors), all covariant representations of a *dynamical system* $\{\mathfrak{A}, \alpha_G\}$ are in a bijective correspondence with representations of another C^*-algebra $\mathfrak{A} \otimes_\alpha G$ constructed from functions on the group G with values in $\mathfrak{A}$ with a help of the action of α_G, and called the **crossed product of the dynamical system $\{\mathfrak{A}, \alpha_G\}$**, cf. [196], or also [188].

Let us consider now $G := \mathbb{R}$, i.e. one–parameter automorphism groups. For $\mathfrak{A} = \mathcal{L}(\mathcal{H})$, all one parameter automorphism groups $t \mapsto \alpha_t$ are "covariant", i.e. they are representable in the form (B.4.2), i.e. $\alpha_t(x) \equiv u_t x u_t^*$ for a one–parameter group of unitary operators $u_t,\ t \in \mathbb{R}$. If the group $\alpha_{\mathbb{R}}$ is "sufficiently continuous", e.g. if the functions $t \mapsto Tr(\varrho \alpha_t(x)),\ \varrho \in \mathfrak{T}_s, x \in \mathcal{L}(\mathcal{H})$ are all continuous, then $t \mapsto u_t$ is strongly continuous and, according to Stone's theorem, cf. Theorem C.3.2, there is a selfadjoint operator A on (a dense domain of) $\mathcal{H}$ such, that

$$u_t \equiv \exp(itA). \tag{B.4.3}$$

[100] Jordan automorphisms α of a C^*-algebra are a certain "combinations" of morphisms, cf. Definition B.2.5(i) (satisfying: $\alpha(xy) \equiv \alpha(x)\alpha(y)$), and **antimorphisms** (satisfying: $\alpha(xy) \equiv \alpha(y)\alpha(x)$, with other morphism properties unchanged); hence, by definition, instead of satisfying the property $(ll) : \alpha(xy) \equiv \alpha(x)\alpha(y)$ of the Definition B.2.5 of *–isomorphisms, Jordan automorphisms satisfy the following property: $\alpha(xy + yx) \equiv \alpha(x)\alpha(y) + \alpha(y)\alpha(x)$.

The operator A is determined by the automorphism group $\alpha_{\mathbb{R}}$ up to an additive real constant. The operator A is called a **(selfadjoint) generator of $\boldsymbol{u}_{\mathbb{R}}$**. The **generator of $\boldsymbol{\alpha}_{\mathbb{R}}$** is obtained as a linear operator $\delta_\alpha(x) := i[A, x]$ on (a dense subset of) $\mathcal{L}(\mathcal{H})(\ni x)$. In a general case:

$$\omega\big(\delta_\alpha(x)\big) := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \omega\big(\alpha_t(x)\big) \tag{B.4.4}$$

for all $x \in D(\delta_\alpha) \subset \mathfrak{A}$. The generator **$\boldsymbol{\delta_\alpha}$ is called the derivation of $\boldsymbol{\alpha}_{\mathbb{R}}$**. Some details of a theory of (unbounded) derivations can be found in [42, 228].

If the group $\alpha_{\mathbb{R}}$ is not "sufficiently continuous", the generator needn't exist. Moreover, some of the covariant representations π of the same $\{\mathfrak{A}, \alpha_{\mathbb{R}}\}$ might be continuous with well defined selfadjoint generators A_π, and in other covariant representations the unitary groups $t \mapsto U_t^\pi$ might be discontinuous (i.e. there is no "Hamiltonian" there). For different continuous covariant representations π the "Hamiltonians A_π" are generally mutually different (e.g., their spectra might be mutually "very different"). In the examples of states describing thermodynamic equilibria for different temperatures the selfadjoint generators describing time evolution of local perturbations are mutually different in known solvable examples, e.g. for simple versions of the BCS model of superconductivity, cf. [119, 261, 32]. In this last mentioned example, the representations of the ("quasilocal") algebra of observables corresponding to different equilibrium states are all faithful, they mutually differ, however, in representing "macroscopic quantities" of the described infinite quantal system by different operators (resp. numbers). Also in more general cases, mutually disjoint representations are distinguished by values of some "macroscopic quantities".

