acta physica slovaca vol. 49 No. 6, 915 – 919 **December 1999**

ON THE QUANTUM VIRASORO ALGEBRA

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Received 6 August 1999, in final form 27 September 1999, accepted 20 October 1999

We give the quantum algebra structure on the q and (pq) deformed Virasoro algebra with its realization in term of the infinite matrix algebra elements.

PACS: 02.10.S, 03.65.Fd, 11.30.Pb

The quantum groups have found wide interest among theoretical physicists and mathematicians, they arose in the study of the algebraic aspects of quantum integrable systems [1, 2]. Their popularity has increased especially by their relation in rational conformal field theories and topological Chern Simon theory [3, 4, 5].

A quantum algebras(Quantized universal enveloping algebras) [6, 7] are generalization of the classical Lie algebras to which they reduce for appropriate values of the deformation parameter. From the mathematical point of view they are Hopf algebras [8], in physical terms this means that there should be a way to taking the tensor products of two representations and get a new representation. In the case of infinite dimensional algebras [9], the deformed Virasoro algebra[10-17]$U_q(vir)$ might lead to news aspects of physics with infinitely many symmetries. However the lack of a quantum algebra structure of the $U_q(vir)$ is a snag in the study of the representation theory. Recently, using the fact that the R-matrix of unit square satisfies the quantum Yang Baxter equation, a non trivial structure of $U_q(vir)$ has been given in [18].

In the present paper we will give a realization of the q and (p,q)-deformed Virasoro algebras in terms of the infinite matrix algebra $gl(\infty, \mathbb{C})$ elements and we will define a non trivial Hopf algebra structure on these deformed algebras: $U_q(Vir)$ and $U_{p,q}(Vir)$

First, let us recall that the classical Virasoro algebra (Vir) is defined as an infinite algebra generated by the generators $\{L_n, n \in Z\}$ satisfying the defining relations

$$[L_n, L_m] = (n-m)L_{n+m}, \tag{1}$$

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and that the Hopf algebra structure on the enveloping algebra of this classical algebra $U(Vir)$ is trivial. An infinite matrix realization of the classical Virasoro algebra (1) is given as follows [19]

$$L_m = \sum_{i \in Z} (i+m) E_{i,i+m} \tag{2}$$

where the (E_{ij}) are the infinite matrices having only a finite number of non vanishing diagonals, with

$$\begin{array}{l} (E_{ij})_{kl} = \delta_{ik}\delta_{jl} \\ E_{ij}E_{kl} = \delta_{jk}E_{il} \end{array} \tag{3}$$

and the following commutation relations:

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}. \tag{4}$$

A q-deformed version of the Virasoro algebra as defined by Curtright and Zachos [20] is given by the following q-deformed relation

$$q^{\mathrm{l-k}} L_l L_k - q^{\mathrm{k-l}} L_k L_l = [l-k]_q L_{k+l} \tag{5}$$

where the symbol $[x]_q$ is defined as

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}.$$

The q-deformed Virasoro algebra (5) can be realized in terms of the infinite matrices (E_{ij}) [21] as follows

$$L_m = \sum_{i \in Z} q^{m-i} [i+m]_q E_{i,i+m} \tag{6}$$

Now we define the quantum algebra structure on the q-deformed enveloping algebra of the q-deformed Virasoro algebra $U_q(Vir)$ as follows:

$$\begin{array}{l} \Delta(L_i) = L_i \otimes T_i + T_i \otimes L_i, \\ \epsilon(L_i) = 0 \\ S(L_i) = -S(T_i) L_i T_i^{-1}, \end{array} \tag{7}$$

where the operators $\{T_i,\ i \in Z\}$ are additional invertible elements of $U_q(Vir)$ which reduce to the unity operator when $q = 1$.
From the consistence of the co-structures on q-deformed Virasoro algebra generators with the basic axioms of the Hopf algebra structure (co-associativity, co-unity and antipode), we deduce that the co-structures on the additional elements are given by

$$\begin{array}{l} \Delta(T_i) = T_i \otimes T_i, \\ \epsilon(T_i) = 1, \end{array} \tag{8}$$

$$S(T_i) = T_i^{-1},$$

Further, the consistence of the Hopf algebra structure (7) with the deformed commutation relations (5), i.e. using the fact that:

$$q^{\mathrm{l-k}}\Delta(L_l)\Delta(L_k) - q^{\mathrm{k-l}}\Delta(L_k)\Delta(L_l) = [l-k]_q\Delta(L_{k+l})$$

$$\epsilon(L_lL_k) = \epsilon(L_l)\epsilon(L_k) = 0$$

$$q^{\mathrm{l-k}}S(L_k)S(L_l) - q^{\mathrm{k-l}}S(L_l)S(L_k) = [l-k]_qS(L_{k+l})$$

we obtain the following relations between the q-deformed Virasoro algebra generators and the additional elements:

$$L_kT_l = q^{2(k+1)\,l}T_lL_k\,, \tag{9}$$

where the additional generators must satisfy the following relations between them:

$$T_kT_l = T_lT_k = T_{k+l} \tag{10}$$

From the equations (9) and (6) we deduce that the relations between the infinite matrices elements and the additional ones are given by

$$E_{j,j+k}T_l = q^{2(k+1)\,l}T_lE_{j,j+k}\,, \quad \forall j \in Z \tag{11}$$

Particularly

$$E_{j,j-1}T_l = T_lE_{j,j-1}$$

and

$$E_{j,j+k}T_0 = T_0E_{j,j+k}\,, \quad \forall j \in Z.$$

Considering now the two-parameters (p,q)-deformation version of the Virasoro algebra given by [22]

$$q^{\mathrm{l-k}}L_lL_k - p^{\mathrm{k-l}}L_kL_l = [l-k]_{(q,p)}L_{k+l}, \tag{12}$$

where the symbol $[X]_{(q,p)}$ is given by:

$$[X]_{(q,p)} = \frac{q^X - p^{-X}}{q-p^{-1}}$$

which can be realized in term of the infinite matrices as

$$L_m = \sum_{i\in Z} q^{(m-i)}[i+m]_{(q,p)}E_{i,i+m} \tag{13}$$

As in the one parameter deformed Virasoro algebra, the quantum group structure on the (p,q)-deformed Virasoro algebra is given by:

$$\Delta(L_i) = L_i \otimes T_i + T_i \otimes L_i,$$

$$\epsilon(L_i) = 0, \tag{14}$$
$$S(L_i) = -S(T_i)L_iT_i^{-1},$$

In this case, using the fact that:

$$q^{\mathrm{l-k}}\Delta(L_l)\Delta(L_k) - p^{\mathrm{k-l}}\Delta(L_k)\Delta(L_l) = [l-k]_{(p,q)}\Delta(L_{k+l})$$
$$\epsilon(L_lL_k) = \epsilon(L_l)\epsilon(L_k) = 0$$
$$q^{\mathrm{l-k}}S(L_k)S(L_l) - p^{\mathrm{k-l}}S(L_l)S(L_k) = [l-k]_{(p,q)}S(L_{k+l})$$

we obtain the following relations between the (q,p)-deformed Virasoro algebra generators and the additional elements.

$$L_kT_l = (pq)^{(k+1)\,l}T_lL_k, \tag{15}$$

where the additional generators $\{T_i,\ i \in Z\}$ must satisfy the relations (10) as in the one parameter deformation case.

In this case, the relations between the infinite matrices algebra elements and the additional elements are given by:

$$E_{j,j+k}T_l = (pq)^{(k+1)\,l}T_lE_{j,j+k}, \tag{16}$$

We see from the above relations (10),(16) the additional invertible elements are centrals if $p = q^{-1}$

$$L_kT_l = T_lL_k, \qquad \forall k,l \in Z$$
$$E_{j,j+k}T_l = T_lE_{j,j+k}, \qquad \forall j,l \in Z$$

and for $p = q$ we come back to one parameter deformed Virosoro algebras case.

Summing up, we have given the realization of the q and (pq)-deformed Virasoro algebra through the $gl(\infty,\mathbb{C})$, and we have carry out on it a quantum structure by adding an extra invertible elements, in limit $q = 1$ it reduce to the unity operator.

It would be very interesting to speculate on the realization of the invertible elements T_i.

Acknowledgments M.E.F. is thankful to Prof. Randjbar-Daemi for the kindinvitation to visit the Abdus Salam iCTP. We are grateful to the referee for the useful remarks.

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