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COHERENT STATE PROPAGATOR FOR A CLASS OF QUADRATIC TIME-DEPENDENT SYSTEMS

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An exact coherent state path integral treatment of a class of quadratic time-dependent systems is proposed. The introduction of auxiliary phase space path integrals maps the original problem to that of a particle in a constant magnetic field and an external time-dependent force. The degenerate parametric amplifier with classical pumping mode is considered to illustrate the formalism.

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1 Introduction

In the past few years an extensive effort has been made to obtain exact and explicit expressions for the propagators of time-dependent systems. These systems have invoked a great interest in various branch of physics like quantum optics, quantum chemistry and quantum field theory and have been investigated using various methods such that the dynamical invariant method [1, 2, 3, 4, 5], the gauge transformation approach [6], the algebraic methods [7, 8], the integrals of motion method [9] and the path integral technique in various representation [10, 11, 12].

Our purpose in this paper is to present an alternative method for the calculation of the coherent state propagator of a class of quadratic time-dependent systems. The paper is organized as follow. In section II we recapitulate the basic properties of the coherent state representation and develop our alternative method of calculation. The linearization of the quadratic terms in the action introduces two auxiliary phase space path integrals which map the original system to that of a fictitious particle moving in two-dimensional space under the action of a constant magnetic field and an external time-dependent force. The new system is then solved with the aid of generalized canonical transformations. Finally, some concluding remarks are given in section III.

2 Coherent state propagator

To begin with let us consider the quadratic time-dependent Hamiltonian

$$H(t) = \omega(t)a^{+}a + \kappa\left(e^{-if(t)}a^{+2} + e^{if(t)}a^{2}\right) + g(t)a + g^{*}(t)a^{+}, \tag{1}$$

where $\omega(t),\ f(t)$ and $g(t)$ are arbitrary time-dependent c-numbers and a^{+}, a are respectively the creation and destruction operators. The Hamiltonian (1) is, up to the factor $\omega(t)/2$, a coherence-preserving Hamiltonian in quantum optics that belongs to the $ops(1/2, \mathbf{R})$ Lie superalgebra in the boson representation [13].

The propagator of the system described by Hamiltonian (1) in the coherent state representation is defined by

$$K(Z_f, T; Z_i, 0) = \langle Z_f \mid \mathrm{T}\exp\left(-\mathrm{i}\int_0^{\mathrm{T}} \mathrm{H}(\tau)\mathrm{d}\tau\right) \mid \mathrm{Z_i}\rangle, \tag{2}$$

where we have set $\hbar = 1$ and T is the Dyson time-ordering symbol. The coherent state $\mid Z\rangle$ is an eigenstate of the destruction operator a with eigenvalue Z at $t = 0$. The main properties of the coherent states are non-orthogonality and resolution of unity given by

$$\langle Z \mid Z'\rangle = \exp\left(-\frac{1}{2}\mid Z\mid^{2} - \frac{1}{2}\mid Z'\mid^{2} + Z^{*}Z'\right), \tag{3}$$

$$\int \frac{d^{2}Z}{\pi} \mid Z\rangle\langle Z \mid = 1. \tag{4}$$

To construct the propagator we follow the canonical method [14, 15]. Accordingly, discretizing the time interval $T = (N+1)\varepsilon$, using Trotter's formula and inserting N-times the completeness relation (4) between each pair of the evolution operator at time ε, we arrive to the following representation for the propagator

$$K(Z_f, T; Z_i, 0) = \lim_{N\longrightarrow\infty} \int \prod_{j=1}^{N} \frac{d^{2}Z_j}{\pi} \prod_{j=1}^{N+1} \langle Z_j \mid \exp(-i\varepsilon H) \mid Z_{j-1}\rangle. \tag{5}$$

Now using the Hamiltonian (1) and the non-orthogonality relation (3) we obtain

$$\begin{aligned} K(Z_f, T; Z_i, 0) = &\lim_{N\longrightarrow\infty} \int \prod_{j=1}^{N} \frac{d^{2}Z_j}{\pi} \exp\left(\frac{1}{2}\mid Z_f\mid^{2} - \frac{1}{2}\mid Z_i\mid^{2}\right) \\ &\times \prod_{j=1}^{N+1} \exp\Bigg\{ - Z_j^{*}\Delta Z_j - i\epsilon\Bigg[\omega(j)Z_j^{*}Z_{j-1} + \kappa\left(e^{-if(j)}Z_j^{*2} + e^{if(j-1)}Z_{j-1}^{2}\right) \\ &+ g^{*}(j)Z_j^{*} + g(j-1)Z_{j-1}\Bigg]\Bigg\}, \end{aligned} \tag{6}$$