Thermodynamic equilibrium states (also of infinite systems, corresponding to the "thermodynamic limit", [224]) can be defined for any "sufficiently continuous" one–parameter automorphism group $\alpha_{\mathbb{R}}$ of a C^*-algebra. This fact is interesting as such, from the point of view of traditional techniques for statistical–mechanical description of thermodynamic equilibria by Gibbs statistical ensembles, because for time evolutions $(t; x) \mapsto \alpha_t(x)$ of an infinite system there is no global Hamiltonian operator H to be inserted into the expression of a "statistical sum", e.g.[101] into

$$Z(T, H) := Tr \exp\left(-\frac{1}{kT} H\right).$$

The definition of the thermodynamic equilibrium states $\omega = \omega_\beta$ for a temperature $T =: (k\beta)^{-1}$ (k is here the Boltzmann constant) of infinite (and other) systems is expressed by the **KMS condition** for states ω , [117, 42, 196]:[102]

$$\omega(\alpha_\lambda(y)x) \equiv \omega(x\alpha_{\lambda+i\beta}(y)),\ \forall \lambda \in \mathbb{C}, x \in \mathfrak{A}, y \in \mathfrak{A}_a, \tag{B.4.5}$$

where $\mathfrak{A}_a \subset \mathfrak{A}$ is the set of analytic elements with respect to α (i.e. $x \in \mathfrak{A}_a \Leftrightarrow \lambda \mapsto \alpha_\lambda(x)$ is an entire–analytic $\mathfrak{A}$–valued function, cf. [218, Chap.VI]). States ω_β satisfying the condition (B.4.5) are the **$\boldsymbol{\beta}$–KMS states for** α, with a given $0 < \beta < +\infty$.

[101] Even in some "traditional" cases, when the Hamiltonian H is a well defined selfadjoint operator, the trace in the following formula does not exist. Take, e.g. a hydrogen atom in a box.

[102] KMS is for Kubo, Martin, and Schwinger.

It is an interesting result of the **Tomita–Takesaki theory** of modular Hilbert algebras, [253, 196, 42], that for a class of states ω of any C^*-algebra $\mathfrak{A}$ one can find a canonical one–parameter automorphism group (called the **modular group for $\{\mathfrak{A}, \omega\}$**) of the weak closure $\pi_\omega(\mathfrak{A})''$ of the GNS–representation of $\mathfrak{A}$ such, that the chosen state ω is a KMS–state of that automorphism group at $\beta = 1$ (this finite nonzero value of β is chosen arbitrarily). The condition for the class of states ω allowing this "creation of dynamics from states" is, that ω is faithful for $\pi_\omega(\mathfrak{A})''$, i.e. that for any positive (nonzero) operator $B \geq 0$ in this W^*-algebra, its diagonal matrix element with the cyclic vector ψ_ω is strictly positive: $(\psi_\omega, B\psi_\omega) \neq 0$ (hence, the W^*-algebra $\pi_\omega(\mathfrak{A})''$ does not contain any "annihilation operators" with respect to the "vacuum vector" ψ_ω).

Let us mention also the phenomenon of "spontaneous symmetry breaking" in a stationary state ω with respect to "dynamical evolution group" $\tau_\mathbb{R} \subset Aut(\mathfrak{A})$. Assume that there is another automorphism group α_G commuting with the time evolution τ:

$$\tau_t \circ \alpha_g = \alpha_g \circ \tau_t, \quad \forall t \in \mathbb{R},\ g \in G.$$

This situation "corresponds", e.g. to commutativity of the Hamiltonian as the selfadjoint generator of the unitary group implementing the time evolution $\tau_\mathbb{R}$ with generators of the transformation group α_G for a Lie group G.

The notion of states with "broken" symmetry comes from expectations that a certain states will have larger symmetry than they really have, [69]. Let us assume that, e.g., in the usual formulation of QM, the Hamiltonian H is invariant with respect to a unitary representation U_G of a finite–dimensional Lie group G in $\mathcal{H}$: $H \equiv U_g H U_g^*$. If there is an eigenvector $\psi_\varepsilon \in \mathcal{H}$ of H: $H\psi_\varepsilon = \varepsilon\psi_\varepsilon$, then also all the vectors $\{U_g\psi_\varepsilon : g \in G\}$ are eigenvectors of H with the same eigenvalue ε . Then a nondegenerate eigenvector $\psi_\varepsilon \in \mathcal{H}$ is proportional to all the vectors $U_G\psi_\varepsilon \in \{\lambda\psi_\varepsilon : \lambda \in \mathbb{C}\}$, hence the state

$$x \mapsto (\psi_\varepsilon, x\psi_\varepsilon) \equiv (U_g\psi_\varepsilon, xU_g\psi_\varepsilon),\ x \in \mathcal{L}(\mathcal{H})$$

is also "G–invariant". If the eigenvalue ε is of higher multiplicity, the G–invariance of ψ_ε might be "broken". Similar considerations apply to equilibrium states at fixed temperature: If, in the above situation, there is only one KMS–state for a given β, then it is invariant also with respect to α_G. The phenomenon of **phase transitions** is usually considered as equivalent to existence of several KMS states for any temperature of "phase coexistence", e.g. below the critical temperature of a ferromagnet. In the last mentioned case, e.g., the group G might be the Euclidean group in $\mathbb{R}^3$ (or only its rotation subgroup $O(3)$) with respect to which the Hamiltonian of the ferromagnet is invariant. Different (extremal) KMS states correspond to different directions of the magnetization of the ferromagnet, hence the rotation symmetry α_G is broken; translation symmetry is broken in states of any crystal state of many–particle systems (with translation invariant Hamiltonians). The stationary *and G–invariant* states always exist, but in the mentioned "degenerate" situations they are not "extremal": they have nontrivial convex decompositions to states (e.g. equilibrium) with lower, hence "broken", symmetry. These situations are considered in the above mentioned decomposition theory, resp. in a part of the ergodic theory, [224, 42, 271, 153, 228, 68].

B.4.1. **Note**. Let us add several words on possible structures of physically relevant C^*-algebras, resp. W^*-algebras. It is useful to classify C^*-algebras according to the sets of projections contained in (the W^*-algebras obtained by) the weak closures of their (e.g. GNS) representations.