where $\Delta Z_j = Z_j - Z_{j-1}$. In order to remove the time-dependence of the quadratic terms we apply the following change of variables

$$Z_j = e^{-\frac{i}{2}f(j)}\beta_j. \tag{7}$$

Substituting in (6) and keeping only terms which contribute in the limit $\epsilon \to 0$ $(N \to \infty)$ we obtain

$$K(\beta_f, T; \beta_i, 0) = \lim_{N \longrightarrow \infty} \int \prod_{j=1}^{N} \frac{d^2\beta_j}{\pi} \exp\left(\frac{1}{2} \mid \beta_f \mid^2 - \frac{1}{2} \mid \beta_i \mid^2\right)$$
$$\times \prod_{j=1}^{N+1} \exp\Bigg\{ - \beta_j^* \Delta\beta_j - i\epsilon \Bigg[\left(\omega(j) - \frac{1}{2}\Delta f(j))\right) \beta_j^* \beta_{j-1} + \kappa \left(\beta_j^{*2} + \beta_{j-1}^2\right)$$
$$+ g^*(j) e^{\frac{i}{2}f(j)} \beta_j^* + g(j-1) e^{-\frac{i}{2}f(j-1)} \beta_{j-1} \Bigg]\Bigg\}. \tag{8}$$

We proceed now to the linearization of the quadratic terms β_j^{*2} and β_j^2 via the introduction of two auxiliary phase-space path integrals. Indeed we write

$$K(\beta_f, T; \beta_i, 0) =$$
$$\exp\left(\frac{1}{2} \mid \beta_f \mid^2 - \frac{1}{2} \mid \beta_i \mid^2\right) \int_{-\infty}^{+\infty} dx_f \int_{-\infty}^{+\infty} dy_f \int \mathrm{D}p_x \mathrm{D}x|_0^{x_f} \int \mathrm{D}p_y \mathrm{D}y|_0^{y_f}$$
$$\times \prod_{j=1}^{N+1} \exp\left\{ i \left[(p_{xj}\Delta x_j + p_{yj}\Delta y_j) - \frac{\varepsilon}{2}\left(p_{xj}^2 + p_{yj}^2\right)\right]\right\} \int \prod_{j=1}^{N} \frac{d^2\beta_j}{\pi}$$
$$\times \prod_{j=1}^{N+1} \exp\Bigg\{ - \beta_j^* \Delta\beta_j - i\epsilon \Bigg[h(j) \beta_j^* \beta_{j-1} + \left(g^*(j) e^{\frac{i}{2}f(j)} + \sqrt{2\kappa}\frac{\Delta x_j}{\epsilon}\right) \beta_j^*$$
$$+ \left(g(j-1) e^{-\frac{i}{2}f(j-1)} + \sqrt{2\kappa}\frac{\Delta y_j}{\epsilon}\right) \beta_{j-1} \Bigg]\Bigg\}, \tag{9}$$

where

$$h(j) = \omega(j) - \frac{1}{2}\frac{\Delta f(j)}{\epsilon}. \tag{10}$$

Expression (9) is seen to be correct since the integration over (x, p_x) and (y, p_y) will generate in the exponent the terms $\left(-i\kappa \int_0^T \beta^{*2} dt\right)$ and $\left(-i\kappa \int_0^T \beta^2 dt\right)$ respectively. After some algebra we put (9) in the following form

$$K(\beta_f, T; \beta_i, 0) =$$

$$\exp\left(-\frac{1}{2}\left(|\beta_f|^2+|\beta_i|^2\right)\right)\int_{-\infty}^{+\infty}dx_f\int_{-\infty}^{+\infty}dy_f\int \mathrm{D}p_x\mathrm{D}x|_0^{x_f}\int \mathrm{D}p_y\mathrm{D}y|_0^{y_f}$$

$$\times\exp\left[i\int_0^T\left(p_x\dot{x}+p_y\dot{y}-\frac{1}{2}(p_x^2+p_y^2)\right)dt\right]\exp\left[-i\left(\sqrt{2\kappa}(y_1-y_i)+\epsilon g_i e^{-\frac{i}{2}f_i}\right)\beta_i\right.$$

$$\left.-i\left(\sqrt{2\kappa}(x_f-x_N)+\epsilon g_f^* e^{\frac{i}{2}f_f}\right)\beta_f^*\right]K_{xy}(\beta_f,T;\beta_i,0). \tag{11}$$