Let us concentrate on a von Neumann classification of W^*-algebras. Let $\mathfrak{M}$ be a W^*-algebra. Let us denote by $\mathfrak{P}(\mathfrak{M})$ the set of all projections in $\mathfrak{M}$. Two projections $p_j \in \mathfrak{P}(\mathfrak{M}), j = 1, 2$ are **equivalent**: $\boldsymbol{p_1 \sim p_2}$, if $\exists u \in \mathfrak{M} : p_1 = uu^*, p_2 = u^*u$. This allows us to introduce an ordering between projections in $\mathfrak{M}$: $p \prec q \Leftrightarrow \{\exists p' \leq q \;\&\; p \sim p'\}$. If $\{p \sim q \leq p \Rightarrow p = q\}$, then p is **finite**. If $\exists q < p \; (q \neq p) \;\&\; q \sim p$, then p is **infinite**. If $0 \neq q \prec p \Rightarrow q$ is infinite, then p is **purely infinite**. A projection $p \in \mathfrak{P}(\mathfrak{M})$ is **abelian**, if $p\mathfrak{M}p := \{p \cdot x \cdot p : \; x \in \mathfrak{M}\} \subset \mathfrak{M}$ is an abelian algebra; p is **minimal**, if $p\mathfrak{M}p \sim \mathbb{C}$. We call $\mathfrak{M}$ **finite** (resp. **infinite**, resp. **purely infinite**), if its identity $e := \mathrm{id}_{\mathfrak{M}}$, as a projection, is finite (resp. infinite, resp. purely infinite). $\mathfrak{M}$ is **continuous**, if $\forall p \in \mathfrak{P}(\mathfrak{M})$ there are $q, q' \in \mathfrak{P}(\mathfrak{M}) : \; p = q + q', \; q \cdot q' = 0, \; q \sim q'$. Now we can introduce the types of W^*-algebras: $\mathfrak{M}$ **is of type I** $\Leftrightarrow$ $\forall p \in \mathfrak{P}(\mathfrak{M}) \; \exists$ abelian $q \leq p$ $\Leftrightarrow$ $\mathfrak{M}$ is isomorphic to a W^*-algebra with abelian commutant. $\mathfrak{M}$ **is of type II**, iff it is continuous and its center $\mathcal{Z}(\mathfrak{M})$ does not contain purely infinite projection. $\mathfrak{M}$ **is of type III**, iff it is purely infinite ($\Rightarrow \mathfrak{M}$ is continuous, and each nonzero $p \in \mathfrak{P}(\mathfrak{M})$ is purely infinite). Any $\mathfrak{M}$ can be written as $\mathfrak{M}_I \oplus \mathfrak{M}_{II} \oplus \mathfrak{M}_{III}$, with $\mathfrak{M}_\alpha$ of type $\alpha \in \{I, II, III\}$, cf. [68, 227, 254].

Let $\mathbb{I}_{\mathcal{H}} \in \mathfrak{M} \subset \mathcal{L}(\mathcal{H})$. Then the type (I, II, or III) of the commutant $\mathfrak{M}' =$ the type of $\mathfrak{M}$. For $\mathfrak{M}$ of type III, no pure state is normal (hence no vector–state given by $\psi \in \mathcal{H}$ is pure). Von Neumann even doubted existence of type III algebras, [190]. Now we know, that perhaps "most" of W^*-algebras occuring in QT are of type III: Many KMS states lead to type III representations, and also many algebras of observables "localized" in restricted domains of Minkowski space in relativistic QFT are of type III, cf. [43, 238, 140, 120]. Such a "wild" structure of the physical C^*-algebras is (also) a consequence of imposed symmetries. $\heartsuit$

C Notes on Unbounded Operators in Hilbert Space

Unbounded operators usually appear in QM as selfadjoint generators A of one–parameter unitary groups $t \mapsto u_t \equiv \exp(-itA)$ which are not continuous in norm topology of $\mathcal{L}(\mathcal{H})$, but they are operator–weakly continuous. Such generators seem to be unavoidable in the present–day formalism of QM, since their presence is a consequence of usage of "nontrivial" unitary representations of noncompact Lie groups G "of motions", such as Galileo, or Poincaré groups. Hence, necessary unboundedness of some operators in QM can be connected, e.g. with our common models of noncompact space–times.

Unbounded linear operators A are also characterized by their domains of definition $D(A)$ which are, as a rule, dense but not equal to the B-space, on which the operators A act. This is especially a property of unbounded symmetric operators on an infinite–dimensional Hilbert space $\mathcal{H}$, and these will be the object of our interest in this Section. The reason for a necessity of dealing with unbounded symmetric operators in some details in framework of papers on physical applications is that ignorance of several basic facts can lead to serious ambiguities in obtained results. Several methods and results presented in the following subsections can be generalized to other spaces and operators than Hilbert spaces and operators acting on them.

C.1 Unbounded operators, their domains and adjoints

Let $\mathcal{H}$ be an infinite–dimensional Hilbert space with scalar product $(x, y) = \overline{(y, x)}, \; x, y \in \mathcal{H}$, and let A be a linear mapping from a linear subset $D(A) \subset \mathcal{H}$ into $\mathcal{H}$. The linear set $D(A)$ is called the **domain of $\boldsymbol{A}$**, and the mapping A is a **linear operator on (a domain $\boldsymbol{D(A)}$ in) $\boldsymbol{\mathcal{H}}$**.

We shall usually assume (if it will be possible) that $D(A)$ is dense in $\mathcal{H}$, i.e. the norm closure $\overline{D(A)} = \mathcal{H}$. The operator A is **symmetric** if

$$(x, Ay) = (Ax, y), \quad \forall x, y \in D(A), \quad \overline{D(A)} = \mathcal{H}. \tag{C.1.1}$$

If $D(A) = \mathcal{H}$ for a symmetric A (now symmetry means $(x, Ay) \equiv (Ax, y)$), then A is bounded (Hellinger–Toeplitz). We shall introduce now a useful description of operators on $\mathcal{H}$. Let us consider the Hilbert space $\mathcal{H} \oplus \mathcal{H}$ consisting of ordered couples $(x; y),\ x, y \in \mathcal{H}$, with pointwise linear combinations $(x_1; y_1) + \lambda(x_2; y_2) \equiv (x_1 + \lambda x_2; y_1 + \lambda y_2)$, and with scalar product $((x_1; y_1), (x_2; y_2)) \equiv (x_1, x_2) + (y_1, y_2)$. For any operator $A :\ D(A) \mapsto \mathcal{H}$, let us define the **graph $\Gamma(A)$ of A** as a subset of $\mathcal{H} \oplus \mathcal{H}$:

$$\Gamma(A) := \{(x; Ax) : x \in D(A)\}. \tag{C.1.2}$$

If $\Gamma(A)$ is closed in the norm of $\mathcal{H} \oplus \mathcal{H}$, the operator A is **closed**. If the closure of the graph of an operator A is again a graph of a (uniquely defined) operator, we denote this operator $\overline{A}$, it is called the **closure of A**, and that operator A (with $\overline{\Gamma(A)} = \Gamma(\overline{A})$) is called a **closable operator**. The closure of an (closable) operator is a closed operator.

Let A be now a densely defined linear operator on $\mathcal{H}$ (such are, e.g. all bounded operators $A \in \mathcal{L}(\mathcal{H})$). Let us define, for any $x \in \mathcal{H}$, the linear functional

$$f_x^A := (x, A\cdot) : y(\in D(A)) \mapsto f_x^A(y) := (x, Ay)(\in \mathbb{C})$$

on the dense domain of A. If this linear functional is continuous (in the induced topology from the norm–topology of $\mathcal{H}$), hence bounded, it can be uniquely extended by linearity and continuity to the whole Hilbert space $\mathcal{H}$. We shall denote these extensions by the unchanged symbols. In that case $f_x^A \in \mathcal{H}^*$. The dual $\mathcal{H}^*$ of $\mathcal{H}$ is antilinearly isomorphic to $\mathcal{H}$ itself; hence, each its element $f \in \mathcal{H}^*$ is uniquely represented by an element $y_f \in \mathcal{H}$ by the identification $f(x) \equiv (y_f, x)$ (this is the **Riesz' lemma**, [218]). Let us denote, with A fixed, by $\tilde{x} \in \mathcal{H}$ the vector corresponding by the Riesz lemma to (the continuous extension of) $f_x^A \in \mathcal{H}^*$. The **adjoint A^*** of A is a linear operator on $\mathcal{H}$ with the domain

$$D(A^*) := \{x \in \mathcal{H} :\ f_x^A \in \mathcal{H}^*, \text{i.e. there is } \tilde{x} \in \mathcal{H}, (\tilde{x}, y) \equiv (x, Ay)\}, \tag{C.1.3}$$

and with the values

$$A^* x := \tilde{x}, \quad \forall x \in D(A^*).$$

It is seen that the density of $D(A)$ in $\mathcal{H}$: $\overline{D(A)} = \mathcal{H}$, is essential for possibility of definition of the adjoint operator A^*.