The xy-dependent coherent state propagator $K_{xy}(\beta_f,T;\beta_i,0)$ is given by

$$K_{xy}(\beta_f,T;\beta_i,0)=\lim_{N\to\infty}\int\prod_{j=1}^{j=N}\frac{d^2\beta_j}{\pi}\exp\left(-\beta^\dagger\mathbf{M}\beta+\beta^\dagger\mathbf{V}+\mathbf{W}^\dagger\beta\right), \tag{12}$$

where the matrix $\mathbf{M}$ and the vectors $\mathbf{V}$ and $\mathbf{W}$ are defined by

$$M_{ij}=\delta_{i,j}-e^{-i\varepsilon h(i)}\delta_{i,j+1},\quad \mathbf{V}=\begin{pmatrix}\beta_i-i\epsilon V(1)\\ -i\epsilon V(2)\\ \vdots\\ -i\epsilon V(N)\end{pmatrix},\quad \mathbf{W}=\begin{pmatrix}-i\epsilon W(1)\\ \vdots\\ -i\epsilon W(N-1)\\ \beta_f^*-i\epsilon W(N)\end{pmatrix}, \tag{13}$$

and

$$V(j)=\left(\sqrt{2\kappa}\frac{\Delta x_j}{\varepsilon}+F^*(j)\right),\quad W(j)=\left(\sqrt{2\kappa}\frac{\Delta y_{j+1}}{\varepsilon}+F(j)\right),$$

$$F(j)=g(j)e^{-\frac{i}{2}f(j)}. \tag{14}$$

The evaluation of the Gaussian integral in (12) leads to a complicated path integral in the phase-space (x,p_x,y,p_y) which is not exactly solvable. So in the following we just consider a restricted class of time-dependent systems verifying the condition $h(j)=0$. From (10) we obtain in the continuous limit

$$\frac{df(t)}{dt}=2\omega(t). \tag{15}$$

Then $K_{xy}(\beta_f,T;\beta_i,0)$ is given by

$$K_{xy}(\beta_f,T;\beta_i,0)=$$

$$\exp\left\{\beta_f^*\beta_i-i\sqrt{2\kappa}\left((x_N-x_i)\beta_f^*+(y_f-y_1)\beta_i\right)-i\beta_f^*\int_0^T F^*(t)dt\right.$$

$$-i\beta_i\int_0^T F(t)dt-2\kappa y_f(x_f-x_i)-\sqrt{2\kappa}\left(y_f\int_0^T F^*(t)dt-x_i\int_0^T F(t)dt\right)$$

$$+\sqrt{2\kappa}\left(\int_0^T y(t)F^*(t)dt - \int_0^T x(t)F(t)dt\right) + 2\kappa\int_0^T \dot{x}ydt$$

$$\left.-\int_0^T F(t)\int_0^t F^*(\tau)d\tau dt\right\}. \tag{16}$$

Now inserting this expression in (11) and keeping only the contributions in the limit $\epsilon \longrightarrow 0$, we obtain the following expression for the propagator

$$K(\beta_f, T; \beta_i, 0) =$$

$$\exp\Bigg[-\frac{1}{2}\left(|\beta_f|^2 + |\beta_i|^2 - 2\beta_f^*\beta_i\right) - i\sqrt{2\kappa}\left((x_f - x_i)\beta_f^* + (y_f - y_i)\beta_i\right)$$

$$-2\kappa y_f(x_f - x_i) + \sqrt{2\kappa}\left(x_i\int_0^T F(t)dt - y_f\int_0^T F^*(t)dt\right) - i\beta_i\int_0^T F(t)dt$$

$$-i\beta_f^*\int_0^T F^*(t)dt - \int_0^T F(t)\int_0^t F^*(\tau)d\tau dt\Bigg]\int_{-\infty}^{+\infty} d\mathbf{r}_f K(\mathbf{r}_f, T; 0, 0) \tag{17}$$

where $K(\mathbf{r}_f, T; \mathbf{r}_i, 0)$, $\mathbf{r} = (x, y)$, is the following phase-space path integral (assuming $\mathbf{r}_i \neq 0$)

$$K(\mathbf{r}_f, T; \mathbf{r}_i, 0) = \exp\left(\kappa(x_f y_f - x_i y_i)\right)\int \mathrm{D}x\mathrm{D}y\int \mathrm{D}p_x\mathrm{D}p_y \exp\Bigg[i\int_0^T\Bigg(p_x\dot{x} + p_y\dot{y}$$

$$-\frac{1}{2}(p_x^2 + p_y^2) + i\kappa(p_y x - p_x y) + \frac{\kappa^2}{2}(x^2 + y^2) + i\sqrt{2\kappa}\left(F(t)x - F^*(t)y\right)\Bigg)dt\Bigg]. \tag{18}$$