For $D(A) = \mathcal{H}$, this definition of adjointness is the "usual one", valid also for the bounded A's. It is easily seen that A^* is a linear operator (hence $D(A^*)$ is a linear subset of $\mathcal{H}$), but it needn't be densely defined.

The reader can check that this definition of A^* can be expressed in terms of graphs as follows: Let V be the unitary operator on $\mathcal{H} \oplus \mathcal{H}$ defined by $V : (x; y) \mapsto (-y; x)$. Then the graph of A^* is expressed as an orthogonal complement

$$\Gamma(A^*) = [V\Gamma(A)]^{\perp}, \tag{C.1.4}$$

hence it is closed. It follows that the adjoint operator is always closed.

For two operators A, B on $\mathcal{H}$, we write $A \subset B$ iff $D(A) \subset D(B)$, and $Ax = Bx, \forall x \in D(A)$, i.e. $A \subset B \Leftrightarrow \Gamma(A) \subset \Gamma(B)$. In this case, $\boldsymbol{B}$ **is an extension of** $\boldsymbol{A}$, or $\boldsymbol{A}$ **is a restriction of** $\boldsymbol{B}$. It is clear from this that a restriction of any closed operator is closable.

C.2 Symmetric operators and their (selfadjoint ?) extensions

A symmetric operator A is **selfadjoint** if $A = A^*$, i.e. if for the above defined domain (C.1.3) we have $D(A^*)$=$D(A)$. If " $i \cdot$ " is the multiplication by the imaginary unit $i \in \mathbb{C}$ in $\mathcal{H}$, and an operator A on $\mathcal{H}$ is selfadjoint, its multiple $\boldsymbol{i \cdot A}$ is called **antiselfadjoint**. Only (anti-)selfadjoint operators can determine one parameter weakly–continuous unitary groups uniquely. e.g. generators of time evolution (Hamiltonians) in QM should be selfadjoint, and not just symmetric.

It is seen from the definition (C.1.1) of a symmetric operator A that the definition is equivalent to the condition $A \subset A^*$. The **Hellinger–Toeplitz theorem** states, [218], that if a symmetric operator A is everywhere defined : $D(A)$=$\mathcal{H}$, then it is continuous: $A \in \mathcal{L}(\mathcal{H})$. This shows, that an unbounded symmetric operator cannot be defined on the whole Hilbert space $\mathcal{H}$. Most of the Hamiltonians H of particle systems in models of QM are unbounded symmetric operators, e.g. formally defined second order differential operators $\sum a_{jk}(q)\partial_j\partial_k + v(q)$ on $\mathcal{H} := L^2(\mathbb{R}^n, \mathrm{d}^n q)$, where an "initial domain" can be chosen such that H is symmetric, e.g. $D(H) := C_0^\infty(\mathbb{R}^n)$, but it is not there selfadjoint. The natural question arises, whether there is a selfadjoint extension of such a H. The answer needn't be, in a general case, positive: Besides an "ideal possibility" of existence of a unique selfadjoint extension, one can have, for some H's, infinitely many (physically distinct) possibilities, or also there could be no selfadjoint extension of some H's. The theory analyzing this situation was formulated by J. von Neumann, known sometimes as **deficiency indices theory**. Let us describe briefly its results.

Let A be symmetric, hence densely defined with densely defined adjoint A^*. Then there is defined the second adjoint A^{**} of A, and from the graph formulation (C.1.4) of definition of the adjoint operator one can see that

$$A \subset A^{**} \subset A^*, \tag{C.2.1}$$

and that $\overline{A} = A^{**}$. If $A^* = A^{**}$, the operator A is called **essentially selfadjoint**, and this is the only case, when A has a unique selfadjoint extension which is then equal to A^*=$\overline{A}$. Since any symmetric operator A is closable, we can assume, that we have a closed $A = A^{**} \subset A^*$ from the beginning. Our present problem is about classification of conditions for existence of possible selfadjoint extensions of a (generally not essentially selfadjoint) closed symmetric operator A.