In deriving (18) we have made an integration by parts such that we deal with the propagator of a fictitious particle of mass unity moving in a two dimensional space under the action of a constant imaginary magnetic field plus an external time-dependent imaginary force.

Let us then consider the following generalized canonical transformation which is a generalization of the one in ref. [11]

$$(x, y, p_x, p_y, t) \to (q_1, q_2, p_1, p_2, s)\,, \tag{19}$$

such that

$$\begin{aligned}
x &= \rho(t)\left(q_1\cosh(\kappa t) - iq_2\sinh(\kappa t)\right),\\
y &= \rho(t)\left(q_2\cosh(\kappa t) + iq_1\sinh(\kappa t)\right),\\
p_x &= \frac{1}{\rho(t)}\left(p_1\cosh(\kappa t) - ip_2\sinh(\kappa t)\right) + \frac{\dot{\rho}(t)}{\rho(t)}x,\\
p_y &= \frac{1}{\rho(t)}\left(ip_1\sinh(\kappa t) + p_2\cosh(\kappa t)\right) + \frac{\dot{\rho}(t)}{\rho(t)}y,
\end{aligned} \tag{20}$$

The auxiliary function $\rho(t)$ verify the following equation

$$\ddot{\rho}(t) - \kappa^2 \rho(t) = 0, \tag{21}$$

whose particular solution is given by $\rho(t) = \cosh(\kappa t)$. It is easy to show that this transformation is generated by the function

$$F_2(x, y, p_1, p_2, t) =$$

$$\frac{1}{\rho(t)} \left[(xp_1 + yp_2) \cosh(\kappa t) - i(xp_2 - yp_1) \sinh(\kappa t) \right] + \frac{\dot{\rho}(t)}{\rho(t)} \frac{x^2 + y^2}{2}. \tag{22}$$

However the canonical transformation (20) introduces a time-dependent mass in the kinetic term that we bring to unity by the introduction of a new time defined by

$$ds = \frac{dt}{\rho^2(t)} \tag{23}$$

Following similar calculations to that in ref.[11], we show that the measure in (18) transforms to

$$\mathrm{D}x \mathrm{D}y \mathrm{D}p_x \mathrm{D}p_y = (\rho_i \rho_f)^{-1} \mathrm{D}q_1 \mathrm{D}q_2 \mathrm{D}p_1 \mathrm{D}p_2. \tag{24}$$

The propagator (18) is now the product of two propagators of particles moving in the field of time-dependent imaginary forces

$$\begin{aligned} K(\mathbf{r}_f, T; \mathbf{r}_i, 0) &= (\rho_i \rho_f)^{-1} \exp\left(\kappa (x_f y_f - x_i y_i) \right) \exp\left[\frac{i}{2} \left(\rho_f \dot{\rho}_f (q_{2f}^2 + q_{1f}^2) \right.\right. \\ &\left.\left. - \rho_i \dot{\rho}_i (q_{2i}^2 + q_{1i}^2) \right) \right] \cdot \prod_{j=1}^{2} K(q_{jf}, s_f; q_{ji}, s_i), \end{aligned} \tag{25}$$

where

$$K(q_{jf}, s_f; q_{ji}, s_i) = \int \mathrm{D}q_j \mathrm{D}p_j \exp\left[i \int_{s_i}^{s_f} \left(p_j q_j^{'} - \frac{p_j^2}{2} + \mathbf{F}_j(t(s)) q_j \right) ds \right]. \tag{26}$$