Let us introduce, for a given $A = A^{**} \subset A^*$ two linear subsets $\mathcal{K}_\pm^A := \mathrm{Ker}(A^* \pm iI_\mathcal{H})$ of $D(A^*) \subset \mathcal{H}$.[103] Their dimensions $n_\pm(A)$ (finite, or not) are called the **deficiency indices of** $\boldsymbol{A}$. A closed symmetric operator A is selfadjoint iff both of its deficiency indices are equal to zero: $n_+(A) = n_-(A) = 0$, i.e. if the adjoint operator A^* has no eigenvalues equal to $\mp i$. The domain $D(A^*)$ can be endowed with the scalar product

$$(x, y)_A := (x, y) + (A^*x, A^*y),\ \forall x, y \in D(A^*), \tag{C.2.2}$$

[103] Remember that $\mathrm{Ker}(F)$ for a linear operator F is the subset of its domain on which its values vanish.

and it becomes a new Hilbert space $\mathcal{H}_A = D(A^*)$ in this way. The three linear subspaces $D(A)$, $\mathcal{K}^A_\mp$ are closed, mutually orthogonal subspaces of $\mathcal{H}_A$ providing its orthogonal decomposition. This "reorganization" of the dense subspace $D(A^*)$ of $\mathcal{H}$ allows us to find an elegant expression for all closed symmetric extensions of A; this is done with a help of the antisymmetric bilinear form σ_A on $\mathcal{H}_A$ defined by:

$$\sigma_A[x,y] := (A^*x, y) - (x, A^*y), \quad \forall x, y \in D(A^*).$$

Closed symmetric extensions A_W of A are exactly all the restrictions of A^* onto arbitrary closed linear subspaces D_W of $\mathcal{H}_A$ that contain $D(A)$, and annihilate the form σ_A:

$$\sigma_A[x,y] = 0, \quad \forall x, y \in D_W. \tag{C.2.3}$$

From these results, one is able to construct domains D_W of the symmetric extensions A_W with a help of linear isometries W (in the original Hilbert space $\mathcal{H}$) from closed linear subspaces S_W of $\mathcal{K}^A_-$ into $\mathcal{K}^A_+$, $\dim S_W \leq \min\{n_-(A), n_+(A)\}$. The domain D_W is

$$D_W := \{y + x + Wx : y \in D(A), x \in S_W\}, \tag{C.2.4}$$

and the wanted symmetric closed extension A_W of A is:

$$A_W(y + x + Wx) := Ay + ix - iWx, \quad \forall y \in D(A),\ x \in S_W. \tag{C.2.5}$$

The deficiency indices of this A_W are $n_\pm(A_W) = n_\pm(A) - \dim S_W$, if $\dim S_W < \infty$. We see that selfadjoint extensions of A exist iff it is $n_-(A) = n_+(A)$. In that case, all the selfadjoint extensions are in the easily definable bijective correspondence with all linear isometries W of $\mathcal{K}^A_-$ onto $\mathcal{K}^A_+$. Hence, the selfadjoint extensions A_W of a symmetric operator A with equal deficiency indices $n_\mp(A) =: n$ are in bijective correspondence with the elements of the Lie group $U(n)$ of all unitary operators of an n–dimensional complex Hilbert space onto itself. The action of A_W's on the corresponding domains is given by (C.2.5), where $S_W := \mathcal{K}^A_-$.