Here the prime means derivative with respect to the new time s and the forces $\mathbf{F}_j(t(s))$ are given by

$$\mathbf{F}_1 = i\sqrt{2\kappa} \cosh^3(\kappa t(s)) \left[F(t(s)) \cosh(\kappa t(s)) - iF^*(t(s)) \sinh(\kappa t(s)) \right]$$

$$\mathbf{F}_2 = \mathbf{F}_1^*. \tag{27}$$

The propagator (26) is the well-known result of Feynman and Hibbs and it is given by [10, 16]

$$K(q_{jf}, s_f; q_{ji}, s_i) = \frac{1}{\sqrt{2\pi i S}} \exp\left[i \left(\frac{1}{2S} (q_{jf} - q_{ji})^2 + q_{jf} \int_{s_i}^{s_f} ds \mathbf{F}_j(t(s)) \frac{s - s_i}{S} \right.\right.$$

$$+q_{ji}\int_{s_i}^{s_f} ds\mathbf{F}_j(t(s))\frac{s_f-s}{S}-\int_{s_i}^{s_f} ds\int_{s_i}^{s_f} d\tau\frac{\mathbf{F}_j(t(s))\mathbf{F}_j(t(\tau))}{2S}G(s,\tau)\Bigg)\Bigg], \tag{28}$$

with $S = s_f - s_i$ and

$$G(s,\tau)=\begin{Bmatrix}(s-s_i)(s_f-\tau)/S & \text{if} s\leq\tau\\ (\mathrm{s_f}-\mathrm{s})(\tau-\mathrm{s_i})/\mathrm{S} & \text{if} s\geq\tau\end{Bmatrix}. \tag{29}$$

The substitution of (28) in (25) along with $x_i = 0$ and $y_i = 0$ give

$$\begin{aligned} K(\mathbf{r}_f,T;0,0) &= \frac{\kappa}{2\pi i\sinh(\kappa T)}\exp\left[\frac{i}{2}\kappa\coth(\kappa T)\left(x_f^2+y_f^2\right)+\kappa x_f y_f\right] \\ &\quad\times\exp\left[i\sqrt{2\kappa}\left(A(T)x_f+B(T)y_f\right)-\frac{i\kappa}{2}\coth(\kappa T)\,C(T)\right], \end{aligned} \tag{30}$$

where we have defined the following quantities

$$A(t) = \sqrt{\frac{\kappa}{2}}\int_{s_i}^{s_f} ds(s-s_i)\left[\coth(\kappa T)\mathbf{F}_1(t(s))+i\mathbf{F}_2(t(s))\right], \tag{31}$$

$$B(t) = A^*(T)+i\int_0^T F^*(t)dt, \tag{32}$$

$$C(T) = \sum_{j=1}^{2}\int_{s_i}^{s_f} ds\int_{s_i}^{s_f} d\tau\mathbf{F}_j(t(s))\mathbf{F}_j(t(\tau))G(s,\tau). \tag{33}$$

According to expression (17) the propagator is given by

$$\begin{aligned} K(\beta_f,T;\beta_i,0) = \exp\Bigg[&-\frac{1}{2}\left(|\beta_f|^2+|\beta_i|^2-2\beta_f^*\beta_i\right)-i\beta_i\int_0^T F(t)dt \\ &-i\beta_f^*\int_0^T F^*(t)dt-\int_0^T F(t)\int_0^t F^*(\tau)d\tau dt-\frac{i\kappa}{2}\coth(\kappa T)\,C(T)\Bigg] \\ &\times\int_{-\infty}^{+\infty} dx_f\int_{-\infty}^{+\infty} dy_f\exp\Bigg[\frac{i}{2}\kappa\coth(\kappa T)\left(x_f^2+y_f^2\right) \\ &+i\sqrt{2\kappa}\left(A(T)-\beta_f^*\right)x_f+i\sqrt{2\kappa}\left(B(T)-\beta_i\right)y_f-\kappa x_f y_f\Bigg]. \end{aligned} \tag{34}$$

Using Fresnel integral formula to perform the integrations over x_f and y_f and going back to the old variables (Z, Z^*), we finally obtain an exact and explicit expression for the coherent state propagator

$$\begin{aligned} K(Z_f,T;Z_i,0) = \sqrt{\mathrm{sech}(2\kappa\mathrm{T})}\exp\Bigg\{&-\frac{1}{2}(|Z_f|^2+|Z_i|^2)+Z_f^*Z_ie^{-\frac{i}{2}(f(T)-f(0))} \\ &-iZ_ie^{\frac{i}{2}f(0)}\int_0^T g(t)e^{-\frac{i}{2}f(t)}dt-iZ_f^*e^{-\frac{i}{2}f(T)}\int_0^T g^*(t)e^{\frac{i}{2}f(t)}dt \end{aligned}$$