C.3 The spectral theorem. Stone's theorem

The resolvent set and spectrum of a selfadjoint unbounded operator A is defined essentially in the same way as it was done for bounded operators in Subsection B.1: The resolvent set $\rho(A) := \{\lambda \in \mathbb{C} : (\lambda\mathbb{I} - A)^{-1} \in \mathcal{L}(\mathcal{H})\}$, but the spectrum $\sigma(A) := \mathbb{C} \setminus \rho(A) \subset \mathbb{R}$ is not compact now. Also in this case, however, it is possible to associate unique projection measure E_A on the real line (supported by the spectrum $\sigma(A)$) to any selfadjoint A, and to formulate the corresponding spectral theorem expressed by the same formula, as it was done in the "bounded case", cf. Theorem B.1.3. This projection measure provides a transparent representation of the **functional calculus** also for unbounded A, cf. Subsection B.1, and Subsection B.2. It is now natural, however, to use also unbounded real Borel functions f on $\mathbb{R}$ for construction of other unbounded operators $f(A)$ from the given one, cf. Theorem B.1.3. In the case of unbounded functions $f \in \mathcal{F}(\mathbb{R}, E_A)$:=*the set of measurable, E_A–almost everywhere finite (i.e. $E_A\big(f^{-1}(\{\infty\})\big) = 0$) real functions on* $\mathbb{R}$, the domain questions arise. One has (cf. [20])

C.3.1. Proposition. *Let A be a selfadjoint (generally unbounded) operator, and let E_A be its canonical spectral (projection valued) measure. Let $f \in \mathcal{F}(\mathbb{R}, E_A)$, and let*[104]

$$f(A) = \int_{\mathbb{R}} f(\lambda) E_A(\mathrm{d}\lambda).$$

The operator $f(A)$ is selfadjoint, with the (dense) domain

$$D(f(A)) := \{x \in \mathcal{H} : \int_{\mathbb{R}} |f(\lambda)|^2 (x, E_A(\mathrm{d}\lambda)x) < \infty\}.$$

For any two functions $f, h \in \mathcal{F}(\mathbb{R}, E_A)$, and for $0 \neq \lambda \in \mathbb{R}$, one has
(i) $D(f(A) + \lambda h(A)) = D(f(A)) \cap D(h(A)) \subset D((f + \lambda h)(A))$;
(ii) $D(f(A)h(A)) = D((f \cdot h)(A)) \cap D(h(A))$. All these operators $\{f(A) : f \in \mathcal{F}(\mathbb{R}, E_A)\}$ mutually commute, i.e. their projection measures commute. ♣

Clearly, the special choice $f(\lambda) \equiv \lambda$ gives $f(A) = A$. Another (bounded, but complex) choice $f(\lambda) \equiv \exp(it\lambda)$ gives a one–parameter unitary group $U(t)$:

$$t \mapsto U(t) := \exp(itA) = \int_{\mathbb{R}} \exp(it\lambda) E_A(\mathrm{d}\lambda).$$

This group is strongly continuous, and it is also norm–continuous iff A is bounded. Different operators A give different groups $U(t)$.

The converse statement is the celebrated **Stone's theorem**, [220, 218]:

C.3.2. Theorem (Stone). *Let $t(\in \mathbb{R}) \mapsto U(t)$ be a weakly continuous one–parameter unitary group on a Hilbert space $\mathcal{H}$, i.e. $U(t_1 + t_2) \equiv U(t_1)U(t_2) \in \mathcal{U}(\mathcal{H})$ $(\forall t_1, t_2 \in \mathbb{R})$, and all the complex–valued functions $t \mapsto (x, U(t)y), \forall x, y \in \mathcal{H}$ are continuous. Then there is a unique selfadjoint operator A such, that*

$$U(t) \equiv \exp(itA).$$

(Let us note, that strong and weak continuity of the unitaries $U(t)$ are equivalent.) ♣

This theorem has a natural generalization to many–dimensional commutative locally compact groups of continuous unitary transformations of $\mathcal{H}$ known as the **SNAG theorem** (by Stone–Najmark–Ambrose–Godement), cf. [220, Chap. X.140], [103, Chap. IV], [218, Theorem VIII. 12]. The SNAG theorem can be used naturally also for construction of "macroscopic (classical) subalgebras" of large quantal systems determined by a group action, [31].

[104] We skip here details on exact meaning of the integral in the spectral representation of $f(A)$.

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