$$
\begin{aligned}
&-2\sinh^2(\kappa T)\mathrm{sech}(2\kappa \mathrm{T})\left(\mathrm{B(T)}-\mathrm{Z_i e^{\frac{i}{2}f(0)}}\right)\left(\mathrm{A(T)}-\mathrm{Z_f^* e^{-\frac{i}{2}f(T)}}\right)\\
&-\frac{i}{2}\tanh(2\kappa T)\left[\left(B(T)-Z_i e^{\frac{i}{2}f(0)}\right)^2+\left(A(T)-Z_f^* e^{-\frac{i}{2}f(T)}\right)^2\right]\Bigg\}\\
&\times\exp\Bigg[-\int_0^T g(t)e^{-\frac{i}{2}f(t)}\times\\
&\int_0^t g^*(\tau)e^{\frac{i}{2}f(\tau)}d\tau dt-\frac{i\kappa}{2}\coth(\kappa T)\,C(T)\Bigg]. \qquad (35)
\end{aligned}
$$

As an application of the present expression let us consider the example of the degenerate parametric amplifier with classical pumping mode. This system is governed by the Hamiltonian

$$H(t)=\omega a^+a+\kappa\left(e^{-2i\omega t}a^{+2}+e^{2i\omega t}a^2\right), \qquad (36)$$

such that Eq. (15) is verified. From (35) the propagator for the degenerate parametric amplifier is then given by

$$
\begin{aligned}
K(Z_f,T;Z_i,0) \;=\; &\sqrt{\mathrm{sech}(2\kappa \mathrm{T})}\exp\Bigg[-\frac{1}{2}(|Z_f|^2+|Z_i|^2)+Z_f^* Z_i e^{-i\omega T}\mathrm{sech}(2\kappa \mathrm{T})\\
&-\frac{i}{2}Z_f^{*2}e^{-2i\omega T}\tanh(2\kappa T)-\frac{i}{2}Z_i^2\tanh(2\kappa T)\Bigg]. \qquad (37)
\end{aligned}
$$

This is exactly the result obtained by Lo using the Lie algebraic approach [8, 12].

3 Conclusion

In this paper, we have presented a new derivation of the propagator of a quadratic time-dependent Hamiltonian using the path integral formalism in the coherent state representation. The method of calculation is based on the linearization of the quadratic terms in the action via the introduction of auxiliary phase-space path integrals. The resulting phase space path integration is then achieved with the aid of a generalized canonical transformation.

References

[1] H.R. Lewis, Jr., W.B. Riesenfeld: *J. Math. Phys.* **10** (1969) 1458

[2] K.H. Cho, S.P. Kim: *J. Phys.* **A27** (1984) 1387

[3] S.P. Kim: *J. Phys.* **A27** (1994) 3927

[4] I.A. Pedrosa: *Phys. Rev.* **A55** (1997) 3219

[5] C.I. Um, S.M. Shin, K.H. Yeon, T.F. George: *Phys. Rev.* **A58** (1998) 1574

[6] F.L. Li, S.J. Wang, A. Weiguny, D.L. Lin: *J. Phys.* **A27** (1994) 985

[7] Q. Wang: *J. Phys.* **A20** (1987) 5041

[8] C.F. Lo: *EuroPhys. Lett.* **24** (1993) 319

[9] V.V. Dodonov, I.A. Malkin, V.I. Man'ko: *J. Phys.* **A8** (1975) L19

[10] D.C. Khandekar, S.V. Lawande, K.V. Bhagwat: *Path Integral Methods and Their Applications*, (World Scientific, Singapore 1993)

[11] L. Chetouani, L. Guechi, T.F. Hammann: *Phys. Rev.* **A40** (1989) 1157

[12] M. Hillery, M.S. Zubairy: *Phys. Rev.* **A26** (1982) 451

[13] Xin Chen, Le-Man Kuang: *J. Phys.* **A27** (1994) L685

[14] J.R. Klauder: *Ann. Phys. (NY)* **11** (1960) 123

[15] T. Boudjedaa, A. Bounames, L. Chetouani, T.F. Hammann, Kh. Nouicer: *J. Math. Phys.* **36** (1995) 1602; *Physica Scripta* **54**, (1996) 225; *Physica Scripta* **56** (1997) 545

[16] R.P. Feynman, A.R. Hibbs: *Quantum Mechanics and Path Integrals*, (MaGraw Hill, New-York 1965).